Private Digital Currency and Monetary Sovereignty*

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Abstract

This paper considers an economy where central-bank-issued fiat money competes with a privately-issued digital currency. We find that if the use of the fiat money is sufficiently low, a welfare-maximizing central bank adopts policies that strengthen the digital currency issuer's market power, leading to high inflation and low welfare. Therefore, the digital currency threatens monetary sovereignty in terms of monetary policy. Maintaining the use of a central-bank-issued money commits the central bank to not strengthening market power of the digital currency. This can be achieved by issuing a properly designed central bank digital currency.

Key words: Monetary Policy, Digital Currency, Central Bank Digital Currency, Monetary Sovereignty

JEL classification nos: E5, E52

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1 Introduction

Recently, many central banks are considering issuing a central bank digital currency (CBDC). According to a 2020 survey of the Bank for International Settlements, 86% of central banks are engaging in work regarding a CBDC (see Boar and Wehrli 2021). One motivation for issuing a CBDC is to maintain monetary sovereignty, i.e., the ability to control monetary policy and fulfill the role as the lender of last resort. There is a concern that monetary sovereignty can be undermined by adoption of digital currencies and declining use of central bank money.[1]

This paper studies how adoption of a private digital currency and declining use of central bank money affect monetary sovereignty in terms of monetary policies, and whether a CBDC can improve these policies. We consider an environment where a private digital currency competes with a central-bank-issued fiat money. The fiat money and the digital currency differ in the types of transactions that they can serve. We focus on a policy setting game where the central bank maximizes total welfare while the private digital currency issuer maximizes its profit. We identify an incentive for the central bank to inflate the fiat money that is not present without the private digital currency. This incentive is particularly important if the use of fiat money is low and can dramatically reduce welfare. Therefore, the private digital currency can undermine monetary sovereignty by altering the central bank's optimal policy in an unfavorable way. With the help of a CBDC, the central bank can restore the first best.

Intuitively, the welfare-maximizing central bank wants to encourage agents to hold more digital currency for transactions that require a digital currency. This can be achieved by setting a high inflation on fiat money, making it costly to hold. Then agents substitute out to hold and use more digital currency. If the use of the fiat money is high, this incentive is dominated by the incentive to maintain the value of fiat money, which helps transactions using the fiat money. But as the use of the fiat money declines, the incentive to set a high inflation dominates and leads to a strategic complementary between the central bank and

[1] "Contingency Planning for a Central Bank Digital Currency," Bank of Canada 2020.

the digital currency issuer: the central bank raises inflation in fiat money, which raises the demand of the digital currency and market power of the digital currency issuer. The digital currency issuer then responds by raising inflation of the digital currency to obtain more profit (seigniorage income), which in turn incentivizes the central bank to further increase inflation in fiat money. This strategic complimentary can drive equilibrium far away from the first best and result in high inflation. In particular, if the use of fiat money drop below a threshold, there can be a discrete drop in welfare.

These findings highlight the threat that a private digital currency could pose on monetary sovereignty. To avoid such a situation, the central bank could try to maintain the use of central-bank-issued money above a certain threshold, which serves as a commitment device to not adopt policies that strengthen the market power of the private digital currency issuer. This can be achieved by introducing a CBDC to serve transactions that require a digital currency. We show that with the help of a CBDC, the central bank do not need to rely on the private digital currency to serve certain transactions. As a result, it can implement more allocations and achieve the first best.

There are other findings worth highlighting. First, a private digital currency can reduce welfare even if it is less costly to use than the fiat money. This occurs if the private digital currency does not expand the set of transacntions that can be served. Then it is welfare-improving if the economy coordinates on using only the fiat money, which captures the network externality in payment. Second, total welfare can be higher (but still suboptimal) if the central bank maximizes welfare generated by only transactions using the fiat money than if the central bank maximizes total welfare. The former eliminates the incentive for the central bank to strengthen the market power of the digital currency issuer, which can lead to a better outcome. This sheds light on how a central bank should set its policy target if it does not issue a CBDC. Third, the digital currency restricts the set of feasible policies of the central bank. It imposes a cap on the inflation rate of the fiat money beyond which the fiat money is not valued. This is another sense that the private digital currency undermines monetary sovereignty. However, this cap normally rules out only suboptimal policies and does not constrain the optimal policy. Therefore, it is more important to investigate the effect of the digital currency on the optimal policy.

The economic literature on digital currencies is growing rapidly. Broadly speaking, there are two streams: one on privately issued digital currencies such as cryptocurrencies and the other one on CBDCs. This paper contributes to both streams.

In the first stream, Fernández-Villaverde and Sanches (2019) studies price stability with currency competition. Schilling and Uhlig (2019) analyze pricing of bitcoin. Benigno (2021) shows that crypocurrencies introduces a cap on the nominal interest rate of the central bank if the crypocurrencies and the fiat money are perfect substitutes. Benigno, Schilling and Uhlig (2019) show a global crypto currency can force countries to synchronize their monetary policy. Different from these paper, we focus on how a private digital currency affects the optimal policy of the central bank in a policy setting game. As an intermediate step, we also get a result similar to Benigno (2021) even though our setup is different. Since our focus is on policies, we abstract from the techological design of the digital currency, such as blockchain and proof-of-work. For research in this area, see Biais et al. (2019), Chiu and Koeppl (2021) and the reference therein.

In the second stream, a number of studies focus on the role of CBDCs as a new monetary policy tool. Barrdear and Kumhof (2016) evaluate the macroeconomic consequences of a CBDC in a dynamic stochastic general equilibrium model. Davoodalhosseini (2021) explores using a CBDC for balance-contingent transfers; Brunnermeier and Niepelt (2019) and Niepelt (2020) derive conditions under which introducing a CBDC has no effect on macroeconomic outcomes, including bank intermediation. Jiang and Zhu (2021) discuss how the interest on a CBDC and the interest on reserves interact as two separate policy tools. Our paper adds to this literature by showing how a CBDC can help monetary policy in a world with declining use of fiat money and increasing adoption of a private digital currency.[2]

[2]Research on the effect of a CBDC on the banking sector includes Andolfatto (2020), Chiu et al. (2021), Keister and Sanches (2021) and Garratt and Zhu (2021)

Our work is also related to the large literature on competing currencies based on search theory.[3] Among them, it is most related to Zhang (2014), who studies the adoption decision of foreign money and monetary policy competition between two countries. She also shows that optimal monetary policy may feature some inflation because of the incentive to tax foreigners who hold domestic money. In our model, the mechanism is different because we do not have foreign agents. Our paper also contributes to the literature on money and liquid assets by showing that the Friedman rule may not be optimal if a liquid asset (digital currency) and the fiat money are substitutes. An incomplete list of papers that discuss liquid assets includes Geromichalos and Herrenbrueck (2016); Lester et al.(2012); Williamson (2012); Venkateswaran and Wright (2013); Li and Li (2013); Han (2015); He et al. (2015) and Rocheteau et al. (2018).

The rest of the paper is organized as the following. Section 2 lays out the model and studies the steady state equilibrium given the policies of the central bank and the digital currency issuer. Section 3 studies the optimal central bank policy under an exogenous digital currency policy. It also shows how a digital currency can reduce welfare even if it is less costly to use than the fiat money. Section 4 analyzes a policy setting game in which the central bank and the digital currency issuer move simultaneously. Section 5 shows that a central bank digital currency can help the central bank to achieve the first best. Section 6 studies two extensions and show that our main insight remains robust. The first extension considers a sequential move policy game where the central bank leads and the digital currency issuer follows. The second extension introduces a fixed cost to adopt the private digital currency. Section 7 concludes.

[3] An incomplete list includes Ravikumar and Wallace (2002); Curtis and Waller (2000); Camera et al. (2004); Martin (2006) and Kahn (2013); Zhou (1997); Trejos and Wright (1996); Trejos (2003); Head and Shi (2003); Camera and Winkler (2003); Li and Matsui (2009); Liu and Shi (2010) and Zhang (2014).

2 Benchmark Model

The framework is based on Lagos and Wright (2005). Time is discrete and continues forever. There is a continuum of buyers and a continuum of sellers. At each date t, agents interact sequentially in two settings: a decentralized market (DM); and a centralized market (CM).

In the DM, buyers want to consume a non-storable good y and sellers can produce it on the spot. Buyers and sellers meet and trade bilaterally. Because of the lack of commitment and anonymity, no credit is viable. There exist a central-bank-issued intrinsically worthless physical fiat money f, and a worthless digital token d, which we refer to as a digital currency. They may be used as means of payment. Sellers may be one of three permanent types. With α_1 probability, a buyer meets a type 1 seller, who accepts only the fiat money. With α_2 probability, a buyer meets a type 2 seller, who does not accept the fiat money, but may accept the digital currency. With α_3 probability, a buyer meets a type 3 seller, who accepts both.[4] One interpretation is that these sellers offer the same product but operate in different locations. Type 1 sellers do not have access to the internet and have to rely on the fiat money. Type 2 sellers specialize in online trading and cannot accept the fiat money due to separation in space. Type 3 sellers operate local stores but also have access to the internet. Therefore, they can accept both. Another interpretation is that the sellers sell different products. For example, in type 1 meetings, buyers need government services, which can be paid only in the fiat money. In type 2 meetings, buyers are purchasing DeFi services and only the on chain digital currency can be used.[5] Then we can interpret the αs as the probability that a buyer needs to consume the corresponding product. In either case, a buyer meets an seller at most once in the DM each period. If no trade occurs, he proceeds to the next CM.

In the CM, both buyers and sellers consume a numeraire good x, supply labor ℓ, trade f and d. Buyers do not know which type of sellers they will meet in the next DM. Hence, they hold both monies to insure themselves against different transaction needs. Both buyers and

[4]The exchange rate between the fiat money and digital currency is determined in the equilibrium.

[5]All the following analysis remains valid if the utility and cost functions vary across meetings, which may be more compelling if meetings of different types involve different products.

sellers have access to a technology that translates labor to the numeraire good one-for-one. At date t, the total supply of the fiat money is M_t^f, and that of the digital currency at date t is M_t^d. They are determined by the central bank and the issuer of the digital currency, respectively. They do so by buying or selling their own monies in the centralized market. Any loss or income is given back to buyers through lump-sum transfers or taxes. The government runs a passive fiscal policy and can enforce lump-sum taxes. Therefore, the central bank has the resource to sustain a negative money growth. The issuer of the digital currency cannot finance losses and the growth rate of the digital currency can be only non-negative. Denote the growth rate of the fiat money as $\mu_{t+1}^f = M_{t+1}^f / M_t^f - 1$ and that of the digital currency as $\mu_{t+1}^d = M_{t+1}^d / M_t^d - 1$. For now, we treat them as exogenous and study the outcome of the economy. Sections 3 and 4 endogenize them.

The lifetime utility for a buyer is $\sum_{t=1}^{\infty} \beta^t \left[U\left(x_t\right) - \ell_t + u\left(y_t\right)\right]$, where u and U satisfy the usual monotonicity and curvature conditions. In addition, u'' is continuous and bounded at any $y > 0$. Similarly, a seller has lifetime utility $\sum_{t=1}^{\infty} \beta^t \left[U\left(x_t\right) - \ell_t - c\left(y_t\right)\right]$, where c'' is strictly positive and bounded away from ∞ on any finite interval. One can write the CM problem of a buyer as

$$\begin{aligned} W_t^B\left(f_t, d_t\right) &= \max_{x_t, \ell_t, \hat{f}_{t+1}, \hat{d}_{t+1}} U\left(x_t\right) - \ell_t + \beta V_{t+1}^B\left(\hat{f}_{t+1}, \hat{d}_{t+1}\right), \\ \text{s.t. } x &= \phi_t\left(f_t - \hat{f}_{t+1}\right) + \psi_t\left(d_t - \hat{d}_{t+1}\right) + \ell_t + T_t, \end{aligned}$$

where V_{t+1}^B is the DM value function in period $t+1$; $\hat{f}_{t+1}$ and $\hat{d}_{t+1}$ are the amount of fiat money and digital currency that the buyer brings into the next DM; and ϕ_t and ψ_t are the prices of fiat money and digital currency in terms of the numeraire good x. If any of them equals 0, the corresponding money is not adopted. The first-order conditions (FOCs) are

$$\hat{f}_{t+1} : \phi_t = \beta \frac{\partial}{\partial \hat{f}_{t+1}} V_{t+1}^B\left(\hat{f}_{t+1}, \hat{d}_{t+1}\right), \tag{1}$$

$$\hat{d}_{t+1} : \psi_t = \beta \frac{\partial}{\partial \hat{d}_{t+1}} V_{t+1}^B\left(\hat{f}_{t+1}, \hat{d}_{t+1}\right), \tag{2}$$

$$x_t : 1 = U'\left(x_t\right). \tag{3}$$

Envelope conditions are

$$\frac{\partial}{\partial f_t} W_t^B(f_t, d_t) = \phi_t, \ \frac{\partial}{\partial d_t} W_t^B(f_t, d_t) = \psi_t, \tag{4}$$

which imply that W_t^B is linear in its arguments.

Because a seller does not need to consume in the DM, he does not take any money out of the CM. Then his CM problem is

$$\begin{aligned} W_{j,t}^S(f_t, d_t) &= \max_{x_t, \ell_t} U(x_t) - \ell_t + \beta V_{j,t+1}^S(0,0) \\ \text{s.t. } x_t &= \phi_t f_t + \psi_t d_t + \ell_t + T_t, \end{aligned}$$

where $j = 1, 2, 3$ indicates the type of the seller. The FOC is $U'(x_t) = 1$ and the envelope conditions are the same as (4).

The DM value function for a buyer is

$$\begin{aligned} V_t^B(f_t, d_t) &= \alpha_1 \left[u\left(y_t^1\right) + W_t^B\left(f_t - f_t^1, d_t\right)\right] + \alpha_2 \left[u\left(y_t^2\right) + W_t^B\left(f_t, d_t - d_t^2\right)\right] \\ &\quad + \alpha_3 \left[u\left(y_t^3\right) + W_t^B\left(f_t - f_t^3, d_t - d_t^3\right)\right] + \left(1 - \sum_{i=1}^{3} \alpha_i\right) W_t^B(f_t, d_t). \end{aligned}$$

where y_t^i, $i = 1, 2, 3$ are consumptions in different types of meetings and f_t^1, d_t^2, d_t^3 and f_t^3 are corresponding payments. For example, with α_1 probability, a buyer meets a type 1 seller, where he pays f_t^1 fiat money for y_t^1 DM consumption. In the next CM, he has $f_t - f_t^1$ fiat money left. One can use (4) to write

$$\begin{aligned} V_t^B(f_t, d_t) &= \alpha_1 \left[u\left(y_t^1\right) - \phi_t f_t^1\right] + \alpha_2 \left[u\left(y_t^2\right) - \psi_t d_t^2\right] \\ &\quad + \alpha_3 \left[u\left(y_t^3\right) - \psi_t d_t^3 - \phi_t f_t^3\right] + W_t^B(f_t, d_t). \end{aligned} \tag{5}$$

Similarly, the DM value function of a type-j seller is

$$V_{j,t}^S(0,0) = \alpha_s[p_t^j - c(y_t^j)] + W_{j,t}^S(0,0). \tag{6}$$

where $p_t^1 = \phi_t f_t^1$, $p_t^2 = \psi_t d_t^1$, $p_t^3 = \psi_t d_t^3 + \phi_t f_t^3$ and α_s is the meeting probability of a seller.

Consumption and payment are determined by Kalai bargaining.[6] Let y^* be the efficient DM consumption that satisfies $u'(y^*) = c'(y^*)$, and let $g(y) = \theta y + (1-\theta) u(y)$, where θ is the bargaining power to the buyer. Define z to be the total real value of payment balances that a buyer can use in a transaction. For example, in a type 1 meeting, only the fiat money is accepted and $z = \phi_t f_t$ in period t. Kalai bargaining specifies the trading quantity y and the real payment p as functions of z:

$$y = Y(z) = \begin{cases} g^{-1}(z) & \text{if } z < z^* \\ y^* & \text{otherwise} \end{cases} \quad \text{and} \quad p = P(z) = \begin{cases} z & \text{if } z < z^* \\ z^* & \text{otherwise} \end{cases}, \tag{7}$$

where $z^* = \theta c(y^*) + (1-\theta) u(y^*)$. In words, if a buyer brings more than z^* usable balances, he consumes the efficient quantity y^*; otherwise, he spend all the usable balances.

