

Estimating the Term Structure of Interest Rates

by

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Abstract

This paper examines various techniques used to estimate the term structure of interest rates from the prices of government bonds; in particular comparing the current Bank of England model with two approaches suggested in the academic literature. There are two main aspects of this problem: estimating the relationship between bond yields and maturity, and the relationship between bond yields and coupon. The paper outlines how these problems are approached by the three models, and compares them on both theoretical and practical grounds. It concludes that there is a trade-off between theoretical rigour and practical considerations.

1 Introduction

When pricing financial instruments, agents throughout the financial markets are (either explicitly or implicitly) revealing information on the interest rates that they regard as being appropriate for the particular transactions they are making; but these prices or yields may also reflect other factors such as the effect of taxation rules and the perceived risk of default by the issuer. Isolating the implied interest rates is therefore a far from trivial task. It can reasonably be assumed that a unique (theoretical) underlying rate exists for each maturity, and so when trying to recover these we are aiming to construct a function that describes a single interest rate for each maturity - *the term structure of interest rates*. This is used for a number of purposes. For example, the Bank of England advises HM Treasury on appropriate interest rates to charge local authorities and some nationalised industries who borrow money through the National Loans Fund (NLF) or the Public Works Loan Board (PWLb). Institutions or individuals undertaking financial transactions may want to know how their own opinions relate to 'market' opinions. It is also useful for financial economists; for example, such data are/can be used to estimate the parameters of general equilibrium term structure models, and to test their stability (eg Cox, Ingersoll and Ross 1985, Longstaff and Schwartz 1992).

Government securities are generally used in the estimation of the term structure of interest rates, since they are free of default risk. If there were a 'suitable' government bond (ie single payment, liquid, etc) maturing at every future date we could simply take the interest rate on that bond as the underlying interest rate for that maturity. In the UK, however, government bonds - *gilt-edged securities* - are not equally spaced through the maturity spectrum: there are 'gaps' for which we need some form of interpolation to identify a continuous term structure. Moreover, there are no single payment (zero coupon) UK

government bonds,⁽¹⁾ so the problem is further complicated by the existence of semi-annual⁽²⁾ interest or 'coupon' payments.

This paper examines various techniques used to recover the term structure of interest rates from UK government bond prices. Some fundamental concepts are defined in Section 2, while the rest of the paper compares the Bank's current yield curve model with two commonly used term structure models in the academic literature. Section 3 describes how the various models estimate the fundamental term structure (or yield-maturity relationship), and Section 4 outlines how each model accounts for the complications caused by coupon (interest) payments. Section 5 presents examples of curves produced by the various methods and Section 6 concludes.

2 Notation and some definitions

Before discussing the issues involved in estimating yield curves, it is useful to set out the notation and terminology used in the rest of the paper. Whilst some of the analysis is specific to the *gilt-edged* market,⁽³⁾ the main issues are relevant when estimating the term structure of interest rates for any government bond market.

(1) Other than short-term Treasury bills.

(2) 2 1/2 % Consols pays interest quarterly, but is the exception rather than the rule.

(3) In particular the treatment of taxation (Section 3) is specific to the UK case, the details of which can be found in "British Government Securities: The Market in Gilt-Edged Securities" (pages 24-5) published by the Bank of England.

2.1 The bond price equation

A bond is simply the obligation on the bond's issuer to provide one or more future cashflow(s). For a conventional⁽⁴⁾ UK government bond, the stream of cashflows consists of regular (semi-annual) fixed interest, or 'coupon' payments and a redemption payment which is paid with the final coupon payment on the gilt's maturity date. The market price of a conventional bond is the market valuation of the stream of cashflows associated with that bond.

A *spot interest rate* is the rate at which an individual cashflow (either a coupon or a redemption payment) is discounted. If spot rates for payments at all dates in the future are known, then the price⁽⁵⁾ of a bond maturing in m periods can be equated to the present value of future cashflows:

$$\text{Price} = \frac{C}{(1+r_1)} + \frac{C}{(1+r_2)^2} + \dots + \frac{R+C}{(1+r_m)^m} \quad (1)$$

where: C = coupon
 R = redemption payment
 r_i = the spot rate applicable for a payment in period i
($i=1, \dots, m$)

(4) There are other kinds of UK government bonds: index-linked (with payments linked to the Retail Price Index), irredeemable (with no contractual redemption date), double-dated (with a period, usually of several years, in which the government can repay the bond) and convertible (which give the holder the option to convert into other (conventional) bonds at particular dates).

