Tax specific term structures of interest rates: evidence from the United Kingdom government bond market

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and

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CONTENTS

Abstract

Coupon income and capital gain on UK Government bonds are taxed differently, so some investors do not regard all bonds as perfect substitutes. This paper examines the extent to which term structures of interest rates derived from the UK bond market are tax specific, using a linear programming technique to select the optimal portfolios for investors facing different tax treatment. Despite the tax reforms of the mid-nineteen eighties - designed to reduce arbitrage opportunities in the gilts market - we find that divergences between yield curves of the main categories of tax payers remain.

1. Introduction

Following the work of McCulloch (1975), there has been considerable interest in estimating term structures of interest rates which pay careful attention to the effects of taxation. Earlier Robichek and Niebuhr (1970) had shown how the tax induced bias could substantially alter the shape of the estimated yield curve. McCulloch's empirical procedure for accounting for the effects of differential taxation when estimating yield curves did not, however, estimate separate yield curves for investors who face different tax rates. The resulting yield curve was for a 'representative' tax rate, which may well not be a rate paid by any investor. Schaefer (1981, 1982) subsequently developed a method of measuring the term structure which specifically allowed for the tax dependence of an investor's choice of securities, thereby introducing the idea of 'tax clienteles'.

With differential taxation some investors do not regard all bonds available in the market as perfect substitutes. If dividend (coupon) income and capital gains are taxed at different rates, and furthermore investors' marginal tax rates also differ, there will be an incentive for some investors to concentrate their holdings on particular types of bonds. Those investors in higher rate income tax brackets will tend to prefer bonds with relatively low coupons. Formally, a tax clientele exists if a security is rationally held by investors in only particular tax brackets. The existence of tax clienteles will determine the relevant yields available to different investors; their net returns will depend both on their tax rates and, as a result, on the securities held. This is clearly relevant for investment appraisal. On theoretical grounds also, it is difficult to justify the assumption of a single 'implicit tax rate', as is the case in many empirical studies of the yield curve, even though market prices of government bonds may be determined by an 'average' tax rate.

In the literature on 'tax clientele' effects, two particular issues have been addressed. The theoretical literature has concentrated on identifying conditions under which equilibrium will exist - when all bonds are willingly held and arbitrage opportunities do not exist. Empirical studies have attempted to ascertain the number of tax clienteles by estimating separate term structures for

Erratum

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The first sentence of the third paragraph on page 8 should read as follows:

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The paper is organised as follows: Section 1 considers the theoretical problem of the existence of equilibrium in the presence of distortionary taxation, and provides a brief description of the taxation of gilts in the United Kingdom. Instead of appealing only to restrictions on short selling, which is the common assumption made in most of the literature [see for example, Dammon and Green (1987), and Litzenberger and Rolfo (1984)], we argue that assymmetries in the tax system also help to preclude infinite arbitrage opportunities and ensure that equilibrium exists. Section 2 outlines the empirical methodology, comparing the two main approaches: McCulloch's regression method and the linear programming method outlined by Schaefer. Our results are presented in Section 3. Section 4 concludes.

2. Equilibrium with differential taxation

Consider a stylised example of a market which consists of only two types of investor, labelled here as 'gross investors' and 'net investors'. Gross investors pay no tax on either income or capital gains, whereas net investors pay each clientele. The number of clienteles is equal to the number of separate term structures that can be found, where each term structure is based on only the set of securities that are rationally held by investors in that particular tax bracket.

In this paper we examine the extent to which term structures of interest rates are tax specific in the UK bond market, using the linear programming method of selecting optimal portfolios outlined by Schaefer. In particular, given the tax reforms in the mid nineteen eighties which were designed to reduce the extent of distortions in the gilts market and reduce arbitrage possibilities, we consider whether the divergences between yield curves of the main categories of tax payers have persisted. Our results suggest significant differences in the optimal portfolios of the main investor categories. Net returns differ not only due to differential taxation, but also because the bonds rationally held are different.

The paper is organised as follows: Section 1 considers the theoretical problem of the existence of equilibrium in the presence of distortionary taxation, and provides a brief description of the taxation of gilts in the United Kingdom. Instead of appealing only to restrictions on short selling, which is the common assumption made in most of the literature [see for example, Dammon and Green (1987), and Litzenberger and Rolfo (1984)], we argue that assymmetries in the tax system also help to preclude infinite arbitrage opportunities and ensure that equilibrium exists. Section 2 outlines the empirical methodology, comparing the two main approaches: McCulloch's regression method and the linear programming method outlined by Schaefer. Our results are presented in Section 3. Section 4 concludes.

