Bank of England

Discussion Papers

Technical Series

No 19

**A multivariate GARCH in mean estimation
of the capital asset pricing model**

by

S G Hall

D K Miles

M P Taylor

November 1988

No 19

A multivariate GARCH in mean estimation
of the capital asset pricing model

by

S G Hall

D K Miles

M P Taylor

November 1988

Contents

ABSTRACT

This paper pursues the idea that the relevant distributional moments for the Capital Asset Pricing Model (CAPM) are the conditional, rather than the unconditional, covariances of returns.   Thus, asset Betas may be time-varying and random rather than constant.   The model is parameterised and estimated on monthly UK data by an application of the Autoregressive Conditional Heteroscedasticity (ARCH) formulation of Engle and its generalisation (GARCH) due to Bollerslev.   We also estimate a more general model which nests the consumption-based CAPM of Breeden.   The results suggest that perceptions of risk, ie conditional variances and covariances, are time-varying but that memories are long and agents update their perceptions relatively slowly. Despite this, measured asset Betas show substantial short-term variation. Estimates of an extended CAPM, nesting both traditional and consumption-based variants, suggest that whilst significant time-variation in risk premia is still evident, no single measure of risk appears adequate.

1 Introduction

According to the Capital Asset Pricing Model (CAPM) developed originally by Sharpe (1964) and Lintner (1965), the required excess return on a risky asset is proportional to its non-diversifiable risk, for which a sufficient statistic is the covariance of the asset return with the return on the market portfolio. In the case where this covariance is zero, the risk is completely diversifiable and the required excess return over the safe rate of return is zero. Empirical tests of the CAPM have generally yielded results unfavourable to the model in its simplest form. In particular, variables other than the covariance of the return with the market return have been found to be significant in explaining the excess return - variables ranging from the own return variance to seasonal dummies (see eg Jensen (1972) or Schwert (1983) for surveys).

Following doubts cast on such empirical studies by Fama (1977, Ch 9), Roll (1977) demonstrated that a theoretically correct implementation of the CAPM is practically impossible. This is because in the theoretical derivation of the CAPM, the market portfolio is assumed to be mean-variance efficient in the sense of Markowitz (1952). In particular, such a portfolio would involve investment in every individual existing asset with an optimal weight. In practice, researchers have generally employed stock market indices in which the weights are almost certainly sub-optimal. Many assets which one would expect to be in an optimal portfolio are excluded - eg housing, human capital. According to Roll's critique, virtually no empirical study is capable of rejecting the CAPM.

Another explanation for these empirical findings is simply that the CAPM is incorrect and that another pricing theory should be used such as the Arbitrage Pricing Theory (APT) of Ross (1976) or the consumption based CAPM due to Breeden (1979).

In this paper, we pursue a third line of inquiry into the empirical failure of the CAPM - namely that the relevant distributional moments for the CAPM are not the unconditional covariances of returns, but the conditional covariances. Thus, asset Betas - the ratio of the covariance of the asset return with the market return to the variance of the market return - may be random and time-

varying rather than constant as in the traditional formulation. It seems reasonable, since agents must form expectations concerning these moments, that conditional covariances, given information up to the time the expectation is formed, should be the relevant measures of risk for the CAPM. If, in addition, excess returns are characterised by conditional heteroscedasticity (see eg Klemkosky and Martin, 1975), time-variation in asset Betas is implied.

In what follows, we model the conditional heteroscedasticity of excess returns using the Autoregressive Conditional Heteroscedasticity (ARCH) model of Engle (1982) and its generalisation - the Generalised ARCH (GARCH) formulation - due to Bollerslev (1986). The models are estimated on monthly UK data for the period March 1975 to June 1987. To some extent our analysis is similar to the US study of Bollerslev et al (1988). However, we also test this version of the CAPM against a more general model which allows expected returns to depend on time-varying conditional covariances with consumption. We are able to reject both the traditional CAPM and the consumption based CAPM of Breeden in favour of a model which makes the risk premia depend on a composite measure of non-diversifiable risk.

2   The Capital Asset Pricing Model

The Sharpe-Lintner formulation of the CAPM states that the expected or required excess return on an asset over the safe asset return is proportional to the required excess return on the market portfolio, the factor of proportionality being equal to the asset Beta:

$$E(R_i) - r = \beta_i[E(R_m) - r] \tag{1}$$

$$\beta_i = Cov(R_i, R_m)/Var(R_m) \tag{2}$$

Where R_i and R_m are the one-period returns on the asset and the market portfolio respectively and r is the one-period safe rate of return. A natural extension of the model is to condition the moments of (1) and (2) on information available to agents at the end of period t-1 when formulating required returns during period t:

$$E(R_{it}|\Omega_{t-1}) - r_{t-1} = \beta_{it} [E(R_{mt}|\Omega_{t-1}) - r_{t-1}] \tag{3}$$

$$\beta_{it} = Cov(R_{it}, R_{mt}|\Omega_{t-1})/Var(R_{mt}|\Omega_{t-1}) \tag{4}$$

Where Ω_{t-1} is the information set at time t-1.

Apart from the problem of parameterising the conditional covariances in (4), which we shall address in the next section, (3) is non-operational because of the lack of an observed series for the required or expected market rate of return one period ahead. If, however, we assume that the 'market price of risk', λ, defined as:

$$\lambda = [E(R_{mt}|\Omega_{t-1}) - r_{t-1}]/Var(R_{mt}|\Omega_{t-1}) \tag{5}$$

is a constant, then we have:

$$E(R_{mt}|\Omega_{t-1}) - r_{t-1} = \lambda[Var(R_{mt}|\Omega_{t-1})] \tag{6}$$

so that parameterisation of the process generating the conditional second moments will be sufficient.

If the conditional variance of the market return is a sufficient statistic for the risk attached to the market portfolio, then λ simply measures how many units of excess market return are required to compensate for one unit of market risk. It is in this sense that it measures the market price of risk. Under certain regularity conditions, Merton (1980) shows that the market price of risk is the weighted harmonic mean of investors' coefficients of relative risk aversion, the weights given by each investor's share in aggregate wealth. Thus, in a simple representative agent model, λ would itself be the coefficient of relative risk aversion. Throughout this paper, we assume λ to be a constant. This considerably simplifies the analysis and seems reasonable given the relatively short data period - roughly ten years - which we examine.

Using (3), (4) and (6) we can write:

$$R_{it} = r_{t-1} + \lambda Cov(R_{it}, R_{mt}|\Omega_{t-1}) + \epsilon_{it} \tag{7}$$

$$R_{mt} = r_{t-1} + \lambda Var(R_{mt}|\Omega_{t-1}) + \upsilon_t \qquad (8)$$

Where we have replaced expected with actual values by augmenting the right hand side of each equation with the rational expectations forecast errors:

$$\epsilon_{it} = R_{it} - E(R_{it}|\Omega_{t-1}) \qquad (9)$$

$$\upsilon_t = R_{mt} - E(R_{mt}|\Omega_{t-1}) \qquad (10)$$

From (9) and (10) it is clear that the relevant conditional second moments are themselves equal to the forecast error variances and covariances:

$$Var\ (R_{mt}|\Omega_{t-1}) = E(\upsilon_t^2|\Omega_{t-1}) \qquad (11)$$

$$Cov\ (R_{it},\ R_{mt}|\Omega_{t-1})] = E(\epsilon_{it}\ \upsilon_t|\ \Omega_{t-1}) \qquad (12)$$

Equations (7), (8), (11) and (12) form the basis of the variant of the CAPM estimated in this paper.