(5) Where 'price' is the sum of the quoted ('clean') price and accrued interest - see section 2.3 below.

2.2 Discount factors and the discount function

The bond price equation (1) describes how the price of a bond can be calculated if all the spot rates r_i ($i=1, \dots, m$) are known. This equation is often written in terms of *discount factors*, so that the present value of each cashflow is written as the product of its nominal value and its discount factor:

$$\text{Price} = d_1 C + d_2 C + \dots + d_m (C+R)$$

or:

$$\text{Price} = C \sum_{i=1}^m d_i + d_m R \quad (2)$$

where d_i is the discount factor for period i ($i=1, \dots, m$) and is simply a transformation of the i^{th} period spot rate:

$$d_i = \frac{1}{(1+r_i)^i} \quad i = 1, \dots, m \quad (3)$$

It is often useful to think of the continuous analogue to the set of discount factors, the *discount function* $\delta(t)$, as a continuous function that maps time t to a discount factor. Equivalently $\delta(t)$ is the present value of £1 receivable at time t , and so given a continuous discount function the present value of a cashflow at any point in the future can easily be calculated. A set of discount factors d_i ($i=1, \dots, m$) can therefore be thought of as discrete points on the continuous discount function $\delta(t)$:

$$d_i = \delta(t_i)$$

where t_i is the time to the end of the i^{th} period. In terms of the discount function, the bond price equation becomes:

$$\text{Price} = C \sum_{i=1}^m \delta(t_i) + \delta(t_m) R \quad (4)$$

2.3 Accrued interest and continuous compounding

The bond price equation (1) is over-simplified since it assumes that the next cashflow is due in exactly one period's time. In fact, while coupon payments on individual bonds are made at fixed dates, bonds can be traded on any working day. Whenever a bond is traded on a day that is not a coupon payment date, the valuation of the bond will reflect the proximity of the next coupon payment date. In the UK, for example, the buyer pays *accrued interest* to compensate the seller for the period since the last coupon payment during which the seller has held the gilt but for which they will receive no coupon payment.⁽⁶⁾ The accrued interest is by market convention calculated simply as the proportion of the coupon foregone by the seller, expressed algebraically in equation (5):

$$ai = t1 \times C \quad (5)$$

where: ai = accrued interest
 $t1$ = time to the next receivable dividend payment (as the actual number of days divided by the number of days in a "standard" year.⁽⁷⁾)

(6) There is a period (usually 37 days) before each coupon date when the bond is traded *ex-dividend*, ie without the right to the next coupon payment, and in this period (between the ex-dividend date and the coupon payment date) accrued interest is negative since it is the buyer who is giving up part of the next coupon payment.

(7) For the United Kingdom, the market convention is to assume that a "standard" year consists of 365 days. In some other countries, such as the United States, accrued interest is instead calculated on a 360 day basis.

A bond's price can therefore be decomposed into two components: the accrued interest and the bond's *clean price*. It is the clean price of a gilt that is usually quoted, since movements in the clean price are independent of the (exactly predictable) changes in accrued interest. The *dirty price* is the actual market valuation of the bond as given by equation (1), at which transactions take place; and is simply the clean (quoted) price plus any accrued interest.

Between coupon payment dates, the bond price equation (1) needs to be modified to allow for the fact that the next coupon payment is not exactly one period in the future.⁽⁸⁾ This is straightforward with either discrete discount factors or a continuous discount function; the latter case (for a bond with m remaining coupon payments) is shown in equation (6) below:

$$P + ai = C \delta(t1) + C \delta(t1+1) + \dots + (C+R) \delta(t1+(m-1)) \quad (6)$$

where: P = clean price
 ai = accrued interest (equation (5))
 $t1$ = time to first coupon payment (as a fraction of a period)
 C = coupon
 R = redemption payment

Although accrued interest calculations are conceptually straightforward, in practice they can be an awkward complication to empirical work. To avoid this, McCulloch (1971, 1975) approximates

(8) There are further complications when considering a bond that has recently been issued. If (as is usually the case) it was not issued on a coupon payment date, the first-ever receivable dividend will be less than the usual coupon payment, reduced to reflect the fact that the holder will not hold the bond for the full coupon period. Furthermore, gilts are often issued partly paid, which reduces the first coupon payment still further and introduces negative cashflows into the right-hand side of the price equation (amounts payable by the holder). The required (algebraic) alterations are reasonably straightforward but are not given here.