2. Equilibrium with differential taxation

Consider a stylised example of a market which consists of only two types of investor, labelled here as 'gross investors' and 'net investors'. Gross investors pay no tax on either income or capital gains, whereas net investors pay different tax rates on capital gains and income.[1] When the income tax rate is higher than the tax rate on capital gains, net investors will prefer to receive more of their return in the form of capital gains. To illustrate the problem of attaining equilibrium, we first consider a scenario where gross investors hold a fully diversified portfolio in the sense that they are holding some of all bonds available in the market. If this represents an equilibrium for gross investors it will be characterised by prices such that they are indifferent between all the bonds. In this situation, however, net investors cannot be indifferent between all bonds. They will have a preference for low coupon bonds. Conversely, if net investors are fully diversified, (ie they are indifferent between all bonds) low coupon bonds must be being discounted at a higher rate than bonds with higher coupons. In this case gross investors cannot be indifferent between all bonds in the market. In either of these scenarios there will be gains from trade between the two types of agents. However, there will never exist a set of prices such that there are no further gains from trade, (ie where both types of investors are indifferent); as one set of investors approach their equilibrium, the other will be moving away from equilibrium.

This problem of attaining equilibrium can be illustrated with the following simple example [see Dammon and Green (1987)]. Suppose there are two assets, a tax-exempt bond which yields r_m, and a taxable government bond yielding a gross return r_g. If t_i is the tax rate, then in equilibrium the after-tax returns should be equalised such that $r_g(1-t_i) = r_m$. Clearly, if investors face different tax rates $(i = 1,..,n)$, then no set of market prices will satisfy this condition for all investors. There will be potentially infinite gains from arbitrage.[2]

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(1) The current system of taxation in the UK differs from this 'simple' characterisation of only two types of investor, in that some 'gross' investors pay the same (non-zero) rate of tax on capital gains and income (see Section 2.1 below).

(2) Exactly the same problem arises if we consider two taxable bonds (with different coupons) and two investors facing different tax rates. The net returns will differ for the two investors and there will be no set of prices which will eliminate arbitrage opportunities for both investors.

In the example above, where net investors are indifferent between all bonds, it must be the case that low coupon bonds are discounted at a lower rate (ie priced higher) than high coupon bonds, since they can obtain the same income stream (coupons and principal) at a lower price. By going short gross investors are effectively borrowing at the low-coupon rate and lending at the high-coupon rate. Gross investors will want to short sell low coupon bonds (to net investors), and with the proceeds buy high coupon bonds. This arbitrage activity will be at the expense of the government (ie the tax authority), since net investors will hold all low coupon bonds, and gross investors hold all high coupon bonds.(3)

In the theoretical literature on tax clientele effects it is assumed that short selling is prohibited. As can be seen from the above example this is sufficient to ensure that gains from trade will not be infinite (Schaefer 1981, 1982), and differential taxation simply leads to differential net returns. However, in practice not all investors are prohibited from short selling.(4) Therefore some additional factor must be responsible for the existence of equilibrium. A plausible candidate in our view is the asymmetry of the tax system.

Investors pay tax on coupons received, but not all investors can claim tax relief on coupon payments that they make.(5) In the example above, if gross investors sell low coupon bonds short, they will then effectively be obliged to make coupon payments to the purchasers (net investors), whereas gross investors will pay tax on the coupons from the high coupon bonds they hold. In practice it is this feature of the tax system which is partly responsible for

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(3) For investors who face short selling restrictions, the arbitrage activity could be done on the basis of 'notional' bonds. In this case some adjustment for credit risk may be necessary.

(4) In the UK only market makers are allowed to short sell. Since all market makers are taxed at the same rate (see Section 2.1), there would be no gain in arbitrage activity amongst market makers, but they could short sell to other net investors.

(5) Investors in the United Kingdom can ordinarily claim tax relief if they borrow to finance the purchase of bonds, providing such borrowing is considered to be part of their normal business. However, in the arbitrage example described here no borrowing is entailed.

curtailing arbitrage activity. The equilibrium that results is analytically equivalent to the equilibrium which exists when it is assumed that all short selling is prohibited, as in most of the academic literature.