In the next section we suggest a tractable parameterisation of the conditional second moments.

3   Parameterising the Conditional Second Moments

Engle, Lilien and Robins (1987) suggest an extension of Engle's (1982) ARCH model whereby the conditional first moment of a time series itself becomes a function of the conditional second moment, which follows an ARCH process:

$$y_t = \alpha'x_t + \delta\ h_t^2 + \epsilon_t$$

$$h_t^2 = E(\epsilon_t^2\ |\ \Omega_{t-1}) = \gamma_0 + \sum_{i=1}^{n}\ \gamma_i\epsilon_{t-i}^2 + \kappa'\ z_t$$

where x_t and z_t are vectors of weakly exogenous conditioning variables. Engle et al (1987) term this kind of model ARCH in mean or ARCH-M.

A straightforward, multivariate extension of the ARCH-M model can be applied to the CAPM formulation of the previous section as follows. We establish the following notation:

$$R_t = (R_{it} \quad -R_{mt})'$$

$$\omega_t = (\epsilon_t \quad \upsilon_t)'$$

$$\iota = (1 \quad 1)'$$

$$e = (0 \quad 1)'$$

$$H_t = \begin{bmatrix} Var(R_{it}|\Omega_{t-1}) & Cov(R_{it}, R_{mt}|\Omega_{t-1}) \\ \\ Cov(R_{it}, R_{mt}|\Omega_{t-1}) & Var(R_{mt}|\Omega_{t-1}) \end{bmatrix}$$

then the ARCH-M formulation of (7), (8), (11) and (12) is

$$R_t = r_{t-1} \, \iota + \lambda \, H_t \, e + \omega_t \tag{13}$$

$$vech \, (H_t) = A_0 + \sum_{i=1}^{n} A_i \, vech(\omega_{t-i} \, \omega_{t-i}') \tag{14}$$

Where 'vech( )' denotes the column-stacking operator of the lower triangular portion of a symmetric matrix and the A_i are (3x3) coefficient matrices, except for A_0 which is a (3x1) coefficient vector. The system (13), (14) describes a multivariate ARCH-M(n) model.

A further extension of the ARCH formulation, which imposes smoother behaviour on the conditional second moments, has been suggested by Bollerslev (1986). In Bollerslev's GARCH formulation, the conditional second moments are functions of their own lagged values as well as the squares and cross-products of lagged forecast errors. Thus, for example, the GARCH-M (n, p) formulation of the above model would consist of (13) and

$$\text{vech}(H_t) = A_0 + \sum_{i=1}^{n} A_i \, \text{vech}(\omega_{t-i} \, \omega'_{t-i}) + \sum_{i=1}^{p} B_i \, \text{vech}(H_{t-i}) \tag{15}$$

where the B_i are (3x3) coefficient matrices. Below, we present estimates of both ARCH-M and GARCH-M formulations of CAPM.

It should be noted that, although there now exists considerable empirical evidence that financial asset prices are characterised by ARCH behaviour (see eg Dickens, 1987) the ARCH parameterisation of the conditional second moments does not appear to have any immediate economic rationale. It should therefore be interpreted in much the same spirit as ARMA time series models - ie as a convenient and parsimonious representation of the behaviour of time series data.

Stacking all of the parameters of the system into a single vector

$$\mu = (\lambda, \, (A_0)', \, \text{vech}(A_1)', \, \ldots, \, \text{vech}(A_n)', \, \text{vech}(B_1)', \, \ldots, \, \text{vech}(B_p)')'$$

and applying Schweppe's (1965) prediction error decomposition form of the likelihood function, the log-likelihood for a sample of T observations (conditional on intitial values) is

$$L(\mu) = \sum_{t=1}^{T} \log |H_t(\mu)| - \sum_{t=1}^{T} \omega'_t H_t^{-1}(\mu) \, \omega_t \tag{16}$$

(where we have assumed normality of the forecast errors).

Although the analytic derivatives of (16) can be computed (see Engle, Lilien and Robins, 1987) variable-metric algorithms which employ numerical derivatives are simpler to use and easily allow changes in specification, and this approach was applied in this paper. Under the usual regularity conditions (Crowder, 1976), maximisation of (16) will yield maximum likelihood estimates with the usual properties.

As they stand, the above formulations are very general and contain a large number of parameters to be estimated, which may be problematic given the non-linearities of the system. As a first step, it seems reasonable to assume that each covariance depends only upon its own past values and surprises. Accordingly, we assume the A_i and B_i matrices to be diagonal. In addition, since inversion of any GARCH model of non-zero order implies an infinitely long memory with respect to past surprises, we limit the estimated GARCH models to first order - ie GARCH-M (1, 1). For the ARCH-M model, we set n=8 and, following Engle (1982), assume that agents linearly discount past surpises in forming expectations of future forecast error variances and, furthermore, that these discount factors are the same in each of the three ARCH equations:

$$vech(H_t) = A_0 + A_1(c_i \; vech \; (\omega_{t-i} \; \omega'_{t-i})) \tag{17}$$

$$c_i = (9 - i)/36 \qquad i = 1, 2, \ldots 8$$

or for the GARCH model:

$$vech \; (H_t) = A_0 + A_1 \; vech(\omega_{t-1} \; \omega'_{t-1}) + B_1 \; vech(H_{t-1}) \tag{18}$$

Apart from estimating the ARCH-M and GARCH-M formulations of the CAPM, we also subject the CAPM specification to a range of diagnostic tests.

4 Data

We used beginning of month data on the share prices of firms quoted on the London Stock Exchange to define one month, ex-post rates of return. Data on dividend payments per share were used to adjust the change in share prices to give the one month holding period returns. Value weighted data on the returns from companies in particular sectors were aggregated to form the returns on four portfolios. The four portfolios were for the following sectors (numbers of individual stocks in each portfolio in parentheses): Mechanical Engineering (58); Financial (122); Electrical (12); and Chemical (20).

The return on the market portfolio was taken as the percentage change in the Financial Times 500 share index (beginning month to beginning month) adjusted for dividend payments. The dividends adjustment is crude because data on the exact timing of dividend payments by each firm in the index is not easily available. Annual average (value weighted) dividend yields of the firms in the index were used to calculate estimates of the monthly dividend yields assuming that the timing of payments were uniformly distributed throughout the year. We used the one month yield on a UK Treasury Bill as our measure of the safe rate of interest.

Data was collected from March 1975 to June 1987. Allowing for lags this gives an estimation period beginning in December 1975, yielding 139 observations.

5   Empirical Results

The model to be estimated comprises the CAPM equation, (13), and the accompanying ARCH and GARCH error process, (17) or (18). We will present two sets of estimates here; first a set of estimates for the four sectors, Chemicals (C), Electricals (E), Mechanical (M) and Financial (F), based on a set of two-equation ARCH-M models. These results do not incorporate the restriction that λ, the price of risk, should be constant across all four sectors. We will then present estimates for a five-equation system, comprising all four sectors and the market equation, using a GARCH-M error process. This imposes the cross-equation parameter restriction on all sectors simultaneously and also allows a free estimate of the rate at which individuals discount observed past errors rather than imposing a fixed eight-month period, as in the case of our ARCH-M estimates.

We begin, in Table 1, by reporting the estimates of the ARCH-M model for the four sectors estimated independently. The lower half of the table presents a range of statistical tests for normality of the errors and for serial correlation. All of the models passed all the tests for normality and for absence of serial correlation in the errors. Only one of the estimated models, however, demonstrates a significant ARCH coefficient; A_1 for the mechancial sector is significantly different from zero while in the other

three sectors it is insignificant and numerically very small, which suggests that the conditional covariance matrix of the error term may not be time-varying.   Apart from this result the performance of the models is fairly statisfactory.

