External MPC Unit

Discussion Paper No.8

Too Much Too Soon: Instability and Indeterminacy with Forward-Looking Rules

by Nicoletta Batini and Joseph Pearlman

External MPC Unit Discussion Paper No. 8*

Too Much Too Soon: Instability and Indeterminacy with Forward-Looking Rules

Nicoletta Batini and Joseph Pearlman

July 2002

ISSN: 1748 – 6203

(*) Disclaimer: These Discussion Papers report on research carried out by, or under supervision of the External Members of the Monetary Policy Committee and their dedicated economic staff. Papers are made available as soon as practicable in order to share research and stimulate further discussion of key policy issues. However, the views expressed are those of the authors and do not represent the views of the Bank of England or necessarily the views of External Members of the Monetary Policy Committee.

"Too Much Too Soon: Instability and Indeterminacy with Forward-Looking Rules"*

Nicoletta Batini† and Joseph Pearlman✧

July 2002

Abstract

This paper extends the existing literature on the uniqueness and stability conditions for an equilibrium under inflation-forecast-based (IFB) rules. It shows that, for a variety of New Keynesian sticky-price and sticky-inflation models, these are a function not just of the degree of responsiveness of the policy instrument to deviations of expected inflation at some horizon *j* from target, but rather of the 'right' combination of that degree and the chosen feedback horizon. In this respect we prove analytically that the determinacy results in Clarida et al (2000) cannot be generalised to the case of IFB rules with longer feedback horizons than one quarter. The paper shows how to identify the feedback/horizon pairs that are associated with unique and stable equilibria and unveils the analytical rationale behind instability or indeterminacy at too long a lag for a given feedback.

JEL Classification: E52, E37, E58

Keywords: inflation targeting, price stability, optimal policy rules, simple feedback rules, multiple equilibria.

† Research Adviser, MPC Unit, Bank of England, Threadneedle Street, London EC2R 8AH, United Kingdom. Tel: +44 20 76014354. Fax: +44 20 76014610
E-mail: nicoletta.batini@bankofengland.co.uk

✧ London Guildhall University,31 Jewry St, London EC3N 2EY, UK.
Tel:+44 20 7320 3069 email:pearlman@lgu.ac.uk

(*) We thank Andrew Levin, Ed Nelson, Steve Nickell and Michael Woodford for comments on an earlier draft. The views expressed herein are those of the authors and not of the Bank of England or the Monetary Policy Committee.

Table of contents

1. The forward-looking dimension of inflation targets

Knut Wicksell's analysis of monetary policy, as described in "The Influence of the Rate of Interest on Prices" (1912), has been influential among policymakers. His account is simple: "[l]ower bank-rates when prices are getting low, and raise them when prices are getting high". Today, numerous central banks follow a milder version of this prescription when setting monetary policy, and use interest rates to feed back on inflation (i.e. the rate of change of the price level, rather than the level itself as Wicksell suggested) according to a strategy known as 'inflation targeting'.

Inflation targeting has some clear advantages over alternative monetary strategies. It shifts the focus of policy directly on to the ultimate goal of stable inflation; it makes monetary policy more accountable by offering ex post measures of inflation performance; and through greater accountability (see Berger et al, 2001) and monitorability, it provides a commitment mechanism that enhances the trust of the public in the anti-inflationary credentials of the monetary authorities. This, in turn, increases the probability of achieving low and stable inflation both in the short and in the medium run.

As is widely recognised by both policymakers and monetary economists, the very success of any inflation target depends on one proviso: that the authorities aim at future—rather than past or current—inflation.[1] This is surely what Wicksell meant when he used the term 'getting': raise interest rates when prices (and/or inflation) are expected to rise, not when they are already high.

But why is it so important that authorities aim at future rather than current or past inflation? The answer is simple. Current inflation is normally predetermined by existing contracts and by inertia in inflation expectations; central banks have little instantaneous control of inflation. So typically, monetary policy can influence inflation only with a lag, whose length and effectiveness are not entirely clear.[2] It follows that the delayed response of inflation to monetary actions obliges monetary authorities to react in a pre-emptive fashion to the behaviour of prices. In fact, a myopic policy that reacts only to present or past events may itself become a source of

[1] See King (1997), Svensson (1997), and Bernanke et al (1999).
[2] See Batini and Nelson (2001a) for a discussion of monetary lags and model-free measures of the delay between policy actions and inflation in the UK and the US.

macroeconomic instability, for the very reasons outlined by Friedman (1960). And even when inflation is not itself sticky, targeting future inflation is still desirable provided that prices are not perfectly flexible, as shown in Clarida, Gali and Gertler (2000)—henceforth CGG (2000).

Among inflation-targeting countries, forward-lookingness in inflation targets is obtained by responding to future expected inflation. The conditional inflation forecast—for a horizon corresponding to the control lag—is the de facto intermediate variable for monetary policy. For example, in New Zealand, Canada and the United Kingdom, monetary policy is based on explicit (and in some cases, published) inflation forecasts. In other inflation-targeting countries, like Sweden, Finland, Australia, Spain and Mexico, inflation forecasts are sometimes less explicit but nevertheless a fundamental part of the monetary policy process.

To examine the implications of the actual decision framework under inflation targeting, economists have proposed two classes of forward-looking rules: inflation forecast targeting rules (Svensson (1997) and Rudebusch and Svensson (1999)) and inflation forecast-based rules (Batini and Haldane (1999))[3].

Inflation forecast targeting (henceforth IFT) rules are those transition paths of the policy instrument that minimise a loss function penalising expected inflation deviations from target subject to a given dynamic model of the economy. Analytically, this amounts to solving for the first-order condition in a standard stochastic linear quadratic regulator problem (cf. Chow (1970), Turnovsky (1977) and Sargent (1987)). Under commitment, these rules are then optimal and efficient in the sense of Ball (1997) and are both deterministically and stochastically stable (Turnovsky, 1977), which makes them immune from the dangers of excessive forward-lookingness.[4]

[3] Rules analogous to inflation forecast-based rules were suggested also in Black et al (1997) .
[4] Evans and Honkapoja (2001, 2002) examine stability and determinacy for optimal policy rules under learning. They find that optimal timeless-perspective-based interest rate setting by the central bank can lead to indeterminacy, so that the economy may not converge to the desired rational expectations equilibrium. This is a consequence of the result that the Riccatti equation for full commitment optimal control is associated with *n* unstable and *n* stable roots. The actual system under control is characterised by each of these roots multiplied by $\delta^{1/2}$, where δ is the discount factor. Thus if one of the unstable eigenvalues is less than $1/\delta^{1/2}$, then it will turn into a stable eigenvalue for the system. Hence it is still possible that there may be too many stable eigenvalues, and hence indeterminacy. A discussion of the behaviour of solutions under learning for simple Taylor-type rules can be found in Bullard and Mitra (2001a, 2001b).

Inflation forecast-based (henceforth IFB) rules are simple rules that respond to deviations of expected inflation from target. From a theoretical point of view, analysis and simulation of IFB rules under the assumption that the authorities are uncertain about the shocks that will hit the economy at any point in time, have shown that these rules share a number of desirable features. This is because they are usually good approximations of optimal feedback rules.[5] However, as these rules are not fully optimal, they can lead to dynamic instability or indeterminacy.[6] A standard result in the literature is that to avoid indeterminacy, the monetary authority must respond aggressively (i.e. with a coefficient above unity, but not excessively large) to expected inflation (see the seminal work of Woodford (1994) and Bernanke and Woodford (1997) followed up by CGG (2000), and Levin, Wieland and Williams (2001), henceforth LWW). The indeterminacy implications of certain parametrisations of IFB rules was also pointed out by other recent contributions including, notably, Chari, Christiano and Eichenbaum (1998), Schmitt-Grohe and Uribe (2000) and Carlstrom and Fuerst (1999, 2000).[7] By contrast, the instability implications of these rules are not often investigated.[8] From an empirical point of view, CGG (2000) found for the US that the Fed appears to have indeed responded to expected inflation at either one quarter or one year ahead. Furthermore, the coefficient for the interest rate response to expected inflation has been considerably greater than 1 during the Volcker-Greenspan era. They also found that the same coefficient was significantly less than 1 in the pre-Volcker era, a possible cause, they argue, of the poor macroeconomic outcomes at the time. Similarly, for the UK, Nelson (2001) finds that the low inflation period from 1992 is indeed characterised by a Taylor rule with a response of the nominal rate to expected inflation above unity, suggesting that the move towards an inflation targeting regime may have enhanced inflation stabilisation in the UK.

This paper extends the existing literature on the uniqueness *and* stability conditions for an equilibrium under IFB rules. In particular, we demonstrate analytically that, for a variety of structural models, both are a function not just of the degree of

---

[5] See, among others, Svensson (1996), Bernanke and Woodford (1997), Clarida, Gali and Gertler (2000) and Batini and Haldane (1999).

[6] In what follows we refer interchangeably to the concepts indeterminate equilibria and sunspot equilibria. In reality, the latter are a particular class of indeterminate equilibria, i.e. where the multiple solutions to the model depend on extraneous random variables called 'sunspots'. For a discussion of the differences between these two concepts, see Benhabib and Farmer (1999).

[7] Other contributions analysing the ability of simple rules to ensure determinate solutions include Benhabib, Schmitt-Grohe and Uribe (2001).