Partially differentiate (5) and then use (7) to obtain

$$\frac{\partial}{\partial f_t} V_t^B(f_t, d_t) = \alpha_1 \lambda(\phi_t f_t)\phi_t + \alpha_3 \lambda(\phi_t f_t + \psi_t d_t)\phi_t + \beta\phi_t \tag{8}$$

$$\frac{\partial}{\partial d_t} V_t^B(f_t, d_t) = \alpha_2 \lambda(\psi_t d_t)\psi_t + \alpha_3 \lambda(\phi_t f_t + \psi_t d_t)\psi_t + \beta\psi_t, \tag{9}$$

where λ is the liquidity premium defined as

$$\lambda(z) = \begin{cases} \frac{u'[g^{-1}(z)]}{g'[g^{-1}(z)]} - 1 > 0 & \text{if } z < z^* \\ 0 & \text{if } z \geq z^* \end{cases}. \tag{10}$$

It captures the fact that more money relaxes the liquidity constraint in the DM and enables buyers to consume more. It is essentially the Lagrangian multiplier on the constraint that buyers cannot spend more than they have. If this constraint is binding, λ is strictly positive and the buyer spends everything. Otherwise, $\lambda = 0$ and the buyer consumes y^*.

We can combine (8)-(9) with (1) to obtain the Euler equations

$$\phi_t = \alpha_1 \beta \lambda\left(\phi_{t+1}\hat{f}_{t+1}\right)\phi_{t+1} + \alpha_3 \beta \lambda\left(\phi_{t+1}\hat{f}_{t+1} + \psi_{t+1}\hat{d}_{t+1}\right)\phi_{t+1} + \beta\phi_{t+1},$$

$$\psi_t = \alpha_2 \beta \lambda\left(\psi_{t+1}\hat{d}_{t+1}\right)\psi_{t+1} + \alpha_3 \beta \lambda\left(\phi_{t+1}\hat{f}_{t+1} + \psi_{t+1}\hat{d}_{t+1}\right)\psi_{t+1} + \beta\psi_{t+1}.$$

[6]The results derived in this paper do not depend on Kalai's solution. They also hold under other solution concepts such as Walrasian pricing and a strategic bargain game analyzed in Zhu (2019).

To characterize the equilibrium, define $z_t^f = \phi_t M_t^f$, $z_t^d = \psi_t M_t^d$ and use the market clearing conditions $\hat{f}_{t+1} = M_{t+1}^f$ and $\hat{d}_{t+1} = M_{t+1}^d$ to obtain

$$\left(1+\mu_{t+1}^f\right) z_t^f = \alpha_1 \beta \lambda \left(z_{t+1}^f\right) z_{t+1}^f + \alpha_3 \beta \lambda \left(z_{t+1}^f + z_{t+1}^d\right) z_{t+1}^f + \beta z_{t+1}^f \tag{11}$$

$$\left(1+\mu_{t+1}^d\right) z_t^d = \alpha_2 \beta \lambda \left(z_{t+1}^d\right) z_{t+1}^d + \alpha_3 \beta \lambda \left(z_{t+1}^f + z_{t+1}^d\right) z_{t+1}^d + \beta z_{t+1}^d. \tag{12}$$

Any non-negative sequence $\left\{\left(z_t^f, z_t^d\right)\right\}_{t=1}^{\infty}$ that solves (11) and (12) constitutes an equilibrium if it satisfies the transversality conditions $\lim_{t\to\infty} \beta^t z_t^f = 0$ and $\lim_{t\to\infty} \beta^t z_t^d = 0$.

Now we study the steady state equilibrium under constant money growth rates: $\mu_t^f = \mu^f$ and $\mu_t^d = \mu^d$ for all t. In the steady state, z_t^f and z_t^d stay unchanged over time and the inflation rates $\pi^f = \phi_t/\phi_{t+1} - 1$ and $\pi^d = \psi_t/\psi_{t+1} - 1$ equal the money growth rates, i.e. $\pi^f = \mu^f$ and $\pi^d = \mu^d$. There always exist equilibria where either or both of the monies are not valued due to self-fulfilling prophecies. Since our goal is to analyze the interaction between the two monies, we focus on the equilibrium where a money is valued if it can be valued. Then the equilibrium conditions reduce to

$$i^f \geq \alpha_1 \lambda \left(z^f\right) + \alpha_3 \lambda \left(z^f + z^d\right) \text{ strict inequality if } z^f = 0, \tag{13}$$

$$i^d \geq \alpha_2 \lambda \left(z^d\right) + \alpha_3 \lambda \left(z^f + z^d\right) \text{ strict inequality if } z^d = 0. \tag{14}$$

Here $i^f = \left(1+\pi^f\right)/\beta - 1$ and $i^d = \left(1+\pi^d\right)/\beta - 1$ are the nominal interest rates of illiquid nominal bonds in the corresponding monies, which are determined by Fisher equation. In each of the equations, the left-hand side is the marginal cost of money, i.e., the forgone interest rate. The right-hand side is the marginal benefit of money, i.e., it enbles more consumption. A money is not valued if its marginal cost outweighs its marginal benefit. Otherwise, buyers bring in enough money such that the cost equals the benefit.

In principle, the equilibrium can have four regimes: (1) no monies are valued; (2) only the fiat money is valued; (3) only the digital currency is valued; and (4) both monies are valued. Regime 1 occurs iff $i^f \geq (\alpha_1 + \alpha_3)\lambda(0)$ and $i^d \geq (\alpha_2 + \alpha_3)\lambda(0)$, i.e., the cost of holding either money exceeds the benefit. In regime 2, holding the digital currency is too costly given that the fiat money is valued. This implies $i^d \geq \bar{\imath}^d \equiv \alpha_2 \lambda(0) + \alpha_3 \lambda(\bar{z}^f)$ where $\bar{z}^f$ satisfies

$$i^f = \alpha_1 \lambda(\bar{z}^f) + \alpha_3 \lambda(\bar{z}^f),$$

or equivalently

$$i^d \geq \bar{\imath}^d = \alpha_2 \lambda(0) + \frac{\alpha_3}{\alpha_1 + \alpha_3} i^f.$$

Symmetrically, in regime 3,

$$i^f \geq \bar{\imath}^f = \alpha_1 \lambda(0) + \frac{\alpha_3}{\alpha_2 + \alpha_3} i^d.$$

Lastly, regime 4 occurs only if $i^f < \bar{\imath}^f$ and $i^d < \bar{\imath}^d$.

Proposition 1 *Equations (13)-(14) defines a steady state unique equilibrium which satisfies*

(a). No money is valued iff $i^f \geq (\alpha_1 + \alpha_3) \lambda(0)$ *and* $i^d \geq (\alpha_2 + \alpha_3) \lambda(0)$.

(b). Only the fiat money is valued iff $i^f < (\alpha_1 + \alpha_3) \lambda(0)$ *and* $i^d \geq \bar{\imath}^d$.

(c). Only the digital currency is valued iff $i^d < (\alpha_2 + \alpha_3) \lambda(0)$ *and* $i^f \geq \bar{\imath}^f$.

(d). Both monies are valued iff $i^f < \bar{\imath}^f$ *and* $i^d < \bar{\imath}^d$.

Proof. See Appendix B. ■

Proposition 1 shows that only one of the four regimes can occur under any set of parameters. Given the fiat money is valued, Proposition 1(b) implies that digital currency can be valued iff $i^d < \bar{\imath}^d$. This condition is more likely to hold if i^f is high and/or α_2, the size of online trading, is large. It implies that if α_2 is not too big, the central bank can deter the adoption of digital currency by setting a low inflation or a low nominal interest rate. In particular, if $i^f = 0$, the digital currency can, at the best, be a niche product that serves only the type 2 meetings. The reverse of that is Proposition 1(c), which suggests that for the fiat money to be valued, i^f cannot be higher $\bar{\imath}^f$ if the digital currency is valued. Because $\bar{\imath}^f$ depends on i^d, the central bank policy i^f is constrained by the digital currency. In other words, the digital currency shrinks the set of feasible polices of the central bank, which may be considered as a form of undermining monetary sovereignty. However, as will be shown later, $\bar{\imath}^f$ normally rules out only suboptimal policies and does not contrain the optimal policy. Therefore, we will be focusing on the effect of a digital currency on the optimal polices in following sections.

	i^f	i^d	α_1	α_2	α_3
z^f	$-$	$+$	$+$	$-$	$+$
z^d	$+$	$-$	$-$	$+$	$+$
$\bar{\imath}^d$	$+$	0	$-$	$+$	$+$
$\bar{\imath}^f$	0	$+$	$+$	$-$	$+$

Table 1: Comparative Statics

Table 1 shows some of the comparative statics, which are derived in Appendix C. Notice that a higher i^f lowers z^f and raises z^d. A higher inflation in the fiat money makes the fiat money more costly to hold. Buyers then demand more digital currency because it is a substitute of the fiat money in type 3 meetings. This increases the demand of the digital currency and the value of the digital currency. As we will see later, this comparative statics is one of the key component to our main result on the optimal policy. Also, an increase in α_2, which captures an increase in the online economy, makes the digital currency more useful. Agents demand more digital currency and less fiat money. As a result, the value of the digital currency increases and that of the fiat money decreases. In other words, agents increases the use of the digital currency and decrease the use of the fiat as the online economy expands.

3 Optimal Policy: Exogenous digital currency Policy

We now study the optimal monetary policy of the central bank if the digital currency grows at an exogenous rate μ^d. The policy variable of the central bank is the time-invariant long-run money growth rates μ^f. Since we focus on the steady state equilibrium, this is equivalent to setting the inflation rate or nominal interest rate i^f under an exogenous i^d. This excerise is interesting for two reasons. First, some digital currencies, such as Bitcoin and Ether, employ exogenous long-run growth rates. Therefore, it is interesting to investigate how the central bank should respond. Second, this analysis helps to trace out the central bank's best response to i^d, which is later used to analyze the policy game. We also show how a digital currency can reduce welfare even if it is less costly to use than the fiat money.

The central bank maximizes the total welfare of the economy, which includes type 2 meetings.[7] The welfare as a function of i^d and i^f is

$$\begin{aligned}\Omega\left(i^f, i^d\right) &= \alpha_1\left[u \circ y\left(z^f\right) - c \circ y\left(z^f\right)\right] + \alpha_2\left[u \circ y\left(z^d\right) - c \circ y\left(z^d\right)\right] \\ &\quad + \alpha_3\left[u \circ y\left(z^d + z^f\right) - c \circ y\left(z^d + z^f\right)\right] + constant,\end{aligned}$$

where z^f and z^d depend on i^f and i^d through (13)-(14). As is standard in the literature, monetary policies affect the welfare only through the social value of production in the DM. We start with the following lemma on the first best, which sets the benchmark for the analysis of the equilibrium outcome.

Lemma 1 *The first best is achieved at $i^f = 0$ and $i^d = 0$. Buyers then consume y^* in all DM transactions.*

Proof. If $i^d = i^f = 0$, (13)-(14) implies that $\alpha_1\lambda\left(z^f\right) = \alpha_2\lambda\left(z^d\right) = \alpha_3\lambda\left(z^d + z^f\right) = 0$. Then (10) implies that $z^f \geq z^*$. Consequently, $Y\left(z^f\right) = y^*$ by (7), which suggest that consumption is efficient in type 1 transactions. Similarly, we can show that consumption is y^* in other two types of transactions. ■

If the central bank can control both i^d and i^f, it can achieve the first best by setting $i^f = i^d = 0$, i.e. by implementing the Friedman rule for both monies. If the central bank can control only i^f, an immediate consequence of Lemma 1 is that the optimal i^f is 0 if $i^d = 0$. However, if $i^d > 0$, the optimal i^f may be different from the Friedman rule.[8]

[7]We treat type 2 transactions to be legitimate transactions instead of crimes. If only a fraction of these transactions involves criminal activities, the central bank may discount surpluses from these transactions. This is equivalent to a reduction in α_2 and our analysis stays unchanged.

[8]If the digital currency issuer does not respond to the central bank policy, the central bank can achieve the first best by conducting open market operation. Specifically, the central bank can buy digital currency and

Example	σ	ε	β	θ	α_1	α_2	α_3
1	0.90	0.05	0.98	0.50	0.03	0.06	0.40
2	0.90	0.1	0.98	1	0.01	0.03	0.40
3	0.90	0.005	0.98	1	10^{-4}	0.1	0.40
4	0.90	0.005	0.98	0.5	0	0.06	0.40
5	0.90	0.005	0.98	1	0	0.06	0.40

Table 2: Parameters for Numerical Examples. $u(y) = \frac{(y+\varepsilon)^{1-\sigma} - \varepsilon^{1-\sigma}}{1-\sigma}$, $c(y) = y$.

To see this, we first study the properties of the welfare function $\Omega\left(i^f, i^d\right)$, which determine the properties of the central bank's optimal policy as a function of i^d (best response to i^d): $B^f(\cdot)$. Figure 1 shows a typical welfare function in i^f under different values of i^d. It is obtained from numerical example 1, whose parametrization and parameters are shown in Table 2. If $i^d = 0$, the welfare function is monotonically decreasing in i^f until a point at which the fiat money is not valued and then it stays constant. It has a unique peak at $i^f = 0$. In this case, buyers in type 3 meetings are not constrained by using only the digital currency. Therefore, an increase in i^f only hurts type 1 meetings, leading to welfare loss.

If i^d is higher, as shown in Figure 1(b), welfare becomes non-monotone in i^f. If i^f is smaller than $\underline{I}_1^f\left(i^d\right) = \alpha_1\lambda\left(z^* - \lambda^{-1}\left(\frac{i^d}{\alpha_2}\right)\right)$, buyers are unconstrained in type 3 meetings. An increasing in i^f only reduces consumption in type 1 meetings, which unambiguously reduces welfare. If i^f is above $\underline{I}_1^f\left(i^d\right)$, liquidity becomes scarce in type 3 meetings. An increase in i^f makes the fiat money more costly to hold. Buyers respond by holding less fiat money and more digital currency, because the latter is a perfect substitute to the fiat money in type 3 meetings. Lower fiat money holdings reduces consumption in type 1 and type 3 meetings. Higher digital currency holdings raises consumption in type 2 meetings. If gains in type 2 meetings dominate losses in type 1 and 3 meetings, total welfare increases with i^f. The welfare function now has two peaks: one at $i^f = 0$ and one at some positive i^f. Since i^d

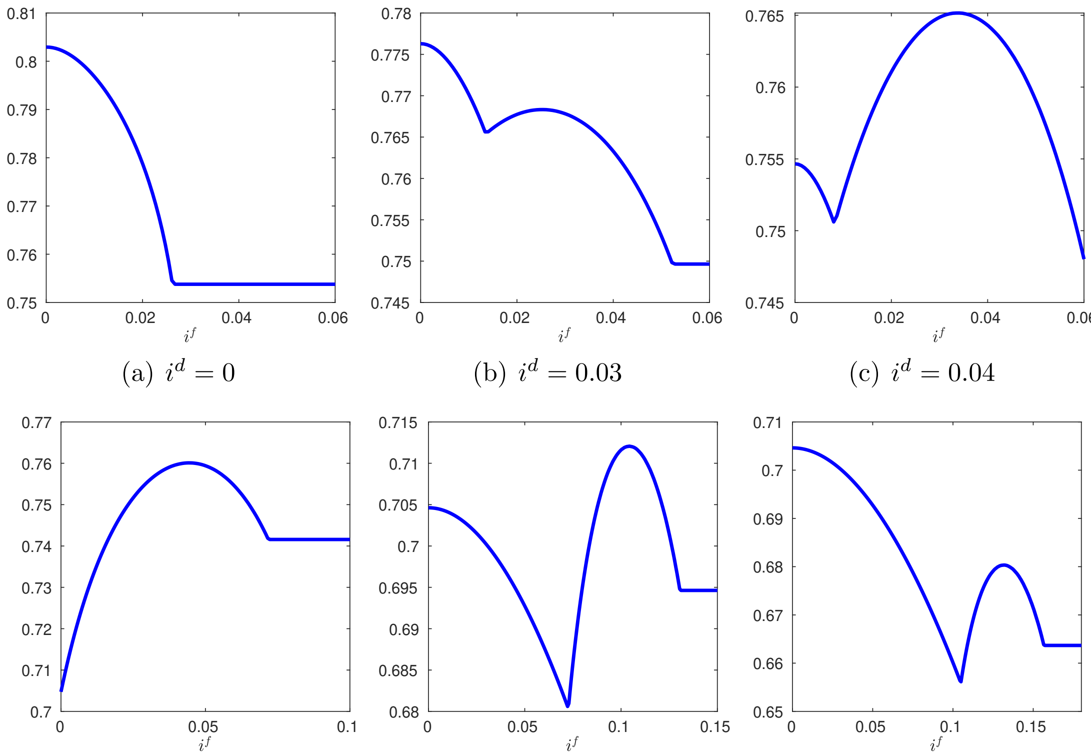
commit to destroying or hoarding forever any purchased amount. This reduces the supply of digital currency and increases its value. If the purchase is so enough that the effective growth rate of digital currency is $1/\beta - 1$, then $i^d = 0$. However, if the digital currency issuer can respond, it would issue more digital currency to off-set the effect of open market operations, which is profit-maximizing. Therefore, we do not focus on open market operations in this paper.

is not very high, $i^f = 0$ maximizes welfare. If i^d further increases, the peak at the positive i^f yields the highest welfare, as shown in Figure 1(c).

