Deep reinforcement learning in a monetary model

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Advanced analytics: new methods and applications for macroeconomic policy

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*Disclaimer: The expressed views are our own and should not be represented as those of the Bank of England (BoE). All errors are ours.

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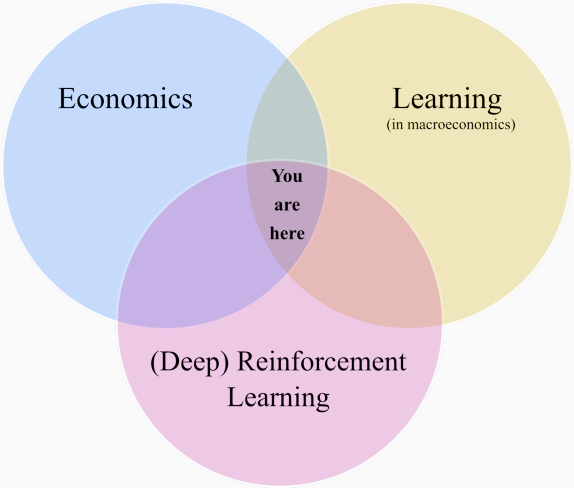
Overview & motivation

Overview


Economics
Learning
(in macroeconomics)
You
are
here
(Deep) Reinforcement
Learning

What we do

We apply deep reinforcement learning (DRL) to solve a general equilibrium model commonly used in the learning literature:

- non-linear Taylor Rule with two steady states
- interaction of fiscal and monetary in different policy regimes

But, first things first ...

What is and why learning in (macro)economics?

- Rational expectations (RE) approach convenience choice to solve a model, but not necessarily how people and businesses actually behave
- Approach to bounded rationality: Specify agent knowledge and behaviour away from RE
- Arguably, every state of the world, e.g. RE equilibrium, must be attainable from some starting point, i.e. learning.
- Policy reaction distinctively different under learning, e.g. forward guidance or stability of Taylor rules

See Eusepi and Preston (2018) and Hommes (2021) for recent reviews.

Example: Adaptive learning (Evans and Honkapohja, 2001)

Agents are "econometricians" trying to estimate expected quantities

$$\mathbb{E}_t[x_{t+1}] = x^e_{t+1} = x^e_t + \phi_t(x_{t-1} - x^e_t)\,, \tag{1}$$

with a gain series ϕ_t. Together with the (optimal) behavioural rules, i.e. linearised FOCs, this leads to a set of ordinary differential equations determining the expectations (E-)stability of the model.

That is, if a steady state is stable under learning, which then serves as a selection criterion.

Deep reinforcement learning

ARTICLE

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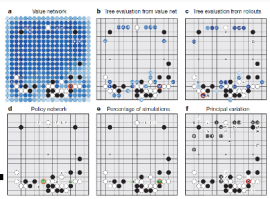
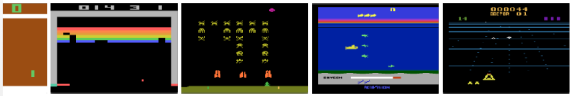
Mastering the game of Go with deep neural networks and tree search

David Silver[1]*, Aja Huang[1]*, Chris J. Maddison[1], Arthur Guez[1], L
Julian Schrittwieser[1], Ioannis Antonoglou[1], Veda Panneershelva
John Nham[2], Nal Kalchbrenner[1], Ilya Sutskever[2], Timothy Lillicr
Thore Graepel[1] & Demis Hassabis[1]



Playing Atari with Deep Reinforcement Learning

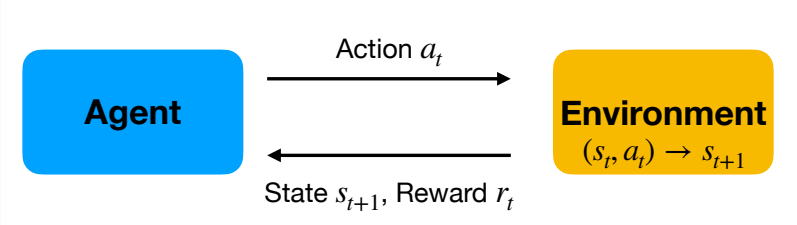
r Mnih Koray Kavukcuoglu David Silver Alex Graves Ioannis Antonoglou
Daan Wierstra Martin Riedmiller



reen shots from five Atari 2600 Games: (*Left-to-right*) Pong, Breakout, Space Invaders, am Rider

Sources: Nature, arXiv, Boston Dynamics

The reinforcement learning (RL) setting



1. Agent observes state of the world s_t
2. Agent takes actions $a_t(s_t)$
3. Agent receives reward r_t from environment
4. Actions and state lead to state transition of the environment s_{t+1}

This setting is very general. See Sutton and Barto (2018) for a comprehensive introduction.