2.1 Taxation of Gilts in the UK

In the United Kingdom, the system of taxation was reformed in the mid-nineteen eighties in order to eliminate some distortions that led to tax avoidance. Before 1985 income tax was charged on coupons that bondholders actually received and capital gains tax (CGT) applied only to bonds held for less than a year. This system enabled investors to avoid income tax payments in some circumstances (ie where their income tax rate was higher than CGT), by purchasing ex-dividend bonds and selling them prior to the next dividend payment. Alternatively investors could avoid all CGT including income that is capitalised, by holding a bond for more than one year, (specifically only a day more than a year).

Since February 1986 income from bond holdings has been taxed on an accruals basis.(6) This implies that higher rate income tax payers should prefer low coupon bonds, since income can no longer be capitalised. Also in July 1985 the Inland Revenue announced that from July 1986 CGT on gilts would be abolished. This change was inevitable once the income tax system had been changed. If coupon income could not be capitalised, CGT would not matter since it did not apply for holdings for more than a year. Furthermore, for holdings of less than one year CGT was used primarily for offsetting losses elsewhere in investors' portfolios. Although these reforms were designed to reduce tax arbitrage, in principle it is possible that in the absence of coupon 'washing' the relative demand for low coupon bonds would be higher and thereby lead to a greater distortion between the prices of high and low coupon bonds. These tax reforms may therefore not be sufficient to eliminate tax clienteles.

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(6) Taxation on an accruals basis implies that the tax liability is imputed on daily coupon accrual rather than the actual twice yearly coupon payments. Thus taxes are paid regardless of whether or not coupons are actually received.

These changes in taxation imply that there are now two main types of investors, those that are exempt from all taxes (eg pension funds), and those that are subject to income tax on interest income at their marginal tax rates. Thus, at present, higher rate individuals pay 40% and lower rate individuals pay 25%. Companies pay the corporation tax rate (at present 33%). No investors pay tax on capital gains on their government bond investments, except market makers in gilts whose profits from trading gilts are subject to corporation tax.

Since investors face different tax rates, the net (after-tax) returns will also differ and this implies that their demands for government bonds will be different. Gross investors will be indifferent between returns in the form of capital gains or coupon payments, whereas net investors will have a strict preference for capital gain returns. This implies that they will be discounting cash flows at different rates. It follows from this that the maximum price an investor is willing to pay for a bond will also differ between investors.

3. Estimation

There are essentially two different methods of estimating the effects of taxes on bond prices. The first, due to McCulloch (1975), is to estimate a term structure of interest rates for a 'representative' investor. The alternative method is that of Schaefer (1981), which finds a separate term structure for each tax bracket (clientele), after first determining the set of bonds which is rationally held by that clientele. McCulloch's regression method equates the present value of a bond to its price by setting i (the tax rate) such that the standard error of the regression is minimised. The present value (PV) of an n period bond to a net investor who pays income tax at rate i and capital gains tax at rate g, is given by,

$$PV(i, g) = [(1-i)C]/(1 + {}_tR_1) + [(1-i)C]/(1 + {}_tR_2)^2 + [(1-i)C + F - g(F - P)]/(1 + {}_tR_n)^n \qquad (1)$$

where: C = Coupon,
$_tR_n$ = The n period discount rate at time t,
P = Current price of bond,
F = Nominal redemption value of bond.

In equilibrium PV = price for those who hold the bond and PV < price for those who do not hold the bond. The problem is that we do not know who is holding the bond. So McCulloch runs a cross-section regression to estimate the term structure using different values of i, setting $g=i/2$.(7) The value of i which minimises the standard error of the regression is then referred to as the effective tax rate (or sometimes known as the market tax rate). Since this is an attempt to find a single effective tax rate, by definition it can only find one clientele. Furthermore, it is not clear what the market tax rate is meant to represent. McCulloch argues that it is a 'fairly accurate indication of the effective marginal tax rate that governs US security prices, and the average tax rebate to the Treasury on its coupon payments.' This, however, will not be the case in a situation where prices are determined by net investors, but most bonds are held by gross investors. Therefore, the average is not the Treasury's tax collection since gross investors do not pay tax. McCulloch's method is only useful when there is a representative investor who determines prices in every period. In this situation it will help ascertain the marginal tax rate, but even in this case there will not be a clear relation between the tax take and the 'market tax rate'.