If i^d increases to $\alpha_2 \lambda(0)$, as shown in Figure 1(d), buyers are constrained in type 3 meetings for all $i^f > 0$. Because the digital currency is not valued if $i^f = 0$, the optimal i^f is positive so that buyers can consume in type 2 meetings. If i^d increases further as in Figure 1(e), the digital currency is not valued if i^f is below $\underline{I}_2^f\left(i^d\right) = (\alpha_1 + \alpha_3)\left[i^d - \alpha_2 \lambda(0)\right]/\alpha_3$. If i^f is less than $\underline{I}_2^f\left(i^d\right)$, an increase in i^f only reduces the holding of the fiat money and hurts all transactions. Therefore, welfare is decreasing in i^f. As i^f moves above $\underline{I}_2^f\left(i^d\right)$, the digital currency starts to be valued. Then an increase in i^f increases the holding of the digital currency, benefiting the type 2 meetings. Therefore, welfare starts to increase with i^f. Again, there are two local maximizers, one at 0 and the other positive. If i^d is not too high, the positive local maximizer yields the highest welfare and is the optimal policy. But if i^d is too high as in Figure 1(f), $i^f = 0$ leads to the maximum welfare.

In general, the welfare function can differ from this example in certain aspects. For example, if $\alpha_1 = 0$, buyers in type 3 meetings are constrained at any $i^f > 0$ unless $i^d = 0$. Then different from Figures 1(b) and 1(c), welfare is always increasing at i^f close to 0 if $0 < i^d < \alpha_2 \lambda(0)$. Also, at certain values of i^d, there can be multiple local maxima in the region with $i^f > 0$.

Figure 2(a) shows the optimal central bank policy, B^f, from example 1. If i^d is sufficiently small, $B^f\left(i^d\right)$ is 0. As i^d increases, it reaches a point at which $B^f\left(i^d\right)$ contains two elements. One is 0 and the other is positive. This occurs because the local maximum of the welfare function at the positive i^f equals that at $i^f = 0$. As i^d further increases, $B^f\left(i^d\right)$ contains only one element, which is positive. In this region, the optimal policy of the central bank is to deviate from the Friedman rule and can potentially lead to high inflation. If i^d is sufficiently high, the central bank finds it too costly to support the digital currency. It then reverts to the Friedman rule and the digital currency is not valued. As pointed out before, the digital currency restricts the central bank policy by putting a cap. In this example, the cap



(a) $i^d = 0$

(b) $i^d = 0.03$

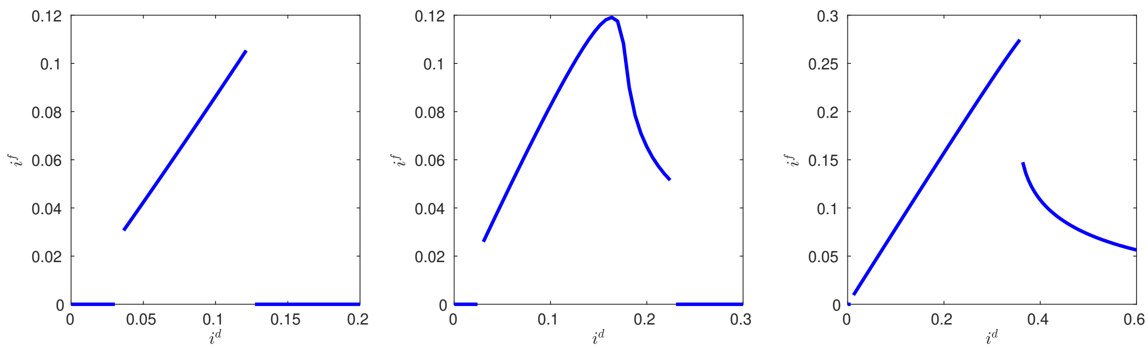
(c) $i^d = 0.04$

(d) $i^d = \alpha_2 \lambda(0) = 0.052$

(e) $i^d = 0.12$

(f) $i^d = 0.15$

Figure 1: Example 1: Welfare as a Function of i^f



(a) Example 1 (b) Example 2 (c) Example 3

Figure 2: Best Response of the Central Bank

only rules out suboptimal policies. In general, the central bank's optimal policy is either an interior point or to make the fiat money worthless. In either case, the cap does not affect the central bank's ability to implement the optimal policy. But the digital currency can change the optimal policy of the central bank. Without the digital currency, the optimal policy is $i^f = 0$. But with the digital currency, the optimal policy can change to some high i^f, leading to high inflation. As we will see later, this can strengthen the market power of the digital currency issuer and lead to low welfare in a policy setting game.

In this example, $B^f\left(i^d\right)$ is single-valued, continuous and increasing when it is positive. But this may not be the case in general. This is illustrated in examples 2 and 3 where $B^f\left(i^d\right)$ can be non-monotone and contain multiple values in the region where it is positive. Theoretically, we can show that Proposition 2 holds. Because $B^f(i^d)$ can be set-valued, we say that $B^f(i^d)$ is bigger than some number if the infimum of all its elements is bigger than that number.

Proposition 2 *If $\alpha_2, \alpha_3 > 0$ and $\lambda(0) < \infty$, then there exist three cut-offs $\tilde{\imath}_1^d$, $\tilde{\imath}_2^d$ and $\tilde{\imath}_3^d$ such that $0 \le \tilde{\imath}_1^d \le \tilde{\imath}_2^d < \alpha_2\lambda(0) < \tilde{\imath}_3^d$; $\tilde{\imath}_1^d$, $\tilde{\imath}_2^d > 0$ if $\alpha_1 > 0$ and converge to 0 if $\alpha_1 \to 0$. The central bank's optimal policy, $B^f\left(i^d\right)$, satisfies*

(a). $B^f\left(i^d\right) = \{0\}$ if $i^d < \tilde{\imath}_1^d$ or $i^d > \tilde{\imath}_3^d$ and $B^f\left(i^d\right) > 0$ for all $i^d \in \left(\tilde{\imath}_2^d, \tilde{\imath}_3^d\right)$.

(b). Suppose there exists $\underline{z} < z^*$ such that λ is log-concave on $[\underline{z}, z^*)$. Then if $\alpha_1 = 0$ and $i^d > 0$ but not too big, $B^f\left(i^d\right) = [\bar{\imath}^f, \infty)$ and the fiat money is not valued at $B^f(i^d)$.

Proof. See Appendix D. ■

This proposition has two important implications. First, $B^f\left(i^d\right)$ is more likely to be positive if α_1 is small. In other words, the optimal policy of the central bank is more likely to be changed by the digital currency if the use of the fiat money is low. As α_1 approaches 0, $\tilde{\imath}_2^d$ decreases, making it easier for i^d to be above $\tilde{\imath}_2^d$. Intuitively, if $i^f = 0$ and $i^d > 0$, consumption is efficient in type 1 and type 3 meetings, but inefficiently low in type 2 meetings. An increase in i^f from 0 results in a second-order welfare loss in type 3 meetings and a very small first-order welfare loss in type 1 meetings if α_1 is small, but can lead to a substantial first-order gain by raising consumption in type 2 meetings. As a result, deviating from the Friedman rule leads to a welfare gain. This mechanism relies only on the fact that consumption is efficient in type 1 and type 3 meetings under $i^f = 0$. Therefore, it holds under other trading mechanisms that share this property, such as competitive pricing and the strategic game analyzed in Zhu (2019).

Second, if $\alpha_1 = 0$, i.e. no meetings requires the fiat money, it is optimal for the central bank to drive its own money out of market as long as i^d is not too big. This captures the network externality in payment: it is optimal that all agents coordinate on the same payment method if it can serve all meetings. As a result, the fiat money only reduces welfare if it is valued. This result requires that λ is log-concave on $[\underline{z}, z^*)$, which is not demanding. It is satisifed, for example, if $u(y) = [(y+\varepsilon)^{1-\sigma} - \varepsilon^{1-\sigma}]/(1-\sigma)$, $c(y) = y$ and buyers make take-it-or-leave-it offers as long as ε is not too big. Notice that the fiat money and the digital currency are symmetric in the equilibrium and in the welfare function. Therefore, we can obtain the following corollary.

Corollary 1 *Suppose that λ is log-concave on $[\underline{z}, z^*)$ for some $\underline{z} < z^*$. If $\alpha_1, \alpha_3 > 0$, $\alpha_2 = 0$ and $i^f > 0$ not too big, then a digital currency reduces welfare if it is valued.*

This corollary implies that a private digital currency can only reduce welfare if it does not expand the types of meetings that can be served. The digital currency, if valued, reduces buyers' holding of the fiat money. This hurts type 1 meetings, which dominates the potential gains in type 3 meetings. Notice that the digital currency is valued only if $i^d < i^f$, i.e. it is less costly to use than the fiat money. Despite of the lower cost, it reduces welfare. By continuity, the same holds true if α_2 is sufficiently small. Again, network externality in payment is at work here: if the digital currency is not very useful, it is better that everyone coordinates on using only the fiat money.

4 A Simultaneous Move Policy Game

This section studies a two-stage policy-setting game. In the first stage, the central bank sets i^f to maximize welfare while the issuer of the digital currency sets i^d to maximize profit (seigniorage income). In the second stage, the economy figures out the steady state equilibrium given the policy choices.[9] This section focuses on a setup where i^f and i^d are set simultaneously. This setup is appropriate if the central bank cannot commit to not responding to the digital currency issuer's policy. Section 6 extends the analysis to a setup where the central bank leads and the digital currency issuer follows.

We focus on the pure strategy Nash equilibrium. We denote the equilibrium polices as $\left(i^f_*, i^d_*\right)$. As is standard, solving for the equilibrium takes two steps. In the first step, we characterize the best response functions of the central bank and the digital currency issuer. In the second step, we look for the fixed points. One complication arises because, as illustrated in the previous section and will be shown later, these two best responses may not be continuous. As a result, it is difficult to obtain a general result on the existence of a pure strategy equilibrium. For our purposes, it suffices to focus on the properties of the equilibria

[9] In other words, we restrict the action space to be policies that have constant long-run money growth rates. We do not consider time-varying policies and assume full commitment. This policy-setting game is related to Zhang (2014) and Geromichalos and Herrenbrueck (2016). The former considers a game where two central banks compete to set policies to maximize welfare of their own citizens. The latter considers an optimal asset issuance problem with two competing asset issuers.

given the existence, which is the focus of this section. It is worth mentioning that in all of the numerical examples considered in this paper, at lease one pure strategy equilibrium exists. Moreover, under certain parametrization, we can derive easy-to-check conditions for the existence of a pure strategy equilibrium. We relegate this discussion to Appendix F.

4.1 Best Response of the Digital Currency Issuer

Section 3 has characterized the best response of the central bank. We only need to obtain that of the digital currency issuer. The digital currency issuer sets a constant growth rate and sells the newly issued digital currency in the CM at the market price. The per-period profit at the steady state is

$$\Pi\left(i^f, i^d\right) = \psi_t \mu^d d_t = \left[\beta\left(i^d+1\right)-1\right] z^d,$$

where z^d depends on i^f and i^d through (13)-(14).[10] The profit is positive if and only if $\beta\left(i^d+1\right)-1>0$. Throughout the paper, we assume that both $\Pi(0, i^d)$ and $\Pi(\infty, i^d)$ are unimodal in i^d.[11]

A typical example of the digital currency issuer's profit as a function of i^d is shown in Figure 3, constructed under the same parameters as in example 1. We only show the region with $i^d \geq 1/\beta - 1$ because the digital currency issuer does not set i^d below $1/\beta-1$. If $i^f=0$, shown in Figure 3(a), buyers are unconstrained in type 1 and 3 meetings with only the fiat money. The digital currency is a niche money that only serves type 2 meetings. The profit has a maximum at i^d around 0.05. If i^f increases as in Figure 3(b), the profit function has a kink at $\underline{I}_1^d\left(i^f\right)=\alpha_2\lambda\left(z^*-\lambda^{-1}\left(i^f/\alpha_1\right)\right)$. To the left of the kink, buyers are not constrained in type 3 meetings, implying $\Pi\left(i^f, i^d\right)=\Pi\left(0, i^d\right)$. To the right of the kink, buyers are constrained in these meetings. There are two local maxima. One is achieved to the left of

[10] An alternative formulation is that at the first stage, the digital currency issuer initially issues digital currency and then chooses i^d to maximize its total revenue, which includes both the revenue from initial issuance and the total seigniorage income, i.e., $\Pi^d\left(i^f, i^d\right)=z^d+\frac{\beta}{1-\beta}\left[\beta\left(i^d+1\right)-1\right] z^d$. The results are similar qualitatively.

[11] This is guaranteed, for example, if $\theta=1$ and $u(y)=y^{1-\sigma}/(1-\sigma)$ with $\sigma<1$. Notice that $\Pi(i^f,\cdot)$ can have multiple peaks for values of i^f other than 0 and ∞.

the kink and coincides with the maximum when $i^f = 0$. The other is achieved to the right of the kink when buyers are constrained in type 3 meetings. In this case, the maximum to the left of the kink is higher and the profit-maximizing i^d remains unchanged. If i^f further increases, as shown in Figure 3(c), the maximum to the right of the kink is higher and then the maximizer changes.

If i^f inceases further, shown in Figure 3(d), the fiat money is valued if and only if i^d is above $\underline{I}_2^d\left(i^f\right) = (\alpha_2 + \alpha_3)\left[i^f - \alpha_1\lambda(0)\right]/\alpha_2$, where the kink in the profit function occurs. In this case, the maximum is attained at a point to the right of the kink. At the optimal i^d, both the digital currency and the fiat money are valued. If i^f further increases, shown in Figure 3(e), the maximum is attained right at the kink. Then, the optimal policy of digital currency issuer is to set a i^d just low enough to drive the fiat money out of circulation. Lastly, if i^f is sufficiently high as in Figure 3(f), the maximum is attained to the left of the kink, where the fiat money is not valued. The digital currency issuer behaves as if it is the only money issuer and further increases in i^f do not change the maximizer.

These properties carry over to the digital currency issuer's best response $B^d(\cdot)$ shown in Figure 4(a). If i^f is sufficiently small,

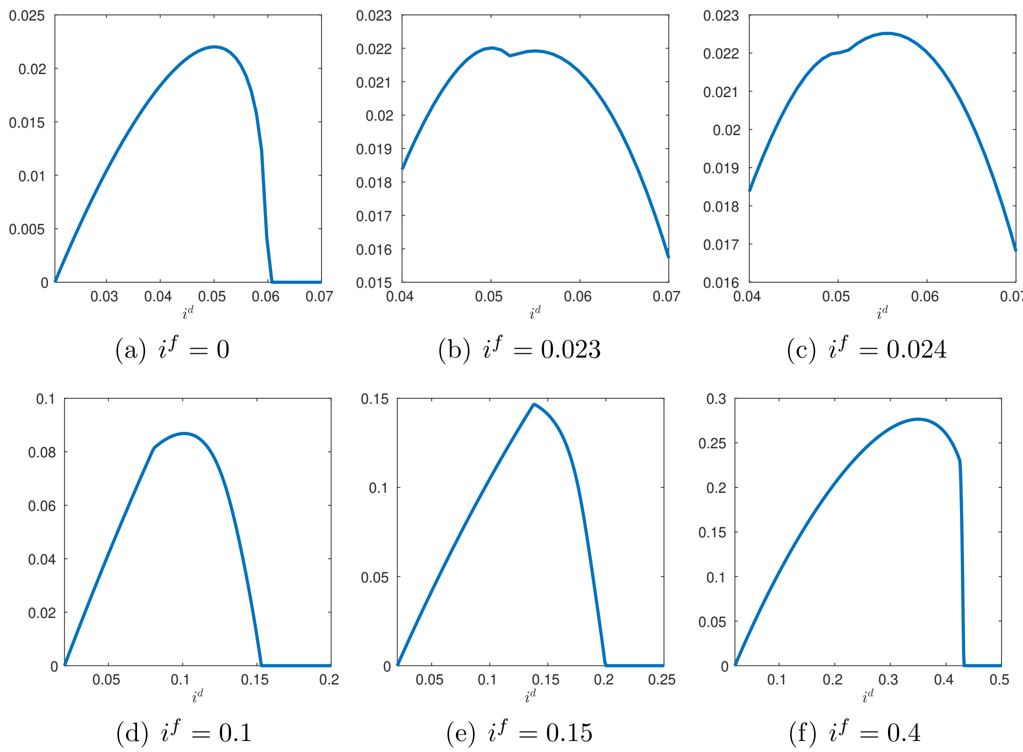
$$B^d\left(i^f\right) = B^d(0) = \arg\max_i \Pi(0,i) = \arg\max_i \left[\beta(1+i) - 1\right]\lambda^{-1}(i/\alpha_2).$$

If i^f is sufficiently large,

$$B^d\left(i^f\right) = B^d(\infty) = \arg\max_i \Pi(\infty,i) = \arg\max_i \left[\beta(1+i) - 1\right]\lambda^{-1}\left(\frac{i}{\alpha_1+\alpha_2+\alpha_3}\right).$$

In both regions, $B^d(\cdot)$ does not change with i^f. Between these two regions, the best respond function has one jump and two kinks.