the bond price equation (1) by assuming that coupon payments are made continuously rather than at discrete points in time, so interest does not accrue. This assumption of *continuous compounding* means that the price equation can be slightly simplified:

$$P = C \int_0^m \delta(\mu) d\mu + R \delta(m) \quad (7)$$

where: P = clean price
 m = maturity of the bond
 $(C, R$ and δ as defined before)

The continuous compounding approximation can significantly alter estimates of the discount function (and of the derived yield curves), so this approximation error should be weighed against the perceived benefit from simplifying the calculations if continuous compounding is to be considered. The following sections describe the methodology for both the continuous and discrete compounding cases, but all results in Section 5 were produced using only the (more precise) discrete method.

2.4 Yields

Since the coupon and the redemption payment are known, it is straightforward to measure the return on a gilt trading at a particular price. There are two measures commonly used: the *flat yield* (sometimes referred to as the *current* or *running* yield) and the *redemption yield*.

The flat yield is analogous to the 'dividend yield' on an equity, and is defined as:

$$\text{Flat Yield} = \frac{\text{Coupon}}{\text{Clean Price}} \quad (8)$$

The flat yield is essentially used to value the return from holding a bond for a short period - and is often thought of as the *income* from the bond. Common market practice is to compare the flat yield on a bond with a short-term interest rate - if the flat yield is below the short-term interest rate, the holder is *ceteris paribus* incurring a short-term cost by holding the bond.

The *redemption yield* (or *yield to maturity*) corresponds to the internal rate of return on the bond. As such, it can be seen that the redemption yield is derived from the bond price equation (1) with all cash flows discounted at the same rate:

$$P + ai = \frac{C}{(1+y)} + \frac{C}{(1+y)^2} + \dots + \frac{R + C}{(1+y)^m} \quad (9)$$

where y = (gross) redemption yield

(P, ai, C and R are defined as before)

Given a price, equation (9) is solved for the redemption yield y using some form of non-linear iteration technique (eg Newton-Raphson). If the bond is to be held to redemption, the redemption yield is clearly a better measure of return than the flat yield. However, it rarely equals the realised return since it assumes that all future coupon payments can on average be reinvested at the internal rate of return.

Of the two measures, the redemption yield is the more widely used. For the rest of the paper the term 'yield' will specifically refer to the redemption yield.

2.5 Yield curves

The discount function $\delta(t)$ can be uniquely transformed into other useful functions, such as the spot rate (or zero coupon) curve, par yield curve and implied forward rate curve. Similarly, a set of regularly spaced *discrete* discount factors d_i ($i=1, \dots, m$) can be transformed into corresponding discrete spot rates, par yields and implied forward rates which, if sufficiently closely spaced, can be plotted as a continuous curve. This section describes how, given a discount function or set of discount factors, the other curves can be derived. It is important to note that all these transformations are unique, so given any one of the four curves the other three can be derived.

Implied forward rates

In equation (3) the discount factor for period i (in discrete time), d_i , is given in terms of the corresponding spot rate, r_i , by the relationship:

$$d_i = (1 + r_i)^{-i}$$

The spot rate r_i can be thought of as an average⁽⁹⁾ of all the implied one period forward rates f_1, f_2, \dots, f_i so that:

$$1/d_1 = (1+r_1) = (1+f_1)$$

$$1/d_2 = (1+r_2)^2 = (1+f_1)(1+f_2)$$

$$1/d_i = (1+r_i)^i = (1+f_1)(1+f_2) \dots (1+f_i) \quad (10)$$

$$1/d_m = (1+r_m)^m = (1+f_1)(1+f_2) \dots (1+f_m)$$

(9) From equation (10) it is clear that $(1+r_i)$ is the *geometric mean* of $(1+f_1), (1+f_2), \dots, (1+f_i)$.