[8] Batini and Haldane (1999) provide an intuitive interpretation to the instability problem with IFB rules.

responsiveness of the policy instrument to deviations of inflation expected at some horizon *j* from target, but rather of the 'right' combination of that degree and the chosen feedback horizon[9]. In this sense we advance beyond the results in LWW (2001), concerning the determinacy implications of longer-horizon forecast-based rules, but which were restricted to numerical investigation. An analytical derivation of the conditions for uniqueness and stability of equilibria is important because numerical results are subject to the particular choice of parameters in the model and in the rule, and so may not be general. More specifically, we find that reacting too aggressively to events that lie too far in the future may deliver results that are as bad as those obtained by responding to events that lie too far into the past. In this respect, we prove analytically, that the finding in CGG (2000) that a response coefficient greater than unity to next quarter's expected inflation in an IFB rule is a necessary and sufficient condition for determinacy, does not generalise to the case when longer feedback horizons are used. This is an important warning for inflation-targeting central banks, because real-world procedures typically involve stabilising inflation in the medium-run, one to two years out, so they are potentially vulnerable to this drawback.

The plan of the paper is as follows. Section 2 investigates when IFB rules can lead to instability or indeterminacy by discussing the stabilising properties of IFB rules with different degrees of forward-lookingness. To do so we start by calculating inflation-output volatility frontiers for a small AS-AD closed-economy structural model with sticky inflation as in Clarida, Gali and Gertler (1999) (henceforth CGG (1999)) and similar to the Fuhrer and Moore—henceforth FM—model in LWW (2001) under various shocks for alternative parametrisations of these rules. We also calculate upper and lower bounds of parameter values associated with determinate and stable equilibria. In Section 3 we explain the analytics of why excessive policy forward-lookingness may trigger dynamic instabilities or involve multiplicity in equilibria as suggested by the results in Section 2. For this purpose, using this same model, we conduct a standard root-locus analysis under the alternative forward-looking rules. This illustrates a general method with which to identify the feedback/horizon pairs that are associated with unique and stable equilibria, and we also offer an economic interpretation of the results. Section 4 applies this method to the small AS-AD

[9] Woodford (2002) also warns of the use of forecasts too far ahead, a warning that we extend below to a more general interest rate rule.

structural sticky-price model used by CGG (2000) and LWW (2001). It also theoretically demonstrates that the determinate outcome from aggressive policy responses to next period's expected inflation does not generalise to ever-increasing forward-looking rules. Section 5 offers some concluding remarks and discusses the policy implications of our findings. A Technical Appendix proves the theorem of Section 4.

2. Can inflation-forecast-based rules be destabilising or lead to multiple equilibria?

Symbolically, an IFB rule takes the following generic form:

$$R_t = \gamma R_{t-1} + \theta(1-\gamma)[E_t\pi_{t+j} - \pi^{\mathrm{T}}] \qquad (1)$$

where R_t denotes the short-term nominal interest rate, π_t is consumer price inflation, π^{T} is the inflation target, and E_t is the expectational operator based on information available at time t. Rule (1) assumes that policymakers set the nominal interest rate so as to respond to deviations of the inflation term from target and of output from potential. In addition, it assumes that policymakers have a tendency to smooth rates, in line with the idea that central banks adjust the short-term nominal interest rate only partially to eliminate the gap between the previous period nominal rate and the current target level. Thus equation (1) includes a lagged interest rate term on the right hand side.[10] Given our notation, the larger the parameter γ, the greater the degree of interest rate smoothing.

j is the feedback horizon of the central bank. When $j = 0$ the central bank feeds back from current dated variables only. When $j > 0$, the central bank feeds back instead from deviations of forecasts of variables from target. This is a proxy for actual policy in inflation targeting countries, notably the United Kingdom, that apparently respond to deviations of current inflation from its short or medium forecast. Finally θ, the feedback parameter, is greater than 0. The bigger is θ, the faster is the pace at which the central bank acts to eliminate the gap between expected inflation and its target value. As we show later, the stabilizing characteristics of (1) depend both on the magnitude of θ and the length of the feedback horizon j.

[10] Alternatively, one can just accept that empirical estimates of (1) support a non-zero value of γ.

IFB rules have various desirable properties.[11] Because they embody transmission lags, they generally help improve inflation control (lag-encompassing). These rules can be designed to smooth the path of output as well as inflation, despite not feeding back from the former explicitly (output-encompassing). Finally, IFB rules deliver clear welfare improvements over Taylor-type rules, which respond to a more restrictive subset of information variables (information-encompassing).

2.1 Necessary and sufficient conditions for uniqueness and stability of equilibria

As mentioned above, depending on the precise combination of the pair (j, θ), IFB rules can lead the economy into instability or indeterminacy. To understand better how this can happen, think of the model economy as one being governed by the behaviour of a set of variables x_1 and x_2 and a set of shocks u. Let us also assume that the vector x_2 is a set of forward-looking, i.e. non-predetermined variables, while x_1 represents pre-determined variables. Assume that the equilibrium dynamics of the economy can be written as:

$$\begin{bmatrix} x_{1,t+1} \\ E_t x_{2,t+1} \end{bmatrix} = C \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + Bu_t \qquad (2)$$

Technically, the condition for a stable and unique equilibrium depends on the magnitude of the eigenvalues of the C matrix. If the number of eigenvalues outside the unit circle is equal to the number of non-predetermined variables, the system (2) has a unique equilibrium which is also stable (see Blanchard & Kahn (1980), Farmer (1999), Evans and Honkapoja (2001)).

Instability occurs when the number of eigenvalues of C outside the unit circle is larger than the number of non-predetermined variables. In practice, this implies that when the economy is pushed off its steady state following a shock, it cannot ever converge back to it, but rather finishes up with (e.g.) explosive inflation dynamics (hyperinflation or hyperdeflation). In the particular case of instability triggered by a specific parametrisation of the policymakers' IFB rule, this arises because the choice of the pair (j, θ) is not the 'right' one. Typically this means that the parameter pair does not imply a strong enough response to induce those changes in the short-term

[11] See Batini and Haldane (1999).

real interest rate, which are necessary to offset movements in inflation following a shock. In this case the rule is said to be ‘destabilising’.

By contrast, indeterminacy occurs when the number of eigenvalues of *C* outside the unit circle is smaller than the number of non-predetermined variables. Put simply, this implies that when a shock displaces the economy from its steady state, there are many possible paths leading back to equilibrium, i.e. there are multiple well-behaved rational expectations solutions to the model economy. With forward-looking rules this can happen when policymakers respond to private sector’s inflation expectations and these in turn are driven by non-fundamental exogenous random shocks (i.e. not based on preferences or technology), usually referred to as ‘sunspots’. If policymakers set the coefficients of the rule so that this accommodates such expectations, the latter become self-fulfilling.[12] This implies that the rule is unable to pin down the behaviour of one or more real and/or nominal variables, making many different paths compatible with equilibrium (see Kerr and King (1996), Chari, Christiano and Eichenbaum (1998), CGG (2000), Calstrom and Fuerst (1999, 2000), Svensson and Woodford (1999), and Woodford (2000)). The fact that the rule itself may introduce indeterminacy and generate so called ‘sunspot equilibria’, is of interest because sunspot fluctuations—i.e. persistent movements in inflation and output that materialise even in the absence of shocks to preferences or technology—are typically welfare-reducing and can potentially be quite large.[13]

[12] To see why this happens, consider a simple example of ‘sunspots’ generated by the following model, expressed in deviation form:

$$\pi_t = E_t\pi_{t+1} + \chi_1 y_t \quad \text{(i)} \qquad\qquad y_t = -\alpha(R_t - E_t\pi_{t+1}) \quad \text{(ii)}$$

where (i) represents an expectations-augmented Phillips curve; Roberts (1995) derives this, based on either the presence of price adjustment costs, or staggered contracts (as in Calvo, 1983). Equation (ii) represents a static IS relationship. Suppose the central bank reacts aggressively to expected inflation by setting nominal interest rates at $R_t = \theta E_t\pi_{t+1}$ where $\theta > 1$. Consider a scenario in which the private sector anticipates that inflation next period will be equal to 1. This will lead to an increase in real interest rates, with a consequent reduction in demand of $\alpha(\theta - 1)$. From (i), this implies that price-setting behaviour will lead to a current inflation rate of $1 - \chi_1\alpha(\theta - 1)$, which we define as π_0. Assuming that this is positive, it follows that $0 < \pi_0 < 1$. If we then lead equations (i) and (ii) forward in time and take expectations, consistency requires that the sequence of successive inflationary expectations is given by 1, $1/\pi_0$, $1/\pi_0^2$, $1/\pi_0^3$, However, these inflation expectations tend to infinity, which is not credible, implying that the scenario of non-zero inflationary expectations is also not credible. On the other hand, suppose that the central bank is not aggressive, and $\theta < 1$. The interest rate then does not even react one-for-one to inflation It follows that $\pi_0 > 1$, and hence that the sequence of inflationary expectations tends to zero. This sunspot sequence is then credible, so that private sector expectations have become self-fulfilling.

