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Discussion Paper No 21

Deriving and testing rate of growth and higher order growth effects in dynamic economic models

by

K D Patterson

and

J Ryding

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and

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Introduction[1]

1 In a recent paper, Currie (1981) drew attention to the importance of assessing the dynamic long-run properties of estimated equations. He showed, in a number of empirical examples, that the equilibrium value of the dependent variable was sensitive to the rates of growth of the explanatory variables, and the magnitude and speed with which these effects were transmitted may be such as to be of concern when using the equations for short and medium-term forecasting. In view of this, Currie (page 706) suggested the possibility that a test for the validity of the constraint that the long-run dynamic effects are zero[2] be integrated within the modelling framework, associated particularly with Hendry and Mizon (1978) and Davidson et al (1978), which proceeds from a general unrestricted dynamic model to a specific restricted model.

2 One of the results of the analysis of this paper is that the imposition of the rate of growth (and higher order growth) constraints may so change the lag profiles, relating explanatory and dependent variables, and the short-run and steady state characteristics of an equation, as to cause considerable doubt as to whether an insignificant test statistic should lead, in the absence of any other rationalisation, to imposition of the constraint.[3]

3 In this analysis, we stay within the single equation framework of Currie (1981) and Hendry and Mizon (1978), and consider the autoregressive distributed lag model given by:[4]

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[1] The authors would like to thank Iain Saville and Nigel Jenkinson for many enlightening discussions; and Simon Babbs and Jeremy Richardson for writing the computer programs and providing many helpful comments. All are members of the Bank's Economics Division.

[2] Equivalently referred to as the rate of growth (or first order growth) constraint.

[3] We have noticed that a commonly held, if only vaguely formulated, prior is that lag profiles should be smooth, and that a lag distribution with weights changing sign is often viewed with consternation.

[4] The disturbance terms, assumed independent and identically distributed (i.i.d.) are left implicit in this and similar equations.

$$y_t = \alpha + \sum_{j=1}^{J} \beta_j x_{t-j} + \sum_{\ell=1}^{M} \gamma_1 y_{t-\ell} \qquad (1)$$

Equation 1 should be interpreted as a reduced form equation which, if necessary, has taken into account the simultaneous dependencies of a larger structural form.[1] Further explanatory variables could be added to the specification in equation 1 but this would complicate the notation without further analytical insight. All variables are assumed to be logarithms; hence, Δx_t is interpreted as the rate of growth of x at time t. It is well known that the static equilibrium version of equation 1 is given by:

$$y = \alpha / (1 - \Sigma\gamma_\ell) + x\ (\Sigma\beta_j)/(1-\Sigma\gamma_\ell)$$

where the coefficient on x is the long-run multiplier (or elasticity) for a unit change in the level of x. Currie (1981) derived the equilibrium version of equation 1 for the case of x, and hence y, growing at a constant rate. We may interpret his results as showing that in dynamic equilibrium there will be the usual static long-run multiplier and a long-run multiplier associated with the constant rate of growth of x. It is this latter multiplier that has been termed the rate of growth coefficient,[2] and we endorse the view that calculating and commenting on this coefficient should become a routine part of reporting estimated model results.

4 In Section 2, we show how Currie's result may be obtained as a special case of a more general approach which will allow the calculation of growth effects for any order of growth. We derive the implicit distributed lag of the equilibrium value of y on all orders of growth of the explanatory variable. The third section analyses the effects of imposing the constraints that the rate of growth coefficient and the second order growth coefficient are zero in a number of simple, but commonly occurring, distributed lag models.

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[1] For a further distinction and illustration of the difference between structural and reduced form steady state relationships, see Patterson and Ryding (1982).

[2] The rate of growth effect is the product of the rate of growth coefficient and the rate of growth of x.

In the fourth section, we consider the problem of testing the hypothesis that the k-th order growth coefficient is zero. This complements Currie's paper in which the choice of an appropriate test statistic was not considered. We show that the Wald principle, which only requires unrestricted estimates, allows the easy calculation of test statistics. In the fifth section, we present some empirical examples which illustrate the results of the earlier sections. The last section contains our concluding comments. In the appendix, a proof is given of the lemma and result used in the second section.

The order of growth coefficients

5 Given equation 1, which we rewrite in terms of the lag operator L as:

$$y_t = \alpha + w(L)x_t = \alpha + \sum_{i=0}^{\infty} w_i L^i x_t \qquad (2)$$

where: $w(L) = \beta(L)/\gamma(L)$

$$\beta(L) = \sum_{j=0}^{J} \beta_j L^j$$

$$\gamma(L) = 1 - \sum_{\ell=1}^{M} \gamma_\ell L^\ell$$

We are interested in deriving the lag weights in the potentially infinite distributed lag of y_t on the rates of growth of x_t, these latter being defined by $(1-L)^k x_t$, $k = 0,1.....\infty$. That is, we seek:

$$\frac{\partial y_t}{\partial[(1-L)^k x_t]} = \frac{\partial w(L)x_t}{\partial \pi_{k,x,t}}, \quad k = 0,1...\infty \text{ and } \pi_{k,x,t} \equiv (1-L)^k x_t \qquad (3)$$

6 We denote these lag weights as λ_k to distinguish them from $w(L) = \sum_{i=0}^{\infty} w_i L^i$, and write the distributed lag of y_t on the rates of growth of x_t as:

$$y_t = \alpha^* + \sum_{i=0}^{\infty} \lambda_i \pi_{i,x,t} \qquad (4)$$

7 From, for example, Dhrymes (1971, Chapter 1) we know that, provided certain convergence criteria are met, the lag weights associated with $L^k x_t$, $k = 0,1...\infty$, are obtained from:

$$w_k = \frac{\partial^k w(L)}{\partial L^k / L=0} / k!$$

8 Hence, the lag weights associated with $(1-L)^k x_t$, $k = 0,1\ldots\infty$, are given by:

$$\lambda_k = \frac{\partial^k w(L)}{\partial (1-L)^k / L=1} / k! \tag{5}$$

the k-th order derivative being evaluated at L=1 implying (1-L)=0. These lag weights are easily evaluated using the following result which is proved in Appendix 2:

$$\frac{\partial^k w(L)}{\partial (1-L)^k / L=1} = (-1)^k \frac{\partial^k w(L)}{\partial L^k / L=1} \tag{6}$$

9 The proof of this result rests on the following lemma which is stated here and proved in Appendix 2:

$$1-L^j = \sum_{i=1}^{j} {}_jC_i (-1)^{i+1} (1-L)^i \tag{7}$$

where ${}_jC_i$ is the binomial coefficient for choosing i from j.

10 It is of some interest to note that the lemma allows any x_{t-j} to be expressed in terms of x_t and a linear combination of the j i-th order differences, ie the rates of growth, of x_t.

11 For purposes of reference, Table A shows the order of growth effects for k = 0,1,2,3; higher order effects are easily obtained using equation 6.

12 The zero order growth coefficient is seen to be just the static long-run multiplier or elasticity; the first order growth coefficient has been termed the rate of growth coefficient or dynamic multiplier, and the second order growth coefficient might be called the acceleration effect.

Table A

Order of growth coefficients (a)

0	$(-1)^{0}\,(\sum_{j=0}^{J}\beta_j\,{}_jC_0)(1-\sum_{\ell=1}^{M}\gamma_\ell)^{-1}$
1	$(-1)^{1}\Big[(\sum_{\ell=1}^{M}\gamma_\ell\,{}_\ell C_1)(\sum_{j=0}^{J}\beta_j\,{}_jC_0)(1-\sum_{\ell=1}^{M}\gamma_\ell)^{-2}$
	$+\,(\sum_{j=1}^{J}\beta_j\,{}_jC_1)(1-\sum_{\ell=1}^{M}\gamma_\ell)^{-1}\Big]$
2	$(-1)^{2}\Big[(\sum_{\ell=1}^{M}\gamma_\ell\,{}_\ell C_1)^{2}\,(\sum_{j=0}^{J}\beta_j\,{}_jC_0)(1-\sum_{\ell=1}^{M}\gamma_\ell)^{-3}$
	$+\,(\sum_{\ell=1}^{M}\gamma_\ell\,{}_\ell C_1)(\sum_{j=1}^{J}\beta_j\,{}_jC_1)(1-\sum_{\ell=1}^{M}\gamma_\ell)^{-2}$
	$+\,(\sum_{\ell=2}^{M}\gamma_\ell\,{}_\ell C_2)(\sum_{j=0}^{J}\beta_j\,{}_jC_0)(1-\sum_{\ell=1}^{M}\gamma_\ell)^{-2}$
	$+\,(\sum_{j=2}^{J}\beta_j\,{}_jC_2)(1-\sum_{\ell=1}^{M}\gamma_\ell)^{-1}\Big]$
3	$(-1)^{3}\Big[\sum_{S=0}^{2}\,(\sum_{\ell=1}^{M}\gamma_\ell\,{}_\ell C_1)^{3-S}\,(\sum_{j=S}^{J}\beta_j\,{}_jC_S)(1-\sum_{\ell=1}^{M}\gamma_\ell)^{-(r+1-S)}$
	$+\sum_{S=0}^{1}\,(\sum_{\ell=3-S}^{M}\gamma_\ell\,{}_\ell C_{3-S})(\sum_{j=S}^{J}\beta_j\,{}_jC_S)(1-\sum_{\ell=1}^{M}\gamma_\ell)^{-2}$
	$+\,2(\sum_{\ell=1}^{M}\gamma_\ell\,{}_\ell C_1)(\sum_{\ell=2}^{M}\gamma_\ell\,{}_\ell C_2)(\sum_{j=0}^{J}\beta_j\,{}_jC_0)(1-\sum_{\ell=1}^{M}\gamma_\ell)^{-3}$
	$+\,(\sum_{j=3}^{J}\beta_j\,{}_jC_3)(1-\sum_{\ell=1}^{M}\gamma_\ell)^{-3}\Big]$

(a) Binomial coefficients are left explicit, though ${}_jC_0=1$, ${}_jC_1=j$, etc.