The implied forward rate f_i for any period can therefore be isolated using:

$$\frac{1/d_i}{1/d_{i-1}} = \frac{(1+f_1)(1+f_2) \dots (1+f_{i-1})(1+f_i)}{(1+f_1)(1+f_2) \dots (1+f_{i-1})}$$

$$\therefore \frac{d_{i-1}}{d_i} = (1+f_i)$$

$$\therefore f_i = \frac{d_{i-1} - d_i}{d_i}$$

$$\therefore f_i = \frac{-\Delta d_i}{d_i} \quad \text{where } \Delta d_i = d_i - d_{i-1} \quad (11)$$

The above is the discrete compounding case. Using the continuous discount function $\delta(t)$ and assuming that interest is compounded continuously we can therefore derive an *instantaneous forward rate curve* $\rho(t)$ by considering equation (11) with periods i and $(i-1)$ infinitesimally close:

$$\rho(t) = \frac{-\delta'(t)}{\delta(t)} \quad (12)$$

The instantaneous forward rate curve is a theoretical construct, providing the interest rate applicable on a future loan that is repaid an instant later. A more useful measure to consider (when using continuous compounding) is the average of $\rho(t)$ over a particular interval $[t_1, t_2]$. This *mean forward rate* $f(t_1, t_2)$ is given by:

$$f(t_1, t_2) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \rho(\mu) d\mu \quad (13)$$

The forward rate $f(t_{i-1}, t_i)$ in equation (13) therefore represents the continuous compounding approximation to the discrete forward rate f_i in equation (11).

Spot (or zero coupon) curve

The spot rate r_t is sometimes called the *zero coupon yield* since it represents the yield to maturity on a (hypothetical) pure discount or zero coupon bond, and can be easily derived from the appropriate discount factor using equation (3). The continuous compounding approximation $\eta(t)$ to the *term structure of spot rates*, or *zero coupon yield curve*, can be derived from equation (13) since the spot rate for payment at time t in the future is the average instantaneous forward rate between now ($t_1=0$) and time t ($t_2=t$). So:

$$\eta(t) = f(0, t)$$

and hence from equation (13):

$$\eta(t) = \frac{1}{t} \int_0^t \rho(\mu) d\mu$$

$$\therefore \eta(t) = \frac{-\ln \delta(t)}{t} \quad (14)$$

(assuming $\delta(0) = 1$).⁽¹⁰⁾

(10) The assumption that the discount function equals unity at time $t=0$ is a sensible restriction, implying that an amount receivable now is not discounted. This, and other, restrictions are discussed in more detail in Section 3.

The equivalent of (14) for the case of discrete compounding is:

$$r_i = \left[\frac{1}{d_i} \right]^{1/i} - 1 \quad (15)$$

The zero coupon yield curve is the construct to which economists usually refer when talking about the term structure of interest rates.

The par yield curve

A (coupon-paying) bond is said to be *priced at par* if its current market price is R , its face (or *par*) value. From equation (9) it can be shown that for a bond to be trading at par, its redemption yield must equal its coupon. Using this fact, the *par yield* y_m can be derived from equations (2) and (9) for any period m (given a series of discrete discount factors d_1, \dots, d_m) by setting the coupon $C = y_m$ and the price $P = R$:

$$R = y_m \sum_{i=1}^m d_i + d_m R$$

$$\therefore y_m = \frac{R(1 - d_m)}{\sum_{i=1}^m d_i} \quad (16)$$

Similarly, the continuous compounding approximation to the par yield curve $y(t_m)$ can be estimated using a rearranged version of equation (7), setting $C = y(t_m)$:

$$y(t_m) = \frac{R(1 - \delta(t_m))}{\int_0^{t_m} \delta(\mu) d\mu} \quad (17)$$

(where t_i is the time to the i^{th} regular coupon payment on the notional m period bond.)

The par yield curve $y(t_m)$ describes the coupon required on a (notional) coupon-paying bond with time to maturity t_m for that bond to trade at par.⁽¹¹⁾

3 Estimating yield curves

The previous section detailed the relationships between different variables and curves on the basis that either a set of discrete discount factors or a continuous discount function is known. Also, since the discount function, par yield curve, zero coupon yield curve and implied forward rate curves are all algebraically related, knowing any one of these four means that we can readily compute the other three. In reality, however, none of the four curves is directly observable; they must instead be derived from bond prices.