[13] Empirical analysis measuring the contribution of sunspot shocks to aggregate fluctuations are of two types. First, calibration exercises that assess how allowing for sunspot shocks can help in matching model properties to business cycle regularities. These include Farmer and Guo (1994), Schmitt-Grohe (1997, 2000), Thomas (1998) and Wu and Zhang (2000). Second, analyses that try to identify sunspot shocks from rational expectations residuals that are left unexplained by exogenous fundamentals. These

For IFB rules, for example, CGG (2000) seem to imply that, assuming an optimising model with Calvo-type price-stickiness, the necessary and sufficient condition for determinacy is that the parameter θ must be greater than 1 when $j = 1$.[14] LWW (2001) by contrast, examined the implications for determinacy when longer horizons than

$j = 1$ are used, by computing numerical conditions for a set of calibrated sticky-price and sticky inflation models. In what follows we also experiment with various forward horizons but, in addition, we show analytically that what matters for determinacy is the combination of the feedback parameter and the chosen horizon, not the former alone. Importantly, we also show that various parametrisations of IFB rules can not only trigger multiple equilibria (sunspots) but can also cause unstable inflationary outcomes. In all cases, an analytical investigation is key because numerical results are subject to the particular choice of parameters in the model and in the rule, and so may not be general.

2.2 Examples of instability and indeterminacy with IFB rules in a small AS-AD optimising model with sticky inflation

To demonstrate that simple forecast-based rules may be destabilising or lead to indeterminate solutions, in this section we evaluate their performance by constructing the implied inflation-output volatility frontiers using a small scale AS-AD structural closed-economy RE model similar that of Batini and Haldane (1999), Batini and Nelson (2001b), CGG (1999) and the FM model of LWW (2001). The assumption that the economy is closed, albeit that most inflation-targeting countries are small open economies, is done for simplicity. Still, as shown in Batini and Haldane (1999), the destabilising characteristics of IFB under specific parametrisations carry over in full to the open economy case and so our analysis is interesting for that case too.

include Farmer and Guo (1995), Salyer (1995) and Salyer and Shreffin (1998). The evidence on the importance of sunspots for macrofluctuations from this research is mixed. However, more recent estimates by Lubik and Schorfheide (2002) obtained by extending the likelihood-based estimation of dynamic stochastic general equilibrium models to account for multiple equilibria for the US suggests these may have been modest before 1979.

[14] Carlstrom and Fuerst (2000) reversed the findings in CGG (2000) by using a model that modifies the traditional MIUF set-up to account for more realistic CIA timing assumptions and imposes finite price stickiness. They showed that to ensure determinacy, the central bank should instead follow a backward-looking policy rule, where the interest rate responds aggressively (i.e. setting $\theta > 1$) to past inflation, rather than future, expected inflation rates.

Once we remove foreign variables and constants in each equation, so that variables are expressed in terms of deviations from equilibrium, the model—excluding the policy rule— can be written as follows:

$$y_t = \alpha_1 y_{t-1} + \alpha_2 E_t y_{t+1} - \alpha_3 [R_t - E_t \pi_{t+1}] + \varepsilon_{ISt} \qquad (3)$$

$$\pi_t = \chi E_t \pi_{t+1} + (1-\chi)\pi_{t-1} + \chi_2 [y_t + y_{t-1}] + \varepsilon_{\pi t} \qquad (4)$$

where y_t is the output-gap, π_t is inflation and R_t is the short-term nominal interest rate. ε_{ISt} and $\varepsilon_{\pi t}$ are aggregate demand and aggregate supply shocks, assumed to be white noise. Equation (3) is the model's IS equation, with real output depending positively on leads and lags of itself and negatively on the ex ante real interest rate. As in Fuhrer (2000), lags of the dependent variable on the right hand side can be rationalised by assuming that preferences over consumption exhibit habit formation. This implies that preferences are not time-separable in consumption, so that households' utility depends not only on the level of consumption in each period, but also on its level in the previous period. Equation (4) is similar to that from an aggregate supply curve based on the assumption of two-period overlapping real wage contracts. It can be derived by assuming intertemporal optimisation on the part of agents with a concern for relative wages, as in Rankin and Ascari (2001); or by assuming that price contracts are set as in Calvo (1983) but in addition are indexed to lagged inflation rates, as in Christiano, Eichenbaum and Evans (2001). The assumption that both output and inflation depend on lags as well as leads of themselves, significantly improves the ability of this model to match UK and US data (for the UK see Batini et al (2001); for the US see Fuhrer (2000) and Fuhrer and Moore (1995)).

In particular, following Batini and Haldane (1999) we set $\alpha_1 = 0.75$, $\alpha_2 = 0$ and $\chi = 0.2$: with this parameterisation, the model has a backward-looking IS function and a partially forward-looking AS equation (as in Fuhrer (1997)). Batini and Haldane (1999) show that with these parameters the model's transmission mechanism is rather sluggish and is broadly in line with simulation responses from VAR-based studies of the effect of monetary shocks in the UK. The rest of the parameters are also calibrated to UK data, following Batini and Haldane (1999). The shock processes are modelled as in Batini and Nelson (2001b), who generated residuals from the model equations (2)-(3) using UK data over the period 1981Q1-1998Q1.

Charts 1 and 2 below plot the loci of output/inflation variability points delivered by rule (1), as the horizon *j* of the inflation forecast is varied, for different values of the feedback parameter, θ. More specifically, Chart 1 plots asymptotic variances obtained by solving the model with a unit variance shock to aggregate demand. Chart 2 does the same for aggregate supply shocks. Along the loci, we vary *j* between zero (current-period inflation targeting) and six (one and a half-year-ahead inflation forecast targeting) periods. Points to the south and west in Charts 1 and 2 are welfare-superior, and points to the north and east inferior.

Chart 1: Demand shock: *j* –locus

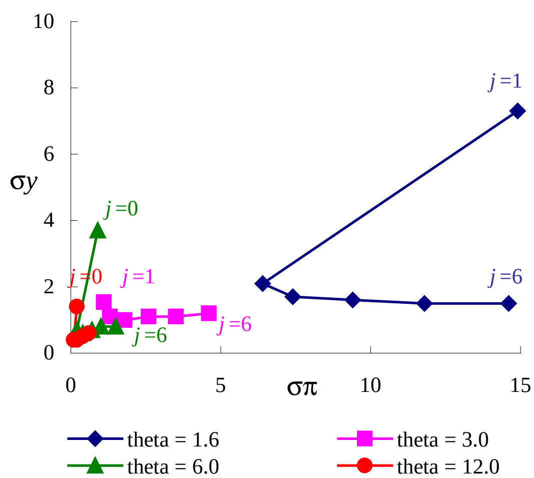
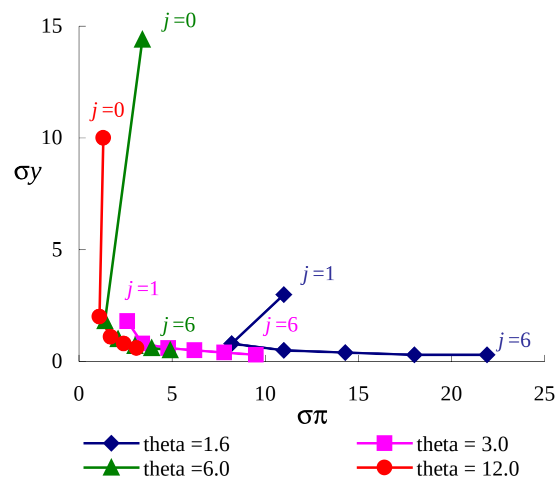


Chart 2: Supply shock: *j* –locus



A few important points emerge from these charts. First, the optimal feedback horizon is positive and lies somewhere between one and three quarters ahead. This forecast horizon secures as good an inflation performance as any other, while at the same time delivering low output variability. This is because for this model, 1-3 quarters is around the horizon at which monetary policy has its largest marginal impact. At shorter horizons than this, the adjustment in monetary policy necessary to return inflation to target is that much greater—the upshot of which is a destabilisation of output. Second, feeding back from a forecast horizon much beyond three quarters leads to worse outcomes for both inflation and output variability. This is the symmetric counterpart of the arguments used above. Just as short-horizon targeting implies reacting too late to the inflationary consequences of economic shocks —thereby requiring a large and output-costly policy response, long–horizon targeting can equally imply that policy reacts too soon in anticipation of those same consequences, thereby setting in train an almost destabilising expectational feedback. Third, higher values of θ, the feedback parameter, shift the j-loci to the left, towards welfare-superior portions of the inflation/output volatility space, for all values of the feedback horizon j. Intuitively, this is due to the fact that more aggressive policy responses (higher θ) are taken into account by price and wage setters in the economy when setting their contracts. These adjust their inflation expectations closer to the central bank's target as they will now expect smaller deviations of inflation from target. As this happens automatically, simply by virtue of model-consistent expectations, in effect interest rates do not have to move as much to correct future inflation gaps as they would have to, had the policymakers announced a less aggressive policy rule (lower θ). The consequence of this spontaneous adjustment in inflation expectations is lower output variability and hence lower inflation variability.