One possible explanation of the finding of a constant conditional covariance matrix may be that the imposed linear weighting pattern over eight months is in fact too restrictive.   In particular, this weighting pattern puts a very large emphasis on the first two or three months in the lag structure;   in fact the mean lag is only about two and a half months.   Thus, in these estimates we are effectively forcing the model to choose between a very short memory which produces a rapidly changing conditional covariance matrix, or a constant covariance matrix, but nothing in between the two.   It may be that the conditional covariance matrix is changing over time but only fairly slowly. To investigate this we need to estimate the type of lag structure in a less restrictive way.   The GARCH-M model allows us to do this in a reasonably parsimonious way.   Furthermore, there is an intuitive interpretation of the GARCH $(1,1)$ model given by (18):   expectations of conditional moments are updated in the light of new information with the weight given to the latest outcome equal to A_1.   In addition, taking advantage of the parismony of the GARCH formulation, we also estimate all four sectors simultaneously so that the price of risk, λ, is constrained to be equal across all the markets.[1]

We impose the restriction that A_1 and B_1 (in (18)) are scalars rather than vectors, but that A_0 is a full vector of fifteen constants.   In words, we assume that across a range of stocks people attach the same relative importance to past events in forming expectations about volatility of prices; ie people use similar forecasting rules for similar forecasting problems.   A common assumption made in GARCH models is that the parameter matrices are diagonal;   this is not appropriate in this model as the covariance terms are central to the analysis.

When we allow unrestricted parameter matrices, the question of positive semi-definiteness of H_t arises.   This is discussed by Baba, Engle, Kraft, and Kroner (1987), who suggest the following restriction to impose positive semi-definiteness on the model:

Table 1: ARCH-M Estimates of the Four Sectors

Sector:	Chemicals		Electrical		Mechanical		Financial	
λ	5.0	(7.8)	4.9	(7.6)	3.1	(4.4)	5.1	(7.8)
A_0 (ϵ_t, ϵ_t)	0.0033	(18.5)	0.0042	(23.8)	0.0002	(0.9)	0.0028	(16.3)
A_0 (ϵ_t, v_t)	0.0022	(8.6)	0.0007	(10.5)	0.0001	(0.3)	0.00044	(7.5)
A_0 (v_t, v_t)	0.0023	(18.6)	0.0024	(23.5)	0.0005	(6.4)	0.0022	(15.8)
A_1	0.047	(1.4)	0.0000001	(0.00)	0.985	(6.0)	0.06	(1.7)
SK_1	0.39		0.09		-0.33		-0.01	
$KURT_1$	0.35		0.18		0.24		0.66	
BJ_1	4.64		0.44		3.1		2.6	
$LB(1)_1$	0.19		1.8		1.9		0.04	
$LB(2)_1$	1.05		1.8		2.4		1.11	
$LB(4)_1$	1.54		3.6		5.5		1.98	
$LB(8)_1$	8.00		6.8		18.1		4.5	
$LB(16)_1$	12.73		18.6		21.6		8.23	
SK_2	-0.2		-0.19		-0.19		-0.2	
$KURT_2$	0.32		0.32		0.24		0.32	
BJ_2	1.63		1.68		1.34		1.63	
$LB(1)_2$	0.32		0.33		0.31		0.32	
$LB(2)_2$	0.70		0.69		0.81		0.71	
$LB(4)_2$	3.05		3.02		3.08		3.06	
$LB(8)_2$	9.62		9.67		8.34		9.59	
$LB(16)_2$	14.59		14.58		13.49		14.6	

SK - coefficient of skewness, critical value (95%) $\approx$ 0.41

Kurt - excess kurtosis, critical value (95%) $\approx$ 0.81

BJ - Bera-Jarque normality test ($\sim\chi^2(2)$ under null)

LB(N) - Ljung-Box statistic for serial correlation ($\sim\chi^2(N)$ under null)

Subscript 1,2 - sector equation and market equation respectively.

Figures in parentheses are asymptotic t-ratios.

$$H_t = A_0' A_0 + A_1' \omega_{t-1} \omega'_{t-1} A_1 + B_1' H_{t-1} B_1$$

where A_0, A_1 and B_1 are symmetric parameter matrices.   In our case A_1 and B_1 are scalars rather than vectors, so that we only need restrict A_0.

Estimation of this five-equation (four sectors and the market index) model by maximum likelihood then gave the following set of parameter estimates:

$$\hat{\lambda} = 3.24 \quad (3.1)$$
$$\hat{A}_1 = 0.027 \ (4.1)$$
$$\hat{B}_1 = 0.956 \ (119.6)$$

$$\hat{A}_0 = \begin{pmatrix} -.006 \ (7.5) & & & & \\ .001 \ (0.9) & .005 \ (4.0) & & & \\ -.0004 \ (0.1) & .0003 \ (0.4) & .005 \ (8.4) & & \\ -.0009 \ (0.5) & .000006 \ (0.0) & -.0005 \ (1.1) & .003 \ (2.6) & \\ -.001 \ (22.2) & .0002 \ (0.9) & .0003 \ (1.7) & -.02 \ (1.0) & -.002 \ (4.6) \end{pmatrix}$$

(asymptotic t-ratios in parenthesis).

Table 2 presents evidence on the properties of the model's five sets of residuals:

Table 2: Diagnostics for multivariate GARCH residuals

	Sector				
	Chemical	Electrical	Mechanical	Financial	Market
SK	0.37	0.06	-0.35	-0.18	-0.26
KURT	0.38	0.11	0.21	0.39	0.14
BJ	4.33	0.19	3.3	1.83	1.88
LB(1)	0.33	1.49	2.0	0.05	0.63
LB(2)	0.91	1.51	2.7	0.39	0.76
LB(4)	1.37	3.77	5.6	0.94	2.33
LB(8)	6.34	7.40	18.0	2.77	7.09
LB(16)	11.08	20.03	22.2	5.50	12.42

See Table 1 for definitions of diagnostic test statistics.

In this model the estimate of λ, the price of risk, is reasonably close to those in the simpler ARCH model, though somewhat lower than the average across the four sectors.   Our estimates of the price of risk are also reasonably close to those obtained on US data by French et al (1987) - who in effect estimate only the market equation, (8), allowing for univariate GARCH in mean. They are, however, at variance with the point estimate of 0.499 obtained by Bollerslev et al using US data, although their estimate does seem intuitively rather small.   The residual diagnostics show that all the error processes are consistent with normality with no serious problems of serial correlation.   We now find a highly significant role for the time varying part of the error process.   This is demonstrated by the t-statistics on A_1 and B_1.   The sum of A_1 and B_1 is close to, but beneath, unity which gives finite long-run forecasts for moments (Engle, 1987).   The mean lag is estimated to be around 22 months which confirms the suggestion made earlier that a much longer time horizon was needed in the ARCH process.