The jump occurs because there can be multiple local maxima for certain values of i^f, as shown in Figures 3(b) and 3(c). One local maximum is in the region where buyers are unconstrained in type 3 meetings and one is in the region where buyers are constrained in these meetings. If i^f is sufficiently small, the first maximum is larger. There exists an i^f



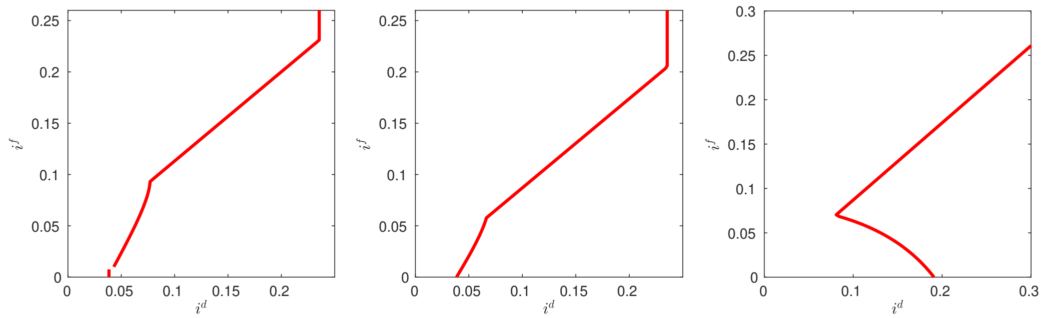
(a) $i^f = 0$ (b) $i^f = 0.023$ (c) $i^f = 0.024$

(d) $i^f = 0.1$ (e) $i^f = 0.15$ (f) $i^f = 0.4$

Figure 3: Example 1: Profit of the Digital Currency Issuer

at which both local maxima have the same value. After this point, the second maximum is larger as i^f, leading to a discoutinuous point in the best response funcion.

The kinks of the best response function occur because the digital currency issuer changes its policy regime. To the left of the first kink, i^f is not too high and the digital currency issuer finds it optimal to allow the fiat money to co-exist with the digital currency. To the right of the kink, i^f is sufficiently high. It is then optimal for the digital currency issuer to drive the fiat money out of the market. However, the digital currency issuer cannot behave as if it is the only money issuer because if it further increases i^d, fiat money starts to circulate. In this region, the central bank's monetary policy stays effective although the fiat money is not valued in equilibrium. It serves as a tool to discipline the digital currency issuer's behavior. This is similar in spirit to Lagos and Zhang (2018) and Chiu et al. (2021). The second kink marks the value of i^f beyond which the digital currency issuer can behave as if it is the only money issuer in the economy.



(a) Example 1 (b) Example 4 (c) Example 5

Figure 4: Policy Game: Best Response Functions

Again, the best response may look slightly different under different parameters. If $\alpha_1 = 0$ as shown in Figure 4(b), there is no jump because buyers are constrained in type 3 transactions if $i^f > 0$. Also, notice that $B^d(\cdot)$ is not necessarily increasing as shown in Figure 4(c), which is also shown analytically in an exmaple in Appendix F.

Proposition 3 *If $\alpha_2, \alpha_3 > 0$, $\alpha_2\lambda(0) > 1/\beta - 1$, $\lambda(0) < \infty$, and λ''/λ' is decreasing on $(0, z^*)$, then there exist three cut-offs $\tilde{\imath}_1^f$, $\tilde{\imath}_2^f$ and $\tilde{\imath}_3^f$ such that $0 \le \tilde{\imath}_1^f \le \alpha_1\lambda(0) \le \tilde{\imath}_2^f < \tilde{\imath}_3^f$; $\tilde{\imath}_1^f > 0$ if $\alpha_1 > 0$ and converges to 0 if $\alpha_1 \to 0$. The best response of the digital currency issuer $B^d(\cdot)$ satisfies*

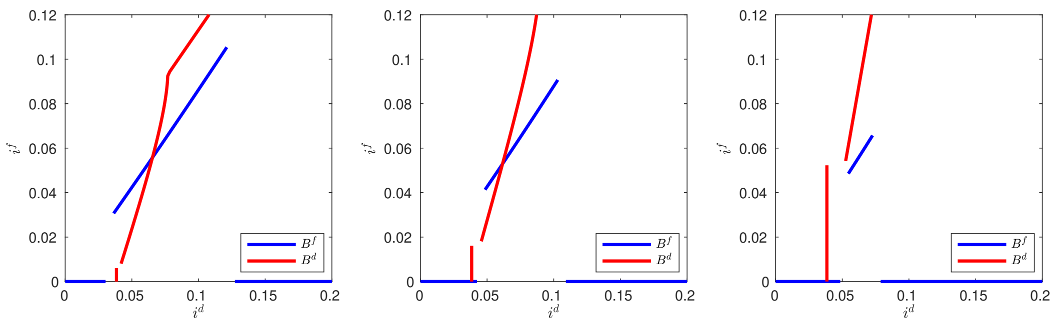
(a). $B^d(i^f) = B^d(0)$ if $i^f < \tilde{\imath}_1^f$ and $B^d(i^f) = B^d(\infty)$ if $i^f > \tilde{\imath}_3^f$;

(b). $B^d(i^f) = \left[i^f - \alpha_1\lambda(0)\right](\alpha_2 + \alpha_3)/\alpha_3$ if $i^f \in \left[\tilde{\imath}_2^f, \tilde{\imath}_3^f\right]$ and the fiat money is not valued.

Proof. See Appendix E. ■

4.2 Equilibrium

Figure 5 shows the best responses of the central bank and the digital currency issuer under different values of α_1, where all the other parameters are the same as Example 1. Any intersection of the best responses corresponds to a pure strategy equilibrium. If α_1 is low as in Figure 5(a), the best responses intersect only once which implies a unique pure strategy equilibrium. In this equilibrium, i_*^f and i_*^d are both far away from the Friedman, leading to high inflation and low welfare. This is because of a strategy complementarity. The digital currency issuer wants a high i^d to get a high profit. The central bank, which cares about the total welfare, then responds by increasing i^f to make digital currency more valuable, which benefits type 2 meetings at the cost of the type 1 and type 3 meetings. Because type 1 meetings are not important and the fiat money is not very useful, this improves aggregate welfare. However, this also strengthens the market power of the digital currency issuer. It then responds by further increasing i^d to capture more profit, which motivates the central bank to further increase i^f. This strategic complementarity drives the equilibrium away from the first best.



(a) Example 1: $\alpha_1 = 0.03$ (b) Example 1: $\alpha_1 = 0.07$ (c) Example 1: $\alpha_1 = 0.3$

Figure 5: Equilibrium in the Simultaneous Move Game

If α_1 is higher, as shown in Figure 5(b), the best response functions intersect twice, resulting in two pure strategy equilibria. One equilibrium is qualitatively similar to the one in Figure 5(a). But in the other equilibrium, the central bank runs the Friedman rule, i.e. $i^f_* = 0$ and the digital currency issuer sets a low i^d_*, resulting in low inflation and high welfare. Intuitively, if α_1 is not too small, the central bank realizes that it needs to maintain the value of the fiat money so that buyers can consume more in type 1 meetings. This weakens the incentive to make the digital currency more valuable, leading to the welfare superior equilibrium.

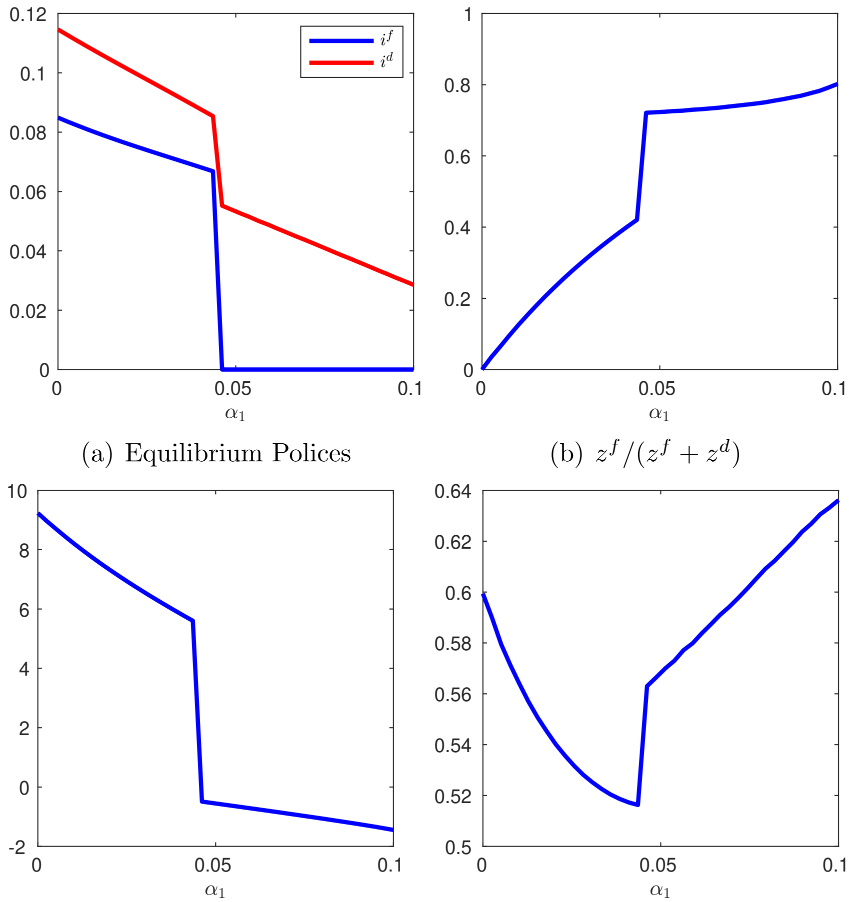
If α_1 is sufficiently high, as shown in Figure 5(c), there exists a unique equilibrium where $i^f_* = 0$ and i^d_* is low. Now type 1 meetings are very important, which makes fiat money very useful. The central bank then finds it optimal to reduce the cost of using fiat money so that households hold enough to obtain the efficient consumption in type 1 meetings. In other words, a high α_1 serves as a commitment device for the central bank to not adopt policies that benefits the market power of the digital currency issuer, ruling out the welfare inferior equilibrium. Theoretically, the following proposition holds.

Proposition 4 *If α_1 is sufficiently small, $i_*^f > 0$ in the simultanous move game.*

Proof. Notice that by proposition 2, $B^f\left(i^d\right)$ contains only positive elements if $i^d \in \left(\tilde{\imath}_2^d, \tilde{\imath}_3^d\right)$ and $\tilde{\imath}_2^d \to 0$ if $\alpha_1 \to 0$. Therefore, if α_1 falls below a cut-off, $\tilde{\imath}_2^d$ falls below $1/\beta - 1$. Then $i_*^d \in \left(\tilde{\imath}_2^d, \tilde{\imath}_3^d\right)$ because B^d is always bigger than $1/\beta - 1$. Consequently, $i_*^f > 0$ because $B^f\left(i_*^d\right) > 0$ by proposition 2. ■

The above analysis highlights the fact that a private digital currency can undermine monetary sovereignty by changing the optimal monetary policy in an unpleasant way if the use of central bank money is sufficiently low. To further stress this point, we next show that such a change in optimal policy can cause an upward jump in inflation and a downward jump in welfare as the use of the fiat money decreases gradually. To this end, we conduct a numerical example where $\alpha_1 + \alpha_2 = 0.14$ and other parameters are the same as Example 1. We study how the equilibrium outcomes change as α_1 declines and α_2 increases. One interpretation is that more and more offline vendors move to sell online.[12] Figure 6(a) shows the equilibrium policies as functions of α_1. If α_1 is high, the economy is at the uique equilibrium where $i_*^f = 0$ and i_*^d is low. And as α_1 decreases, i_*^d becomes higher and i_*^f keeps at 0. The economy stays at the low inflation and high welfare equilibrium although there may be another welfare inferior equilibrium. The central bank's optimal policy has not been changed by the digital currency. If, however, α_1 falls below some cut-off, the low inflation and high welfare equilibrium ceases to exist. Then both i_*^f and i_*^d jumps up and the optimal policy of the central bank is changed by the digital currency. From that point on, both i_*^d

[12]One can obtain similar results if α_1 decreases and α_3 increases.



(a) Equilibrium Polices

(b) $z^f/(z^f + z^d)$

(c) Total Inflation

(d) Welfare

Figure 6: Effects of Declining Fiat Money Usage

and i^f_* increase as α_1 further decreases. In this region, the central bank's optimal policy is to encourage the use of the digital currency by making fiat money more costly to use.

Figure 6(b) shows the total value of the fiat money in circulation as a fraction of the total value of both monies. It declines as α_1 decreases and approaches 0 if α_1 approaches 0. If α_1 is sufficiently low, the slope becomes steeper. This is because in that region, the central bank encourages the use of the digital currency, further reducing the value of fiat money. Figure 6(c) shows the aggregate inflation calculated by the average of fiat money inflation and digital currency inflation weighted by their total value. It increases as α_1 decreases and there is a discrete jump if α_1 falls below the cut-off. Lastly and perhaps most importantly, Figure 6(d) shows the welfare. It is measured by ω that satisfies $\Omega\left(i^f_*, i^d_*\right)$ $= (\alpha_1 + \alpha_2 + \alpha_3)\left[u(\omega y^*) - c(\omega y^*)\right]$, i.e. the fraction of efficient DM consumption that yields the same welfare as the equilibrium outcome. If α_1 is above the cut-off, the total welfare declines as α_1 decreases. And if α_1 falls below the cut-off, the total welfare declines sharply as the low inflation and high welfare equilibrium ceases to exist. Then it increases as α_1 further decreases. This is due to the network externality: it is more beneficial if more people use one means of payment. However, even at $\alpha_1 = 0$ the welfare is still lower than at $\alpha_1 = 0.1$.

It is worth noting that at $\alpha_1 = 0$, the fiat money is not used in the equilibrium. However, the welfare is much higher than the case without the fiat money, which attains only 25% of the efficient consumption and features $i^d_* = 0.275$. This is becasue that the existence of fiat money disciplines the digital currency issuer's policy. If, off the equilibrium, the digital currency issuer raises i^d, the fiat money will be valued, which reduces its profit. This is related to Lagos and Zhang (2018) and Chiu et al. (2021). The former shows that monetary policies stay effective even if cash usage approaches 0 because it serves as an outside option. The latter shows that a central bank digital currency can discipline banks' behaviors even if it is not used. Here, we show that if α_1 is 0, the optimal policy of the central bank is indeed to set i^f such that the fiat money is not valued but would be valued if the digital currency issuer increases i^d. On the one hand, if the fiat money is worthless, everyone has to use the digital currency. This is beneficial because of the network externality. On the other hand, the central bank wants to make the fiat money a good outside option to discipline the market power of the digital currency.

Lastly, welfare is weakly higher if the central bank cares about only meetings where the fiat money is used. In this case, the optimal policy of the central bank is $i^f = 0$. If α_1 is high as in Figure 5(c), it leads to the same equilibrium as in the case where the central bank cares about aggregate welfare. If α_1 is intermediate as in Figure 5(b), this picks out the welfare-superior equilibrium. If α_1 is low as in Figure 5(a), this leads to an equilibrium that dominates the unique equilibrium when the central bank maximizes aggregate welfare. Intuitively, if the central bank cares only about meetings using the fiat money, it does not have any incentive to strengthen the market power of the digital currency issuer. Therefore, the force that drives the equilibrium away from the first best disappears.

5 Central Bank Digital Currency

Now suppose that the central bank introduce a CBDC. The CBDC has a total supply of M_t^a at period t and bears an interest i^a. For simplicity, we consider the case where the CBDC is designed to be a perfect substitute to digital currency in type 2 meetings but cannot be used in other meetings. At the end of this section, we briefly discuss alternative designs. The central bank stands ready to exchange the CBDC and the fiat money at par in the CM. It sets a constant growth rate on the total supply of the fiat money and the CBDC, i.e. $M_{t+1}^a + M_{t+1}^d = (1+\mu)(M_t^a + M_t^d)$, but lets the market determine the composition. The central bank also chooses i^a. We show that the central bank can achieve the first best by setting $i^a = 0$ and $i^d = 0$.