3.1 The Linear Programming Approach

The tax clientele hypothesis (with no short selling) implies that holdings of all bonds should be non-negative, and should be zero for bonds where the present value is less than the price. All coupons used to calculate the present value are net of tax. Thus for each tax bracket (clientele) the set of relevant bonds is those bonds for which PV = price. There are then two problems that need to be resolved simultaneously: (i) finding the set of relevant bonds for each tax bracket by solving the present value equation with a strict equality; and (ii) at

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(7) This was the relationship between income tax and capital gains tax in the US in 1975.

the same time ensuring that for the remaining set of bonds there is a strict inequality. In other words the discount rates which equate PVs to prices for these 'relevant' bonds are not equal to the discount rates which equate the present values of other bonds to their prices.

The linear programming approach considers the above problem in terms of minimising total expenditure on a portfolio of bonds which will yield a given sequence of cash flows in each period. The optimal portfolio (say x^*) is one which provides cash flows $(S = s_1, s_2,..,s_T)$ at minimum cost. Formally the optimisation problem is written as:

$$Min \ \Sigma p_i x_i$$

st 1) $\sum_i a_{ij} > S_j \qquad j = 1,...T$ **LP (1)**

2) $x_i > 0 \qquad i = 1,...m$

where, p_i is the price of bond i
x_i number of units of bond i held
a_{ij} is the after tax coupon payment of bond i in period j.

The choice of S, the required cash flows, is determined by an iteration technique which is outlined in the appendix.

An alternative way to specify the same problem is to write it in terms of the dual to the linear program **LP (1)**, in which the present value of the cash flows is maximised. The dual values of **LP (1)** will yield the cost of a marginal increase in cash flows in each period. If one thinks of the cost in terms of the present value of an increase in cash flow in period j, then the dual value is simply the discount factor applicable to period j. If $d_j = (1+R_j)^{-j}$ is the discount factor on the j^{th} period cash flow, then the one period forward rate between period j-1 and j is given as $d_{j-1}/d_j = (1+r_j)$. The problem can then be written as,

$Max\ \Sigma\, s_j\, d_j$

$$\text{st} \quad 1) \quad \sum_j a_{ij}\, d_j \le p_i \qquad i = 1,\ldots,m \qquad \textbf{LP (2)}$$

$$2) \quad d_j \ge 0 \qquad j = 1,\ldots,m$$

The solutions to both formulations of the **LP** are conditional on the choice of *S*. If an arbitrarily selected set of bonds can provide any *S* at minimum cost, then the choice of S does not determine the term structure of interest rates. The term structure is conditional on *S* in general because the choice of *S* determines which set of bonds and how many have to be held in order to minimise costs.

The minimum cost solution **LP (1)** has an optimal feasible solution provided that for every period *j* there is at least one bond that provides a positive cash flow in that period. However, if we do not make any allowance for carrying excess cash flows forward (or discounting them back), the cost of meeting the cash flow constraints in **LP (1)** may be very high. Note that the 'shadow price' of each cash flow constraint is the corresponding discount factor for that period.(8) Also, if a cash flow constraint is not binding then the cash flow in that period is greater than that which is actually required. Decreasing this cash flow constraint will therefore have no effect on the optimal value of x^*, since this value is determined by cash flows in other periods, and so the corresponding discount factor (shadow price) is zero. This clearly cannot be a realistic description of the term structure. To get around this problem we treat the discount function as continuous rather than discrete, by employing continuous approximation functions to the discrete discount function.

The actual specification of the approximation functions and the final linear program that we solve are outlined in the appendix. The iteration technique which is used to specify the required cash flows is also explained in the appendix. Next we turn to a discussion of our empirical results.

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(8) The 'shadow price' for the j^{th} constraint is the amount that x^* is decreased when the cash flow s_j is decreased by 1 unit, which might be interpreted as the opportunity cost of receiving 1 unit in period *j*.