Formal RL definition

The agent aims to maximise expected cumulative lifetime reward, or expected return,

$$\max_{\mathcal{P}} \mathbb{E}_t[G_t] \quad \text{with} \quad G_t \equiv \sum_{k=0}^{\infty} \beta^k r_{t+1+k}(s)\,, \tag{2}$$

following a behavioural policy $\mathcal{P} : s_t \rightarrow a_t$, with $s_t \in \mathcal{S} \subset \mathbb{R}^{n_s}$ (state space) and $a_t \in \mathcal{A} \subset \mathbb{R}^{n_a}$ (action space).

The environment the agents interaction with returns a reward and a new state, i.e. $\mathcal{E} : (s_t, a_t) \rightarrow (s_{t+1}, r_t)$, with $r_t \in \mathbb{R}$.

The state transitions is modelled as a Markov decision process (MDP) $\mathcal{T} : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \rightarrow Pr(s_{t+1}|s_t, a_t) \in [0, 1]$.

Problem: Writing down $\mathcal{T}$ is simple, knowing $\mathbb{E}_t[G_t]$ and $Pr(s_{t+1}|s_t, a_t)$ is hard (dynamic programming, value function iteration, etc.).

State and action values

The expected return is maximised by finding the policy $\mathcal{P}^*$, which maximises the values function

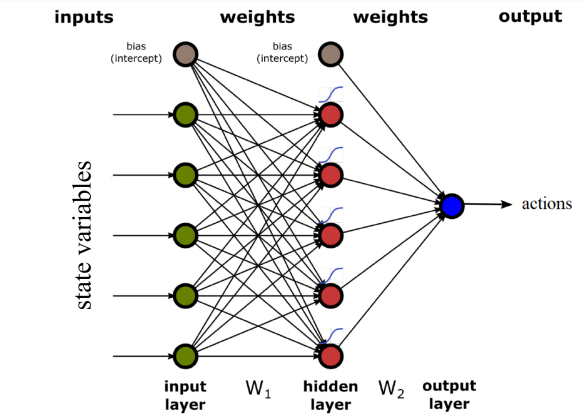
$$
\begin{aligned}
V_{\mathcal{P}}(s) &= \mathbb{E}_{\mathcal{P}}\big[G_t | s = s_t\big] && (3)\\
&= \max_{a \in \mathcal{A}} \mathbb{E}_{\mathcal{P}}\big[G_t | s = s_t, a = a_t\big] \\
&= \max_{a \in \mathcal{A}} Q(s, a)\,, && (4)
\end{aligned}
$$

with $Q(s, a)$ the action-value function. We are done if we know $\mathcal{P}^*$ and V^*/Q^*.

There are different ways to address this problem, which is an area of active AI research.

Deep reinforcement learning (DRL)

In DRL, functions $\mathcal{P}$ and V/Q are parameterised using deep artificial neural networks (Goodfellow et al., 2016), i.e. neural nets with several hidden layers, $\mathcal{P}_\phi$ and Q_θ:



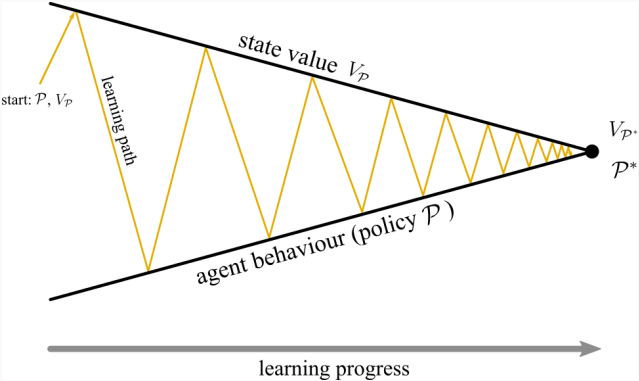
Source: Chakraborty and Joseph (2017). DRL approach

General DRL setting for (macro)economics

- Write down model (environment and state)
- Specify learning agents, e.g. households, firms, etc., and their actions
- Specify state transitions as MDP
- Learning using DRL algorithm (e.g. Haarnoja et al. (2018)): learning protocol
  1. sample state transition(s) and store in memory
  2. train $\mathcal{P}_\phi$ and Q_θ from memory
  3. test $\mathcal{P}_\phi$ and Q_θ with new state transitions and metric of choice

Generalised policy iteration (GPI)

GPI connects economics and learning, and conventional learning approaches with RL



V^*: steady state values, $\mathcal{P}^*$: FOC.