13 The result (equation 6) which is used to derive the order of growth coefficients can now be seen to motivate the interpretation that the absence of a rate of growth effect implies the mean lag in equation 1 is zero, see Currie (1981, page 707). This is so because the mean lag is given by:

$$\frac{\partial w(L)}{\partial L/L=1} \; / \; w(1) = (-1)\lambda_1/w(1)$$

where w(1) is w(L) evaluated at L=1, and λ_1 is the rate of growth coefficient. Such an interpretation could also be pursued for the higher order growth coefficients.[1] For example, if the variability of the normalised w_i sequence is defined, see Dhrymes (1971, Chapter 1), as:

$$\mathrm{Var}\,\{w_i^*\} = w''(1)/w(1) + w'(1)/w(1) - [w'(1)/w(1)]^2$$

$$= 2\lambda_2/w(1) - \lambda_1/w(1) + \lambda_1^2/w(1)^2 \qquad \text{from (5)}$$

where $\{w_i^*\} = \{w_i\}/w(1)$, and $w''(1) \equiv \partial^2 w(L)/\partial L^2$ evaluated at L=1, then the absence of rate of growth (first order), λ_1, and acceleration (second order) coefficients, λ_2, implies $\mathrm{Var}\{w_i^*\} = 0$; and if the first k growth coefficients are zero then the k-th order moment about the mean is zero. However, we should bear in mind, in such interpretations, that the signs of the lag coefficients, w_i, are not all constrained to be the same, hence they cannot be interpreted as probabilities even if normalised by dividing through by w(1).

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[1] This interpretation can lead to some difficulties. For example, if w(1) is positive, the mean lag and rate of growth coefficient are of opposite sign; thus, if the latter is positive, the mean lag is negative. In his analysis of Hendry and Mizon's (1978) demand for money functions, Currie (1981, page 709) finds in their preferred equation rate of growth coefficients on income and prices of -6.1 and -1.7 respectively, the sign of the first of these being difficult to rationalise; these coefficients imply positive mean lags of 3.8 and 1.7 quarters respectively, whereas in the 'general' version of the demand for money function [Currie (1981, page 712)], the coefficients are -17.8 and 2.5, implying mean lags of 6.2 and -7.8 quarters respectively.

Constraining the growth coefficients to be zero: an analysis of some simple models

14 We saw earlier that the mean lag is $-\lambda_1/w(1)$, where λ_1 is the rate of growth coefficient; thus, provided $w(1)^{-1}$ is not equal to zero, the mean lag will be zero if, and only if, $\lambda_1 = 0$. For the mean lag to be zero there must be at least one change of sign in the lag weights after w_0. Thus, a monotonic lag distribution, with lag weights all of the same sign, apart perhaps from w_0, can only constrain λ_1 to zero at the cost forcing the w_i, $i > 0$, to be zero. In this section, we illustrate this and other implications with some simple models.

15 We use the notation AD(M,J) to denote an autoregressive distributed lag model of order M in $\gamma(L)$ and J in $\beta(L)$.

<u>AD(1,0)</u> $$y_t = \alpha_0 + \beta_0 x_t + \gamma_1 y_{t-1}$$

This is a simple Koyck lag with $0 < \gamma_1 < 1$ if the implied lag distribution is to be monotonic. In this case $\lambda_1 = -\gamma_1\beta_0/(1-\gamma_1)^2$, which can only be constrained to zero by setting $\gamma_1 = 0$, but this implies $w_0 = \beta_0$ and $w_i = 0$, $i > 0$.

16 <u>AD(1,1)</u> $$y_t = \alpha_0 + \beta_0 x_t + \beta_1 x_{t-1} + \gamma_1 y_{t-1}$$

This is an immediate generalisation of the Koyck lag[1] which allows non-monotonicity, for $0 < \gamma_1 < 1$, in the sense that there may be a difference of sign between w_0 and the sequence $\{w_i\}$, $i > 0$. However, this is not sufficient to allow imposition of $\lambda_1 = 0$ and still obtain a non-degenerate lag distribution; it is the pattern in the w_i, $i > 0$, which is important as w_0 receives a zero weight in the

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[1] Note that the first order error correction model is AD(1,1) with the homogeneity constraint, $w(1) = (\beta_0 + \beta_1)/(1-\gamma_1) = 1$, thus this constraint serves to set the first order growth coefficient equal to the negative of the mean lag.

mean lag. Formally, the mean lag is $\frac{\gamma_1}{1-\gamma_1} + \frac{\beta_1}{\beta_0+\beta_1}$;

setting this to zero implies $\gamma_1 = -\beta_1/\beta_0$, and hence:[1]

$$w_0 = \beta_0, \ w_i = 0, \ i > 0.$$

17 AD(2,0) $\quad y_t = \alpha_0 + \beta_0 x_t + \gamma_1 y_{t-1} + \gamma_2 y_{t-2}$

In this model $\lambda_1 = -\beta_0(\gamma_1 + 2\gamma_2)(1 - \gamma_1 - \gamma_2)^{-2}$, which if set equal to zero implies $\gamma_1 = -2\gamma_2$. Chart 1[2] illustrates the effect of this constraint on the roots of the difference equation. Imposing the constraint forces the difference equation to have either a pair of complex roots, or one positive and one negative root with the negative dominant. The case of stable complex roots occurs for $-1 < \gamma_2 < 0$, and implies that the lag distribution generates damped sinusoidal oscillations. The necessary condition for stability $|\gamma_1 + \gamma_2| < 1$, implies that $-1 < \gamma_2 < \frac{1}{3}$ and $-\frac{2}{3} < \gamma_1 < 2$. For $0 < \gamma_2 < \frac{1}{3}$, we have the second case with one root of either sign and the dominant negative root inducing alternating signs in the lag weights. Charts 2 and 3 illustrate two possible lag distributions.

18 Chart 1 is also useful in analysing the effects of imposing the constraint that the second order growth coefficient, λ_2, is zero. In an AD(2,0) model:

$$\lambda_2 = \beta_0(\gamma_1 + 2\gamma_2)^2 (1 - \gamma_1 - \gamma_2)^{-3} + \beta_0\gamma_2 (1 - \gamma_1 - \gamma_2)^{-2}$$

Setting this equal to zero implies the constraint:[3]

---

[1] $w_0 = \beta_0$, $w_1 = \gamma_1 w_0 + \beta_1 = 0$, $w_2 = \gamma_1 w_1 = 0$, ... etc.

[2] Chart 1 and all other charts appear in Appendix 1.

[3] In deriving this and other constraints, we assume $1-\gamma_1 - \gamma_2 \neq 0$; that is, we rule out unit roots, since the growth coefficients are not defined in this case; however, it is convenient, particularly in presenting the diagrammatic illustrations, to treat the constraints as continuous functions of γ_1 and γ_2 and make any such necessary qualifications in the text.

$$\gamma_1^2 + 3\gamma_2^2 + 3\gamma_2\gamma_1 + \gamma_2 = 0$$

19 This constraint, which is an ellipse, is plotted in Chart 1. The ellipse has points of tangency with the complex roots boundary at $\gamma_1 = \gamma_2 = 0$, and $\gamma_1 = 2$, $\gamma_2 = -1$ and is otherwise inside the complex roots region, hence imposition of the second order constraint forces the roots of the difference equation to be a complex pair if the lag distribution is to be admissible and non-degenerate.

20 Imposition of both first and second order constraints is shown diagrammatically by the intersection of the linear constraint, $\gamma_1 = -2\gamma_2$, and the ellipse.[1] This occurs at the origin and at two unit roots, hence only one of the constraints can be imposed.

21 AD(2,1) $\quad y_t = \alpha_0 + \beta_0 x_t + \beta_1 x_{t-1} + \gamma_1 y_{t-1} + \gamma_2 y_{t-2}$

In this model the first order growth coefficient is:

$$\lambda_1 = -\left[(\beta_0 + \beta_1)(\gamma_1 + 2\gamma_2)(1 - \gamma_1 - \gamma_2)^{-2} + \beta_1(1 - \gamma_1 - \gamma_2)^{-1}\right]$$

Setting this to zero gives the constraint:

$$(\gamma_1 + 2\gamma_2)(1 + \gamma_2)^{-1} = -\beta_1/\beta_0$$

The first order constraint is now a ray from the fixed point $\gamma_1 = 2$, $\gamma_2 = -1$ with slope depending on $C \equiv -\beta_1/\beta_0$, where stability requires that C lies in the interval $-\infty < C < 1$; the ray cuts the γ_1 axis at C, hence the first order constraint in the AD(2,0) model is reproduced at $C = 0$. Note that for $0 < C < 1$ the constraint passes through the regions of two positive roots. However, in an AD(2,1) model, roots in this region are not sufficient to ensure that the lag

[1] Alternatively note that:

$$\lambda_2 = -(\gamma_1 + 2\gamma_2)(1 - \gamma_1 - \gamma_2)^{-1}\lambda_1 + \beta_0\gamma_2(1 - \gamma_1 - \gamma_2)^{-2}$$

Setting this to zero and solving simultaneously with $\lambda_1 = 0$ gives $\gamma_1 = \gamma_2 = 0$, and $\gamma_1 = 2, \gamma_2 = -1$.

distribution has weights, apart from w_0, of the same sign;[1] indeed, since the constraint ensures the mean lag is zero, we know there is at least one change of sign after w_0.