When the model is unstable or indeterminate, inflation-output volatility pairs cannot be computed and hence do not appear in Charts 1 and 2. For this reason, Table 1 below lists critical values for θ, the feedback parameter, for given values of j, associated with unique and stable equilibria. This helps to trace viable rule parametrisations in (θ, j) space. The table indicates that, in general, too low values of θ are associated with unstable outcomes; whereas too high values are associated with indeterminate outcomes. In particular, for all $j > 1$, stability requires a response coefficient θ above 1. This is in line with the conventional wisdom in the literature that nominal rates have to rise (or fall) more than inflation, to elicit corresponding moves in the real interest rate (see, among others, Woodford (2000), Taylor (2000), CGG (2000), and LWW (2001)). The table also reveals that upper and lower values of the bounds for θ change with j, in line with the idea that it is the combination of feedback size and length of horizons which matters for stability and uniqueness of equilibria under IFB rules. For instance, in this set-up, when $j = 1$, stable and unique equilibria can be generated by a wide range of θ (1.4 to infinity). However, as j increases, the range of θ ensuring determinacy (as well as stability) gets increasingly narrow. With $j = 8$, for instance, the case in which the central bank responds to deviations of two-years-ahead expected inflation from target, the maximum value of θ that is associated with determinate solutions is 3.2. In practice, this means that the more forward-looking the policymaker is in responding to inflation gaps, the more care should be taken in adopting aggressive responses. With stronger responses, private sector's inflation expectations condition on the pre-emptiveness of the central bank behaviour. For too large values of θ, this can set in train a self-fulfilling expectational sunspot sequence, leading to indeterminacy. This result contradicts the general statement in LWW (2001) that in models where inflation and output are highly persistent, forward-looking policy (reacting to expected inflation up to four years ahead) is relatively immune to indeterminacy problems.[15]

The existence of indeterminacy for higher values of θ at longer horizons explains why the j-loci for $\theta = 6$ and $\theta = 12$ in Chart 1, for example, are shorter than the other j-loci in the chart. Determinate solutions for these values only exist for j up to 5 and 3, respectively.

Table 1: Critical values of θ for stability (lower) and indeterminacy (upper)

Feedback horizon j in eq(1)	Lower θ bound	Upper θ Bound
0	4.2	Infinity
1	1.4	Infinity
2	1.0	148
3	1.0	36
4	1.0	15.8
5	1.0	8.8
6	1.0	5.4
7	1.0	4.0
8	1.0	3.2

To summarise, when compared to those in CGG (2000) and LWW these results are revealing in two important respects. First, they indicate that below unity, values of the feedback parameter θ do not necessarily generate multiple solutions. Rather, in this model they lead to unstable outcomes. Also, the lower threshold for avoiding instability is greater than 1 when either current or one-quarter-ahead inflation forecasts is used in the rule (i.e. it is equal to 4.2 and 1.4 when $j = 0$ and $j = 1$, respectively).

[15] LWW (2001) find that determinacy in the FM, sticky-inflation closed-economy model obtains for all combinations of $0 < \theta \leq 10$ and $0 \leq \gamma \leq 1.5$ in (1) with $j \leq 16$. The FM model assumes that the weights on the backward and forward-looking components of inflation expectations are both equal to 0.5, whereas in this exercise we place a higher weight on backward than forward-looking inflation components in equation (4). This is sufficient to reverse LWW (2001)'s result. More on this later, where we experiment with model calibrations that are similar or identical to those used by LWW (2001).

Second, our findings indicate that the conclusion in CGG (2000) that an aggressive response to forward inflation is necessary and sufficient for determinacy is not valid for this model when the feedback horizon is longer than one quarter. For $j > 1$, determinacy in our set-up exists only for values of θ below a certain threshold. We find that this, in turn, is a function of the inflation horizon to which the rule responds.

Third, in contrast to LWW (2001), we show that it is not necessarily true that (any) persistence in inflation and output minimises the risks of indeterminacy at longer lags for a wide range of feedback and smoothing parameters.

In practice, the possibility that indeterminate solutions arise with IFB rules for any given θ, is in general not merely a function of j. Rather, it will depend on the overall or economy-wide degree of forward-lookingness in the model economy. So in model (3)-(4) it will depend, for example, also on the parameter χ, which dictates the extent by which wage and price-setters look forward when deciding their current contracts. The larger this parameter, the more volatile are wages and prices, and hence the lower the upper critical value of θ for each horizon j. We explore this issue in more detail in the next section, where we provide an analytical explanation by means of the root-locus technique. In Section 4 we apply this technique to the model used by CGG (2000) and show how our results apply to that model.

3. Root-locus analysis of the stability and determinacy properties of forward-looking rules

In order to gain further insight into the stabilising properties of IFB rules, we analyse their performance by using the 'root locus' technique, an approach that is commonly used in the control engineering literature. This allows us to identify precisely the range of stabilising feedback parameters associated with these rules, and also to measure how long a lead is suitable for conventional sticky-price/sticky-inflation models before indeterminacy sets in. So it advances upon the numerical analysis in LWW (2001) because it enables us to write down parametrically the conditions for stability and determinacy under inflation forecast-based rules and so draw more general conclusions.

In practice, the root locus technique illustrates diagrammatically how the roots of the characteristic equation describing the dynamics of the model vary when the feedback term θ changes. As the conditions for stability and determinacy of the model hinge on the value of these roots, from these diagrams we can infer which regions of the (θ, j) parameter space are associated with well-behaved solutions. In our case the technique entails deriving a separate diagram for each value of j. However, in the majority of cases a clear pattern emerges quickly, so in what follows we only draw these diagrams at most for $j = 0,...,4$.

3.1 Ex ante Real Interest Rate Rules

For didactic reasons, we start off with an ex ante real interest rate rule rather than a rule for the nominal interest rate. This rule is different from the one used in CGG (2000) and LWW (2001), so we do not compare our results with those yet, but leave that to subsection 3.2 where we employ a nominal rule. For ease of exposition we assume that target inflation is 0, and we also assume that $\gamma = 0$, so that (1) becomes:

$$R_t - E_t\pi_{t+1} = \theta E_t\pi_{t+j} \tag{5}$$

Furthermore, we assume that there is no forward-looking term in the IS equation (3) and also assume initially that $\chi = 0$, so that wage setting and inflation are purely backward-looking. We can now rearrange model (1), (3), (4) to obtain a reduced form expression for inflation as:

$$\pi_t = \pi_{t-1} - \chi_2\alpha_3\theta(E_t\pi_{t+j} + E_{t-1}\pi_{t+j-1}) + \varepsilon_{3t} + \chi_2(\varepsilon_{1t} + \varepsilon_{1t-1}) \tag{6}$$

The solution to this is characterised by the roots of the equation

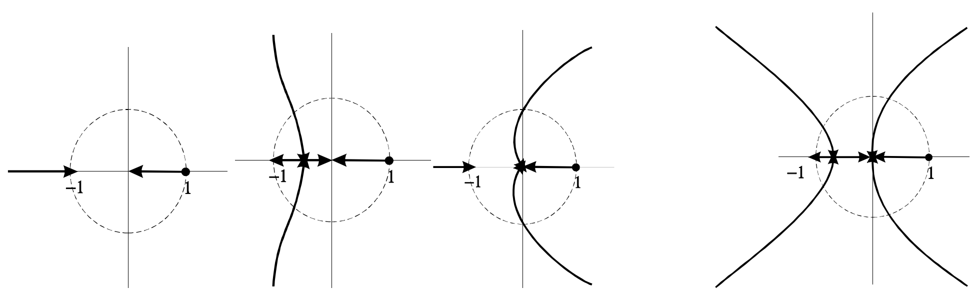
$$\chi_2\alpha_3\theta z^j(z + 1) + z - 1 = 0 \tag{7}$$

where z is the forward operator. It is clear that the reduced form solution of (6) depends only on π_{t-1} and no other lags of π. Thus, the only admissible values of θ will be those where there is exactly one root of (7) inside the unit circle (since the number of stable roots must equal to the number of predetermined variables for a unique and stable equilibrium to exist). As explained in Section 2, if all solutions of z lie outside the unit circle, then the system is unstable; whereas if more than one lies inside the unit circle, then the solution will be indeterminate.

We are now in a position to apply the root locus technique. In this respect, Chart 3 depicts the complex plane,[16] and shows how the $j + 1$ roots of (7) change as θ varies between 0 and ∞. Also shown is the unit circle.

Consider firstly the case $j = 1$. When $\theta = 0$, there is only one root of (7), at $z = 1$ (indicated by the bold dot '•'). When θ is very small, equation (7) reveals that there is another root at $-\infty$. As θ becomes very large, we can again see from (7) that there are two roots, at 0 and –1. For $j = 1$, therefore, a first portion of the root locus starts at 1 and move leftwards in the direction of the arrow, ending at $z = 0$. A second portion of the locus starts at $-\infty$, and also heads in the direction of the arrow, this time rightwards, eventually ending at $z = -1$.

The next diagram to the right in Chart 3 shows the case when $j = 2$. With this feedback horizon there are two complex roots at infinity for very small θ. From there, as θ increases,the two curved arms of the root locus then move towards the real axis. When they meet, one root heads for $z = 0$, and the other for $z = -1$.



(a) $j = 1$ (b) $j = 2$ (c) $j = 3$ (d) $j = 4$

Chart 3. Characteristic roots for the model with $\alpha_2 = \chi = 0$

More generally, for arbitrary values of j, there will be j arms (asymptotically at equal angles to their neighbour) of the root locus all converging in from infinity, as θ increases, towards the roots at $z = 0$ and $z = -1$, plus the portion of the locus from $z = 1$ to $z = 0$.