The estimates of the matrix of constants (A_0) has the interesting property that it is almost diagonal; all the diagonal elements are significant while only one of the off-diagonal elements is significant.   This suggests that the unconditional covariance matrix is actually diagonal.   This amounts to the statement that agents expect a non-zero variance on all sectors even in the long run but that they do not have a prior expectation about the long-run covariances of the system.   The model actually performs fairly well if the complete A_0 matrix is dropped although this restriction is clearly rejected by the data (likelihood ratio test statistic of 77.5 $\sim\chi^2(15)$ under the null). We decided however to investigate the possibility of a diagonal constant matrix (A_0).   The likelihood ratio test for restricting the A_0 matrix to be diagonal is 6.8 ($\sim\chi^2$ (10) under null) which is easily accepted.   So this form of the GARCH error process forms our preferred model.   The parameter estimates for this restricted model are given below:

$\hat{\lambda} = 3.25$ (3.0)

$\hat{A}_1 = 0.031$ (4.8)

$\hat{B}_1 = 0.955$ (142.0)

$$\hat{A}_0 = \begin{bmatrix} 0.004\ (5.2) & & & \\ 0.002\ (4.4) & 0.006\ (8.8) & & \\ & 0.005\ (5.5) & & \\ & & 0.005\ (8.1) \end{bmatrix}$$

and the diagnostics for the model residuals are given in Table 3.

Table 3: Diagnostics for restricted multivariate GARCH residuals

Sector	Chemical	Electrical	Mechanical	Financial	Market
SK	0.37	0.07	-0.34	-0.2	-0.26
KURT	0.38	0.11	0.22	0.4	0.15
BJ	4.37	0.2	3.3	1.8	1.86
LB(1)	0.33	1.47	2.0	0.04	0.62
LB(2)	0.92	1.50	2.7	0.4	0.75
LB(4)	1.38	3.79	5.6	0.94	2.34
LB(8)	6.34	7.42	18.0	2.78	7.11
LB(16)	11.9	20.0	22.2	5.52	12.46

See Table 1 for definitions of diagnostic test statistics.

The residual dianostics again indicate that the model is free of serial correlation and has a normally distributed error process.   The parameter estimates have changed very little from the unrestricted model.   This model shows an important element of time variation in the determination of the covariance matrix and again the mean lag is just under two years.

Figures 1-4 show the time varying Betas for each of the four sectors, that is the ratio of the expected covariance between the sector error and the market error divided by the expected variance of the market error at each point in time (equation (2)).   Figure 5 shows the expected variability of our diversified portfolio - the FT 500.   These figures show that despite the rather long lag structure the conditional covariance matrix can still change in a fairly sudden fashion, the actual range of the fluctuations is, however, not very great.   Figure 5 reveals that 1977 and 1981/82 were perceived as being particularly risky periods with the expected variances from holding the diversified portfolio around twice as high as the average for the whole period.   In the 1980s there appears to have been a significant reduction in the correlation between returns in the financial sector and for the whole portfolio.   Further, by the end of the period the risk from holding a financial sector portfolio is more easily diversified than with the other sectors.   This may reflect the recent importance of shocks specific to the financial sector - eg LDC debt developments - which have little impact on the non-financial sectors of the economy.

6   More General Measures of Risk

The CAPM model estimated above takes the covariance of the return on a
portfolio with the market return as a sufficient measure of the risk attached
to that portfolio.   This is only valid if the market portfolio is the
efficiently diversified portfolio.   In practice, it is unlikely that the
volatility of the stock market index is a good proxy for the risk of an
efficiently diversified portfolio; we noted above that many assets in
personal sector portfolios are omitted from the stock market portfolio - eg
housing and human capital.   Breeden (1979) develops a model in which a proxy
for the volatility of the (unobserved) value of the portfolio of wealth of
individuals is used.   His insight is that with efficient capital markets,
intertemporal optimising decisions by individuals would make consumption
expenditure closely linked to the present value of the total wealth of
individuals.   (Here wealth would include human capital and the expected
productivity of investments at future dates.)   With diminishing marginal
utility of consumption, individuals will want to smooth expenditure through
time and changes in real consumption will be linked to changes in the
perceived value of wealth.   Indeed, under certain strong conditions, there is
an exact relation between changes in consumption and unexpected fluctuations
in the value of portfolios of assets (see Breeden 1979 for further details).
A natural measure of the risk of particular asset then becomes the covariance
between the value of that asset and changes in consumption.

Breeden derives an equilibrium condition, analogous to our equation (7), which
we can write as:

$$E(R_{it}|\Omega_{t-1}) = r_{t-1} + \lambda \ Cov \ (R_{it}, \ \Delta c_t|\Omega_{t-1}) \tag{19}$$

where Δc_t is the change in the natural logarithm of aggregate real consumption
and λ is now a weighted average of individual coefficients of relative risk
aversion whose weights depend on consumption levels.   The major difference
between (19), which we might call a consumption based CAPM, and the model
discussed in the previous section is in the measure of risk.

We estimated a general model of returns on assets using both measures of risk.
The model we estimate can be written:

$$R_{it} = r_{t-1} + \lambda_m (E \text{ Cov}(R_{it}, R_{mt}|\Omega_{t-1})) + \lambda_c (E\text{Cov} (R_{it}, \Delta c_t|\Omega_{t-1})) + \epsilon_{it} \quad (20)$$

where λ_m is the price of market related risk (which is the same as λ above) and λ_c is the price of consumption related risk.

In practice, neither covariance term in (20) will be likely to prove a sufficient measure of risk; one can interpret the coefficients λ_m and λ_c as revealing the contribution of each proxy to predicting actual, but unobservable, perceived risk.   We model expected covariances using the same GARCH (1, 1) specification as above.   We need to append to our earlier model an equation describing the expected change in consumption so as to derive proxies for conditional covariances between consumption and asset prices. The Hall (1978) consumption function is in the spirit of the consumption CAPM and suggests a natural specification of Δc_t - a random walk with drift:

$$\Delta c_t = c_0 + u_t \quad (21)$$

c_0 = constant;
u_t is the unexpected element of the change in consumption.

We now have a six equation system with c_o and the risk parameters λ_c and λ_m to estimate.   As above, we impose equality of these risk parameters across equations.   We also adopt a similar model of the processes generating expected variances and covariances, with key parameters A_1 and B_1 as above. Writing out the model in full we have:

$$R_t = r_{t-1} \iota + \lambda_m H_t e_1 + \lambda_c H_t e_2 + w_t$$

$$\Delta c = c_0 + u_t$$

$$\text{vech}(H_t) = A_0' A_0 + A_1 \text{vech}(w_{t-1} w_{t-1}') + B_1 \text{vech}(H_{t-1})$$

notation is as before except:

$$e_1 = (0\ 0\ 0\ 0\ 1\ 0)'$$

$e_2 = (0\ 0\ 0\ 0\ 0\ 1)'$

H_t is a (6 x 6) matrix

$\tilde{w}_t = (\epsilon_{1t}\ \ \epsilon_{2t}\ \ \epsilon_{3t}\ \ \epsilon_{4t}\ \ v_t\ \ u_t)'$

$\tilde{w}_t$ is a vector of residuals with the first four elements the residuals from the asset portfolios, the fifth the total stock market residual and the final element the consumption shock.

$w_t = (\epsilon_{1t}\ \ \epsilon_{2t}\ \ \epsilon_{3t}\ \ \epsilon_{4t}\ \ v_t)'$, as before.

7   Empirical Results for the General CAPM

Consumption data is not available monthly and we were forced to use the percentage change in an index of real monthly retail sales as a proxy for Δc_t. Other data is as above.   Tables 4 and 5 show estimated parameter values and the diagnostics for the model residuals.