4. Results

On the basis of the LP method outlined in the previous section we calculated the net (after tax) yield curves for three classes of investors;

a) pension funds - 0% tax rate
b) companies - 33% marginal tax rate and
c) individuals - 40% marginal tax rate

The yield curves were calculated for different dates. Since capital gains tax was abolished in the mid-nineteen eighties, we selected one date prior to the abolition and compared it with a period after. In principle this should enable us to examine whether CGT made any significant difference, although no firm conclusions can be drawn from this exercise as we are not actually modelling the determinants of the yield curve. In addition, as a check on whether the LP method generates results which accord with some stylised facts, we calculated the same yield curves for the two days on either side of the date that the United Kingdom entered the Exchange Rate Mechanism of the EMS. The evidence from other measures of yield curves and also from changes in the price of index-linked gilts indicates that there was a significant downward revision of inflation expectations following ERM entry. Therefore such 'event' studies may offer some tentative evidence on whether our procedure is consistent with other results.

For determining the optimal portfolios of the three categories of investor the data used in the LP consisted of approximately sixty government bonds, (prices and their associated after tax coupons). Only a small number of bonds were omitted - those that were deemed to be sufficiently illiquid, index-linked stocks and irredeemables.

Charts 1 and 2 below show the tax adjusted yield curves for pension funds and individuals for December 1983 and December 1988. The yield curve for companies is not shown, since it turned out to be very similar to that for individuals.(9)

Chart 1 **Tax-adjusted yield curves** (1983)

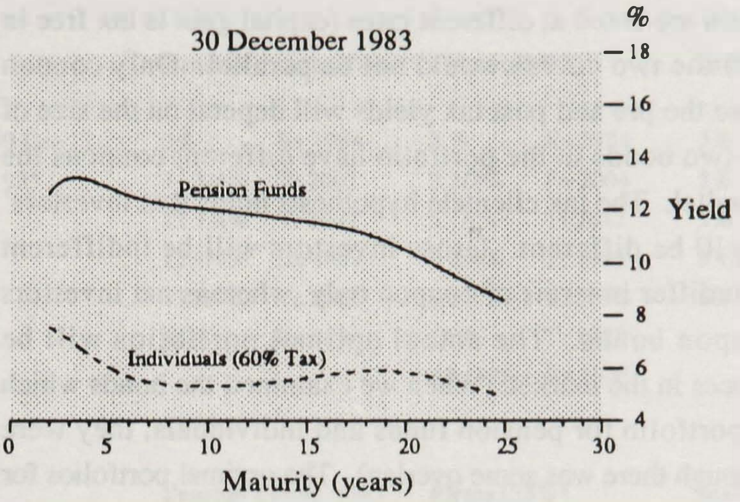
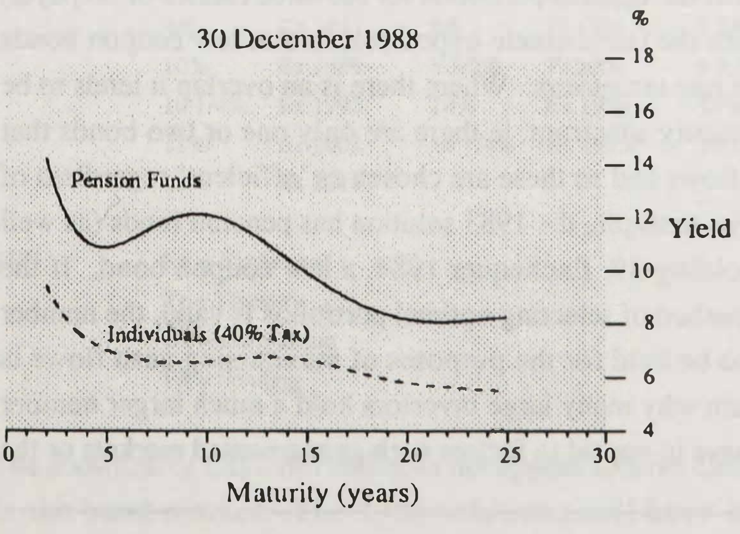


Chart 2 (1988)



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(9) The difference in their respective marginal tax rates (33% and 40%) was not sufficiently large to constitute a difference in their optimal portfolios.