22 Some of the potential lag distributions are illustrated in Charts 5 and 6 for the regions of two positive roots - for example, B, C, E - and one positive dominant and one negative root - A, D.

23 The lag distributions for the examples with two positive roots are approximately monotonic. For example, in B the lag weights w_0 to w_8 are positive only thereafter becoming negative and small (the smallest is $w_{12} = -0.01668$ compared with the long-run response of 2.5). Examples C and E are very similar to Koyck lag distributions. In these cases, the mean lag is forced to zero by multiplying up the small opposite signed weights which would otherwise be thought negligible. However, whilst such approximate monotonicity may be thought desirable, it is necessary to note that the area of the parameter space which reproduces this property is small compared with the total area compatible with stability, necessary conditions being $0 < C < 1$, $-1 < \gamma_2 < 0$, and $\gamma_1^2 > -4\gamma_2$.

24 The constraint that the second order growth coefficient is zero in an AD(2,1) model is, after some rearrangement:

$$\gamma_1^2(1 - Z) + 3\gamma_2^2(1 - \tfrac{2}{3}Z) + 3\gamma_2\gamma_1(1 - Z) + \gamma_2(1 + 2Z) + \gamma_1 Z = 0$$

where $Z = \beta_1/(\beta_0 + \beta_1)$. The constraint is an ellipse for $-3 < Z < 1$, ie $-\infty < C < 0.75$, a parabola at $Z = -3$, and otherwise an hyperbola. [For $Z = 0$ we reproduce the second order constraint in an AD(2,0) model.] Chart 4[2] illustrates the nature of this constraint for $Z = -1$, $C = 0.5$, and $Z = 2/3$, $C = -2$. The conclusions

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[1] See Feller (1950, Chapter 11); the lag weights are functions of the roots of $\gamma(L)$ and, inter alia, $\beta(L)$ evaluated at these roots.

[2] The following are useful in sketching this constraint: when $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_2 = -(1 + 2Z)/(3 - 2Z)$; when $\gamma_2 = 0$, $\gamma_1 = 0$, $\gamma_1 = -Z/(1 - Z) = C$, and there is a fixed point at $\gamma_1 = 2$, $\gamma_2 = -1$, through which first and second order constraints pass for all admissible values of Z.

to be drawn, as far as the potential lag distributions are concerned, are broadly the same as in the case of a first order constraint.

25 The first and second order constraints are simultaneously satisfied[1] at $\gamma_1 = 2$, $\gamma_2 = -1$ and $\gamma_1 = C$, $\gamma_2 = 0$. The first set corresponds to two unit roots, and the second reduces the model from AD(2,1) to AD(1,1); however, we know that the imposition of the first order constraint in the latter model results in a degenerate lag distribution.

26 AD(2,2) $$y_t = \alpha_0 + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \gamma_1 y_{t-1} + \gamma_2 y_{t-2} \qquad (8)$$

We consider this model only briefly in view of the earlier analysis. After some simplification, the constraint that $\lambda_1 = 0$ is:

$$v \equiv (\gamma_1 + 2\gamma_2)(\beta_0 + \beta_1 + \beta_2) + (\beta_1 + 2\beta_2)(1 - \gamma_1 - \gamma_2) = 0 \qquad (9)$$

which implies the following linear relation between γ_1 and γ_2:

$$\gamma_1 = a + b\gamma_2, \quad a = -(\beta_1 + 2\beta_2)/(\beta_0 - \beta_2)$$

$$\text{and } b = -(2\beta_0 + \beta_1)/(\beta_0 - \beta_2)$$

The second order constraint is:

$$(\gamma_1 + 2\gamma_2)v + \gamma_2(\beta_0+\beta_1+\beta_2)(1-\gamma_1-\gamma_2) + \beta_2(1-\gamma_1-\gamma_2)^2 = 0 \qquad (10)$$

Solving this simultaneously with the first order constraint gives the two solutions $\gamma_1 = 2$, $\gamma_2 = -1$, and $\gamma_1 = -\beta_1/\beta_0$, $\gamma_2 = -\beta_2/\beta_0$.

---

[1] The second order constraint, after some simplification, is:
$$(\gamma_1 + 2\gamma_2)v + \gamma_2(\beta_0 + \beta_1)(1 - \gamma_1 - \gamma_2) = 0$$
where $v = (\gamma_1 + 2\gamma_2)(\beta_0 + \beta_1) + \beta_1(1 - \gamma_1 - \gamma_2)$. The first order constraint implies $v = 0$, hence, to find γ_1 and γ_2 such that both constraints are satisfied, solve:
$\gamma_2(\beta_0 + \beta_1)(1 - \gamma_1 - \gamma_2) = 0$ and $v = 0$ simultaneously; see also footnote [2] on page 15.

The first solution corresponds to two unit roots, and the second solution [1] leads to the degenerate lag distribution $w_0 = \beta_0$, $w_i = 0$, $i > 0$.

27 We conclude that a third order model, when the model order is M provided $J \leq M$, is required if both first and second order constraints are to be imposed; and, in general, in an AD(M,J) model, $J \leq M$, $M - 1$ constraints on the order of growth coefficients may be imposed simultaneously.

---

[1] $w_0 = \beta_0$, $w_1 = \gamma_1 w_0 + \beta_1 = 0$, $w_2 = \gamma_1 w_0 + \beta_2 w_0 + \beta_1 = 0$,

$w_3 = \gamma_1 w_2 + \gamma_2 w_1 = 0$, ... etc.

Testing the hypothesis that the k-th order growth effect is zero

(i) The Wald test statistic

28 Of the three principles which can be used to construct test statistics (Wald, likelihood ratio, Lagrange multiplier), the Wald principle which utilises estimates from the unrestricted model is the easiest to apply in this context. The likelihood ratio and Lagrange multiplier principles would both require estimation of the restricted model.

29 The Wald test statistic is given by:

$$W = h(\hat{\theta})' \, (H\hat{\phi}H')^{-1} \, h(\hat{\theta}) \qquad (11)$$

where a ^ indicates an unrestricted estimator; $h(\theta)$ is the vector of constraints on the coefficient vector θ expressed in the form $h(\theta) = 0$; H is the matrix (a vector if one constraint) of derivatives of $h(\theta)$ with respect to θ evaluated at $\hat{\theta}$; and $\hat{\phi}$ is a consistent estimator of the asymptotic covariance matrix of $\hat{\theta}$. Under the null hypothesis, $h(\theta) = 0$ and certain regularity conditions, W is distributed asymptotically as $\chi^2(r)$, where r is the number of restrictions, that is, the dimension of $h(\theta)$.

30 The hypothesis of most recent concern has been that there is no first order growth effect, see the examples in Currie (1981); there are occasions, however, when hypotheses may be formulated on second order growth effects, and, presumably, the magnitude of all reasonable order growth effects will give some insight into the verisimilitude of an estimated model.

31 For the Wald test to be applicable, certain regularity conditions have to be satisfied. A pertinent one, in the present context, is that the first and second partial derivatives of $h(\theta)$ should be uniformly continuous functions of θ, see assumption VIb in Wald

(1943, page 463).[1] From Table A (on page 10), we see that all the growth coefficients, including the static long-run multiplier, include terms in the inverse of $[1-\Sigma\gamma_\ell]$, and hence so will their partial derivatives which will not be continuous, switching from $+\infty$ to $-\infty$ as $\Sigma\gamma_\ell$ crosses unity from below. However if $\Sigma\gamma_\ell = 1$, the order of growth coefficients are not defined; therefore, to be consistent with the stability of the difference equation and the existence of the order of growth coefficients, we restrict the parameter space such that $|\Sigma\gamma_\ell| < 1$.

(ii) Calculation of the test statistics

32 By way of illustration, we show how the test statistic is calculated for a single rate of growth coefficient.[2]

33 The constraint is:

$$h(\theta) = \left(\sum_{\ell=1}^{M} \gamma_\ell \ell\right)\left(\sum_{j=0}^{J} \beta_j\right)(1-\Sigma\gamma_\ell)^{-2} + \left(\sum_{j=1}^{J} \beta_j j\right)(1-\Sigma\gamma_\ell)^{-1} = 0$$

the relevant partial derivatives are:

$$\frac{\partial h(\theta)}{\partial \beta_j} = \left(\sum_{\ell=1}^{M} \gamma_\ell \ell\right)(1-\Sigma\gamma_\ell)^{-2} + j(1-\Sigma\gamma_\ell)^{-1} \qquad j=0,\ldots J$$

$$\frac{\partial h(\theta)}{\partial \gamma_\ell} = 2\left(\sum_{j=0}^{J} \beta_j\right)\left(\sum_{\ell=1}^{M} \gamma_\ell \ell\right)(1-\Sigma\gamma_\ell)^{-3} + \ell\left(\sum_{j=0}^{J} \beta_j\right)(1-\Sigma\gamma_\ell)^{-2} + \left(\sum_{j=1}^{J} \beta_j j\right)(1-\Sigma\gamma_\ell)^{-2} \qquad \ell=1,\ldots M$$

---

[1] The discussion in Theil (1971, page 372) is also relevant in establishing the existence of a limiting distribution for a discrete random variable which takes the zero value with positive probability.

[2] Reference to Table A suggests that as $\Sigma\gamma_\ell \neq 1$, the hypothesis that the r-th order growth effect is zero is mathematically equivalent to that obtained by multiplying through by $[1-\Sigma\gamma_\ell]^{r+1}$, and this latter approach is implicitly adopted in Currie's (1981) formulation of the rate of growth constraint; however, such an approach is not statistically equivalent and we expect the tests based on the original formulation to be asymptotically more efficient. The difference in the case of the static multiplier is between basing the statistic alternatively on $\Sigma\beta_j = 0$ or $\Sigma\beta_j/(1 - \Sigma\gamma_\ell) = 0$.