Table 4:   Parameter estimates for general CAPM

$$\hat{\lambda}_m = 4.23\ (3.5)$$
$$\hat{\lambda}_c = 26.54\ (3.6)$$
$$\hat{A}_1 = .083\ (5.1)$$
$$\hat{B}_1 = .390\ (5.7)$$
$$\hat{c}_0 = .0026\ (3.0)$$

	Chemicals	Electricals	Mechanical	Financial	Market	Consumption
	.34 (5.6)					
	.15 (2.7)	.31 (6.6)				
$\hat{A}_0 =$	.19 (3.0)	-.0003 (.01)	.21 (4.8)			
	.02 (0.4)	.26 (7.5)	.28 (8.6)	.12 (1.5)		
	.13 (10.1)	.14 (9.6)	.14 (9.3)	.13 (8.1)	.24 (14.2)	
	.00 (0)	.001 (.2)	-.004 (.1)	-.002 (.2)	.01 (2.2)	.11 (11.7)

Asymptotic t-ratios in parentheses.

Table 5:   Diagnostic Tests on General Model Residuals

Sector: Chemicals Electricals Mechanical Financial Market

	Chemicals	Electricals	Mechanical	Financial	Market
SK	.42	.08	-.33	-.05	-.19
KURT	.40	.17	.27	.62	.31
BJ	5.19	.33	3.14	2.41	1.56
LB(1)	.13	1.52	1.37	.03	.33
LB(2)	1.47	1.59	1.55	1.27	.79
LB(4)	2.13	3.66	5.10	2.16	3.01
LB(8)	8.42	6.87	17.74	4.41	9.53
LB(16)	13.52	19.33	21.58	8.16	14.65

See Table 1 for definitions of diagnostic test statistics.

Both λ_m and λ_c are positive and significantly different from zero; our empirical versions of both the original Sharpe-Lintner CAPM and the consumption CAPM can therefore be rejected in favour of a more general model where the two different measures of risk are both relevant.   It is noteable that in this extended CAPM model the parameters of the GARCH process are significantly different than in the earlier models.   The weight attached to past history is smaller than in earlier models whilst the matrix of constants (A_0) reveals a large number of significant off diagonal elements implying that unconditional covariances are generally well determined and significantly different from zero.   Nonetheless A_1 and B_1 are both significantly different from zero revealing clear signs of time-variation in conditional variances and covariances and hence in risk premia.   Model residuals from the portfolios of assets appear consistent with the hypotheses that they follow serially uncorrelated and normally distributed processes.

8   Conclusion

In this paper we have estimated versions of the CAPM which allow expectations of the risk characteristics of portfolios to change over time.   The results suggest that perceptions of risk are time-varying but that agents update their perceptions relatively slowly; memories are long and a fairly small weight is placed on the most recent random element in returns.   Despite this, and

because of the high volatility of one-month returns, natural measures of perceived risk - Betas - do show significant short-term variation.   Extended versions of the CAPM which allow different measures of undiversifiable risk were estimated and suggested that whilst significant time-variation in risk premia still exists, no single measure of risk appears adequate.

Notes

1   In order to interpret the GARCH CAPM model (13), (18) for this multi-equation system, simply re-interpret some of the notation.   In particular:

$R_t = (R_{1t} \ R_{2t} \ R_{3t} \ R_{4t} \ R_{mt})'$
$\omega_t = (\epsilon_{1t} \ \epsilon_{2t} \ \epsilon_{3t} \ \epsilon_{4t} \ \nu_t)'$
$\iota = (1 \ 1 \ 1 \ 1 \ 1 \ )'$
$e = (0 \ 0 \ 0 \ 0 \ 1)'$
$H_t = \text{Cov} \ (R_t | \Omega_{t-1})$

where the numbers in subscript refer to the four sectors examined.