The test of whether we have uncovered tax clientele effects is that the yields available to investors in different tax brackets should not differ simply to reflect their different tax rates. Note that the difference in yield curves shown in Chart 2 is not sufficient to indicate the existence of tax clienteles. Even in the absence of tax clienteles, ie if optimal portfolios were identical, since income and capital gain are taxed at different rates (capital gain is tax free in the United Kingdom) the two curves would not be parallel. Only coupon payments are taxed, so the pre and post tax yields will depend on the size of the coupons - if any two bonds in the portfolio have different coupons the curves will not be parallel. The tax clientele hypothesis holds that investors' optimal portfolios will be different. Gross investors will be indifferent between bonds which differ in terms of coupon only, whereas net investors will prefer low coupon bonds. The actual optimal portfolios will be determined by the prices in the market. When we examined the bonds which formed the optimal portfolio for pension funds and individuals, they were indeed different, (although there was some overlap). The optimal portfolios for all the three categories of investor that we examined each consisted of less than ten bonds.(10)

In Table 1 below we list the optimal portfolios for the three classes of taxpayer. The results accord with the tax clientele hypothesis in that low coupon bonds are held by the higher rate tax payers. Where there is an overlap it tends to be due to gaps in the maturity spectrum, ie there are only one or two bonds that can provide the cash flows and so these are chosen as 'efficient' regardless of tax considerations. For example, the 1983 solution has pension funds (as well as other investors) holding 3% Exchequer 1984, a low coupon bond. If the linear programming method of selecting optimal portfolios is valid, the number of bonds that needs to be held for the purposes of maximising cash flows is quite small. To explain why many large investors hold a much larger number of bonds one would have to appeal to factors such as segmented markets or the

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(10) The number of bonds in the optimal portfolio is theoretically constrained to be less than or equal to the number of approximating functions used. However, we found that increasing this number beyond ten did not increase the number of 'efficient' bonds, suggesting that our choice of twenty approximation functions did not unduly constrain the number of bonds in the optimal portfolio.

liquidity characteristics of the market being different at different points of the yield curve, or possibly transactions costs exceeding the gain from portfolio adjustment.

TABLE 1: OPTIMAL PORTFOLIOS

	Pension Funds (0%)		Firms (40%)		Individuals (60%)	
30 Dec	3%	Ex 1984	3 %	Ex 1984	3%	Ex 1984
1983	11 3/4%	Tr 2007	3 1/2%	Fn 2004	3%	Tr 1997
	12 3/4%	Tr 1992	5 1/2%	Tr 2012	3%	Ex 1995
	13 1/4%	Ex 1987	6 3/4%	Tr 1998	3 1/2%	Fn 2004
	13 3/4%	Tr 2003	8 1/4%	Tr 1990	6%	Fn 1993
	14%	Ex 1996	10%	Tr 1987		
			10 1/4%	Ex 1995		

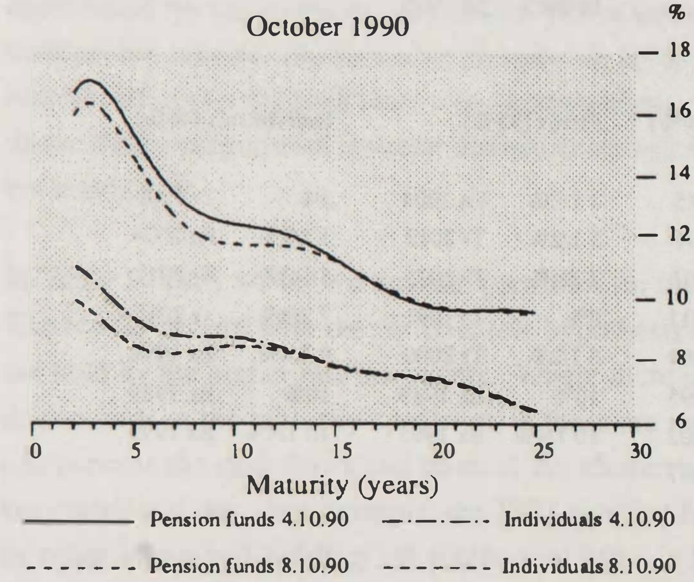
	Pension Funds (0%)		Firms (33%)		Individuals (40%)	
30 Dec	7 3/4%	Tr 2015	3 1/2%	Fn 2004	3%	Tr 1992
1988	8%	Tr 1991	5 1/2%	Tr 2012	3 1/2%	Fn 2004
	8 1/2%	Tr 2007	7 3/4%	Tr 2015	5 1/2%	Tr 2012
	9%	Cn 2011	8%	Tr 1991	7 3/4%	Tr 2015
	10%	Ex 1989	8 1/2%	Tr 2000	9 3/4%	Ex 1998
	10 1/4%	Ex 1995	10%	Ex 1989	10%	Ex 1989
	12%	Ex 2002	10 1/4%	Ex 1995	10 1/4%	Ex 1995
	12%	Ex 2017				