Denote the row vector of such derivatives as h'_1 and h'_2 respectively, then the test statistic, distributed as $\chi^2(1)$ under the null hypothesis, is:

$$h(\hat{\theta})^2/\Lambda$$

with $\Lambda = h'_1 D(\hat{\beta}) h_1 + 2h'_1 C(\hat{\beta},\hat{\gamma}) h_2 + h'_2 D(\hat{\gamma}) h_2$, where D( ) and C( ) denote, respectively, the dispersion and covariance matrices. The square root of this statistic is asymptotically normally distributed[1] given that Λ is the estimated variance of the asymptotic distribution of $h(\hat{\theta})$.

[1] See, for example, Theil (1971, pages 372-8).

Empirical examples

34 In this section, we use two examples - a transactions demand for money equation and a consumption function - to illustrate and amplify the results of the previous sections.

35 The first example is a conventional autoregressive distributed lag model of the demand for money [see, for example, Hendry and Mizon (1978) and Hendry (1979)] given by:

$$m_t = \alpha_0 + \Sigma\beta_j r_{t-j} + \Sigma\tau_k p_{t-k} + \Sigma\delta_n y_{t-n} + \Sigma\gamma_\ell m_{t-\ell} + u_t \qquad (12)$$

where all variables are natural logarithms except the nominal interest rate which is entered as $r_t = \ln(1 + R_t/100)$; R is the local authority three-month rate; p is the implicit deflator for total final expenditure; y is total final expenditure at 1975 prices; and m is the M1 definition of money. All data were seasonally unadjusted for the period 1964 Q2 to 1979 Q4, and the regression included three seasonal dummies. Maximum lags were set at 4 on each variable, and for notational convenience we refer to such a model with a uniform lag length as AD(4,.). Applying a uniform reduction in the lag lengths, we found both AD(3,.) and AD(2,.) models consistent with the data; the nested F-statistics being 0.81 and 1.28 respectively, with no indication of autocorrelation in any specification using a Lagrange multiplier test for up to fourth order autocorrelation.

36 The steady state associated with the AD(2,.) model is given by:

$$m = \underset{}{-11.27} - \underset{(1.48)}{10.41r} + \underset{(18.40)}{0.9345p} + \underset{(2.12)}{2.183y} - \underset{(0.909)}{11.41}\pi_{1,p} - \underset{(0.430)}{36.22}\pi_{1,y} \qquad (13)$$

where we have assumed that second order growth in p and y is absent, and hence first order growth in r is also absent.[1] The figure in

[1] Patterson and Ryding (1982) consider in greater detail the link between orders of growth of prices and the nominal interest rate.

parentheses below a coefficient is the Wald statistic for the hypothesis that the coefficient is zero, distributed as $\chi^2(1)$ under the null. All Wald statistics use the divisor T-k rather than T, where T is the sample size and k is the total number of coefficients. The Wald statistic for the joint hypothesis that all dynamic multipliers is zero, distributed as $\chi^2(3)$ under the null, is 1.205.

37 The dynamic multipliers on prices and expenditure are negative and insignificant at conventional levels of significance. Currie (1981) similarly reports negative dynamic multipliers on prices and expenditure for Hendry and Mizon's (1978) preferred sterling M3 equation. Currie (1981, page 709) suggests a rationale for the negative dynamic multiplier on prices in terms of the substitution between money balances and real assets as inflation alters their relative returns. A negative dynamic multiplier on expenditure is, however, difficult to explain and Currie notes, of Hendry and Mizon, that they 'offer no rationale for its entry' [Currie (1981, page 709)]. In view of this we consider the model resulting from re-estimation with the constraint that the dynamic multiplier on expenditure is equal to zero. We also note[1] from Section 3 earlier that an AD(2,.) is the last stage at which this constraint can be imposed without implying a degenerate lag distribution on expenditure. expenditure.

38 Tables B and C opposite summarise the effects of imposing the constraint that the dynamic multiplier on expenditure is zero.

39 Inspection of each column of Table B reveals that the effects of imposing the constraint on one variable are pervasive; and this is so as Table C reveals, because the constraint is effected through changes in the coefficients on the lagged dependent variable as well as through the distributed lag coefficients on expenditure.

40 From Table B we see that both the static and dynamic multipliers undergo very marked changes. Indeed, the changes are such as to suggest that the constrained estimates are uniformly unacceptable. The mean lags have been extended on the price level

[1] This assumes in the notation of Section 3 that $M \geqslant J$, that is, the implied rational lag polynomial is 'proper'.

Table B

Demand for M1: AD(2,.) model with λ_1 (expenditure) = 0

Variable	Static multiplier (λ_0)		Dynamic multiplier (λ_1)		'Mean' lag		Percentage response at 10 years	
	U	C	U	C	U	C	U	C
r	-10.41	-36.4	171.21	2410.0	16.45	66.21	90.3	45.2
p	0.9345	2.785	-11.41	-152.7	12.21	54.83	93.0	54.7
y	2.183	0.233	-36.22	0	16.59	0	90.2	100

U = unconstrained.

C = constrained.

Table C

The demand for M1: effects of imposing the constraint λ_1 (expenditure) = 0

		Unconstrained	Constrained
Coefficients on lagged dependent variable	γ_1	0.6994	0.8213
	γ_2	0.2305	0.1615
	$\Sigma\gamma$	0.9299	0.9828
Sum of coefficients on expenditure	$\Sigma\delta$	0.1531	0.0040
Roots of the difference equation		0.9437	0.9852
		-0.2442	-0.3278

and the interest rate from 3 or 4 years to an implausible 13 1/2 and 16 1/2 years respectively.

41 The lag distributions on expenditure before and after imposition of the constraint are shown in Charts 7 and 8 respectively. In both cases, as the roots of the difference equations indicate, there will be non-monotonicity in the cumulated response; however, in the unconstrained lag distribution, Chart 7, this is very minor compared with the distribution in Chart 8, which is implied by the constraint that the mean lag is set to zero.

42 This example, and the analysis of Section 3, suggests that imposition of the constraint, that a dynamic multiplier be set equal to zero, has far-reaching effects which may well outweigh the problems associated with non-zero, but insignificant, dynamic multipliers. We therefore regard with some caution the suggestion in Currie (1981) that setting insignificant dynamic multipliers to zero should become an integral part of the model building strategy which considers increasingly restricted versions of a general model.

43 Rather than imposing a dynamic multiplier constraint in the AD(2,.) model for M1, we considered whether a further reduction in the lag length was acceptable, a path which would rule out setting a dynamic multiplier to zero without implying a degenerate lag distribution. The data would accept an AD(1,.) model with the maximum cumulated type 1 error for lag simplification set[1] at 7 1/2%. Having accepted this simplification, we also found the data would accept homogeneity of degree one with respect to the price level and expenditure,[2] such a restriction in turn implying the error correction model of Davidson et al (1978). The resulting steady state is given by:

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[1] The results would have been unchanged had a 5% or 10% significance level been used at each stage; using 2 1/2% the Sidak inequality actually implies an upper limit on the cumulated type 1 error of $1 - (1 - \alpha)^n$, equal to 7.3% in this case.

[2] The relevant F-statistics being 1.69 for a reduction from AD(2,.) to AD(1,.), and 1.27 for the restriction of AD(1,.) to an error correction mechanism.

$$m - p - y = \underset{}{0.375} - \underset{(10.31)}{4.66r} - \underset{(17.36)}{10.69\,\pi_{1,p}} - \underset{(26.14)}{9.87\,\pi_{1,y}} \qquad (14)$$

44 The static multiplier on the interest rate and the dynamic multipliers on the price level and expenditure are all now significant. Moreover, the lag distributions indicate a monotonic approach to a new equilibrium; Chart 9 gives the lag distribution on expenditure to illustrate this point. The mean lags are given by the negative of the dynamic multipliers for the price level and expenditure. This example would suggest that ignoring insignificant dynamic multipliers may lead to a preferred model which has more acceptable properties than one which has imposed constraints on the dynamic multipliers.

45 The second example is a consumption function of the kind estimated by Davidson et al (1978). The dependent variable is consumers' expenditure on non-durable goods and services, C; the explanatory variables are real personal disposable income, Y, and the implicit price deflator for non-durable goods and services, P. The data are annual for the period 1950-1975, with real variables defined in 1975 prices, and all variables entered in natural logarithms indicated by lower case letters. Estimation[1] of an AD(2,.) model resulted in the following steady state:

$$c = \underset{(283.9)}{1.011y} + \underset{(0.009)}{0.0293p} - \underset{(0.362)}{7.471\,\pi_{1,y}} - \underset{(0.089)}{2.737\,\pi_{1,p}} \qquad (15)$$

The number in parentheses below a coefficient is the Wald statistic for the hypothesis that the coefficient is zero. The level of prices has almost no effect on the level of consumption, whereas consumption is negatively related to the rate of inflation, $\pi_{1,p}$, and the rate of growth of real disposable income, $\pi_{1,y}$. However, neither of these dynamic multipliers are individually or jointly significantly different from zero; the Wald statistic for the joint hypothesis that both dynamic multipliers are zero being 4.213.

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[1] Because of its illustrative nature, we used OLS in this example; Davidson et al (1978, page 689) found little change in their coefficient estimates and goodness of fit using instrumental variables in a quarterly model.