Now the buyer problem is

$$\begin{aligned} W_t^B\left(f_t, d_t, a_t\right) &= \max_{x_t, \ell_t, \hat{f}_{t+1}, \hat{d}_{t+1}, \hat{a}_{t+1}} U\left(x_t\right) - \ell_t + \beta V_{t+1}^B\left(\hat{f}_{t+1}, \hat{d}_{t+1}, \hat{a}_{t+1}\right) \\ \text{s.t. } x &= \phi_t\left(f_t - \hat{f}_{t+1}\right) + \phi_t\left[\left(1+i^a\right)a_t - \hat{a}_{t+1}\right] + \psi_t\left(d_t - \hat{d}_{t+1}\right) + \ell_t + T_t, \end{aligned}$$

where $\hat{a}_{t+1}$ is the holding of the CBDC next period. Because the central bank exchanges the CBDC and the fiat money at par, the real price of the CBDC is the same as that of the fiat money. The DM value function then becomes

$$\begin{aligned} V_t^B(f_t, d_t, a_t) &= \alpha_1\left[u\left(y_t^1\right) - \phi_t f_t^1\right] + \alpha_2\left[u\left(y_t^2\right) - \psi_t d_t^2 - \phi_t\left(1+i^a\right)a_t^2\right] \\ &\quad +\alpha_3\left[u\left(y_t^3\right) - \psi_t d_t^3 - \phi_t f_t^3\right] + W_t^B(f_t, d_t, a_t). \end{aligned} \tag{15}$$

After some calculations, one can show that M_t^a and M_t^f grows at the same rate at the steady state and the equilibrium is determined by

$$i^f \geq \alpha_1\lambda\left(z^f\right) + \alpha_3\lambda\left(z^d + z^f\right) \quad \text{strict inequality if } z^f = 0,$$

$$i^d \geq \alpha_2\lambda\left(\left(1+i^a\right)z^a + z^d\right) + \alpha_3\lambda\left(z^d + z^f\right) \quad \text{strict inequality if } z^d = 0,$$

$$\tfrac{1+i^f}{1+i^a} - 1 \geq \alpha_2\lambda\left(\left(1+i^a\right)z^a + z^d\right) \quad \text{strict inequality if } z^a = 0,$$

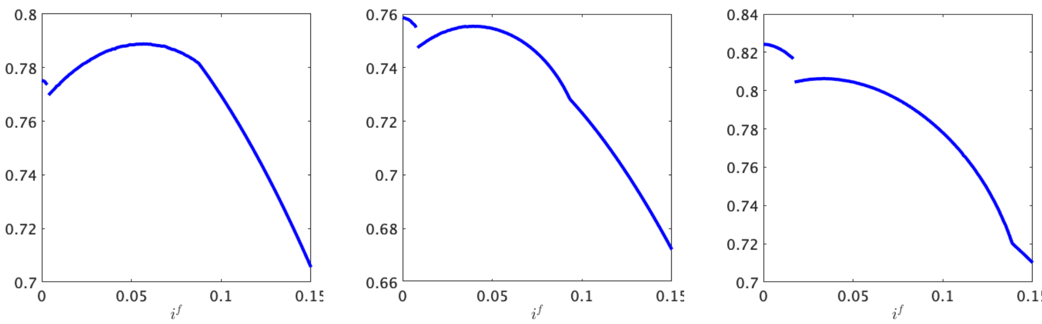
where $z^a = \phi_t M_t^a$. If $i^f = 0$ and $i^a = 0$,

$$\alpha_1\lambda\left(z^f\right) = \alpha_3\lambda\left(z^f + z^d\right) = \lambda\left(z^d + \left(1+i^a\right)z^a\right) = 0.$$

The second steady state condition holds as strict inequality if $i^d > 0$. Therefore, the private digital currency is not valued, because the digital currency issuer only sets $i^d \geq 1/\beta - 1 > 0$. In addition, by (7) and (10), buyers consume y^* in all DM meetings. Then the first best is achieved by Lemma 1.

Proposition 5 *If the CBDC can be used only in type 2 meetings, first best is achieved under* $i^f = 0$ *and* $i^a = 0$.

Here we consider a particular design of CBDC, i.e. it can only be used in type 2 meetings. Of course, there can be other designs, such as a CBDC that serves both type 2 and type 3 meetings or all three meetings. In these cases, $i^f = 0$ and $i^a = 0$ still deliver the first best. In fact, what is crucial for efficiency is that the CBDC is a perfect substitute to the private digital currency in type 2 meetings.



(a) $\alpha_1 = 0.01$ (b) $\alpha_1 = 0.03$ (c) $\alpha_1 = 0.07$

Figure 7: Welfare as a Function of i^f with Sequential Move

6 Extensions

6.1 A Sequential Move Policy Game

We next consider a sequential game in which the central bank first sets its monetary policy and then the digital currency issuer follows. This game is suitable for cases where the central bank can commit to not responding to the digital currency policy.

In this game, the central bank takes the digital currency issuer's best response as given and sets a policy that maximizes the total welfare. As shown in the previous section, the best response of the digital currency issuer can contain multiple elements. We assume that the digital currency issuer always picks the smallest i^d in such a situtation. Then one can show that the total welfare function is upper semi-continuous in i^f. Consequently, there exists at least one solution to the central bank's problem. In addition, we can obtain a result analogous to Proposition 4. Therefore, the digital currency still changes the optimal policy of the central bank if use of the fiat money is sufficiently low.

Proposition 6 *The sequential move game has at least one pure strategy equilibrium. If α_1 is sufficiently small, the equilibrium central bank policy, i^f_*, is positive.*

Proof. Proof omitted because it is similar to that of Proposition 4. ■

Figure 7 shows the total welfare as a function of i^f in the sequential move game as α_1

changes. Other parameters are the same as in Example 1. In principle, it can have four regions separated by kinks or discontinuities, which correspond to the four regions of the digital currency issuer's best response function. Figure 7 shows only the first three regions. In the first region, i^f is sufficiently small and buyers are unconstrained in type 3 meetings. Then, welfare is strictly decreasing because a higher i^f only hurts type 1 meetings. As i^f moves above a threshold, the digital currency issuer's best response jumps up. Consequently, welfare drops discontinuously, which starts the second region. In this region, a higher i^f first increases welfare because gains from type 2 meetings dominate and then decreases welfare because losses from type 1 and 3 meetings dominate. In the first two regions, the digital currency and the fiat money co-exist. As i^f further increases, a kink marks the start of the third region. In this region, the digital currency issuer finds it optimal to drive out fiat money. If i^f is higher, the digital currency issuer can drive out fiat money at a higher i^d. Hence, a higher i^f unambiguously decreases welfare. If i^f further increases, another kink occurs and the fourth region starts. In this region, the fiat money is not valued even if the digital currency issuer acts as if it is the only money issuer in the economy. Then, welfare stays constant as i^f increases.

Compared to the simultaneous move game, the outcome can be more efficient because the central bank moves first. This is true under $\alpha_1 = 0.03$ and $\alpha_1 = 0.07$, as shown in Figures 7(b) and 7(c). In both cases, the equilibrium in the sequential game is $i^f = 0$. If $\alpha_1 = 0.03$, the simultaneou move game has a unique pure strategy equilibrium, which is dominated by the equilibrium in the sequential move game. If $\alpha_1 = 0.07$, the simultaneous move game has two equilibria and the sequential move game picks out the more efficient one. However, if α_1 is sufficiently small, the optimal i^f can still be positive, as shown in Figure 7(a).

To sum up, in the sequential move game, the central bank has more ability to control inflation because it commits to not responding to the private issuer. This yields better outcome. But again, as the use of the fiat money declines, it is optimal for the central bank to tolerate more inflation and strengthen the digital currency issuer's market power.

Therefore, our main insight remains robust: the private digital currency still negatively impacts monetary sovereignty if the use of the central bank money is sufficiently low. It is worth noting that unlike in the simultaneous game, welfare is higher if the central bank maximize total welfare than if the central bank maximize welfare from transactions using the fiat money, i.e. type 1 and type 3 meetings. In the former case, the central bank can always set $i^f = 0$, which is the optimal policy in the latter case. Therefore, the central bank always achieve weakly higher welfare if it maximizes total welfare.

6.2 Fixed Adoption Costs

This section endogenizes the adoption decisions of the digital currency through a fixed cost. Suppose that type 3 sellers can accept the fiat money at no cost and may accept digital currency in the next DM if and only if they pay a per-period cost κ in the current CM. Let τ be the acceptance rate of digital currency in type 3 meetings. Buyers take τ as given and decide how much fiat money and digital currency to hold. Given i^d, i^f and τ, the steady state equilibrium is determined by

$$i^f = [\alpha_1 + \alpha_3 (1 - \tau)] \lambda \left(z^f\right) + \alpha_3 \tau \lambda \left(z^d + z^f\right) \quad (16)$$

$$i^d = \alpha_2 \lambda \left(z^d\right) + \alpha_3 \tau \lambda \left(z^d + z^f\right). \quad (17)$$

These two equations implicitly define z^f and z^d as functions of τ. Denote these functions as $Z^f(\tau)$ and $Z^d(\tau)$, respectively.

If a type 3 seller chooses to accept the digital currency, he can sell more upon meeting a buyer. This benefit is

$$\begin{aligned}\Sigma(\tau) = {} & \beta\alpha_s \left\{P\left[Z^d(\tau) + Z^f(\tau)\right] - c \circ Y\left[Z^d(\tau) + Z^f(\tau)\right]\right\} \\ & -\beta\alpha_s \left\{P\left[Z^f(\tau)\right] - c \circ Y\left[Z^d(\tau)\right]\right\},\end{aligned}$$

where $\circ$ means the composition of the two functions. This benefit depends on other sellers' adoption decisions through τ because these decisions determine buyers' holding of the digital currency, which in turn changes the benefit from accepting the digital currency. If the benefit is higher than κ, the seller pays the cost and accept the digital currency. If the benefit is lower than κ, he does not adopt. And if the benefit equals κ, he is indifferent between adopting or not. Let φ be the adoption probability of a type 3 seller as a function of τ. Then we have

$$\varphi(\tau) = \begin{cases} 1 & \text{if } \kappa < \Sigma(\tau) \\ (0,1) & \text{if } \kappa = \Sigma(\tau) \\ 0 & \text{if } \kappa > \Sigma(\tau) \end{cases},$$

and the equilibrium is characterized by (16)-(17) and $\varphi(\tau) = \tau$. To analyze the equilibrium, we first derive properties of $\Sigma(\tau)$.

Lemma 2 *(a) $\Sigma(\tau)$ is weakly increasing in τ. (b) $\Sigma(\tau)$ is weakly increasing in i^f for a given τ. (c) $\Sigma(\tau)$ is weakly increasing in α_2 for a given τ.*

Proof. Under Kalai bargaining,

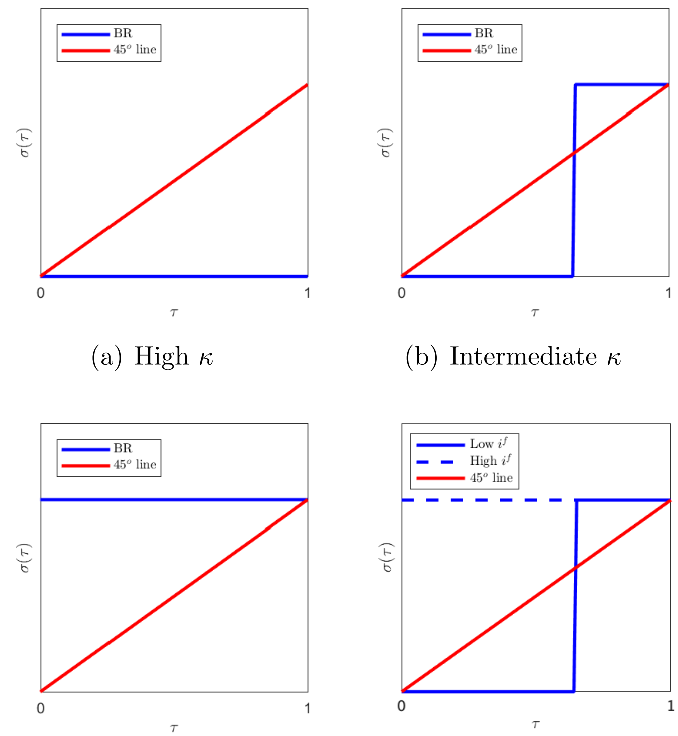
$$\Sigma(\tau) = \alpha_s \beta(1-\theta) \int_{Z^f(\tau)}^{Z^d(\tau)+Z^f(\tau)} \lambda(z) dz.$$

If τ increases, $Z^d(\tau)$ increases and $Z^f(\tau)$ decreases. Because λ is decreasing, this implies that $\Sigma(\tau)$ increases with τ. This proves the first claim. The second and the third claims follow in the same way because a higher i^f or a higher α_2 increases $Z^d(\tau)$ and decreases $Z^f(\tau)$ ■

Intuitively, an increase in τ, or i^f, or α_2 makes digital currency more attractive and buyers hold more of it. As a result, the benefit from accepting digital currency increases.

Proposition 7 *The steady state has three possibilities. (1) If $\Sigma(0) > \kappa$, digital currency is accepted in all type 3 meetings. (2) If $\Sigma(1) < \kappa$, digital currency is not accepted in type 3 meetings. (3) If $\Sigma(0) < \kappa < \Sigma(1)$, there exist three equilibria. The digital currency may be accepted in all type 3 meeting, or in no type 3 meetings, or in a fraction $\tau \in (0,1)$ of type 3 meetings where $\kappa = \Sigma(\tau)$.*

Proof. By Lemma 2, $\Sigma(\tau)$ is weakly increasing, which implies that if $\Sigma(0) > \kappa$, $\Sigma(\tau) > \kappa$ for any τ. Hence, the only equilibrium is $\tau = 1$. Similarly, $\tau = 0$ if $\Sigma(1) < \kappa$. If $\Sigma(0) <$



(a) High κ (b) Intermediate κ

(c) Low κ (d) Different i^f

Figure 8: Best Response

$\kappa < \Sigma(1)$, $\varphi(0) = 0$, which implies $\tau = 0$ is an equilibrium. Similarly, $\tau = 1$ is another equilibrium. In addition, by continuity of $\Sigma(\tau)$, there exists a $\tau \in (0, 1)$ such that $\kappa = \Sigma(\tau)$, which is the third equilibrium. ■

Notice that that if $i^f = 0$, $\Sigma(\tau) = 0$ for any τ because buyers are not constrained in type 3 meetings with only the fiat money. Therefore, the digital currency is used only as a niche money for any κ. However, if $i^f > 0$, $\Sigma(0) > 0$ and type 3 sellers accept digital currency if κ is sufficiently low.

Figure 8 plots the best response (BR) of a type 3 seller, $\varphi(\tau)$, and the 45^o line. Any intersection between them constitutes an equilibrium. Figures 8(a)-8(c) are under different values of κ. If κ is high, $\varphi(\tau)$ intersects the 45^o line only at the origin and the only possible equilibrium is $\tau = 0$. If κ is intermediate, $\varphi(\tau)$ cuts the 45^o line at $\tau = 0$, $\tau = 1$ and some $\tau \in (0, 1)$. As a result, there are three equilibria: (1) all type 3 sellers do not accept digital currency; (2) all of them accept digital currency; and (3) some accept and some do not. Lastly, if κ is sufficiently low, the only equilibrium is at $\tau = 1$.

This analysis illustrates the discontinuities that can arise in adoption of the digital currency. If κ is high, the digital currency is a niche product that is used only in type 2 meetings. As κ decreases, digital currency may continue to be a niche product, i.e., agents coordinate on the equilibrium with $\tau = 0$. But if κ becomes sufficiently small, the equilibrium with $\tau = 0$ disappears and suddenly, the economy switches to the equilibrium with $\tau = 1$. Inappropriate monetary policy by the central bank can expedite this process. Figure 8(d) illustrates this. If i^f is low, $\varphi(\tau)$ is depicted by the blue solid line. There exists equilibria with $\tau = 0$ and $\tau \in (0, 1)$. But if i^f is high, $\varphi(\tau)$ changes to the dashed line. Then the only equilibrium is $\tau = 1$. Intuitively, a higher i^f makes buyers more constrained by using only the fiat money and raises the benefits of using digital currency. As a result, type 3 sellers are more willing to accept digital currency.

The previous analysis on the optimal central bank policy remains valid with a fixed adoption cost. To see this, notice that if $\kappa = 0$, the equilibrium is the same with and without the adoption decision. Therefore, previous results on the optimal policy hold if κ is 0. By continuity, the same conclusions hold under any sufficiently small κ. Figure 9 illustrates that i^f_* can be positive given an exogenous i^d. If i^f is low, the digital currency is not accepted by type 3 sellers and increasing i^f reduces welfare. If i^f is sufficiently high, digital currency is accepted by all type 3 sellers and buyers consume more.[13] This benefit outweighs the fixed cost to accept the digital currency as long as κ is not too high. Therefore, welfare jumps up. Once the digital currency is used in type 3 meetings, it starts to be a substitute for the fiat money. Then the effect of i^f on digital currency, which is described in previous sections, becomes active. A higher i^f can raise welfare and i^f_* can be positive. This happens if κ is low, as shown by the magenta curve. Notice that this also requires α_1 to be small. Indeed, $\alpha_1 = 0$ in this example. Therefore, the digital currency can still undermine monetary sovereignty by

[13] We pick the equilibrium with $\tau = 0$ as long as it exists. If it does not exist, the only equilibrium is $\tau = 1$.