[16] In this plane, the horizontal axis depicts real numbers, and the vertical axis depicts imaginary numbers. If a root is complex, i.e. $z = x + iy$, then its complex conjugate x-iy is also a root. Thus the root locus is symmetric about the real axis.
All root-loci diagrams are based on output from Matlab, using the package Linear System Learner, available from http://users.ece.gatech.edu/~bonnie/education/LSLNR/).

So what can we infer from these diagrams regarding the type of equilibrium for each (θ, j) pair?

Let us consider again the case $j = 1$ in Chart 3. In this case, as θ increases, there is always one stable and one unstable root[17] (i.e. one value of z is outside and the other inside the unit circle at all times), so that a unique solution for inflation always exists: there is no risk of incurring either unstable or indeterminate outcomes.[18] For larger values of j, again instability never arises—as one might expect for a real interest rate rule—as there is always at least one stable root. However, beyond a certain value of θ (different for each j, and in line with results in Table 1) there are too many stable roots (converging on 0 and – 1 from the point where the two root locus arms meet, within the unit circle), so that the solution is indeterminate. As explained earlier in our discussion of the determinants of sunspot equilibria (see footnote 7), when the rule implies a direct link between the interest rate and inflation expectations, any non-fundamental shock to inflationary expectations will imply a decline in the real interest rate, which stimulates demand. In turn, this induces a rise in actual inflation, which confirms expectations making them 'self-fulfilling'. However for smaller values of θ, these 'sunspot shocks' are exponentially increasing in their effects, implying that the same non-fundamental shocks involve declines in the real rate of interest, that are incompatible with the central bank inflation targeting strategy. As private agents reject this scenario as illogical, inflation expectations are reined in, which drives the system to its unique saddle path solution. By contrast, when the value of θ is large, the final effect on real rates of sunspot shocks is negligible, so agents incorporate the effects of the shocks in their expectations. As any type of shock can be potentially accommodated this way by agents and confirmed by the central bank via the rule, the solution to the model and thus the ensuing inflation path is not unique (too many stable roots).

Another conclusion that emerges clearly from these diagrams is that there exists, in practice, a trade-off between volatility and indeterminacy. In other words, by choosing a large feedback parameter the central bank can reduce inflation and output volatility to their minima, but at the same time risk ending up with sunspot

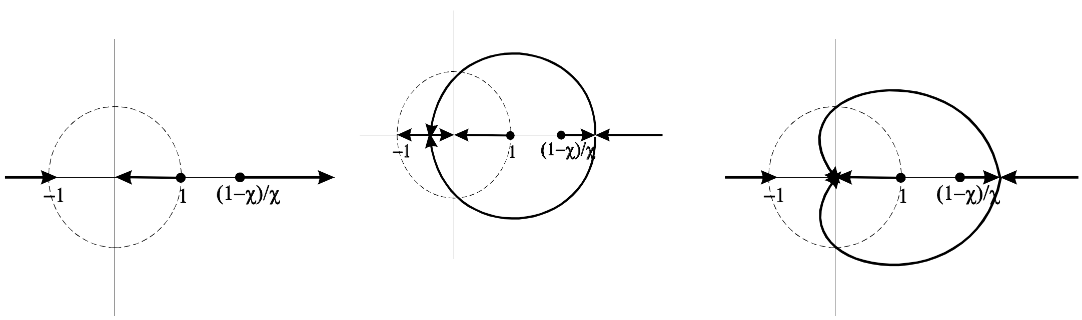
[17] Note that this can easily be shown by the more conventional method of checking the roots of the quadratic equation obtained from (7).

[18] The reason for the difference between this and the results of Section 2 is the different parameter values.

fluctuations and multiple equilibria, which are ultimately welfare-reducing. On the other hand, by choosing small values of θ (so that the stable root may be close to 1) and therefore ruling out indeterminate solutions, it risks higher inflation and output volatility (unstable outcomes).[19]

We now turn to the case where wage formation depends on forward expectations, so that $\chi > 0$. In this case, again assuming that the central bank sets the ex ante real interest rates, it follows that the characteristic equation for the reduced form inflation expression is given by:

$$\chi_2 \alpha_3 \theta z^j (z + 1) - (z - 1)(\chi z - (1 - \chi)) = 0 \qquad (8)$$



(a) $j = 1$ (b) $j = 2$ (c) $j = 3$

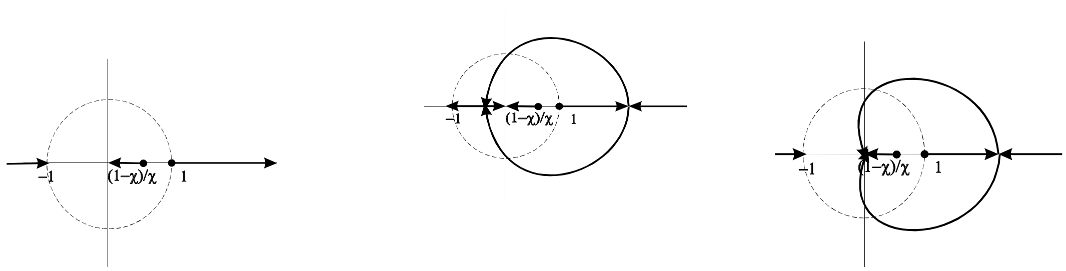
Chart 4. Characteristic roots for the model with $\alpha_2 = 0$, $0.5 > \chi > 0$

Chart 4 shows the root locus diagrams, as θ increases, for the case when wage formation is mainly backward-looking, i.e. when $\chi < 0.5$ (as in Batini and Haldane (1999) and Batini and Nelson (2001a)). For brevity, we only show the root loci for the cases where $j = 1, 2, 3$. In general, higher values of j would give similar diagrams, but with more arms of the loci heading in to the roots at 0 from infinity. As in Chart 3 for the case when $\chi = 0$, the solution is unique if there is exactly one stable root. It follows that for $j = 1$ there is stability and determinacy for all values of θ. However, when $j > 1$, there is always a value of θ beyond which there are too many stable roots and therefore, here as well, there is a trade-off between volatility and indeterminacy.

[19] Of course, this trade-off is distinct from the effect on the interest rate created by having large values of θ; although the volatility of inflation may be reduced, it follows that because interest rates are proportional to inflation, it is possible that as θ increases, the volatility of interest rates may increase. This is an issue that is beyond the scope of this paper, and we do not pursue it here.

Thus large θ reduces volatility, but risks multiple equilibria, while small θ (with the stable root close to 1) results in higher volatility, but no indeterminacy.

When wage formation is mainly forward-looking, i.e. when $1 \geq \chi > 0.5$, the analysis is somewhat different. This is in line with our intuition that what matters for stability and uniqueness is not just the forward-looking behaviour of the policymakers, but rather, the 'economy-wide' degree of forward-lookingness (i.e. the extent to which the private sector forward-looking behaviour combines with the pre-emptiveness of the central bank to give an overall measure of forward-lookingness in the economy).



(a) $j = 1$ (b) $j = 2$ (c) $j = 3$

Chart 5. Characteristic roots for the model with $\alpha_2 = 0$, $1 > \chi > 0.5$

Chart 5 shows a similar set of root loci to Chart 4 for this case. However, here we see that for small values of θ, the stable characteristic root is less than $(1-\chi)/\chi$ (with values of this root being located in the portion of the root locus to the left of $(1-\chi)/\chi$ on the real axis). In practice, this means that the larger is χ, the less volatile is inflation. For example, if $\chi = 0.8$, then the stable root is less than 0.25, so there is no need to be overly aggressive in order to reduce volatility. It follows that here, contrary to the previous cases, there is no longer a trade-off between volatility and indeterminacy, unless the monetary authority is resolved to reduce volatility to its absolute minimum. This is because when the private sector is very forward-looking when setting wage and price contracts, the central bank need only be mildly forward-looking to ensure both low inflation and output variability and uniqueness of the equilibrium.

In summary, this analysis suggests that the trade-off between volatility and indeterminacy with forward-looking rules is strictly related to the extent by which wage and price setters in the economy are forward-looking. The same forward horizon can give determinacy for large regions of the feedback parameter space if agents are predominantly forward-looking. The same is not true when these are predominantly backward-looking in setting their contracts. In this case the set of feasible feedback parameter values shrinks progressively as the feedback horizon lengthens.