Potential advantages of DRL in (macro)economics

- Global solution technique with no need of linearisation
- Principled way to (bounded) rationality, i.e. agent behaviour and knowledge (this paper)
- General approach to handle heterogeneity, e.g. household income or age distribution. See AA colleagues (Hill et al., 2021).

Examples of RL in economics and finance

- Charpentier et al. (2020): Brief introduction to RL in a economics and finance background
- Zheng et al. (2020): Learning in large-scale geographic ABM
- Calvano et al. (2020): Investigate algorithmic collusion in financial markets
- Chaudhry and Oh (2020): Extract high-frequency expectations in financial markets to measure information effects
- Castro et al. (2021): Learn policy rules of banks participating in a high-value payments system

The model environment

Households

A single representative household maximises its expected lifetime utility, subject to an inter-temporal budget constraint:

$$\max_{c_t, m_t, n_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, m_t, n_t) \text{ s.t.} \tag{5}$$

$$M_t + B_t + C_t = M_{t-1} + B_{t-1} R_{t-1} + W_t n_t - P_t \tau_t, \tag{6}$$

with P_t the price level at time t, $x_t = \frac{X_t}{P_t}$, $x \in \{M_t, B_t, C_t, W_t\}$ relate real and nominal money, government bonds, consumption and wages, and τ_t is a real lump-sum tax to the government each period.

We take the utility (Evans and Honkapohja, 2005)

$$U(c_t, m_t, n_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + \chi \frac{m_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi}. \tag{7}$$

Firms

A single representative firm produces according to

$$y_t = \varepsilon_t^y n_t, \tag{8}$$

with technology (shock) ε_t^y, maximising profits

$$\max_{w_t} \; y_t - w_t n_t \,, \tag{9}$$

by setting setting the optimal wage[1]

$$w_t = \varepsilon_t^y \,. \tag{10}$$

[1] This could be replaced by DRL resulting in a multi-agent learning problem.

Market clearing

Markets clear every period, i.e.

$$y_t = c_t \qquad (goods)\,, \tag{11}$$

and

$$c_t^{\sigma}\, n_t^{\varphi} = \varepsilon_t^{y} \qquad (labour). \tag{12}$$

Government

The government issues interest-bearing bonds and non-interesting bearing currency (money), and collects taxes under the real inter-temporal *government budget constraint* (GBC)

$$m_t + b_t + \tau_t = \frac{m_{t-1}}{\pi_t} + R_{t-1}\frac{b_{t-1}}{\pi_t}\,, \tag{13}$$

subject to the transversality condition

$$\lim_{j\to\infty} \prod_{k=0}^{j} \left(\frac{\pi_{t+k}}{R_{t+k-1}}\right) b_{t+j} = 0\,. \tag{14}$$

Fiscal policy takes the linear tax rule as in Leeper (1991)

$$\tau_t = \gamma_0 + \gamma b_{t-1} + \varepsilon_t^{\tau}\,, \tag{15}$$

where ε_t^{τ} is an exogenous random shock that is assumed to be i.i.d. with mean zero, and $0 \leq \gamma \leq \beta^{-1}$. We follow Leeper (1991) to define fiscal policy as being active if $\gamma < \beta^{-1} - 1$ (AFP) and passive if $\gamma > \beta^{-1} - 1$ (PFP).