46 At this stage, on the basis of the above test statistics, a researcher might reasonably drop prices altogether and constrain the dynamic multiplier on real income to be zero. An alternative, which is illustrated here, is to keep the inflation effect in view of its importance in accounting for observed changes in the savings ratio, and to constrain to zero the insignificant dynamic multiplier on real income. Imposing this constraint results in the roots of the estimated difference equation being forced outside the unit circle. The sum of the coefficients on the lagged dependent variables is 0.9180 for the unconstrained case and 1.0067 with the imposition of the rate of growth constraint. Examining the lag distributions and summary lag statistics for the unconstrained and constrained estimation reveals a marked change not only for expenditure but also for prices; the cumulative responses for prices being given in Charts 10 and 11 respectively, with the latter clearly illustrating the explosive nature of the estimated model.[1]

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[1] As an alternative modelling strategy, in this example we also estimated the basic error correction model, that is, an AD(1,.) model with a long-run marginal propensity to consume of unity. In error correction form this was:

$$\underset{}{\Delta c_t} = \underset{(10.00)}{0.534}\, \Delta y_t - \underset{(3.36)}{0.130}\, \Delta p_t - \underset{(4.02)}{0.101}\, (c - y)_{t-1}$$

('t' statistics in parentheses.)

The 't' statistic on Δp indicates that prices do have a role to play in the consumption function. The steady state is given by:

$$c = y - \underset{(28.90)}{4.61}\, \pi_{1,y} - \underset{(46.45)}{1.29}\, \pi_{1,p}$$

(Wald statistics in parentheses.)

Conclusions

47 We have shown that, with a suitable interpretation, a rational distributed lag function W(L) is the generating function of the lag weights associated with the growth rates of an explanatory variable, x. The equilibrium value of the dependent variable, y, can then easily be written as a function of the growth rates of x up to any desired order.[1]

48 The analysis of a number of simple, but commonly occurring, autoregressive distributed lag models illustrated the effects of constraining either, or both, the first and second order growth coefficients to zero. We noted that imposing either of these constraints was likely to have a substantial impact on the profile of the lag distribution, in particular on whether or not such a profile was smooth.

49 The test statistics for hypotheses on the order of growth coefficients can be simply calculated using the Wald principle, which only requires estimation of the unrestricted model. Two empirical examples were used to illustrate the calculation of these test statistics and the effects of imposing a rate of growth constraint. In these examples, it was clear that the imposition of such a constraint had a very marked effect not only on the variable which was being constrained, but also, because the constraint implied changes to the coefficients on the lagged dependent variables, on the other explanatory variables in the equation. For example, in the demand for M1 equation, the imposition of the rate of growth constraint on expenditure led to marked changes in the constrained steady state, and a severe attenuation in the lag distributions on the interest rate and price level. In the consumption expenditure example, the roots of the estimated difference equation were forced outside the unit circle by the imposition of a rate of growth

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[1] In evaluating the importance of order of growth effects, it is necessary to bear in mind that the effect on equilibrium will depend on the order of growth coefficient and the magnitude of growth.

constraint. We consider, therefore, that it is important to treat with some caution the suggestion to integrate the testing and imposition of rate of growth constraint(s) within the framework of nesting increasingly restricted specifications of a general model.

Appendix 1

Charts

Chart 1

The constraints $\lambda_1 = 0$, $\lambda_2 = 0$ in an AD(2,0) model

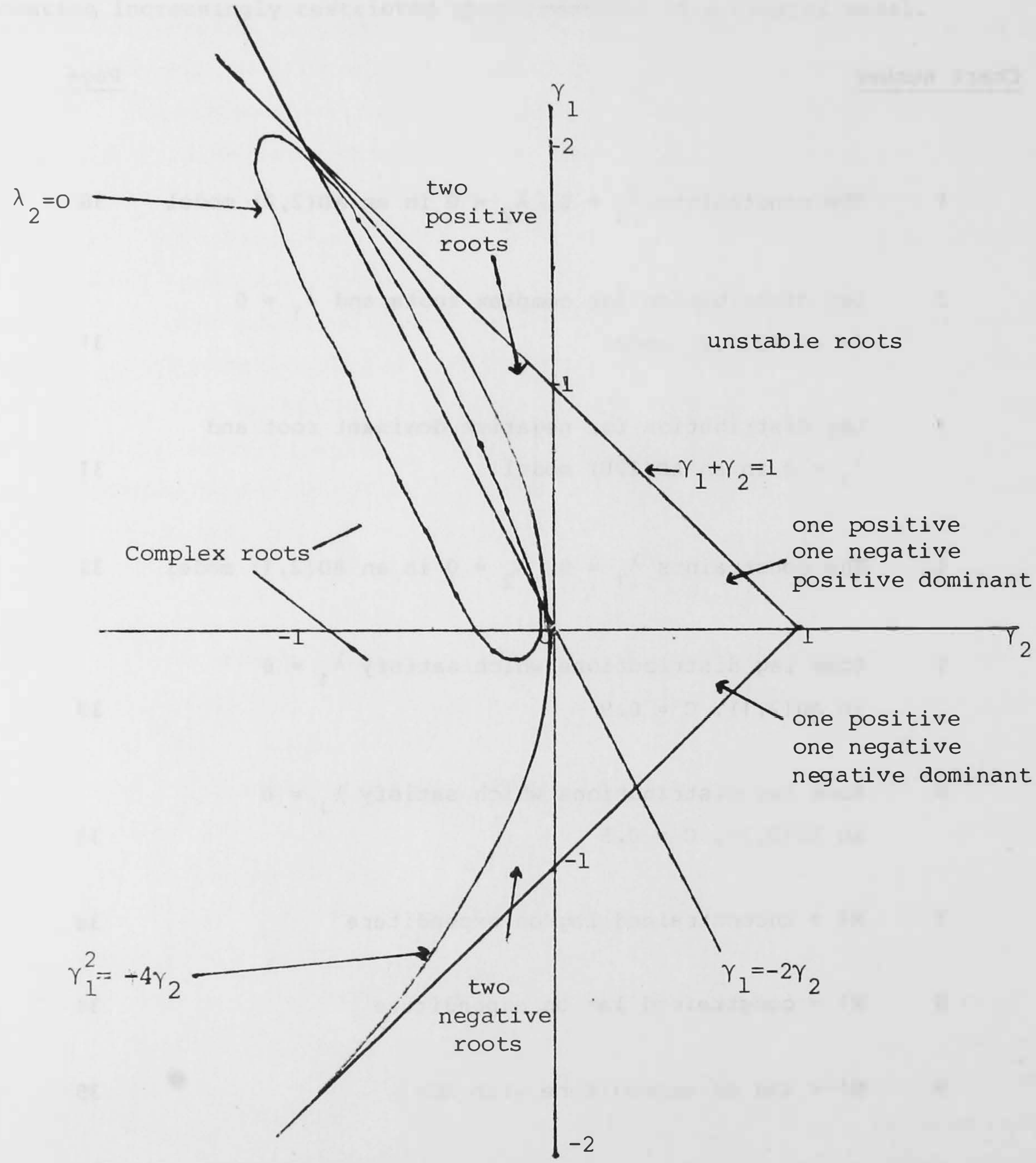


Chart 2

Lag distribution for complex roots and $\lambda_1 = 0$ in AD(2,0) model

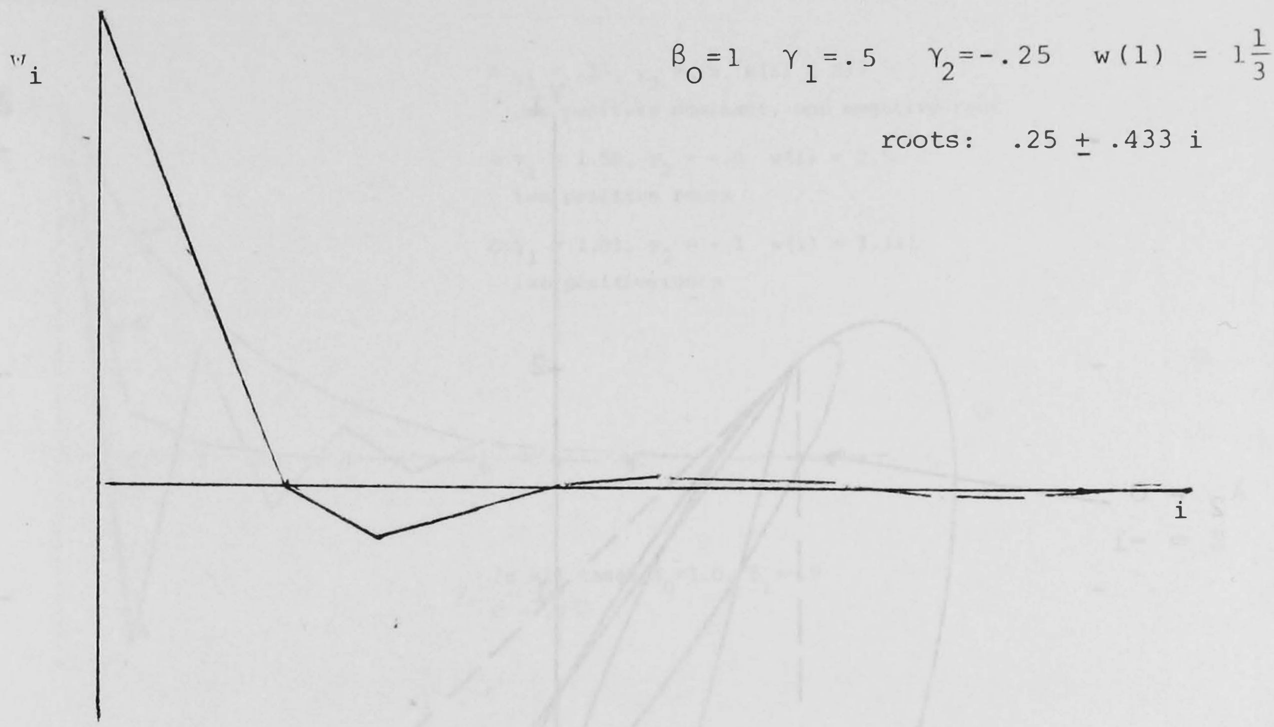


Chart 3

Lag distribution for negative dominant root and $\lambda_1 = 0$ in an AD(2,0) model