KEY:
Ex = Exchequer
Tr = Treasury
Cn = Conversion
Fn = Funding

The abolition of CGT in 1986 does not appear to have eliminated tax clienteles in the bond market. The 1988 solutions still have individuals holding predominantly low coupon bonds and pension funds holding high coupon bonds. It is more likely that the abolition affected the viability of certain trading strategies, but not optimal portfolios. Our analysis cannot of course cast light on the exact effects of the abolition, because the set of bonds that investors could choose from on the two dates is different.

Chart 3 below shows the effects of entry into the ERM on the tax-adjusted yield curves. As pointed out earlier the reason for this part of the exercise was simply to check that our procedure was reasonable. The behaviour of our optimising investors appears as expected. Yields for all investors fell sharply at the short end, most likely due to a combination of a downward revision of inflation expectations and the cut in base rates that accompanied entry into the ERM.

Chart 3 **The Effects of ERM entry** (1990)



5. Conclusion

In this paper we have examined the extent to which taxation affects the yield curves of investors who face different tax rates. The yields differ in a manner which confirms the existence of tax clienteles - optimal portfolios are different. If maximising cashflows is the only objective, then the linear programming method appears to provide quite striking results, in that the optimal portfolio consists of relatively few bonds. In practice there are likely to be a variety of other objectives which determine investment behaviour, but it is nonetheless useful to isolate the required strategy to attain one objective.

The optimal portfolios outlined here are entirely hypothetical; we do not have any information on the actual portfolios of the three categories of investor considered. It is therefore not possible to verify the extent to which our model has any predictive power.[11] The objective here has been to ascertain, in principle, the possible differences in term structures as a result of taxation.

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(11) Data on government bondholdings by the main categories of investor is available by maturity but not by coupon.

Appendix

Approximation functions

The shadow prices d_j, $j=1,...,T$, in **LP (1)** can be calculated explicitly from the dual to this linear program, specified in **LP (2)**.

We now assume that the discount function may be approximated by a linear combination of $n+1$ component functions $f_k(\ )$, $k=0,...,n$, ie:

$$d_j(t) = \sum_{k=0}^{n} \alpha_k f_k(t) \tag{1}$$

where α_k, $k=0,...,n$ are the weights attached to the component functions and t is maturity, now measured continuously rather than in discrete periods as before. Now consider the i^{th} bond, $i=1,...,m$ with $M(i)$ cash flows until maturity. If the j^{th} cash flow from bond i is denoted by a_{ij}, $j=1,...,M(i)$ paid at time t_{ij}, then the present value of the stream of payments from bond i is:

$$\sum_{j=1}^{M(i)} a_{ij} f_k(t_{ij}) \tag{2}$$

which, using (1), may be approximated by:

$$\sum_{j=1}^{M(i)} a_{ij} \sum_{k=0}^{n} \alpha_k f_k(t_{ij}) \tag{3}$$

and this can be expressed as a linear combination of the weights:

$$\sum_{k=0}^{n} b_{ik} \alpha_k \tag{4}$$

where:

$$b_{ik} = \sum_{j=1}^{M(i)} a_{ij} f_k(t_{ij}) \qquad (5)$$

(Note that for computational purposes, $M(i)$, can be replaced by a general constant T, say, that is the same for all m bonds and corresponds to the number of time periods in the discount function to be estimated. After-tax cash flows a_{ij} are then set to zero for each period in which there is no cash flow on bond i.)

In a similar manner the objective function of **LP (2)** can be rewritten as:

$$\sum_{k=0}^{n} \sigma_k \ \alpha_k \qquad (6)$$

where:

$$\sigma_k = \sum_{j=1}^{T} s_j f_k(t_j) \qquad (7)$$

Before turning to the final linear program, it is useful to discuss what shape the approximating functions will take. It is obviously desirable that the estimated discount function has sensible economic properties, such as:

(a) being equal to 1 at time $t=0$ (ie $d_j(0)=1$), inferring that the value of £1 today is exactly £1,

(b) being monotonic non-increasing, so that the present value of receiving £1 in the future is never more than £1. This avoids the possibility of negative forward rates, and

(c) being non-negative. A negative discount function would infer money having negative value in the future.