Central bank

We follow Benhabib et al. (2001) and Evans and Honkapohja (2005) with a global non-linear interest rate rule

$$R_t - 1 = \varepsilon_t^R f(\pi_t) \qquad (\textit{Taylor rule}), \tag{16}$$

with $f(\pi)$ assumed to be non-negative and nondecreasing, and ε_t^R is an exogenous, i.i.d. and positive random shock with a mean of one:

$$f(\pi_t) = (R^* - 1)(\frac{\pi_t}{\pi^*})^{\frac{AR^*}{R^*-1}}, \tag{17}$$

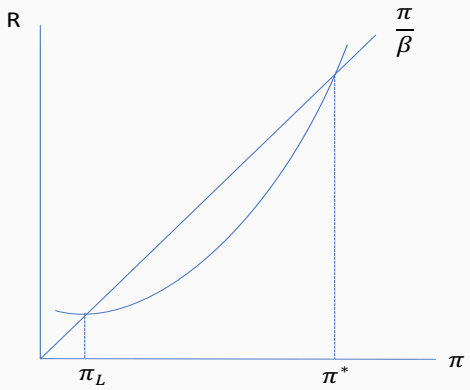
where $A > 1$, and $\pi^* > 1$ is the inflation target.

Steady states

The Taylor rule (16) implies two steady states at the intersection with the Euler/Fisher equation

$$\frac{\pi}{\beta} = 1 + (R^* - 1)\left(\frac{\pi}{\pi^*}\right)^{\frac{AR^*}{R^*-1}}. \qquad (18)$$

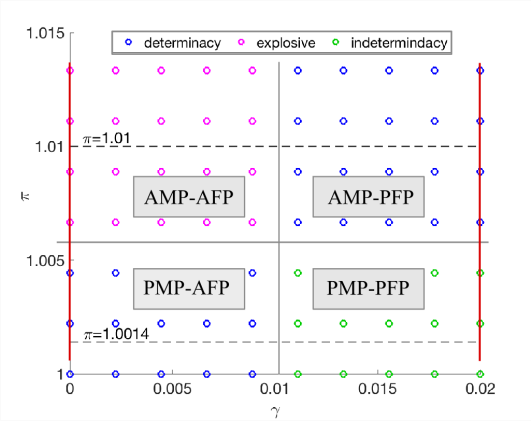
Monetary policy (MP) is said to be active at π^* ($f'(\pi_t) > 1$; AMP) and passive at π_L ($f'(\pi_t) < 1$; PMP).



This situation is very general and commonly investigated in learning in macroeconomics.

Policy regimes

Using a standard parameterisation and local stability analysis we obtain four policy regimes



	AMP (π^*)		PMP (π_L)	
	PFP	AFP	PFP	AFP
π_{ss}	1.0100	1.0100	1.0014	1.0014
m_{ss}	1.7157	1.7157	2.0614	2.0614
$c_{ss}/n_{ss}/y_{ss}$	1	1	1	1
b_{ss}	4	4	4	4
u_{ss}	-1.0170	-1.0170	-1.0118	-1.0118
γ_0	-0.0566	0.0234	-0.0426	0.0375

Joining the model and RL

State representation

$$s_t = \left(m_{t-1}, b_{t-1}, \pi_{t-1}, c_{t-1}, n_{t-1}, \epsilon_t^{\tau}, \epsilon_t^{R}, \epsilon_t^{y} \right) . \tag{19}$$

Household agent actions

$$a_t = \left(c_t^{act}, b_t^{act}, n_t \right) , \tag{20}$$

where $x_t^{act} = X_t / P_{t-1}$, $x \in \{c, b\}$. Information flow and market clearing

$$\pi_t = c_t^{act} / y_t \, , \tag{21}$$

$$c_t = c_t^{act} / \pi_t \, , \tag{22}$$

$$b_t = b_t^{act} / \pi_t \, . \tag{23}$$

Model environment: Production, market clearing, pricing, GBC, FP, MP.

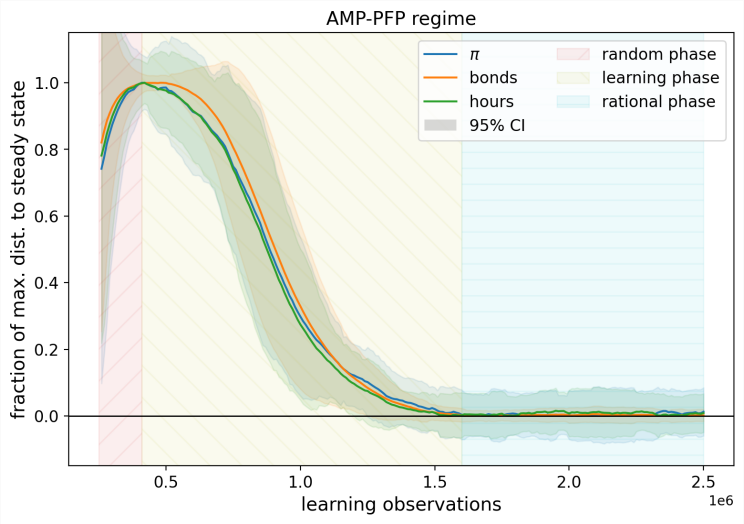
No first-order conditions (FOC)