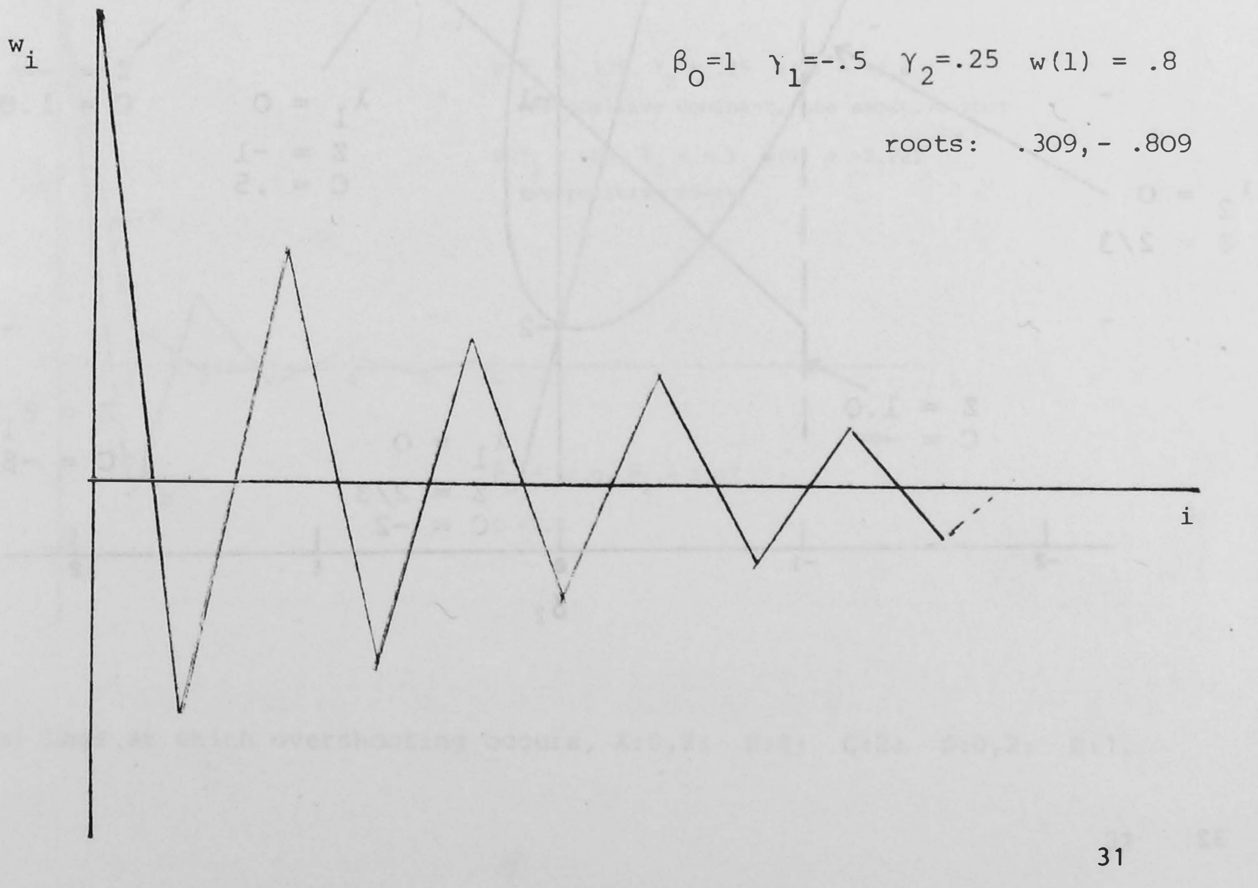


Chart 4

The constraints $\lambda_1 = 0$, $\lambda_2 = 0$ in an AD(2,1) model

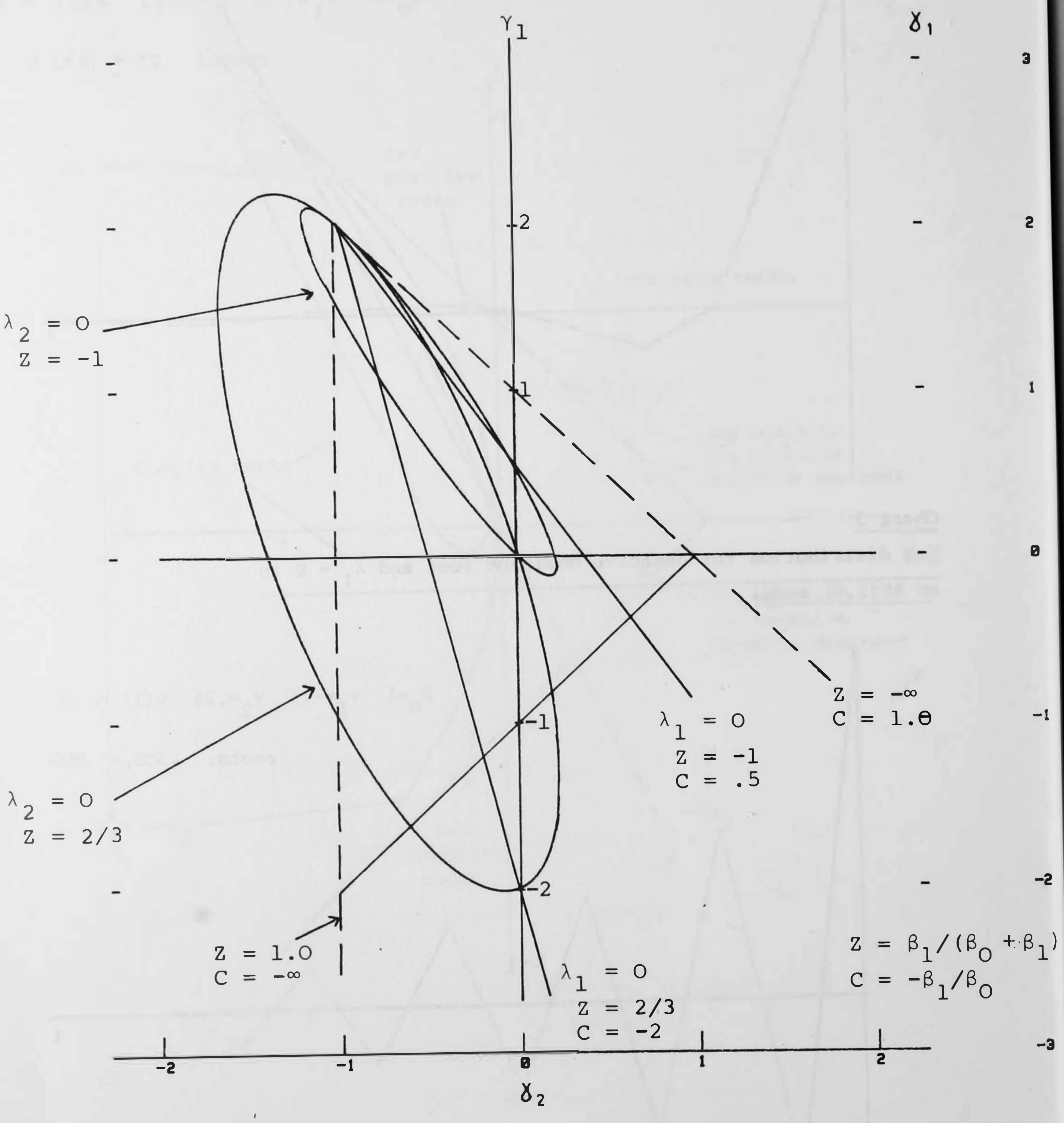


Chart 5

Some lag distributions (a) which satisfy $\lambda_1 = 0$ in an AD(2,1), C = 0.9

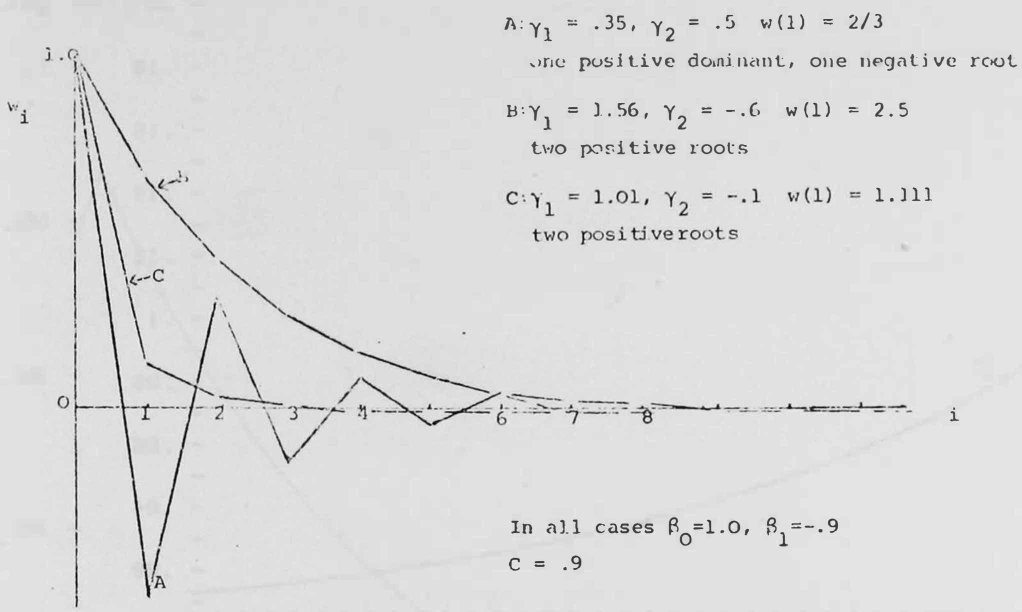
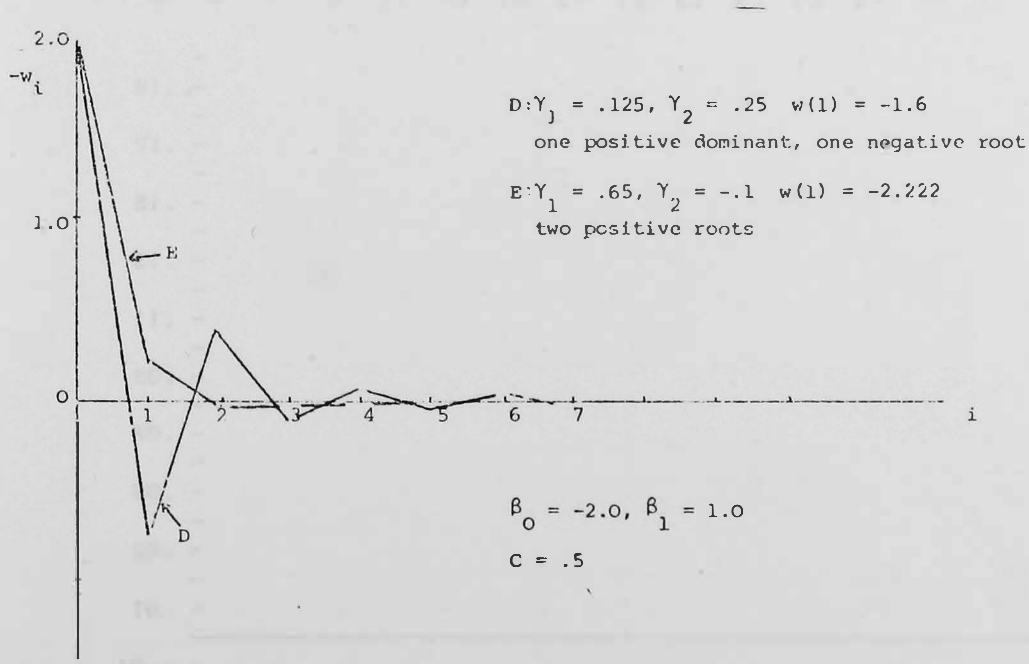


Chart 6

Some lag distributions (a) which satisfy $\lambda_1 = 0$ in an AD(2,1), C = 0.5



(a) Lags at which overshooting occurs, A:0,2; B:4; C:2; D:0,2; E:1.