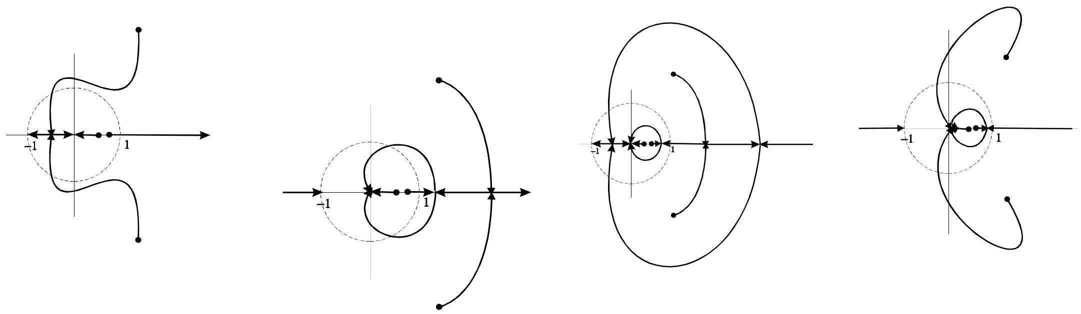
3.2 Nominal Interest Rate Rules

We now investigate a nominal interest rate rule (as used in CGG (2000) and LWW (2001)) for the model (1), (3), (4), corresponding to the simulations of Section 2, and including the assumption of nominal interest rate smoothing, which requires $0 < \gamma < 1$. This is somewhat more difficult to analyse using root locus techniques than the ex ante real interest rate rule, so we conduct a more general analysis of this rule only later in Section 4. After some effort, it is possible to show that the characteristic equation for this rule is given by

$$(z-\gamma)[(z-1)(z-\frac{1-\chi}{\chi})(z-\alpha_1)+\alpha_3\frac{\chi_2}{\chi}z^2(z+1)]-\theta\alpha_3(1-\gamma)\frac{\chi_2}{\chi}z^{j+2}(z+1)=0 \quad (9)$$

It is clear that a necessary (but not sufficient) condition for stability is that $\theta > 1$, since nominal interest rates must respond at least one-for-one to expected inflation. Since both the supply and demand equations now contain dynamics of order 1, and in addition the interest rate rule has dynamics of order 1 it is now necessary to have exactly three stable roots (one for each non-predetermined variable). In equation (9), the term inside the square brackets has roots that cannot easily be characterised. Nevertheless, it is instructive to investigate its roots for the particular case of the parameters of Section 2:

$$\gamma = 0.5,\ \alpha_2 = 0,\ \alpha_1 = 0.75,\ \chi = 0.2,\ \chi_2 = 0.2,\ \alpha_3 = 0.5$$



(a) $j = 0$ (b) $j = 1$ (c) $j = 2$ (d) $j = 3$

Chart 6. Characteristic roots for the model with nominal interest rate rule
(Diagrams not completely to scale)

The root locus diagrams associated with this parametrisation are shown in the chart above. They are slightly difficult to interpret, so we explain them one by one. First note that for $\theta = 0$, there are four roots, two of which are complex, another one equal to 0.5, and the last slightly larger; in addition for all leads there is a root at $z = 1$ when $\theta = 1$. For lead $j = 0$, at $\theta = 1$, the two complex roots lie outside the unit circle (note that this cannot be inferred from the root locus diagram, and requires separate checking). As θ increases further, it eventually reaches a threshold value (see Table 1) for which there are 3 stable roots, two complex and one real, As θ increases further towards ∞, the real stable root heads for $z = 0$, while the two complex roots meet on the negative part of the real axis, with one root continuing on to 0 and the other to –1. Thus beyond the threshold value of θ, there are always exactly 3 stable and one unstable root, so the system is stable and determinate. For lead $j = 1$, at $\theta = 1$, there is only one stable root, but once the stability threshold of θ is reached, there are always exactly 3 stable and 1 unstable roots (note that there is a value of θ such that the root at + ∞ suddenly switches to a root at – ∞). For lead $j = 2$, once θ is slightly greater than 1, there are 3 stable roots, two of which are complex. However, for very large values of θ, there is indeterminacy, as all 5 roots are stable, with 4 converging on $z = 0$ and one on $z = -1$. For lead $j = 3$, the story is similar, except that as $\theta \to \infty$, there are 5 stable roots converging on $z = 0$, and one unstable root converging on $z = -1$.

For leads greater than 3, the story is much the same, except that indeterminacy kicks in at ever lower values of θ. In this sense, these results are in contrast with results in

LWW (2001), as they find that, using the FM model, determinacy obtains for a wide range of feedback parameter values.[20]

4. Application of the Root Locus technique to a New Keynesian Structural Sticky-Price Model

We now apply the root locus technique to examine the properties of IFB rules in the model used by Bernanke and Woodford (1997) (with expectations conditional on the previous period's information), CGG (2000) and LWW (2001).[21] We will show that this is not merely a mechanical procedure, but that it also provides some insights as to how far ahead one can use inflation forecasts to influence nominal interest rates.

The CGG (2000) analysis is particularly interesting because of their empirical work on US monetary policy rules and their general conclusion that a feedback response above unity to next quarter's expected inflation leads to determinacy.[22] We can now verify whether the results of Section 2 are specific to our previous model or can be generalised to other models with different degrees of nominal rigidities. In practice, in model (3)-(4) inflation depends on a moving average of the output gap, and output is entirely backward-looking, which implies that results will not be identical between the two models.

The inflation and output equations in the model used by CGG (2000) have the following form (in deviations about the mean):

$$\pi_t = \delta E_t \pi_{t+1} + \lambda (y_t - y^*_t) \qquad (10)$$

[20] Note that however, the calibration of the AS equation in the FM model in LWW (2001) implies a higher degree of forward-looking behaviour (χ = 0.5) than our calibration here (where χ = 0.2) and so, in principle, involves a different trade-off between volatility and indeterminacy.
[21] This is a dynamic model and consists of two structural equations. The first is a standard expectational IS curve, as in Kerr and King (1996), Woodford (1996) and McCallum and Nelson (1997). The second, characterising the supply side is based on Calvo (1983), is discussed under different expectational assumptions in Bernanke and Woodford (1997), and is fully derived by Gali (2002).
[22] CGG (2000) found for the US, that there were clear differences in the conduct of monetary policy pre and post 1979. Their estimates indicate that the Fed appears to have indeed responded to expected inflation at either one quarter or one year ahead in both periods. However, the coefficient for the interest rate response to expected inflation has been considerably greater than 1 during the Volcker-Greenspan era, while the same coefficient was significantly less than 1 in the pre-Volcker era, a possible cause, they argue, of macroeconomic instability at the time. More precisely, they find that the feedback coefficient on expected inflation took a value between 2 and 3 after 1979. This contrasted with feedback on the one-quarter-ahead output gap, which although significant in both periods, had a fairly large size only prior to 1979. As a consequence, we continue with the strategy of the previous section, and focus solely on feedback from forward-looking inflation.

$$y_t = E_t y_{t+1} - 1/\sigma(R_t - E_t\pi_{t+1}) + \varepsilon_{1t} \qquad (11)$$

where the notation is as before, and y^* is an exogenous stochastic process interpretable as the log of the natural rate level of output in period t. The third and final equation in the model is the nominal interest rate rule (1). So we assume that there is no feedback on the output gap, broadly in line with CGG (2000) empirical estimates for the US post 1979 period. Finally the inflation target is set at 0 for convenience.

Equation (10) is derived by assuming that firms in the economy are monopolistically competitive and that each has a constant probability of changing its price at each period, as in Calvo (1983). This implies that each firm will set current prices such that the weighted average of future anticipated mark-ups over marginal cost exactly matches a constant mark-up. Equation (11) is derived by combining a standard Euler equation for consumption obtained by assuming that the expected utility of the representative consumer increases in both consumption and leisure and a market-clearing condition, relating output to consumption.[23]

Given equations (1), (10), (11), it is possible to show that the characteristic equation for the system is given by:

$$(1-\gamma)\theta z^{j+1} + (z-\gamma)[\sigma/\lambda(\delta z - 1)(z-1) - z] = 0 \qquad (12)$$

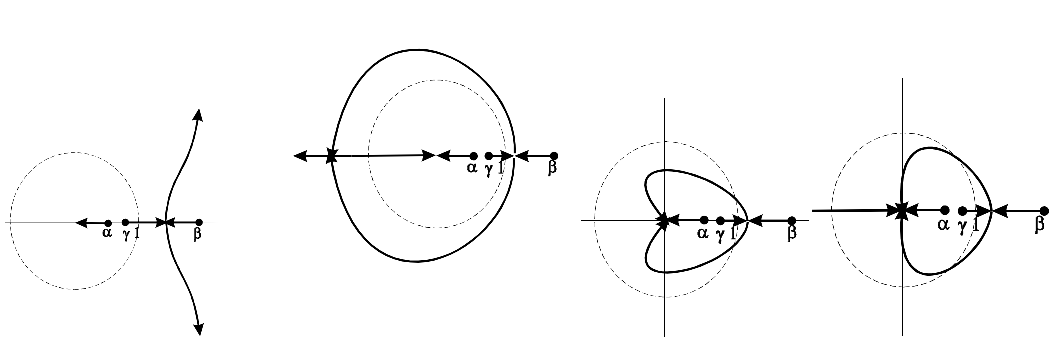
which we rewrite for convenience as:

$$\theta z^{j+1} + A(z-\gamma)(z-\alpha)(z-\beta) = 0 \qquad (13)$$

where $A = \sigma\delta/(\lambda(1-\gamma))$. It is easy to demonstrate that $\alpha < 1$ and $\beta > 1$.

We can now draw a similar set of root locus diagrams to those derived in Section 2 (one for each lead j), for varying θ. Inspection of equations (1), (10) and (11) suggests that stability and uniqueness (no sunspots) occur only if there is exactly one stable root. Furthermore examination of (12) indicates—as pointed out by CGG (2000)—that the potentially critical value of θ, for determinacy, is unity.