State transition

1. Observe state s_t
2. Take actions $\mathcal{P}_\phi(s_t) = a_t = (b_t^{act}, c_t^{act}, n_t)$
3. Production (8) takes place and firm sets wages (10)
4. Markets clear: Inflation π_t is set by (21)
5. This determines real consumption c_t and bond holdings b_t (22)-(23)
6. Policy realisations:
   - The monetary authority sets the current gross interest rate R_t via the Taylor rule (16)
   - The government raises taxes τ_t (15)
7. The money holdings m_t are realised from the GBC (13)
8. Agent obtains reward $r_t = U(c_t, m_t, n_t)$
9. Next periods shocks are realised, $(\epsilon_{t+1}^{\tau}, \epsilon_{t+1}^{R}, \epsilon_{t+1}^{y})$
10. State update $s_t \leftarrow s_{t+1} = \left(m_t, b_t, \pi_t, c_t, n_t, \epsilon_{t+1}^{\tau}, \epsilon_{t+1}^{R}, \epsilon_{t+1}^{y}\right)$

Results

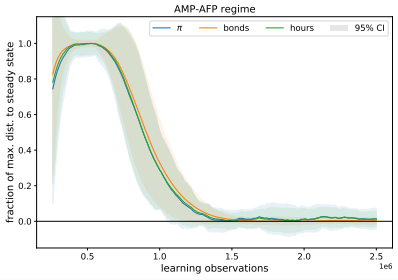
Steady state learning in the AMP-PFP regime

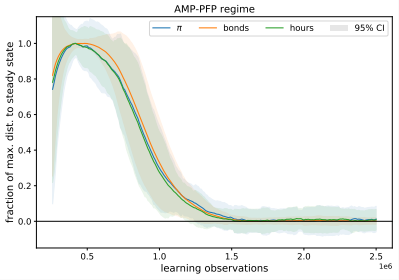


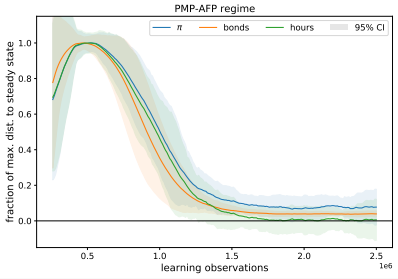
Learning phases

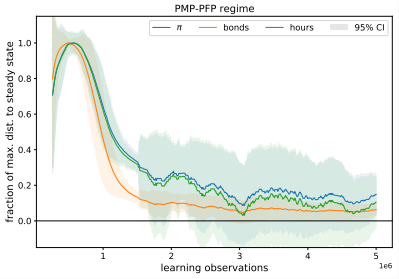
- random (agent initialisation)
- learning
- rational

Steady state learning in all regimes (charts)









Steady state learning in all regimes (table)

	AMP (π^*)		PMP (π_L)	
	PFP	AFP	PFP	AFP
AL	yes	no	no	yes
RL	yes	yes	yes†	yes†
	$\lvert\Delta_{ss}\rvert$ (%) for RL			
π	0.346	0.278	9.217	5.209
b	0.005	0.004	0.038	0.024
n	0.004	0.003	0.009	0.003
m	0.091	0.089	11.569	7.364
u	0.003	0.003	0.346	0.196

†imprecision in learning about inflation at π_L.

Measuring bounded rationality

The household is said to behave rational if it follows FOC. During learning, we define the FOC-distance to measure deviations in a standardised way