Chart 7

M1: unconstrained lag on expenditure

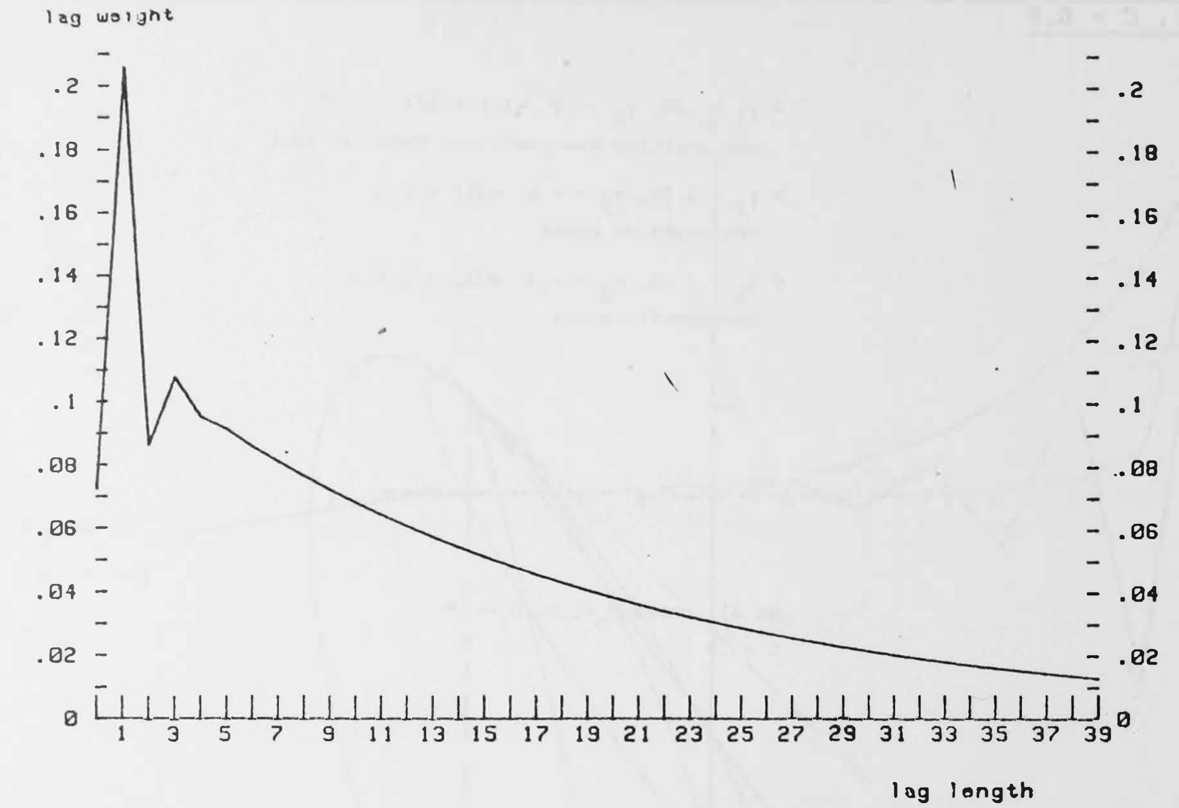


Chart 8

M1: constrained lag on expenditure

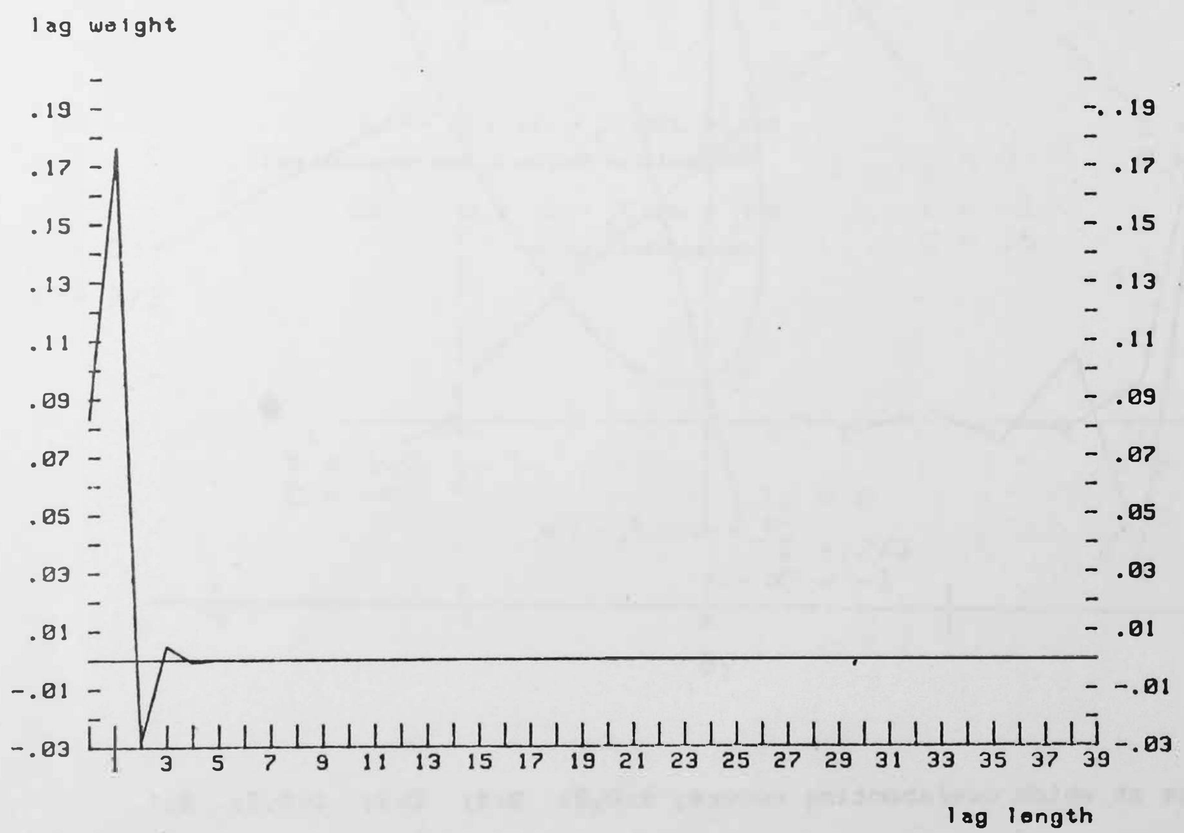


Chart 9

M1: lag on expenditure with ECM

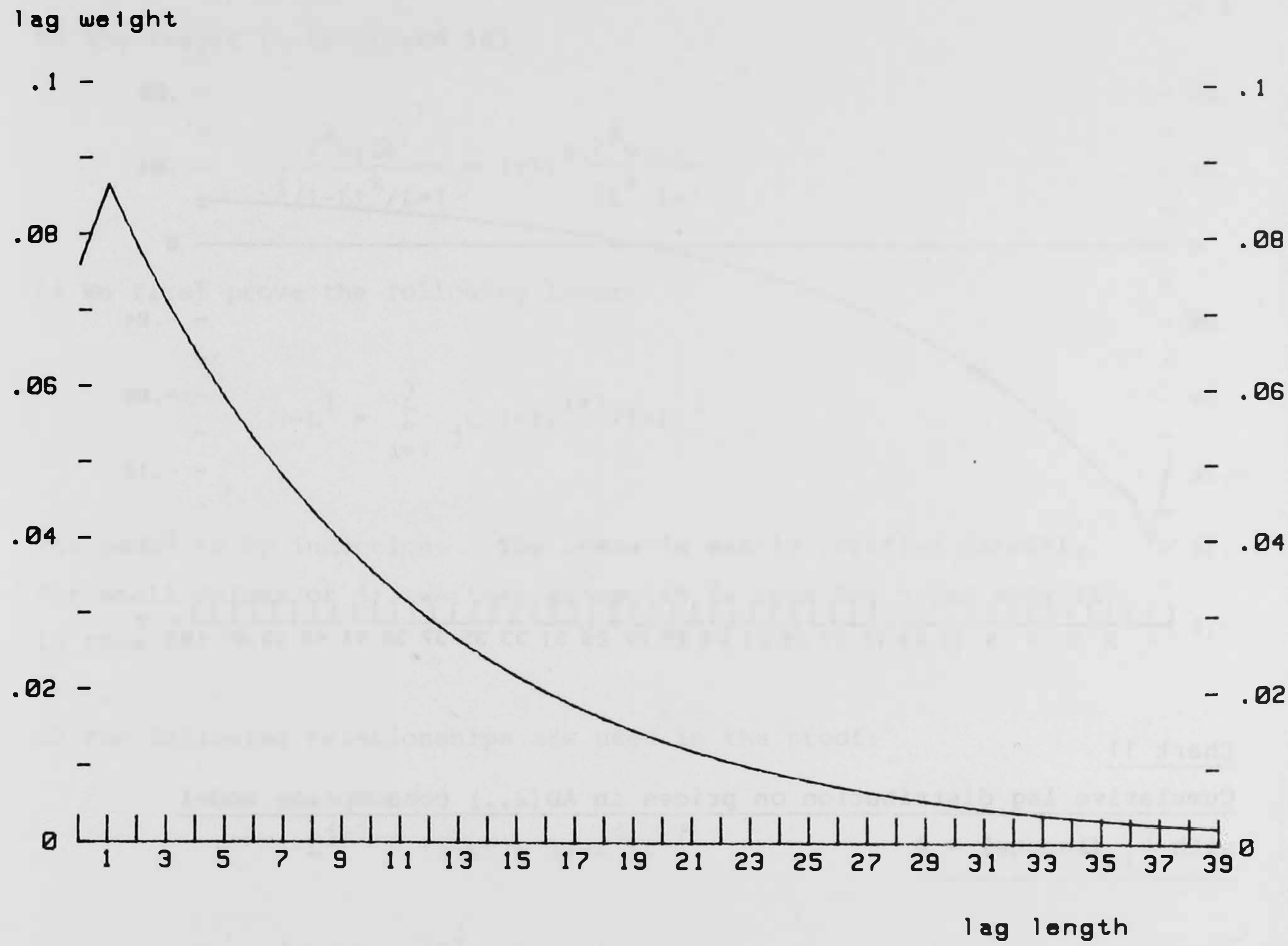
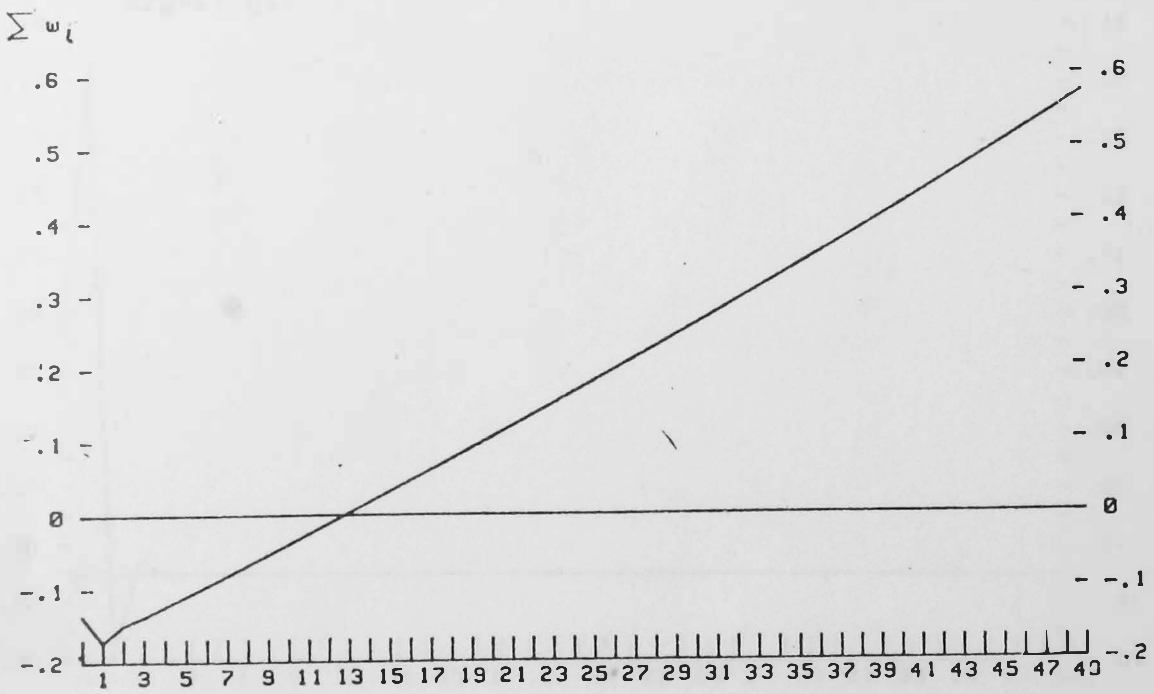


Chart 10

Cumulative lag distribution on prices in AD(2,.) consumption model

Chart 11

Cumulative lag distribution on prices in AD(2,.) consumption model with λ_1 (income) = 0



Appendix 2

Derivation of results used in Section 2

50 The result to be proved is:

$$\frac{\partial^k w(L)}{\partial (1-L)^k / L=1} = (-1)^k \frac{\partial^k w(L)}{\partial L^k / L=1}$$

51 We first prove the following lemma:

$$1-L^i = \sum_{i=1}^{j} {}_jC_i (-1)^{i+1} (1-L)^i$$

The proof is by induction. The lemma is easily verified directly for small values of j; we then assume it is true for j and show it is true for j+1.

52 The following relationships are used in the proof:

$$1-L^{j+1} = (1-L) + (1-L^j)L \qquad (1)$$

$$(1-L)^i - (1-L)^{i+1} = (1-L)^i[1-(1-L)]$$
$$= (1-L)^i L \qquad (2)$$

$${}_{j+1}C_i = {}_jC_i + {}_jC_{i-1} \qquad (3)$$

$${}_{j+1}C_{j+1} = {}_jC_j \qquad (4)$$

$${}_jC_0 \equiv 1 \qquad (5)$$

53 Proof: $1-L^{j+1} = \sum_{i=1}^{j+1} ({}_jC_i + {}_jC_{i-1})(1-L)^i(-1)^{i+1}$, using relationship 3