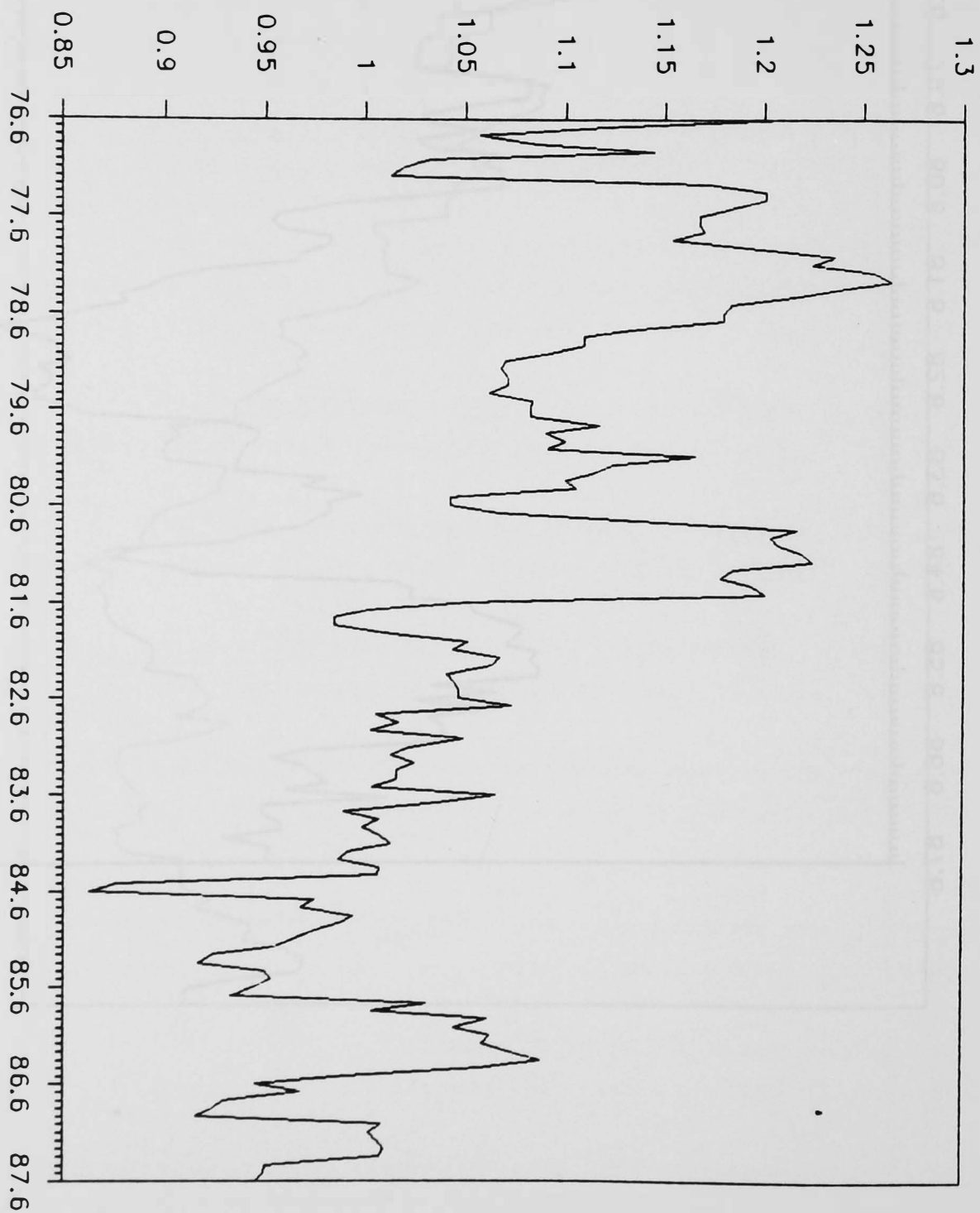FIGURE 1:CHEMICAL SECTOR 'BETA'
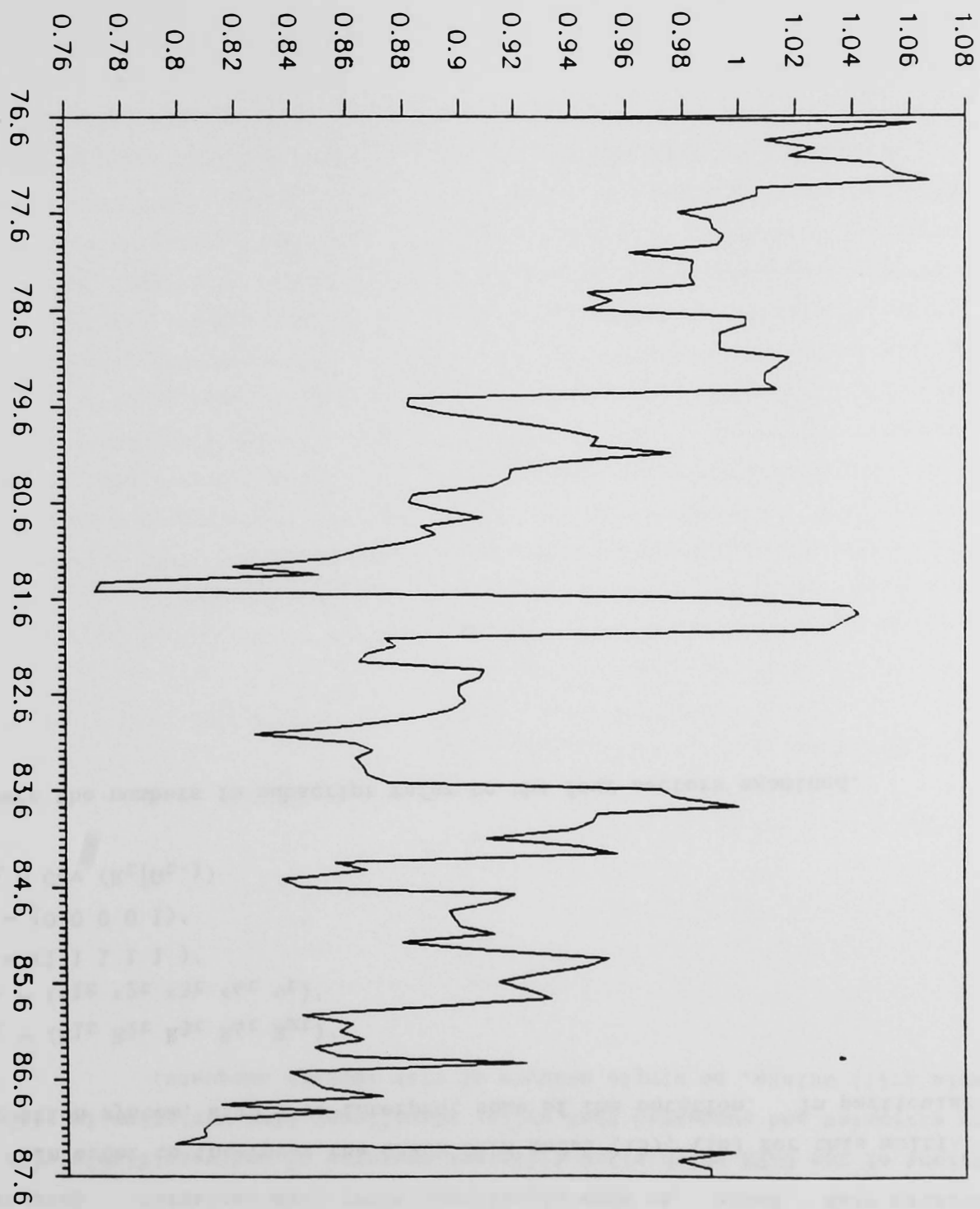
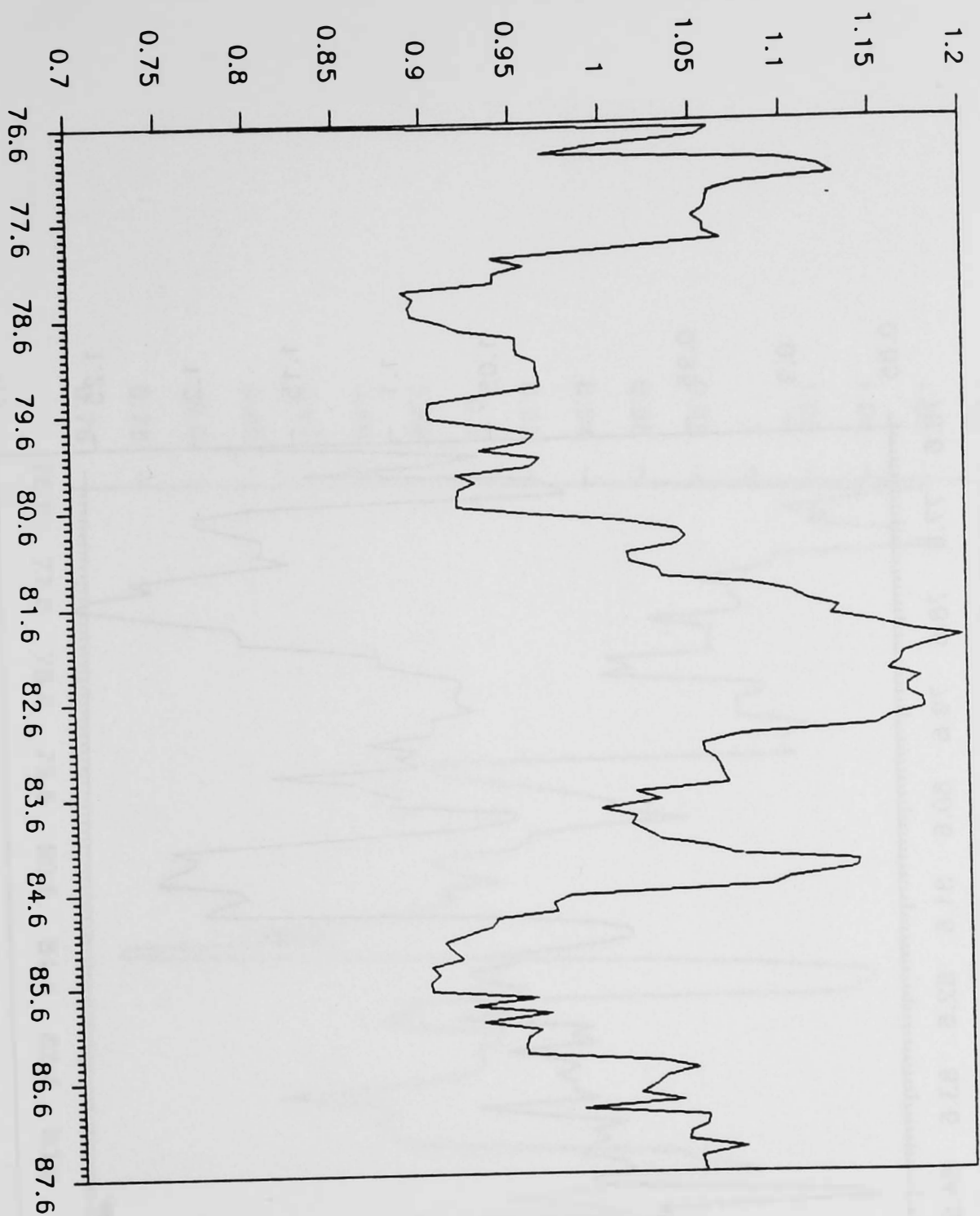FIGURE 2:ELECTRICAL SECTOR 'BETA'