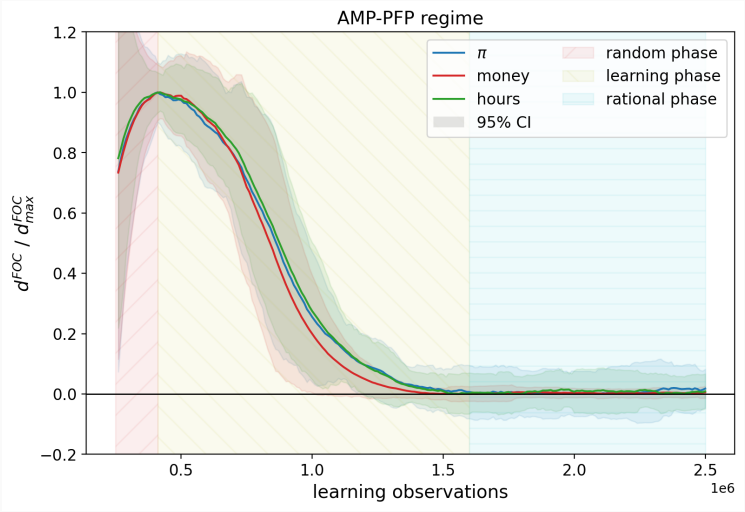
$$d_x^{FOC} \equiv \left| FOC(x) - 1 \right|, \tag{24}$$

The explicit expression for the Euler equation, or *Euler distance*, is

$$d_\pi^{FOC} = \left| \beta \, \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{R_t}{\pi_{t+1}} \right] - 1 \right|. \tag{25}$$

FOC distances evaluate agent actions $\mathcal{P}(s)$. Analogous measures for V/Q can be derived with respect to state values.

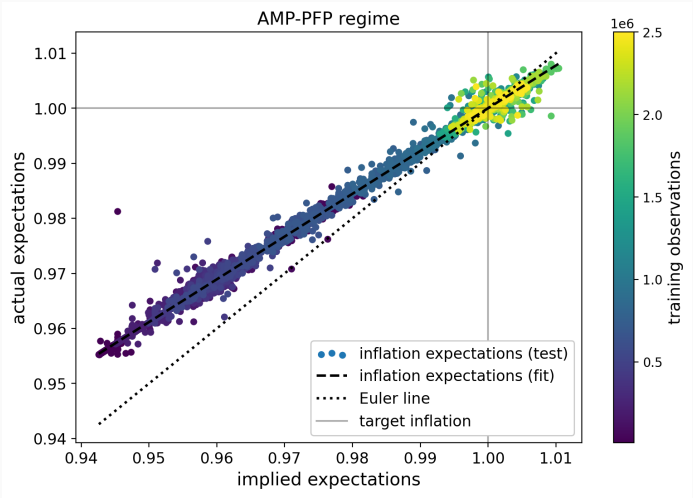
FOC learning in the AMP-PFP regime



The same Learning phases as expected by GPI.

Measuring inflation expectations during learning

Implied agent expectations can be extracted from realised values



Future steps

- Improve and better understand learning robustness
- Aim for truly global learning
- Compare IRF with those from adaptive learning
- Conduct experiments: policy or regimes shifts

Take-away messages

- DRL offers a general approach to solve structural macro models
- Quantify bounded rationality and learning in a principled way: agent behaviour as a free model parameter
- Global solution techniques which can also address heterogeneity
- All policy regimes are learnable under DRL
- Promising toolbox for (macro)economics
- Learning and convergence technically challenging

Thanks for listening

Q& A

Actor-critic DRL setting

$\mathcal{P}$ and Q fulfil the Bellman equation

$$Q(s_t, a_t) = r(s_t) + \beta \mathbb{E}_{\mathcal{P}}\big[Q(s_{t+1}, a_{t+1})\big]. \tag{26}$$

using sampled state transitions as observations, i.e. interactions of the agent and the environment, and standard optimisation techniques like stochastic gradient descent, the policy and action-value function networks can be trained by iteratively minimising the Bellman residuum,

$$L(\phi, \theta) = \mathbb{E}_{s_t, a_t, r_t}\left[\frac{1}{2}\big(Q_\theta(s_t, a_t) - \hat{Q}_\theta(s_t, a_t)\big)^2\right], \tag{27}$$

$$\text{with} \quad \hat{Q}_\theta(s_t, a_t) = r_t(a_t, s_t) + \beta\, \mathbb{E}_{\mathcal{P}}\big[Q_\theta\big(s_{t+1}, \mathcal{P}_\phi(s_{t+1})\big)\big]. \tag{28}$$

We use Haarnoja et al. (2018). The code we used for optimisation is available at `https://github.com/pranz24/pytorch-soft-actor-critic`.