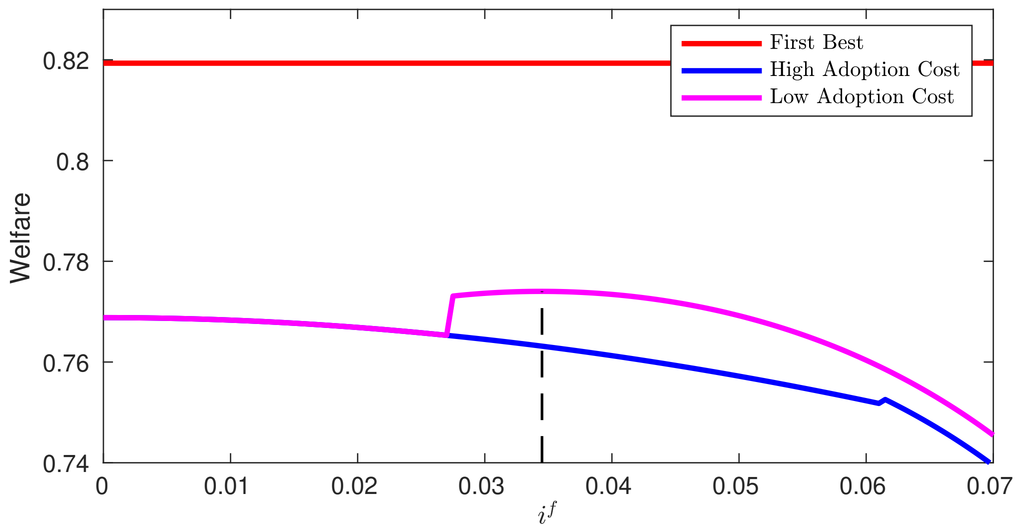


Figure 9: Welfare as a Function of i^f with $\kappa > 0$

changing the optimal policy of the central bank if the use of central bank money is low.

7 Conclusion

Many central banks are concerned that wide adoption of a private digital currency and decline in the use of central bank money may undermine monetary sovereignty. This concern is one of the motivations for issuing a CBDC but is not formally assessed. This paper shows that this concern is valid in terms of monetary policy. A digital currency can change the optimal policy of the central bank in an unpleasant way. This occurs if the use of the central bank money is sufficiently low. In this case, the central bank's optimal policy is to strengthen the market power of the digital currency issuer, which leads to high inflation and low welfare. This insight is robust to various extensions of the model. To defend monetary sovereignty, the central bank should maintain or expand the use of central bank money. One option is to introduce a central bank digital currency. We show that with a properly design, a CBDC an help the central bank to obtain the first best.

Our results also shed light on how a central bank should set its policy target if it decides to not issue a CBDC and a private digital currency is widely used. If the central bank cannot commit to not responding to the private digital currency policy as in the simultaneous move game, it should aim to maintain the value of the fiat money instead to maximizing total welfare. If it can commit, as in the sequantial move game, it should aim to maximize welfare. We think it is an important question whether and how the central bank should reform its policy targets with the presence of private digital currencies. Further research along this line is needed.

Appendix A: Derivations of (8) and (9)

Notice that (7) implies

$$\begin{aligned}
y_t^1 &= \begin{cases} g^{-1}(\phi_t f_t) & \text{if } \phi_t f_t < z^* \\ y^* & \text{otherwise} \end{cases} \text{ and } f_t^1 = \begin{cases} f_t & \text{if } \phi_t f_t < z^* \\ z^*/\phi_t & \text{otherwise} \end{cases}, \\
y_t^2 &= \begin{cases} g^{-1}(\psi_t d_t) & \text{if } \psi_t d_t < z^* \\ y^* & \text{otherwise} \end{cases} \text{ and } d_t^2 = \begin{cases} d_t^2 & \text{if } \psi_t d_t < z^* \\ z^*/\psi_t & \text{otherwise} \end{cases}, \\
y_t^3 &= \begin{cases} g^{-1}(\psi_t d_t + \phi_t f_t) & \text{if } \psi_t d_t + \phi_t f_t < z^* \\ y^* & \text{otherwise} \end{cases} \\
\phi_t f_t^3 + \psi_t d_t^3 &= \begin{cases} \psi_t d_t + \phi_t f_t & \text{if } \psi_t d_t + \phi_t f_t < z^* \\ z^* & \text{otherwise} \end{cases}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\frac{\partial y_t^1}{\partial f_t} &= \begin{cases} \phi_t / g'[g^{-1}(\phi_t f_t)] & \text{if } \phi_t f_t < z^* \\ 0 & \text{otherwise} \end{cases}, \frac{\partial f_t^1}{\partial f_t} = \begin{cases} 1 & \text{if } \phi_t f_t < z^* \\ 0 & \text{otherwise} \end{cases}, \\
\frac{\partial y_t^2}{\partial d_t} &= \begin{cases} \psi_t / g'[g^{-1}(\psi_t d_t)] & \text{if } \psi_t d_t < z^* \\ 0 & \text{otherwise} \end{cases}, \frac{\partial d_t^2}{\partial d_t} = \begin{cases} 1 & \text{if } \psi_t d_t < z^* \\ 0 & \text{otherwise} \end{cases}, \\
\frac{\partial y_t^3}{\partial f_t} &= \begin{cases} \phi_t / g'[g^{-1}(\psi_t d_t + \phi_t f_t)] & \text{if } \psi_t d_t + \phi_t f_t < z^* \\ 0 & \text{otherwise} \end{cases}, \\
\frac{\partial y_t^3}{\partial d_t} &= \begin{cases} \psi_t / g'[g^{-1}(\psi_t d_t + \phi_t f_t)] & \text{if } \psi_t d_t + \phi_t f_t < z^* \\ 0 & \text{otherwise} \end{cases}, \\
\frac{\partial (\phi_t f_t^3 + \psi_t d_t^3)}{\partial f_t} &= \begin{cases} \phi_t & \text{if } \psi_t d_t + \phi_t f_t < z^* \\ 0 & \text{otherwise} \end{cases}, \\
\frac{\partial (\phi_t f_t^3 + \psi_t d_t^3)}{\partial d_t} &= \begin{cases} \psi_t & \text{if } \psi_t d_t + \phi_t f_t < z^* \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

Differentiate (5) with respect to f_t and use (4) to obtain

$$\frac{\partial V_t^B(f_t, d_t)}{\partial f_t} = \alpha_1 \left[u'(y_t^1) \frac{\partial y_t^1}{\partial f_t} - \phi_t \frac{\partial f_t^1}{\partial f_t} \right] + \alpha_3 \left[u'(y_t^3) \frac{\partial y_t^3}{\partial f_t} - \frac{\partial (\phi_t f_t^3 + \psi_t d_t^3)}{\partial f_t} \right] + \phi_t.$$

Substitute in the expressions for the derivatives and use the definition of λ to arrive at (8). A similar derivation leads to (9).

Appendix B: Proof of Proposition 1

First, suppose that $i^f \geq (\alpha_1 + \alpha_3)\lambda(0)$ and $i^d \geq (\alpha_2 + \alpha_3)\lambda(0)$. Then (13)-(14) hold with $z^h = z^f = 0$, which implies that exists an equilibrium in regime 1. Because λ is decreasing, this is the only equilibrium that can exist and the conditions are sufficient for regime 1. Moreover, if regime 1 occurs, (13)-(14) imply $i^h \geq (\alpha_1 + \alpha_3)\lambda(0)$ and $i^d \geq (\alpha_2 + \alpha_3)\lambda(0)$, i.e. these conditions are necessary. This establishes Proposition 1(a).

Next, suppose that $i^f < (\alpha_1 + \alpha_3)\lambda(0)$ and $i^d \geq \bar{\imath}^d$. There exists a unique $z^f > 0$ that satisfies

$$i^d \geq \alpha_2\lambda(0) + \alpha_3\lambda\left(z^f\right) \tag{18}$$

$$i^f = \alpha_1\lambda\left(z^f\right) + \alpha_3\lambda\left(z^f\right). \tag{19}$$

Therefore, there is a unique equilibrium in regime 2. Also notice that no other regimes can occur. First, by the previous argument, regime 1 cannot occur. Second, suppose toward contradiction, regime 3 occurs. There exists an $z^d > 0$ such that

$$i^d = \alpha_2\lambda\left(z^d\right) + \alpha_3\lambda\left(z^d\right) \tag{20}$$

$$i^f \geq \alpha_1\lambda(0) + \alpha_3\lambda\left(z^d\right). \tag{21}$$

Then (18) and (20) imply that

$$\alpha_2\lambda\left(z^d\right) + \alpha_3\lambda\left(z^d\right) \geq \alpha_2\lambda(0) + \alpha_3\lambda\left(z^f\right).$$

Because λ is decreasing, this means $z^d < z^f$. Similarly, (19) and (21) imply that

$$\alpha_1\lambda\left(z^f\right) + \alpha_3\lambda\left(z^f\right) \geq \alpha_1\lambda(0) + \alpha_3\lambda\left(z^d\right),$$

which means $z^f < z^d$. This is a contradiction. Therefore, regime 3 cannot occur. Lastly, if there is an equilibrium in regime 4, there exist positive $\tilde{z}^f$ and $\tilde{z}^d$ that satisfy

$$i^d = \alpha_2\lambda\left(\tilde{z}^d\right) + \alpha_3\lambda\left(\tilde{z}^f + \tilde{z}^d\right), \tag{22}$$

$$i^f = \alpha_1\lambda\left(\tilde{z}^f\right) + \alpha_3\lambda\left(\tilde{z}^f + \tilde{z}^d\right). \tag{23}$$

Comparing (23) with (19), one can conclude $\tilde{z}^f + \tilde{z}^d > z^f$ because λ is decreasing. This implies that (18) and (22) cannot hold at the same time because $\tilde{z}^d > 0$. This rules out regime 4. Therefore, if $i^f < (\alpha_1 + \alpha_3)\lambda(0)$ and $i^d \geq \bar{\imath}^d$, there is a unique equilibrium and it is in regime 2. Conversely, if there is an equilibrium in regime 2, there exists $z^f > 0$ that satisfies (18)-(19). Rearrange and use the fact λ is decreasing to obtain $i^f < (\alpha_1 + \alpha_3)\lambda(0)$ and $i^d \geq \bar{\imath}^d$. Therefore, there is a unique equilibrium and the equilibrium is in regime 2 if and only if these two conditions hold. This proves Proposition 1(b). A similar argument applies to Proposition 1(c).

Lastly, we show Proposition 1(d). We first show the "if" part. The above analysis implies that the equilibrium can only be in regime 4 if $i^d < \bar{\imath}^d$ and $i^f < \bar{\imath}^f$. We only need to show that an equilibrium exists and is unique, i.e. there exists a unique solution to (13) and (14). Equation (13) defines z^f as a decreasing function of z^d, which we denote, with a bit abuse of notation, as $Z^f\left(z^d\right)$. If z^d is sufficiently large, $Z^f\left(z^d\right) = \underline{z}^f$, where $i^f = \alpha_1\lambda\left(\underline{z}^f\right)$ if $i^f < \alpha_1\lambda(0)$ and $\underline{z}^f = 0$ if otherwise. Similarly, (14) defines z^d as a decreasing function of z^f, denoted as $Z^d\left(z^f\right)$, and for z^f sufficiently large, $Z^d\left(z^f\right) = \underline{z}^d$ where $i^d = \alpha_2\lambda\left(\underline{z}^d\right)$ if $i^d < \alpha_2\lambda(0)$ and $\underline{z}^d = 0$ if otherwise. The equilibrium z^f satisfies $Z^f[Z^d\left(z^f\right)] = z^f$. If $i^d < \bar{\imath}^d$ and $i^f < \bar{\imath}^f$, $Z^f[Z^d(0)] = Z^f\left(\bar{z}^d\right) > 0$, where $\bar{z}^d$ is defined in the main text. If z^f is sufficiently large, $Z^f[Z^d\left(z^f\right)] = Z^f(\underline{z}^d)$, which is positive and finite. Therefore, $Z^f[Z^d\left(z^f\right)] - z^f$ is positive if $z^f = 0$ and negative if z^f is sufficiently large. Because Z^f and Z^d are continuous functions, there exists at least one equilibrium by the intermediate value theorem. To see that it is unique, just notice that at the solution of $Z^f[Z^d\left(z^f\right)] - z^f = 0$, the slope of this function in sign to $-D$ where

$$D = \alpha_1\alpha_2\lambda'\left(z^f\right)\lambda'\left(z^d\right) + \alpha_1\alpha_3\lambda'\left(z^f\right)\lambda'\left(z^f + z^d\right) + \alpha_2\alpha_3\lambda'\left(z^d\right)\lambda'\left(z^f + z^d\right) > 0. \quad (24)$$

Therefore, $Z^f[Z^d\left(z^f\right)] - z^f = 0$ has at most one solution and uniqueness follows. This shows that there is a unique equilibrium and the equilibrium is in regime 4 if $i^d < \bar{\imath}^d$ and $i^f < \bar{\imath}^f$. This proves the"if" part. To see the "only if" part, just notice that the previous analysis implies the equilibrium cannot be in regime 4 if either $i^d > \bar{\imath}^d$ or $i^f > \bar{\imath}^f$.

Appendix C: Comparative Statics

The comparative statics of $\bar{\imath}^f$ and $\bar{\imath}^d$ are straightforward from their definition. Therefore, we focus on the other results. We only derive them when both the fiat money and the digital currency are valuded. Applying the Cramer's rule, one can obtain

$$\begin{aligned}
\frac{dz^f}{di^f} &= \frac{\alpha_2\lambda'\left(z^d\right)+\alpha_3\lambda'\left(z^f+z^d\right)}{D}<0,\\
\frac{dz^d}{di^f} &= -\frac{\alpha_3\lambda'\left(z^f+z^d\right)}{D}>0,\\
\frac{dz^f}{di^d} &= -\frac{\alpha_3\lambda'\left(z^f+z^d\right)}{D}>0,\\
\frac{dz^d}{di^d} &= \frac{\alpha_1\lambda'\left(z^f\right)+\alpha_3\lambda'\left(z^f+z^d\right)}{D}<0,\\
\frac{dz^f}{d\alpha_1} &= -\frac{\lambda\left(z^f\right)\left[\alpha_2\lambda'\left(z^d\right)+\alpha_3\lambda'\left(z^f+z^d\right)\right]}{D}>0,\\
\frac{dz^d}{d\alpha_1} &= \frac{\lambda\left(z^f\right)\alpha_3\lambda'\left(z^f+z^d\right)}{D}<0,\\
\frac{dz^f}{d\alpha_2} &= \frac{\lambda\left(z^d\right)\alpha_3\lambda'\left(z^f+z^d\right)}{D}<0,\\
\frac{dz^d}{d\alpha_2} &= -\frac{\lambda\left(z^d\right)\left[\alpha_1\lambda'\left(z^f\right)+\alpha_3\lambda'\left(z^f+z^d\right)\right]}{D}>0,\\
\frac{dz^f}{d\alpha_3} &= -\frac{\lambda\left(z^f+z^d\right)\alpha_2\lambda'\left(z^d\right)}{D}>0,\\
\frac{dz^d}{d\alpha_3} &= -\frac{\lambda\left(z^f+z^d\right)\alpha_1\lambda'\left(z^f\right)}{D}>0.
\end{aligned}$$

Here D is defined in (24).

Appendix D: Proof of Proposition 2

We prove this proposition by establishing a series of lemmas.