FIGURE 3:MECHANICAL SECTOR 'BETA'

FIGURE 4: FINANCIAL SECTOR 'BETA'

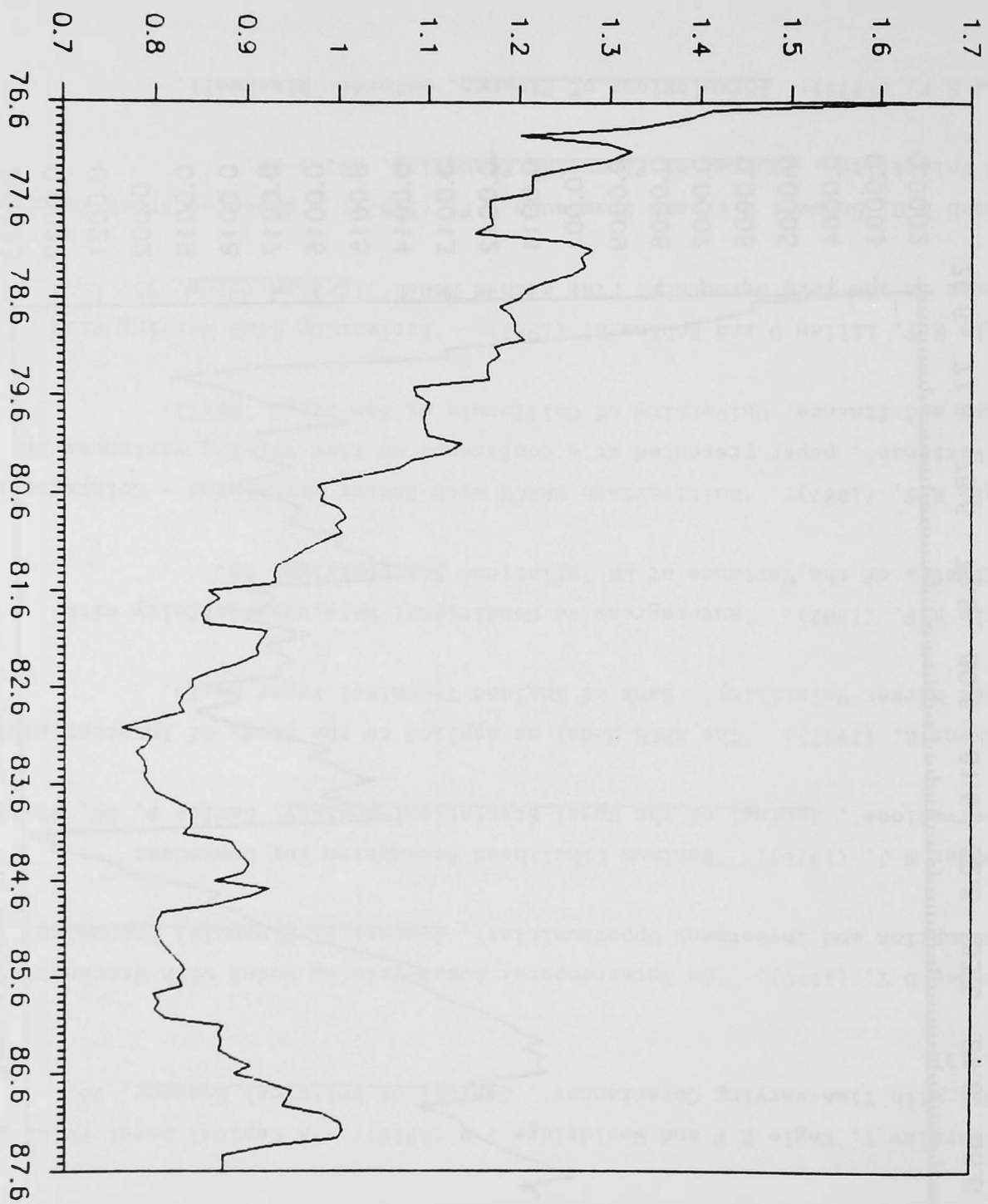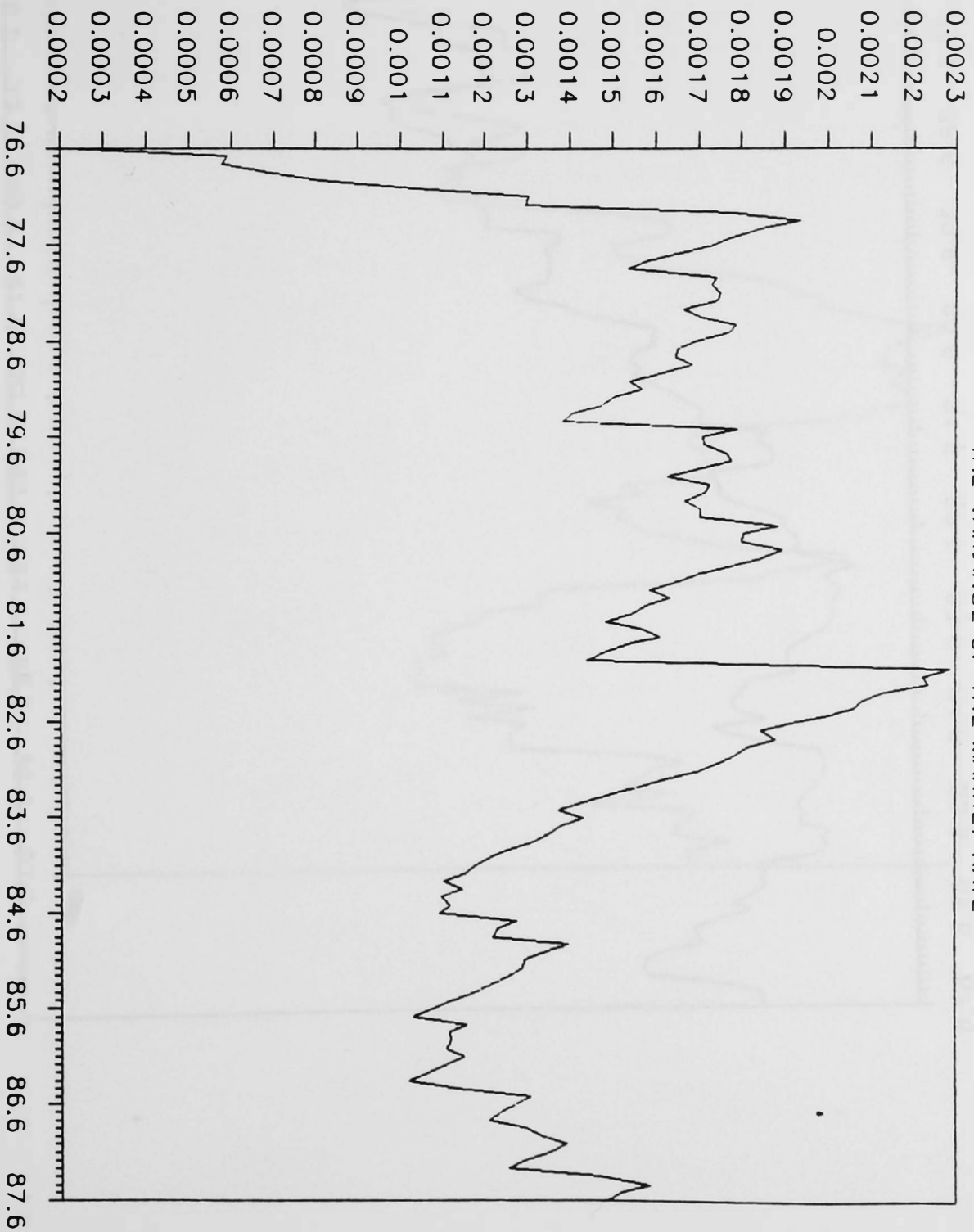# FIGURE 5

THE VARIANCE OF THE MARKET RATE

References

Baba Y, Engle R F, Kraft D F and Kroner K F (1987): 'Multivariate Simultaneous Generalised ARCH' Mimeo University of California San Diego.