With this in mind, we follow Schaefer by using Bernstein polynomials of the form:

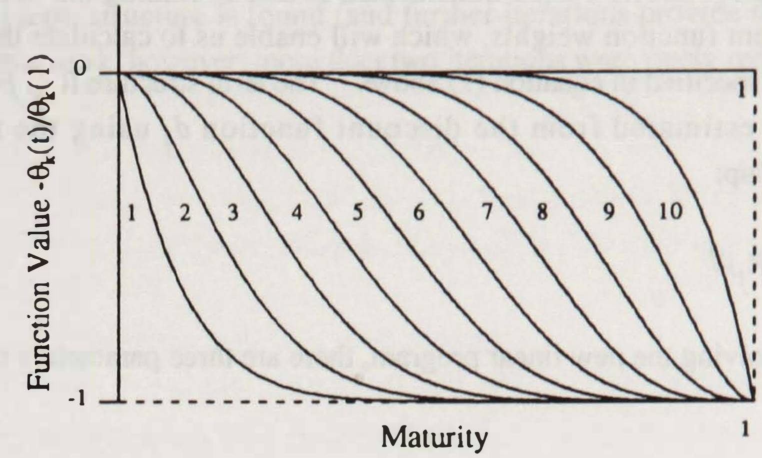
$$\theta_k(t) = \sum_{r=0}^{n-k} (-1)^{r+1} \binom{n-k}{r} \frac{t^{(k+r)}}{(k+r)} \qquad k = 1,..,n \tag{8a}$$

The first function $\theta_0(t)$ is defined as:

$$\theta_0(t) = 1 \tag{8b}$$

In equation (8a), t is measured (without loss of generality) on the interval [0,1]. Figure 1 shows these functions, scaled by $\theta_j(1)$, for $k=1,...,10$.

FIGURE 1 : Approximating Functions $\theta_k(t), k=1,...,10$



The use of this set of polynomials, along with three constraints to ensure that $d(0)$ is unity and that $d(1)$ is non-negative, enable us to define the complete problem as in equations (9a) - (9e) below:

$$\text{maximise} \quad \sum_{k=0}^{n} \sigma_k \alpha_k \tag{9a}$$

subject to:

$$\sum_{k=0}^{n} b_{ik} \alpha_k \leq p_i \qquad i = 1,...,m \tag{9b}$$

$$\alpha_k \geq 0 \qquad k = 0,...,n \tag{9c}$$

$$\alpha_0 = 1 \tag{9d}$$

$$\sum_{k=0}^{n} \alpha_k f_k(1) \geq 0 \tag{9e}$$

The problem has now been transformed to that of finding the optimal set of component function weights, which will enable us to calculate the discount function specified in equation (1) above. The term structure R_j, $j=1,...,T$ can then be estimated from the discount function d_j using the following relationship:

$$d_j = (1+R_j)^{-j} \tag{10}$$

Before solving the new linear program, there are three parameters that need to be set:

(a) n, the number of approximating functions to be used. The choice of n is arbitrary, but Schaefer warns about choosing n too low, as it constrains the number of 'efficient' bonds. For the purposes of this work, n was set to 20,

(b) T, the length of the term structure to be estimated. In practice this is constrained by the longest bond in existence at any particular time (although the term structure could possibly be extrapolated if required). In this work, based on the UK Gilt-Edged Market, T is set to 25 years, and

(c) $\underline{s}$, the vector of after-tax cash flows which defines the objective function in equation (**9a**). It is impossible to make the estimated term structures independent of $\underline{s}$, but it is possible to ensure that the objective function is equally sensitive to all points along the length of the term structure. To do this, set s_j, $j=1,...,T$ using equation (**11**) below, which is essentially the derivative of the linear program:

$$s_j = (\gamma/j)\, exp(jR_j) \tag{11}$$

Using equation (**11**) implies that the term structure R_j is already known. Since we are trying to estimate the term structure, some form of iterative procedure is required. Following Schaefer, R_j is initially set to 10% for all j, $\underline{s}$ is calculated using equation (**11**) and linear program (**9**) is then solved. The resulting term structure is then fed back into equation (11) to create a revised $\underline{s}$ vector, and the linear program is solved again. This procedure is continued until a final term structure is found (and further iterations provide the same result). In this work, however, more than two iterations were rarely required.

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