$$= {}_jC_0(1-L) + \sum_{i=1}^{j} {}_jC_i(1-L)(-1)^{i+1} + (-1)^{j+2}{}_jC_j(1-L)^{j+1}$$

$$+ \sum_{i=2}^{j} {}_jC_{i-1}(1-L)^{i}(-1)^{i+1}, \text{ using relationship 4}$$

$$= {}_jC_0(1-L) + \sum_{i=1}^{j} {}_jC_i\left[(-1)^{i+1}(1-L)^{i} + (-1)^{i+2}(1-L)^{i+1}\right]$$

$$1-L^{j+1} = {}_jC_0(1-L) + \sum_{i=1}^{j} {}_jC_i(-1)^{i+1}\left[(1-L)^{i} - (1-L)^{i+1}\right]$$

$$= {}_jC_0(1-L) + \sum_{i=1}^{j} {}_jC_i(-1)^{i+1}(1-L)^{i}L, \text{ using relationship 2}$$

$$= (1-L) + (1-L^{j})L$$

The last step uses relationships 1 and 5 and the assumption that the lemma is true for j.

54 Returning to the result to be proved, we have:

$$w(L) = \sum_{i=0}^{\infty} w_i L^i \tag{6}$$

Using the lemma to substitute for L^i, i = 0 ...

$$w(L) = \sum_{i=0}^{\infty} w_i - \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} w_i \, {}_iC_j(-1)^{j+1}(1-L)^{j}$$

hence:

$$\frac{\partial^k w(L)}{\partial(1-L)^k / L=1} = (-1)^k \, k! \sum_{i=k}^{\infty} w_i \, {}_iC_k \tag{7}$$

Differentiating w(L), with respect to L, k times and evaluating at L = 1, we obtain:

$$\frac{\partial^k w(L)}{\partial L^k / L=1} = k! \sum_{i=k}^{\infty} w_i \, {}_iC_k \qquad (8)$$

Comparison of relationships 7 and 8 shows the result is proven.

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