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Household learning protocol

Algorithm 1 Training and testing protocol of household agent

Initialise: Environment $\mathcal{E}$ (parameterised model), agent (parameterised by $\mathcal{P}_\phi$, Q_θ)

- **for** steps = 1 to N_{train} **do**
  - initialise training episode with random state s_t
  - **while** training episode is not done **do**
    - **if** steps $\leq N_{burn}$ **then**
      - Take allowed random action a_t
    - **else**
      - Draw exploration action $a_t = \mathcal{P}_\phi^{exp}(s_t)$
    - **end if**
    - Environment returns $(r_t, s_{t+1}) = \mathcal{E}(s_t, a_t)$
    - Add transition (s_t, a_t, r_t, s_{t+1}) to memory
    - Update $\mathcal{P}_\phi$, Q_θ using batch gradient descent from memory
    - **if** $mod(\text{steps}, N_{interval}) = 0$ **then**
      - **for** test episode = 1 to N_{test} **do**
        - Record state transitions (*)
      - **end for**
      - Save current agent $(\mathcal{P}_\phi^{steps}, Q_\theta^{steps})$
    - **end if**
    - State update $s_t \leftarrow s_{t+1}$
    - Test episode termination criteria $(N_{epi}^{max}, d_u^{min})$
  - **end while**
- **end for**
- Save final agent $(\mathcal{P}_\phi^{final}, Q_\theta^{final})$

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Model dynamic properties I

The deterministic steady states in the absence of random shocks is characterised by the following set of equations:

$$\text{Euler / Fisher Equation:} \quad R = \frac{\pi}{\beta} \tag{29}$$

$$\text{Money Demand:} \quad m = y\left(\frac{\pi - \beta}{\chi\pi}\right)^{-1/\sigma} \tag{30}$$

$$\text{Monetary Policy:} \quad R = 1 + (R^* - 1)\left(\frac{\pi}{\pi^*}\right)^{\frac{AR^*}{R^*-1}} \tag{31}$$

$$\text{Fiscal Policy \& GBC:} \quad b = \left(\frac{1}{\beta} - 1 - \gamma\right)^{-1}\left[\gamma_0 + \left(1 - \frac{1}{\pi}\right)m\right] \tag{32}$$

$$\text{Output:} \quad y^{\sigma+\varphi} = 1 \tag{33}$$

Equation (29) and (31) together determine the steady state of inflation:

$$\frac{\pi}{\beta} = 1 + (R^* - 1)\left(\frac{\pi}{\pi^*}\right)^{\frac{AR^*}{R^*-1}} \tag{34}$$

Model dynamic properties II

In the neighbourhood of either steady state, our model can be described by a linear approximation for π_t and b_t of the form

$$\begin{bmatrix} \hat{\pi}_t \\ \hat{b}_t \end{bmatrix} = \mathbf{B} \begin{bmatrix} \hat{E}_t \pi_{t+1} \\ \hat{E}_t b_{t+1} \end{bmatrix} + \mathbf{C} \begin{bmatrix} \hat{\varepsilon}_t^R \\ \hat{\varepsilon}_t^\tau \\ \hat{\varepsilon}_t^y \end{bmatrix}. \tag{35}$$

Proposition:(Evans and Honkapohja, 2007)] In the linear system given by (35),
(i) If fiscal policy is passive, $|\gamma - \beta^{-1}| < 1$, the steady state π^* is locally determinate and the steady state π_L is locally indeterminate.
(ii) If fiscal policy is active, $|\gamma - \beta^{-1}| > 1$, the steady state π^* is locally explosive and the steady state π_L is locally determinate.

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Model parameters

parameter	value	description
β	0.9900	discount factor
σ	3.0000	inverse of intertemporal elasticity of consumption and money holdings
φ	1.0000	inverse of Frisch elasticity of labor supply
χ	0.1000	relative preference weight of money holdings
γ_P	0.0200	passive fiscal policy (PFP) coefficient
γ_A	0.0000	active fiscal policy (AFP) coefficient
A	1.3000	Taylor rule coefficient
π^*	1.0100	target gross high-inflation rate (4% net per annum)
π_L	1.0014	implied gross low-inflation steady state (see Figure **??**)
ϵ_t^τ	0.0005	monetary policy shock (std. dev.)
ϵ_t^R	0.0005	fiscal policy shock (std. dev.)
ϵ_t^y	0.0005	technology shock (std. dev.)

Baseline model parameterisation. The shock series ϵ_t^τ, ϵ_t^R, ϵ_t^y follow log-normal, normal and normal distributions, with means of one, zero and one, respectively.

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