Lemma 3 *If $\alpha_1=0$ and $0<i^d<\alpha_2\lambda(0)$, $B^f(i^d)>0$. Moreover, if i^d is not too big and λ is log-concave on $[\underline{z},z^*)$ for some $\underline{z}<z^*$, $B^f(i^d)=[\bar{\imath}^f,\infty)$.*

Proof. If $0 < i^d < \alpha_2 \lambda(0)$ and i^f sufficiently small, both monies are valued in equilibrium. One can show that

$$\begin{aligned}
\frac{\partial \Omega\left(i^f, i^d\right)}{\partial i^f} &= \alpha_2 \left\{u'[Y\left(z^d\right)] - c'[Y\left(z^d\right)]\right\} Y'\left(z^d\right) \frac{\partial z^d}{\partial i^f} \\
&\quad + \alpha_3 \left\{u'[Y\left(z^d + z^f\right)] - c'[Y\left(z^d + z^f\right)]\right\} Y'\left(z^d + z^f\right) \frac{\partial (z^d + z^f)}{\partial i^f} \\
&= \frac{1}{\theta}\left[\alpha_2 \lambda(z^d) \frac{\partial z^d}{\partial i^f} + \alpha_3 \lambda(z^d + z^f) \frac{\partial (z^d + z^f)}{\partial i^f}\right], \qquad (25)
\end{aligned}$$

where z^d and z^f are determined by

$$i^f = \alpha_3 \lambda\left(z^d + z^f\right), \qquad (26)$$

$$i^d = \alpha_2 \lambda\left(z^d\right) + \alpha_3 \lambda\left(z^d + z^f\right). \qquad (27)$$

This implies

$$\frac{\partial z^d}{\partial i^f} = -\frac{1}{\alpha_2 \lambda'\left(z^d\right)} > 0, \ \frac{\partial (z^d + z^f)}{\partial i^f} = \frac{1}{\alpha_3 \lambda'(z^d + z^f)} < 0.$$

After some algebra, one can show $\partial \Omega\left(i^f, i^d\right) / \partial i^f$ is equal in sign to

$$\frac{\lambda(z^d + z^f)}{\lambda'(z^d + z^f)} - \frac{\lambda(z^d)}{\lambda'(z^d)}. \qquad (28)$$

If $i^f = 0$, $\lambda(z^d + z^f) = 0$ and this expression is positive. Therefore, $B^f(i^d) > 0$ and the first claim of this lemma follows. Moreover, if i^d is sufficiently small, $z^d > \underline{z}$ if $i^f = 0$. Because z^d is increasing in i^f, $z^* > z^d + z^f > z^d > \underline{z}$ for all $i^f > 0$ if $z^f > 0$. Moreover, λ is log-concave on $[\underline{z}, z^*)$ by assumption, implying (28) is positive at all i^f as long as $z^f > 0$. Therefore, the welfare function is increasing until z^f is 0, or equivalently, until $i^f = \bar{\imath}^f$. If $i^f > \bar{\imath}$, the fiat money is not value and welfare is not affected by i^f. Therefore, $B^f(i^d) = [\bar{\imath}^f, \infty)$. ■

Lemma 4 *If $\alpha_1 > 0$, there exists $\tilde{\imath}_1^d > 0$ such that $B^f(i^d) = 0$ for all $i^d \leq \tilde{\imath}_1^d$.*

Proof. First, notice $\Omega(i^f, 0)$ is strictly decreasing in i^f if $i^f < \alpha_1 \lambda(0)$. It is constant in i^f if $i^f > \alpha_1 \lambda(0)$ because the fiat money is not valued. If $i^d < \alpha_2 \lambda(0)$, $\Omega(i^f, i^d)$ is decreasing in i^f

if $i^f < \underline{I}_1^f\left(i^d\right)$ becaus buyers are unconstrained in type 3 meetings in this range. Therefore, $B^f(i^d) = 0$ if and only if

$$\Delta_1(i^d) = \Omega(0, i^d) - \max_{i^f > \underline{I}_1^f(i^d)} \Omega(i^f, i^d) > 0. \tag{29}$$

As discussed above, $\Omega(i^f, 0)$ is decreasing and $\underline{I}_1^f(0) = \alpha_1\lambda(0) > 0$, which implies $\Delta_1(0) > 0$. Moreover, $\Delta_1(i^d)$ is continuous in i^d. Therefore, it is positive if i^d is not too big. We can then define

$$\tilde{\imath}_1^d = \sup\{x : \Delta_1(i^d) > 0 \text{ } \textit{if } i^d \in (0, x)\}. \tag{30}$$

By the previous discussion, $\tilde{\imath}_1^d$ is well-defined and is positive. Then, $B^f(i^d) = 0$ if $i^d < \tilde{\imath}_1^d$ by definition. ■

Lemma 5 *There exists $\tilde{\imath}_3 > \alpha_2\lambda(0) > \tilde{\imath}_2^d \geq \tilde{\imath}_1^d$ such that $\inf B^f(i^d) > 0$ for all $i^d \in \left(\tilde{\imath}_2^d, \tilde{\imath}_3^d\right)$. Moreover, $B^f(i^d) = 0$ if $i^d > \tilde{\imath}_3^d$.*

Proof. If $i^d = \alpha_2\lambda(0)$, digital currency is not valued if $i^f = 0$. Moreover,

$$\left.\frac{\partial\Omega\left(i^f, i^d\right)}{\partial i^f}\right|_{i^f=0} = \alpha_2\left[u'(0) - c'(0)\right] Y'(0)\,\frac{\partial z^d}{\partial i^f} > 0.$$

This suggests $B^f(i^d) > 0$ if $i^d = \alpha_2\lambda(0)$. By continuity, we can conclude that $\Delta_1(i^d) < 0$ if $i^d < \alpha_2\lambda(0)$ and is sufficiently close to i^d. Define

$$\tilde{\imath}_2^d = \inf\{x < \alpha_2\lambda(0) : \Delta_1(i^d) < 0 \text{ } \textit{if } i^d \in (x, \alpha_2\lambda(0))\}. \tag{31}$$

By definition, $\Delta_1(i^d) < 0$ for all $i^d \in (\tilde{\imath}_2^d, \alpha_2\lambda(0))$. This implies that $B^f(i^d) > 0$ if $i^d \in (\tilde{\imath}_2^d, \alpha_2\lambda(0))$. Because $B^f(i^d) = 0$ if $i^d < \tilde{\imath}_1^d$, $\tilde{\imath}_2^d \geq \tilde{\imath}_1^d$ by definition.

Next, if $i^d > \alpha_2\lambda(0)$, $\Omega(i^f, i^d)$ is independent of i^d and strictly decreasing in i^f if $i^f < \underline{I}_2^f\left(i^d\right)$ because the digital currency is not valued. Define

$$\Delta_2(i^d) = \Omega(0, i^d) - \max_{i^f > \underline{I}_2^f(i^d)} \Omega(i^f, i^d).$$

Then $B^f(i^d) = 0$ if $\Delta_2(i^d) > 0$ and $B^f(i^d) > 0$ if $\Delta_2(i^d) < 0$. Notice that $\max_{i^f > \underline{I}_2^f(i^d)} \Omega(i^f, i^d)$ is decreasing in i^d because $\Omega(i^f, i^d)$ is decreasing in i^d and $\underline{I}_2^f(i^d)$ is increasing in i^d. This implies $\Delta_2(i^d)$ is monotonically decreasing if $i^d > \alpha_2\lambda(0)$, because $\Omega(0, i^d)$ is independent of i^d. If i^d is sufficiently large, $\Delta_2(i^d) > 0$ and by the above analysis, $\Delta_2(i^d) < 0$ if $i^d = \alpha_2\lambda(0)$. This implies that there exists a cut-off $\tilde{\imath}_3^d$ such that $\Delta_2(i^d) < 0$ if $i^d \in (\alpha_2\lambda(0), \tilde{\imath}_3^d)$ and $\Delta_2(i^d) > 0$ if $i^d > \tilde{\imath}_3^d$, which concludes the proof. ■

Lemma 6 *If $\alpha_1 \to 0$, then $\tilde{\imath}_2^d \to 0$ and $\tilde{\imath}_1^d \to 0$.*

Proof. Suppose toward the contradiction, there exists a sequence $\alpha_{1n} \to 0$ such that $\alpha_2\lambda(0) > \tilde{\imath}_{2n}^d > \epsilon$ for some $\epsilon > 0$ as $n \to \infty$, where we use subscript n to denote the values along this sequence. This suggests that we can always find $i_n^d > \epsilon$ such that $\Delta_{1n}(i_n^d) \geq 0$ for every n. One can show that Δ_1 is continuous in α_1 uniformly in i^d. This implies that there exists a subsequence n_k such that $\Delta_{1n_k}(i_{n_k}^d)$ converges to $\Delta_{10}(i^{d*})$ for some $i^{d*} > \epsilon$, where Δ_{10} is the value of Δ_1 when $\alpha_1 = 0$. Therefore, $\Delta_{10}(i^{h*}) \geq 0$. However, Lemma 3 and the fact that $B^f(i^d) > 0$ if and only if $\Delta_1(i^d) < 0$ imply $\Delta_{10}(i^{d*}) < 0$ if $i^d \in (0, \alpha_2\lambda(0))$. This creates a contradiction. Then $\tilde{\imath}_2^d \to 0$, and $\tilde{\imath}_1^d \to 0$ because $\tilde{\imath}_1^d \leq \tilde{\imath}_2^d$. ■

All the lemmas together implies that Proposition 2 holds with $\tilde{\imath}_1^d$, $\tilde{\imath}_2^d$ and $\tilde{\imath}_3^d$ defined by (30)-(31) and $\Delta_2(\tilde{\imath}_3^d) = 0$.

Appendix E: Proof of Proposition 3

We establish a series of lemmas that together imply Proposition 3.

Lemma 7 *There exists $\tilde{\imath}_1^f < \alpha_1\lambda(0)$ such that $B^d(i^f) = B^d(0)$ if $i^f < \tilde{\imath}_1^f$.*

Proof. Let $i_1^* = B^d(0) = \arg\max_i[\beta(1+i) - 1]\lambda^{-1}(i/\alpha_2)$ and

$$\Delta_3(i^f) = \Pi(0, i_1^*) - \max_{i^d > \underline{I}_1^d(i^f)} \Pi(i^f, i^d).$$

If $i_1^* < \underline{I}_1^d\left(i^f\right)$, then

$$\Pi(0, i_1^*) = \max_{i^d \in [0, \underline{I}_1^d(i^f)]} \Pi(0, i^d) = \max_{i^d \in [0, \underline{I}_1^d(i^f)]} \Pi(i^f, i^d). \tag{32}$$

The last equality holds because $\Pi(0, i^d) = \Pi(i^f, i^d)$ if $i^d \in [0, \underline{I}_1^d(i^f)]$. These two equations imply that $B^d(i^f) = i_1^*$ if $\Delta_3(i^f) > 0$ and $i_1^* < \underline{I}_1^d(i^f)$.

Next, define

$$\tilde{\imath}_1^f = \sup\{x \geq 0 : \Delta_3(i^f) > 0 \text{ if } i^f \in [0, x]\}. \tag{33}$$

By definition, $\underline{I}_1^d\left(\tilde{\imath}_1^f\right) \geq i_1^*$ because otherwise $\Delta_3(i^f) \leq 0$ for some $i^f < \tilde{\imath}^f$. We now show that $\underline{I}_1^d\left(\tilde{\imath}_1^f\right) > i_1^*$. To see this, suppose toward contradiction $\underline{I}_1^d\left(\tilde{\imath}_1^f\right) = i_1^*$, then by assumption,

$$\Pi(\tilde{\imath}_1^f, i_1^*) = \Pi(0, i_1^*),$$

and buyers are just unconstrained in type 3 meetings if $i^d = i_1^*$ and $i^f = \tilde{\imath}_1^f$. Moreover,

$$\left.\frac{\partial \Pi(\tilde{\imath}_1^f, i^d)}{\partial i^d}\right|_{i^d = i_1^*} = \beta z_1^* + \frac{[\beta(1+i_1^*) - 1][\alpha_1 \lambda'(z^* - z_1^*) + \alpha_3 \lambda'(z^*)]}{\alpha_1\alpha_2\lambda'(z^* - z_1^*)\lambda'(z_1^*) + \alpha_1\alpha_3\lambda'(z^* - z_1^*)\lambda'(z^*) + \alpha_2\alpha_3\lambda'(z^*)\lambda'(z_1^*)},$$

where $z_1^* = \lambda^{-1}(i_1^*/\alpha_2)$. And

$$\left.\frac{\partial \Pi(0, i^d)}{\partial i^d}\right|_{i^d = i_1^*} = \beta z_1^* + [\beta(1+i_1^*) - 1]\frac{1}{\alpha_2 \lambda'(z_1^*)}.$$

Because $\lambda'(z^*) < 0$,

$$\left.\frac{\partial \Pi(\tilde{\imath}_1^f, i^d)}{\partial i^d}\right|_{i^d = i_1^*} > \left.\frac{\partial \Pi(0, i^d)}{\partial i^d}\right|_{i^d = i_1^*} = 0.$$

The second equality holds because i_1^* maximizes $\Pi(0, \cdot)$. This implies that $\Pi(\tilde{\imath}_1^f, i^d)$ is increasing if $i^d = i_1^*$. Consequently,

$$\Delta_3(\tilde{\imath}_1^f) = \Pi(0, i_1^*) - \max_{i^d > \underline{I}_1^d\left(\tilde{\imath}_1^f\right)} \Pi(i^f, i^d) = \Pi(i^f, i_1^*) - \max_{i^d > i_1^*} \Pi(i^f, i^d) < 0,$$

which contradicts the definition of $\tilde{\imath}_1^f$ because Δ_3 is continuous. Therefore, $\underline{I}_1^d\left(\tilde{\imath}_1^f\right) > i_1^*$ and if $i^f < \tilde{\imath}_1^f$, then $i_1^* < \underline{I}_1^d\left(i^f\right)$ because $\underline{I}_1^d\left(i^f\right)$ is decreasing in i^f. This implies $B^d(i^f) = i_1^* = B^d(0)$. Lastly, notice that $\tilde{\imath}_1^f < \alpha_1\lambda(0)$ because $\underline{I}_1^d\left(i^f\right) = 0 < i_1^*$ if $i^f \geq \alpha_1\lambda(0)$. ■

Lemma 8 *There exists $\tilde{\imath}_3^f > \alpha_1\lambda(0)$ such that $B^d(i^f) = B^d(\infty)$ if $i^f \geq \tilde{\imath}_3^f$.*

Proof. Define $i_2^* = \arg\max_{i^d} \Pi(\infty, i^d)$ and

$$\tilde{i}_3^f = \frac{\alpha_2}{\alpha_2 + \alpha_3} i_2^* + \alpha_1 \lambda(0). \tag{34}$$

Notice that at $i^f = \tilde{i}_3^f$, the fiat money is valued if and only if $i^d > i_2^*$. If $i^f \geq \tilde{i}_3^f$, $\Pi(i^f, i^d)$ is equal to $\Pi(\infty, i^d)$ on $[0, \underline{I}_2^d(i^f)]$, where $\underline{I}_2^d(i^f) > i_2^*$ by definition. Therefore,

$$\Pi(i^f, i_2^*) = \Pi(\infty, i_2^*) = \max_{i^d} \Pi(\infty, i^d) \geq \max_{i^d} \Pi(i^f, i^d),$$

where the last inequality follows because a higher i^f increases the profit of the digital currency issuer at any i^d by raising z^d. This concludes the proof. ■

Lemma 9 *Suppose that λ''/λ' is decreasing on $(0, z^*)$. Let $\left(z^d, z^f\right)$ be the equilibrium under i^d and i^f, where $z^f > 0$ and $z^d + z^f < z^*$. And let $\tilde{z}^d$ be the equilibrium under i^d when the fiat money is not valued. Then*

$$\frac{\partial \tilde{z}^d}{\partial i^d} - \frac{\partial z^d}{\partial i^d} > -\frac{\left[\alpha_3 \lambda' \left(z^f + z^d\right)\right]^2}{\left[\alpha_1 \lambda' \left(z^f\right) + \alpha_3 \lambda' \left(z^f + z^d\right)\right] D} > 0.$$

Moreover, $\frac{\partial \Pi(i^f, i^d)}{\partial i^d} < \frac{\partial \Pi(\infty, i^d)}{\partial i^d}$.