[23] It is easy to show that: if (i) output only depends on labour, (ii) real marginal cost depends on the real wage, and (iii) output demand expressed in logs depends approximately linearly on consumption expressed in logs, then the marginal costs, when expressed in real terms, are proportional to the gap between output y_t and its natural rate y_t^*.



(a) $j = 0$ (b) $j = 1$ (c) $j = 2$ (d) $j = 3$

Chart 7. Characteristic roots for the New Keynesian model

Chart 7(a) confirms this by showing that for $j = 0$, i.e. feedback on current inflation, the system is stable and has a unique equilibrium for all values of $\theta > 1$; once the root locus has passed through $z = 1$, there is exactly one stable root, as required for determinacy. In addition, and in contrast to our results for the model in Section 2, for values of $\theta < 1$ there are no unstable outcomes, but rather indeterminacy. This is due to the entirely forward-looking nature of the model, which contrasts with the predominantly backward-looking model of Section 2. As we have argued above, it is the overall degree of forward-lookingness in the economy that matters for the occurrence of self-fulfilling sunspots.[24] When the rule starts responding to inflation expectations at longer horizons ($j \geq 1$), self-fulfilling inflationary expectations and sunspot equilibria are once again possible as θ becomes too large.[25] These manifest themselves as soon as the root locus enters the unit circle. For the case $j = 1$, using one-period ahead forecasts, it is clear from Chart 7(b) that indeterminacy occurs when the root locus enters the unit circle at $z = -1$. Thus the critical upper bound for θ is obtained by substituting $z = -1$into (12), and corresponds exactly to that obtained by Woodford (2002).[26] We note, as he does, that the greater is the degree of inertia γ in the interest rate rule, the less binding is this upper bound. For inflation expectations at

[24] Note the contrast between this model and that of Section 2. The latter had problems of indeterminacy for high values of θ, while for this model indeterminacy occurs at low values of θ. The reason is because the earlier model is predominantly backward-looking, while this model is mainly forward-looking. Were we to use an interest rate rule for current inflation ($j = 0$) in this model, we would also find that there is a problem of instability for high θ, as opposed to low θ for the model of Section 2.

[25] This finding is in line with that in Bernanke and Woodford (1997), which assume that policy responds to private-sector forecasts rather than to model-consistent forecasts as we do here.

[26] Bernanke and Woodford (1997) already derived analytical conditions for indeterminacy for the case when j =1 and $\gamma = 0$, using this model with expectations conditional on the past period's information set.

longer horizons, Charts 7(c) and 7(d) are indicative of the general shape of the root locus in the proximity of the unit circle.

Chart 7 also gives further clues as to whether it is possible to have indeterminacy for certain forward horizons for inflation, independently of the level of the feedback parameter θ. Clearly this will not happen when there is exactly <u>one</u> stable root for some given θ; this is the situation where the arms of the root locus starting from α and β meet to the right of $z = 1$, before they branch off into complex values away from the real, horizontal axis (which is the case drawn in Charts 7(c) and (d)). However, it could happen that the two arms instead meet *inside* the unit circle and then remain within it; this would imply that there was *never* exactly one stable root, which would in turn imply indeterminacy.

Examining the chart, it appears that whenever θ increases beyond 1, the root of the equation increases to a value greater than 1. However, this is no longer true when the root locus from β passes through $z = 1$ from the right. This is because in this case these two complex arms of the locus stay *within* the unit circle. We therefore conclude that there is determinacy for $\theta > 1$ if $\partial z/\partial\theta > 1$ at $z = 1$ i.e. the root locus passes through $z = 1$ from the left. Conversely, there is indeterminacy if $\partial z/\partial\theta < 0$ at $z = 1$. The associated results are summarised in the following theorem, and proved in the Technical Appendix.[27]

Theorem:

Whatever the combination of parameter values, there is always some lead J such that for $j > J$ there is indeterminacy for *all* values of θ.

The implication therefore is that the analysis of CGG (2000) does not generalise to the case when longer feedback horizons are used. From the proof, the relevant values of j for indeterminacy for all θ are characterised by

$$j - (1/(1-\gamma) + (1-\delta)\sigma/\lambda) > 0$$

Therefore, using parameters $\gamma = 0.5$, discount factor $\delta = 0.99$, coefficient of relative risk aversion $\sigma = 1$, this inequality is satisfied for all $j \geq 3$ for values of the output elasticity of inflation λ>0.01. There is substantial empirical evidence on the parameter λ based on US data, the lowest value reported by CGG (2000) being 0.05. For γ = 0.8, as estimated by CGG, this inequality is satisfied for all $j \geq 6$, so that use of forward-looking inflation beyond five quarters would be ruled out; this is therefore a fairly robust upper limit.

Note that these results are consistent with those of LWW in their Figure 2, which were obtained using numerical simulation. In addition, their results in Figure 3 for longer horizons are in general agreement with our theorem, that greater interest rate smoothing (higher γ) gives smaller scope for indeterminacy.

As real-world inflation target procedures typically involve stabilising inflation in the medium-run, one to two years out, corresponding to *j* ranging from 4 to 8, our findings represent an important warning for inflation-targeting central banks. For example, if the model of the US economy used by CGG (2000) were to be truly representative of reality, then our findings indicate that for reasonable parametrisation of monetary policy rules, there is a significant risk of sunspot fluctuations arising from self-fulfilling expectational sequences in the US whenever the Fed responds to expected inflation at horizons beyond one and a quarter years.

5. Conclusions

IFB rules are simple rules that respond to deviations of expected inflation from target. Simulation of IFB rules has shown that they have a number of desirable features because they are usually good proxies of optimal feedback rules. However, as these rules are not fully optimal, they can lead to dynamic instability or indeterminacy. A result in the literature (e.g. Bernanke and Woodford (1997) and CGG (2000)) is that to avoid indeterminacy, the monetary authority must respond aggressively (i.e. with a coefficent above unity, even if not too large) to expected inflation. By contrast, the instability implications of these rules are not often investigated.

This paper has extended the existing literature on the uniqueness *and* stability conditions for equilibrium under IFB rules. Advancing upon numerical results in LWW (2001), we have demonstrated analytically, for a variety of structural New

[27] A similar result is theorised in Giannoni and Woodford (2001), Proposition 5, page 47, but only for the case when γ = 0.

Keynesian models, that both are a function not just of the degree of responsiveness of the policy instrument to deviations of inflation expected at some horizon *j* from target, but also of that chosen feedback horizon. In particular, we have found that reacting too aggressively to events that lie too far into the future, may deliver results that can be as bad as those obtained by responding to events that lie too far into the past. In this respect, we have shown that an above unity response to expected inflation more than one quarter out, is not a necessary and sufficient condition for determinacy. Furthermore, beyond a certain expectation horizon, *any* feedback will produce indeterminacy, and a consequent lowering of welfare. This is an important warning for inflation-targeting central banks, particularly as the model used by CGG (2000) and LWW (2001) is subject to self-fulfilling sunspot sequences for feedback on inflation forecasts as little as six quarters out.

By using the root locus analysis—a technique borrowed from the control engineering literature—we have shown how to identify the feedback/horizon pairs that are associated with unique and stable equilibria for a variety of real business cycle sticky-price and sticky-inflation models. We find that this is a simple and robust way to unveil the analytical rationale behind instability or indeterminacy at too long a lag, for given feedback. Indeed, the root locus method is possibly the only one that is capable of revealing our main theoretical result.

References

Ball, L, (1999), "Efficient Rules for Monetary Policy", *International Finance*, 2(1), pp 63-83.

Batini, N and Haldane, A G (1999), 'Forward-looking rules for monetary policy', in Taylor, J B (ed) (1999), *Monetary Policy Rules*, Chicago: University of Chicago Press, pp 157-202.

Batini, N, Harrison, R, and Millard, S, (2001), "Monetary Policy Rules for Open Economies", *Journal of Economic Dynamics and Control*, forthcoming.

Batini, N and Nelson, E, (2001a), 'Optimal horizons for Inflation Targeting', *Journal of Economic Dynamics and Control*, Vol. 25, pages 891-910.

Batini, N and Nelson, E, (2001b), "The lag from monetary policy actions to inflation: Friedman revisited", *International Finance*, 4(3), pp 481-400.

Benhabib, J and Farmer, R E, (1999), "Indeterminacy and Sunspots in Macroeconomics", in (Taylor and Woodford, 1999), chap. 6, pp. 387-448.

Benhabib, J, Schmitt-Grohe, S, and Uribe, M (2001). Monetary Policy and Multiple Equilibria", *American Economic Review*, 91(1), pp 167-86.

Berger, H, de Haan, J and Eijffinger, S (2001) Central Bank Independence: An Update of Theory and Evidence, *Journal of Economic Surveys*, 15, 3-40.

Bernanke, B and Woodford, M (1997) "Inflation Forecasts and Monetary Policy", *Journal of Money Credit and Banking*, vol. 24, 653-684.

Black, R, Macklem, T and Rose, D, (1997), "On policy rules for price stability", in *Price stability, inflation targets and monetary policy*, ed. Bank of Canada. Ottawa: Bank of Canada.

Blanchard, O J and Kahn, C M, (1980), "The Solution of Linear Difference Models under Rational Expectations", *Econometrica*, 48(5), pp 1305-11.