Bollerslev T, (1986): 'Generalized Autoregressive Conditional Heteroskedasticity', Journal of Econometrics, 31.

Bollerslev T, Engle R F and Wooldridge J M (1988): 'A Capital Asset Pricing Model with Time-varying Covariances', Journal of Political Economy, 96, 116-131.

Breeden D T, (1979): 'An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities', Journal of Financial Economics, 7.

Crowder M J, (1976): 'Maximum Likelihood Estimation for Dependent Observations', Journal of the Royal Statistical Society, Series B, 38, 45-53.

Dickens R, (1987): 'The ARCH Model as Applied to the Study of International Asset Market Volatility', Bank of England Technical Paper No.13.

Engle R F, (1982): 'Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of UK Inflation' Econometrica, 55.

Engle R F, (1987): 'Multivariate GARCH with Factor Structures - Cointegration in Variance', paper presented at a conference on time-varying variances in money and finance, University of California at San Diego, April.

Engle R F, Lilien D and Robins R, (1987): 'Estimating Time Varying Risk Premia in the Term Structure: The Arch-M Model', Econometrica, 55.

French K R, Schwert G W, and Stambaugh R F, (1987): 'Expected Stock Returns and Volatility', Journal of Financial Economics, 19, 3-29.

Fama E F, (1977): Formulations of Finance, Oxford: Blackwell.

Hall R, (1978): 'Stochastic Implications of the Life Cycle - Permanent Income Hypothesis: Theory and Evidence' <u>Journal of Political Economy</u>, 86, 971-88.

Jensen M, (1972): 'Capital Markets: Theory and Evidence' <u>Bell Journal of Economics and Management Science</u>, 3.

Lintner J, (1965): 'The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets', <u>Review of Economics & Statistics</u>, 4.

Klemkosky R and Martin J, (1975): 'The Adjustment of Beta Forecasts', <u>Journal of Finance</u>, xxx, 1123-1128.

Merton R, (1980): 'On Estimating the Expected Return on the Market: An Exploratory Investigation', <u>Journal of Financial Economics</u>, 8.

Markowitz H, (1952): 'Portfolio Selection', <u>Journal of Finance</u>, 7.

Roll R, (1977): 'A Critique of the Asset Pricing Theory's Tests' <u>Journal of Financial Economics</u>, 4.

Ross S, (1976): 'The Arbitrage Theory of Capital Asset Pricing', <u>Journal of Economic Theory</u>, 13.

Sharpe W, (1964): 'Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk', <u>Journal of Finance</u>, 19.

Schwert G W, (1983): 'Size and Stocks Returns, and Other Empirical Regularities', <u>Journal of Financial Economics</u>, 12, 3-12.

Schweppe F, (1965): 'Evaluation of Likelihood Functions for Gaussian Signals', IEEE Transactions on Information Theory, 11, 61-70.

Bank of England Discussion Papers

	Title	Author
1-5, 8, 11-14, 16-17, 19-22	*These papers are now out of print, but photocopies can be obtained from University Microfilms International (see below).*	
6	'Real' national saving and its sectoral composition	C T Taylor A R Threadgold
7	The direction of causality between the exchange rate, prices and money	C A Enoch
9	The sterling/dollar rate in the floating rate period: the role of money, prices and intervention	I D Saville
10	Bank lending and the money supply	B J Moore A R Threadgold
15	Influences on the profitability of twenty-two industrial sectors	N P Williams
18	Two studies of commodity price behaviour: Interrelationships between commodity prices Short-run pricing behaviour in commodity markets	Mrs J L Hedges C A Enoch
23	A model of the building society sector	J B Wilcox
24	The importance of interest rates in five macroeconomic models	W W Easton
25	The effects of stamp duty on equity transactions and prices in the UK Stock Exchange	Mrs P D Jackson A T O'Donnell
26	An empirical model of company short-term financial decisions: evidence from company accounts data	Ms G Chowdhury C J Green D K Miles
27	Employment creation in the US and UK: an econometric comparison	I M Michael R A Urwin
28	An empirical model of companies' debt and dividend decisions: evidence from company accounts data	Ms G Chowdhury D M Miles
29	Expectations, risk and uncertainty in the foreign exchange market: some results based on survey data	M P Taylor
30	A model of UK non-oil ICCs' direct investment	E J Pentecost
31	What has the European Monetary System achieved?	M P Taylor M J Artis
32	The demographics of housing demand; household formations and the growth of owner-occupation	M J Dicks
33	Measuring the risk of financial institutions' portfolios: some suggestions for alternative techniques using stock prices	S G F Hall D K Miles
34	An error correction model of US consumption expenditure	I R Harnett
35	Industrial structure and dynamics of financial markets; the primary eurobond market	E P Davis

Papers presented to the Panel of Academic Consultants[a]

	Title	Author
8	International monetary arrangements the limits to planning*	P M Oppenheimer
9	Institutions in the financial markets: questions, and some tentative answers*	M V Posner
10	The arguments for and against protectionism*	M Fg Scott The Hon W A H Godley
14	The usefulness of macroeconomic models*	Prof W H Buiter T F Cripps Prof Angus Deaton Prof A P L Minford M V Posner
15	Factors underlying the recent recession*	G D N Worswick Dr A Budd
17	Why do forecasts differ?*	Prof M J Artis
19	Bank lending, monetary control and funding policy*	Prof A D Bain
20	The economics of pension arrangements*	Prof Harold Rose J A Kay
22	Monetary trends in the United Kingdom	Prof A J Brown Prof D F Hendry and N R Ericsson
23	The UK economic recovery in the 1930s	G D N Worswick P N Sedgwick Prof Michael Beenstock Dr Forrest Capie Prof Brian Griffiths
24	Employment, real wages and unemployment in the United Kingdom*	Prof J R Sargent Sir Bryan Hopkin

Technical Series

	Title	Author
1-11	*These papers are now out of print, but photocopies can be obtained from University Microfilms International (see below).*	
12	The development of expectations generating schemes which are asymptotically rational	K D Patterson
13	The arch model as applied to the study of international asset market volatility	R R Dickens
14	Modelling the UK economy in a stock-flow consistent manner	E P Davis
15	International comparison of asset market volatility: a further application of the ARCH model	R R Dickens
16	A three sector model of earning behaviour	D J Mackie
17	Integrated balance sheet and flow accounts for insurance companies and pension funds	Raymond Crossley
18	Optimal control of stochastic non-linear models	S G Hall I R Harnett M J Stephenson
19	A multivariate GARCH in mean estimation of the capital asset pricing model	S G Hall D K Miles M P Taylor

---

* These papers are no longer available from the Bank, but photocopies can be obtained from University Microfilms International, at White Swan House, Godstone, Surrey RH9 8LW.

(a) Other papers in this series were not distributed.