Proof. If the fiat money is valued,

$$\alpha_2 \lambda\left(z^d\right) + \alpha_3 \lambda\left(z^d + z^f\right) = i^d. \tag{35}$$

Differentiating with respect to z^f, one can get

$$\frac{\partial z^d}{\partial z^f} = -\frac{\alpha_3 \lambda'\left(z^d + z^f\right)}{\alpha_2 \lambda'\left(z^d\right) + \alpha_3 \lambda'\left(z^d + z^f\right)}.$$

Using the expression, one can obtain after some calculation that

$$\frac{\partial\left[\alpha_2 \lambda'\left(z^d\right) + \alpha_3 \lambda'\left(z^d + z^f\right)\right]}{\partial z^f}$$

is equal in sign to

$$\frac{\lambda''\left(z^d\right)}{\lambda'\left(z^d\right)} - \frac{\lambda''\left(z^d + z^f\right)}{\lambda'\left(z^d + z^f\right)},$$

which is positive by assumption. This implies that

$$\alpha_2\lambda'\left(z^d\right)+\alpha_3\lambda'\left(z^d+z^f\right)$$

is decreasing in z^f if z^d satisfies (35). If the fiat money is not valued,

$$\alpha_2\lambda\left(\tilde{z}^d\right)+\alpha_3\lambda\left(\tilde{z}^d\right)=i^d. \tag{36}$$

The above analysis implies that $\tilde{z}^d > z^d$ and

$$\alpha_2\lambda'\left(z^d\right)+\alpha_3\lambda'\left(z^d+z^f\right)>\alpha_2\lambda'\left(\tilde{z}^d\right)+\alpha_3\lambda'\left(\tilde{z}^d\right).$$

Next, divide both sides by

$$[\alpha_2\lambda'\left(z^d\right)+\alpha_3\lambda'\left(z^d+z^f\right)][\alpha_2\lambda'\left(\tilde{z}^d\right)+\alpha_3\lambda'\left(\tilde{z}^d\right)]>0.$$

Then we obtain

$$\frac{1}{\alpha_2'\lambda\left(\tilde{z}^d\right)+\alpha_3\lambda'\left(\tilde{z}^d\right)}>\frac{1}{\alpha_2\lambda'\left(z^d\right)+\alpha_3\lambda'\left(z^f+z^d\right)}.$$

Notice that

$$\begin{aligned}\frac{\partial z^d}{\partial i^d} &= \frac{\alpha_1\lambda'\left(z^f\right)+\alpha_3\lambda'\left(z^f+z^d\right)}{D},\\ \frac{\partial \tilde{z}^d}{\partial i^d} &= \frac{1}{\alpha_2'\lambda\left(\tilde{z}^d\right)+\alpha_3\lambda'\left(\tilde{z}^d\right)}>\frac{1}{\alpha_2\lambda'\left(z^d\right)+\alpha_3\lambda'\left(z^f+z^d\right)},\end{aligned}$$

where D is defined as in (24). Then

$$\begin{aligned}\frac{\partial \tilde{z}^d}{\partial i^d}-\frac{\partial z^d}{\partial i^d} &> \frac{D-\left[\alpha_1\lambda'\left(z^f\right)+\alpha_3\lambda'\left(z^f+z^d\right)\right]\left[\alpha_2\lambda'\left(z^d\right)+\alpha_3\lambda'\left(z^f+z^d\right)\right]}{\left[\alpha_1\lambda'\left(z^f\right)+\alpha_3\lambda'\left(z^f+z^d\right)\right]D}\\ &= -\frac{\left[\alpha_3\lambda'\left(z^f+z^d\right)\right]^2}{\left[\alpha_1\lambda'\left(z^f\right)+\alpha_3\lambda'\left(z^f+z^d\right)\right]D}>0. \end{aligned}\tag{37}$$

Because the fiat money is not valued if $i^f=\infty$,

$$\frac{\partial\Pi(i^f,i^d)}{\partial i^d}=\beta z^d+[\beta(1+i^d)-1]\frac{\partial z^d}{\partial i^d}<\beta\tilde{z}^d+[\beta(1+i^d)-1]\frac{\partial \tilde{z}^d}{\partial i^d}=\frac{\partial\Pi(\infty,i^d)}{\partial i^d}.$$

The inequality holds because $z^d<\tilde{z}^d$ and (37) holds. This concludes the proof. ■

Lemma 10 *If λ''/λ' is decreasing on $(0, z^*)$, there exist $\tilde{i}_3^f > \tilde{i}_2^f > \alpha_1\lambda(0)$ such that for every $i^f \in \left[\tilde{i}_2^f, \tilde{i}_3^f\right]$,*

$$B^d\left(i^f\right) = \left[i^f - \alpha_1\lambda\left(0\right)\right]\left(\alpha_2 + \alpha_3\right)/\alpha_3.$$

Proof. Define

$$\tilde{i}_2^f = \inf\{i^f : \partial\Pi(x, i^d)/\partial i^d \leq 0 \text{ if } i^d \in [\underline{I}_2^d(x), \infty) \text{ for all } x > i^f\}. \tag{38}$$

Notice that by (34), $\underline{I}_2^d(x) > i_2^*$ for all $x > \tilde{i}_3^f$, where i_2^* is the maximizer of $\Pi(\infty, i^d)$. By the previous lemma, $\partial\Pi(x, i^d)/\partial i^d < \partial\Pi(\infty, i^d)/\partial i^d < 0$ on $[i_2^*, \infty)$. This implies that $\tilde{i}_2^f$ is well-defined and is at most $\tilde{i}_3^f$. We now show $\tilde{i}_2^f < \tilde{i}_3^f$. First notice that (37) implies that $\partial\tilde{z}^d/\partial i^d - \partial z^d/\partial i^d$ is bounded away from 0 for all $i^d \in [\underline{I}_2^d(i^f), i_2^*]$ if $i^f \in (\tilde{i}_3^f - \eta, \tilde{i}_3^f)$ for some $\eta > 0$. This is because λ' is continuous and z^d and z^f are continuous functions of i^d and i^f. This in turn implies that there exists $\eta > 0$, $\epsilon > 0$ such that

$$\frac{\partial\Pi(i^f, i^d)}{\partial i^d} + \epsilon < \frac{\partial\Pi(\infty, i^d)}{\partial i^d}$$

for all $i^d \in [\underline{I}_2^d(i^f), i_2^*]$ if $i^f \in (\tilde{i}_3^f - \eta, \tilde{i}_3^f)$. Because $\frac{\partial\Pi(\infty, i^d)}{\partial i^d}$ is continuous in i^d, there exists a $\delta > 0$ such that $\frac{\partial\Pi(\infty, i^d)}{\partial i^d} < \epsilon/2$ if $i^d \in (i_2^* - \delta, i_2^* + \delta)$. Since $\underline{I}_2^d(i^f)$ is increasing in i^f and equals i_2^* at $\tilde{i}_3^f$, if $i^f < \tilde{i}_3^f$ is sufficiently close to $\tilde{i}_3^f$, then $\underline{I}_2^d(i^f) \in (i_2^* - \delta, i_2^*)$ and $i^f > \tilde{i}_3^f - \eta$. As a result, if $i^d \in [\underline{I}_2^d(i^f), i_2^*]$

$$\frac{\partial\Pi(i^f, i^d)}{\partial i^d} < \frac{\partial\Pi(\infty, i^d)}{\partial i^d} - \epsilon = -\epsilon/2 < 0.$$

Because $\partial\Pi(i^f, i^d)/\partial i^d < \partial\Pi(i^f, i^d)/\partial i^d < 0$ for all $i^d > i_2^*$, there exists $\eta > 0$ such that if $i^f \in (\tilde{i}_3^f - \eta, \tilde{i}_3^f)$, then $\partial\Pi(i^f, i^d)/\partial i^d < 0$ for all $i^d \in [\underline{I}_2^d(i^f), \infty)$. In other words, $\tilde{i}_2^f < \tilde{i}_3^f$.

If $i^f \in \left[\tilde{i}_2^f, \tilde{i}_3^f\right]$, $\Pi(i^f, i^d)$ is strictly increasing in i^d on $[0, \underline{I}_2^d(x)]$ because $\Pi(i^f, i^d) = \Pi(\infty, i^d)$ in this region and $\underline{I}_2^d(i^f) \leq i_2^*$. And it is decreasing in i^d on $[\underline{I}_2^d(x), \infty)$ by the definition of $\tilde{i}_2^d$. This implies that the best response of the digital currency issuer is

$$\underline{I}_2^d(i^f) = \left[i^f - \alpha_1\lambda\left(0\right)\right]\left(\alpha_2 + \alpha_3\right)/\alpha_3.$$

Lastly, notice that by definition $\tilde{\imath}_2^f > \alpha_1\lambda(0)$. Otherwise, $\partial\Pi(x,i^d)/\partial i^d \leq 0$ for all $i^d > \underline{I}_2^d(\alpha_1\lambda(0)) = 0$, i.e. the profit is always decreasing in i^d. This leads to a contradiction because the digital currency issuer can always set $i^d = 1/\beta - 1$ to get a higher profit than $i^d = 0$. This completes the proof of this lemma. ■

The above lemmas together show that Proposition 3 holds with $\bar{\imath}_1^f$, $\bar{\imath}_2^f$ and $\bar{\imath}_3^f$ defined by (33), (38) and (34).

Appendix E: Existence of a Pure Strategy Equilibrium in the Simultaneous Move Game

Throughout this section, we assume that $u(y) = y^{1-\sigma}/(1-\sigma)$, $c(y) = y^{1+\eta}/(1+\eta)$, $\alpha_1 = 0$ and $\theta = 1$. We will derive conditions for the existence of a pure strategy equilibrium that can be easily checked numerically. To start, denote $(1+\eta)/(\sigma+\eta)$ by ξ. The next result charaterizes the best response function of the digital currency issuer.

Lemma 11 *The best response of the digital currency issuer is*

$$B^d\left(i^f\right) = \begin{cases} \frac{\xi(1-\beta)+\beta\left(\alpha_2-i^f\right)}{(\xi-1)\beta} & i^f < i_L \\ \frac{\alpha_2+\alpha_3}{\alpha_3} i^f & i_L < i^f < i_H \\ \frac{\beta(\alpha_2+\alpha_3)+\xi(1-\beta)}{\beta(\xi-1)} & i^f > i_H \end{cases},$$

where

$$\begin{aligned} i_L &= \frac{\xi(1-\beta)+\beta\alpha_2}{\beta\left(\xi-1+\frac{\alpha_3}{\alpha_2+\alpha_3}\right)}\frac{\alpha_3}{\alpha_2+\alpha_3}, \\ i_H &= \frac{\xi(1-\beta)+\beta(\alpha_2+\alpha_3)}{\beta(\xi-1)}\frac{\alpha_3}{\alpha_2+\alpha_3}. \end{aligned}$$

Proof. Given that digital currency is valued, fiat money is not valued iff

$$i^f \geq \alpha_3\lambda\left(z^d\right),\ i^d = \alpha_2\lambda\left(z^d\right) + \alpha_3\lambda\left(z^d\right),$$

where $\lambda(z) = [(1+\eta)z]^{-\frac{1}{\xi}} - 1$. This implies that digital currency and fiat money co-exist iff $\alpha_3 i^d/(\alpha_2+\alpha_3) > i^f$. Consequently, given i^f,

$$\Pi^d\left(i^f, i^d\right) = \begin{cases} \lambda^{-1}\left(\frac{i^d}{\alpha_2+\alpha_3}\right)\left[\beta\left(1+i^d\right)-1\right] & \text{if } i^d < \frac{\alpha_2+\alpha_3}{\alpha_3} i^f \\ \lambda^{-1}\left(\frac{i^d-i^f}{\alpha_2}\right)\left[\beta\left(1+i^d\right)-1\right] & \text{if } i^d > \frac{\alpha_2+\alpha_3}{\alpha_3} i^f \end{cases}.$$

Substitute the expression λ into $\Pi^d\left(i^f, i^d\right)$ and take derivative with respect to i^d. After some algebra, one can show that

$$\frac{\partial \Pi^d\left(i^f, i^d\right)}{\partial i^d} \simeq \begin{cases} -\xi\left[\beta\left(1+i^d\right)-1\right]\frac{1}{\alpha_2+\alpha_3}+\beta\left(1+\frac{i^d}{\alpha_2+\alpha_3}\right) & \text{if } i^d < \frac{\alpha_2+\alpha_3}{\alpha_3}i^f \\ -\xi\left[\beta\left(1+i^d\right)-1\right]\frac{1}{\alpha_2}+\beta\left(1+\frac{i^d-i^f}{\alpha_2}\right) & \text{if } i^d > \frac{\alpha_2+\alpha_3}{\alpha_3}i^f \end{cases}.$$

The first branch is positive (negative) if i^d is smaller (bigger) than i_1^d, where

$$i_1^d = \frac{\xi\left(1-\beta\right)+\beta\left(\alpha_2+\alpha_3\right)}{\beta\left(\xi-1\right)}.$$

Therefore, if $\frac{\alpha_2+\alpha_3}{\alpha_2}i^f > i_1^d$, or equivalently, $i^f > i_H$, $B^d\left(i^f\right) = i_1^d$. The second branch is positive (negative) if i^d is smaller (bigger) than i_2^d where

$$i_2^d = \frac{\xi\left(1-\beta\right)+\beta\left(\alpha_2-i^f\right)}{\beta\left(\xi-1\right)}.$$

Then if $i_2^d > \frac{\alpha_2+\alpha_3}{\alpha_2}i^f$, or equivalently, $i^f < i_L^f$, $B^d\left(i^f\right) = i_2^d$. Lastly, suppose

$$i_2^d < \frac{\alpha_2+\alpha_3}{\alpha_2}i^f < i_1^d,$$

or equivalently $i_L^f < i^f < i_H^f$. Then $\frac{\partial \Pi^d\left(i^f,i^d\right)}{\partial i^d} > 0$ if $i^d < \frac{\alpha_2+\alpha_3}{\alpha_3}i^f$ because we are in the first branch and $i^d < i_1^d$. Moreover, $\frac{\partial \Pi^d\left(i^f,i^d\right)}{\partial i^d} < 0$ if $i^d > \frac{\alpha_2+\alpha_3}{\alpha_3}i^f$ because we are in the second branch and $\frac{\alpha_2+\alpha_3}{\alpha_3}i^f > i_2^d$. Therefore, $B^d\left(i^f\right) = \frac{\alpha_2+\alpha_3}{\alpha_3}i^f$. ■

Notice that if i^f is small, digital currency and fiat money co-exist. While if i^f is intermediate, the best response of the digital currency issuer is to set i^d just low enough to drive the fiat money out of circulation. In this region, even though fiat money is not valued and used in the economy, the central bank monetary policy is still effective as it determines i^d. Also notice that if $\alpha_1 = 0$, the best response function of the digital currency issuer is continuous. Unfortunately, this is not true for the best response function of the central bank in general. However, under certain conditions, it is continuous on relevant regions, which is enough to guarantee existence of a pure strategy equilibrium.

Proposition 8 *A pure strategy equilibrium exists if under $i^d = \left(\alpha_2+\alpha_3\right)i_L/\alpha_3$,*

$$\log\left(\frac{x}{1-x}\frac{\alpha_2}{\alpha_3}\right) < \left(1+\xi\right)\log\frac{\frac{x}{\alpha_2+\alpha_3}+\frac{1}{i^d}}{\frac{1-x}{\alpha_2}+\frac{1}{i^d}},\ \forall x \in \left[0, \alpha_3/\left(\alpha_2+\alpha_3\right)\right). \qquad (39)$$

In particular, there exists an $\zeta > 0$ *such that* $i_*^d = i$ *and* $i_*^f = \frac{\alpha_3}{\alpha_2+\alpha_3} i$ *is an equilibrium for every* $i \in [i_L, i_L + \zeta]$.

Proof. Recall that

$$\frac{\partial \Omega\left(i^f, i^d\right)}{\partial i^f} \simeq \begin{cases} -\frac{i^d - i^f}{\alpha_2 \lambda'\left(z^d\right)} + \frac{i^f}{\alpha_3 \lambda'\left(z^d + z^f\right)} & \text{if } i^f \leq \frac{\alpha_3}{\alpha_2+\alpha_3} i^d \\ 0 & \text{if } i^f > \frac{\alpha_3}{\alpha_2+\alpha_3} i^d \end{cases}.$$

Substitute in the expressions of λ, z^d and z^f and rearrange to obtain that if $i^f < \frac{\alpha_3}{\alpha_2+\alpha_3} i^d$

$$\frac{\partial \Omega\left(i^f, i^d\right)}{\partial i^f} \simeq -\log\left(\frac{x}{1-x}\frac{\alpha_2}{\alpha_3}\right) + (1+\xi)\log\frac{\frac{x}{\alpha_2+\alpha_3} + \frac{1}{i^d}}{\frac{1-x}{\alpha_2} + \frac{1}{i^d}},$$

where $x = i^f / i^d$. If the assumption of the proposition is satisfied, $\frac{\partial \Omega\left(i^f, i^d\right)}{\partial i^f} > 0$ if $i^d = \frac{\alpha_2+\alpha_3}{\alpha_2} i_L$ and $i^f < \frac{\alpha_3}{\alpha_2+\alpha_3} i^d$. By continuity, this means that there exists $\zeta > 0$ such that if $i^d = i$ for any $i \in \left[\frac{\alpha_2+\alpha_3}{\alpha_2} i_L, \frac{\alpha_2+\alpha_3}{\alpha_2} i_L + \zeta\right]$, $i^f = \frac{\alpha_3}{\alpha_2+\alpha_3} i$ is the best response of the central bank. Furthermore, according to Lemma 11, if $i_L + \frac{\alpha_3}{\alpha_2+\alpha_3}\zeta < i_H$, the best response of the digital currency issuer is $i^d = i$ if $i^f = \frac{\alpha_3}{\alpha_2+\alpha_3} i$. As a result, $i_*^d = i$ and $i_*^f = \frac{\alpha_3}{\alpha_2+\alpha_3} i$ is an equilibrium for any $i \in [i_L, i_L + \xi]$. ■

Condition (39) implies that $\Omega\left(i^f, i^d\right)$ is strictly increasing in i^f if fiat money is valued and $i^d = \frac{\alpha_2+\alpha_3}{\alpha_2} i_L$. It is straightforward to check numerically because i_L depends only on parameters. We have experimented with many sets of parameters and condition (39) always holds. Notice that it is not true that (39) holds for all i^d. In fact, $\Omega\left(i^f, i^d\right)$ may have multiple local maximizers. And the global maximizer may jump from one local maximizer to another as i^d changes. As a result, the best response function of the central bank has discontinuities even under this simple parametrization. Interestingly, there is a continuum of equilibria where the digital currency issuer sets i^d to make fiat money just not valued. The central bank is happy with its money not being valued because it is its best policy: it is beneficial to have only digital currency circulating because that maximizes its value, which captures the network externality+. As a result, buyers would be able to consume more. By continuity, Proposition 8 implies that a pure strategy equilibrium exists if α_1 and $1 - \theta$ are both sufficiently small and ε is sufficiently small.

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