Bullard, J and Mitra, K, (2001a), "Determinacy, learnability and monetary policy inertia", working paper, www.stls.frb.org/research/econ/bullard

Bullard, J and Mitra, K, (2001b), "Learning about monetary policy rules", Journal of Monetary Economics, forthcoming.

Calvo, G A, (1983), "Staggered contracts in a utility-maximizing framework", *Journal of Monetary Economics*, 12, 383-98.

Carlstrom, C T and Fuerst, T S, (1999), "Real indeterminacy in monetary models with nominal interest rate distortions", Federal Reserve Bank of Cleveland working paper.

Carlstrom, C T and Fuerst, T S, (2000), "Forward-looking versus backward-looking Taylor rules", Federal Reserve Bank of Cleveland working paper.

Chari,V V, Christiano, L J, and Eichenbaum, M, (1998), "Expectation traps and discretion", *Journal of Economic Theory*, 81(2), pp 462-92.

Chow, G C, (1970), "Optimal Stochastic Control of Linear Economic Systems" *Journal of Money, Credit and Banking*, 2(3), pp 291-302.

Christiano, L J, Eichenbaum, M and Evans, C, (2001), "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy", NBER Working Paper, no. 8403.

Clarida, R, Gali, J, and Gertler, M, (1999), "The Science of Monetary Policy: A New Keynesian Perspective" in *Journal of Economic Literature*; 37(4), pp. 1661-1707.

Clarida, R, Gali, J, and Gertler, M, (2000), "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory", *Quarterly Journal of Economics*, 115(1), pp 147-80.

Evans, G W and Honkapoja, S, (2001), "Expectations and the stability problem for optimal monetary policies", working paper, www.valt.helsinki.fi/RAKA/seppo.html

Evans, G W and Honkapoja, S, (2002), "Monetary policy, expectations and commitment", mimeo.

Farmer, R E A, (1999), *The macroeconomics of self-fulfilling prophecies*. Second edition. Cambridge and London: MIT Press,

Farmer, R E A and Guo, JT, (1994), "Real Business Cycles and the Animal Spirits Hypothesis", *Journal of Economic Theory*, 63, pp. 42-72.

Farmer, R E A and Guo, JT, (1995), "The Econometrics of Indeterminacy: An Applied Study", *Carnegie-Rochester Conference on Public Policy*, 43, pp. 225-27.

Friedman, M (1960), *A Program for Monetary Stability*. New York,: MacMillan.
Fuhrer, J C (1997), 'The (un)importance of forward-looking behavior in price specifications', in *Journal of Money, Credit and Banking*, 29, pp. 338-350.

Fuhrer, J C (2000), 'An optimisation-based model for monetary policy analysis: can habit formation help?', *American Economic Review*, pp.367-390.

Fuhrer, J C, and Moore, G R, (1995), "Inflation Persistence", Quarterly Journal of Economics, 110, pp. 127-59.

Gali, J (2002), 'New Perspectives on Monetary Policy, Inflation, and the Business Cycle', CEPR Discussion Paper No. 3210.

Giannoni, M P and Woodford, M, (2001), "Optimal Interest-Rate Rules", mimeo, Princeton University.

Kerr, W and King, R G, (1996), “Limits on interest rates rules in the IS model”, *Economic Quarterly*, vol 82, 2, pp 47-76.

King, M A, (1997), ‘Monetary Policy and the Exchange Rate’, *Bank of England Quarterly Bulletin*, May, pages 225-27.

Levin, A, Wieland, V and Williams J C, (2001), “The performance of inflation forecast-based rules under model uncertainty”, Federal Reserve Board International Finance Discussion Paper No.

Lubik, T A and Schorfheide, F, (2002), “Testing for Indeterminacy: An Application to US Monetary Policy”, paper presented at the NBER Summer Institute, Monetary Economics Programme, Cambridge Mass, July 2002.

McCallum, B T and Nelson, E, (1997), “An Optimising IS-LM specification for monetary policy and business cycle analysis”, NBER working paper no. 5875.

Nelson, E, (2001), ‘UK monetary policy 1972-97: a guide using Taylor rules’, CEPR Discussion Paper No. 2931.

Rankin, N and Ascari, G, (2001), “Staggered wages and output dynamics under disinflation”, *Journal of Economic Dynamics and Control*, 26 (4), pp. 653-680

Roberts, J M (1995), New Keynesian Economics and the Phillips Curve, *Journal of Money, Credit and Banking*, 27(4), 975-984.

Rudebusch, G D and Svensson, L E O (1999), “Policy rules for inflation targeting”, in Taylor, J B (ed), *Monetary Policy Rule*s, Chicago: University of Chicago Press, pages 203–53.

Salyer, K D, (1995), “The Macroeconomics of Self-Fulfilling Prophecies: A Review Essay”, *Journal of Monetary Economics*, 35, pp. 215-242.

Salyer, K D and Shreffin, S M, (1998), “Spotting Sunspots: Some Evidence in Support of Models with Self-Fulfilling Prophecies”, *Journal of Monetary Economics*, 42, pp. 511-523.

Sargent, T J, (1987), *Dynamic macroeconomic theory*, Cambridge, Mass. and London: Harvard University Press.

Schmitt-Grohe, S, (1997), “Comparing Four Models of Aggregate Fluctuations due to Self-Fulfilling Fluctuations”, *Journal of Economic Theory*, 72, pp. 96-147.

Schmitt-Grohe, S, (2000), “Endogenous Business Cycles and the Dynamics of Output, Hours and Consumption”, *American Economic Review*, 90(5), pp. 1136-1159.

Schmitt-Grohe, S and Uribe, M, (2000), “Price level determinacy and monetary policy under a balanced-budget requirement”, *Journal of Monetary Economics*, 45, pp 211-246.

Svensson, L E O, (1997), “Inflation Forecast Targeting: Implementing and Monitoring Inflation Targets”, *European Economic Review*, 41, pp 1111-1146.

Svensson, L E O and Woodford, M, (1999), “Implementing Optimal Policy through Inflation-Forecast Targeting”, mimeo, Princeton University.

Svensson, L E O and Woodford, M, (2000), “Indicator Variables for Optimal Policy”, IIES Seminar Paper No. 688, NBER Working Paper No. 7953.

Taylor, J B (ed) (1999), *Monetary Policy Rules*, Chicago: University of Chicago Press.

Taylor, J B, (1999), “A Historical Analysis of Monetary Policy Rules”, in Taylor, J B (ed) (1999), *Monetary Policy Rules*, Chicago: University of Chicago Press, pp 319-41.

Taylor, J B, and Woodford, M, (eds.) (1999), *Handbook of Macroeconomics*, Volume1. Elsevier, Amsterdam.

Thomas, Julia (1998), “Do Sunspots Produce Business Cycles?”, mimeo, University of Minnesota.

Turnovsky, (1977) *Macroeconomic analysis and stabilization policies*, Cambridge University Press.

Wicksell, K, (1935), *The Influence of the Rate of Interest on Prices*, Routledge, London.

Woodford, M, (1994),”Nonstandard Indicators for Monetary Policy: Can Their Usefulness be Judged from Forecasting Regressions?”, in *Monetary Policy*, edited by N G Mankiw, Chicago, University of Chicago Press for NBER.

Woodford, M, (1996), “Control of the public debt: a requirement for price stability?”, NBER working paper no. 5684.

Woodford, M, (2000), “Pitfalls of forward-looking monetary policy”, *American Economic Review* 90(2), pp 100-104.

Woodford, M, (2002), “Interest and Prices”, manuscript, Princeton University.

Wu, Y and Zhang, J, (2000), “Monopolistic Competition, Increasing Returns to Scale, and the Welfare Costs of Inflation”, *Journal of Monetary Economics*, 46, pp. 417-440.

Technical Appendix

Proof of Theorem

The derivative $\partial z/\partial\theta$ is obtained from implicit differentiation of (13):

$$z^{j+1} + ((j+1)\theta z^{j} + A[(z-\gamma)(z-\alpha) + (z-\beta)(z-\alpha) + (z-\gamma)(z-\beta)])\partial z/\partial\theta = 0 \quad \text{(A1)}$$

In particular, at the point $\theta = 1$, $z = 1$, it is given by

$$1 + (j+1 + A[(1-\gamma)(1-\alpha) + (1-\beta)(1-\alpha) + (1-\gamma)(1-\beta)])\partial z/\partial\theta = 0 \quad \text{(A2)}$$

Note that if we now use the definition of $A = \sigma\delta/(\lambda(1-\gamma))$, and the implicit values $\alpha\beta = 1/\delta$, $\alpha + \beta = 1 + 1/\delta + \lambda/(\delta\sigma)$, then for j =1, (A2) can be rewritten

$$1 - [\gamma/(1-\gamma) + (1-\delta)\sigma/\lambda]\partial z/\partial\theta = 0$$

so that $\partial z/\partial\theta > 0$; this complements what is shown in Chart 7(a), where we deduce that there is determinacy for a range of θ greater than 1.

For more general j, (A2) can be rewritten as

$$1 + [j - 1/(1-\gamma) - (1-\delta)\sigma/\lambda]\partial z/\partial\theta = 0$$

It is therefore clear that there must exist a minimum value of j above which the term multiplying $\partial z/\partial\theta$ is positive. Hence $\partial z/\partial\theta < 0$ as required when the lead j is large enough.