Two fundamental problems need to be addressed by any model attempting to identify the term structure of interest rates implied by prices of government bonds. The first is the problem of 'gaps' in the maturity spectrum - there is not always a suitable bond, or any bond at all, maturing at a date of interest. Second, the term structure is defined in terms of zero coupon bonds - but all UK government bonds pay

(11) The par yield curve is essentially the same as a swap rate curve (in the absence of default risk), since a par yield represents the fixed interest payments required by the market to match the same number of future (unknown) floating payments. However, there are a number of practical differences in estimating the two curves.

coupons, so a zero coupon yield cannot be inferred directly from the price of a coupon-paying bond.

These two problems lead to further practical estimation problems. First, the problem of filling the gaps - what shapes should the term structure be allowed to take? To answer this question, a decision on the appropriate trade-off between 'smoothness' (removing 'noise' from the data) and 'responsiveness' (flexibility to accommodate a genuine movement in the term structure) is required. For example, it might be felt that the estimated term structure should be smooth, but not to the extent that it is seriously misrepresented. Second, is it preferable to estimate the term structure via the discount function or via the par yield curve? There are other practical hurdles to overcome: for example, in the UK many investors pay income tax on coupon payments whereas any capital gain is tax-free. This differential taxation of coupon payments and capital gains results in taxpayers preferring, and hence paying a premium for, low coupon bonds; so the size of the coupon on a bond will affect its yield. Such coupon effects, along with any other tax effects, need to be removed from any estimate of the term structure.

The rest of this paper describes three models used to estimate the term structure of interest rates: the model currently used by the Bank of England (Mastronikola 1991) and two from the academic literature, due to McCulloch (1971, 1975) and Schaefer (1981). The many problems inherent in any estimation of the term structure can be neatly split into three categories: which curve to estimate (Section 3.1), how the chosen curve should be estimated (Section 3.2), and how to deal with other factors which might influence relative bond prices, such as tax effects (Section 4).

3.1 Yield curve or discount function?

Models used to estimate the term structure of interest rates fall into two distinct categories: those that fit the par yield curve and those that fit a discount function. The Bank's current model (Mastronikola *op cit*) is an example of the former, whereas most of the latter are based on fitting discount functions, pioneered by McCulloch (1971).

Fitting a curve through redemption yields

The Bank's yield curve model essentially fits a curve through redemption yields, derived directly from observed prices using equation (9). This methodology, while simple to understand, has the theoretical drawback that it does not explicitly restrict payments due on the same date to be discounted at the same rate. To see why this is the case consider two bonds; the first, bond A, maturing in one periods' time and the second, bond B, in two periods:

$$\begin{aligned} \text{Price of Bond A} &= \frac{R_a + C_a}{(1+y_a)} \\ \text{Price of Bond B} &= \frac{C_b}{(1+y_b)} + \frac{R_b + C_b}{(1+y_b)^2} \end{aligned} \quad (18)$$

Estimating the yield curve by fitting a curve through the redemption yields on these two bonds does not restrict the first coupon payment on bond B to be discounted at the same rate as the redemption payment on bond A even though both payments are due at exactly the same time. Instead, when estimating a yield curve in this manner the assumption must be made that the first coupon on bond B is discounted using the rate indicated by the yield on bond A, the yield on bond B reflecting the difference in rates between period 1 and period 2. In other words, bond A is assumed to provide all the information required for

inferences about how the earlier coupon payments on bond B are discounted.

Given a specification of the functional form for the yield curve (see Section 3.2), the estimation procedure is simply to fit a curve of the given functional form to minimise the sum of squared differences between the observed and fitted yields. The estimated curve is implicitly a par yield curve. This approach is reasonable if other aspects of the model define this curve explicitly as the par yield curve (eg as in Mastronikola 1991 - see Section 4). However, whether or not a regression of redemption yield against maturity is a realistic approximation to the par yield curve depends on market conditions. If bonds are trading so that the average redemption yield at each maturity - the rate derived from a yield against maturity regression - is close to the par yield at that maturity, then the assumption is reasonable. However, the less well this assumption matches the reality, the worse the approximation.

Fitting a discount function

Most of the academic literature follows McCulloch (1971) in explicitly constraining cashflows from different bonds due at the same time to be discounted at the same rate, and estimates a discount function from which the term structure can be derived.⁽¹²⁾ McCulloch uses the form of the bond price equation with a continuous discount function and makes the assumption of continuous compounding - that the coupon payments are made continuously through time rather than at regular discrete intervals.⁽¹³⁾ Under this assumption interest does not accrue, and equation (7) is used to give the price on bond i ($i=1, \dots, n$) :

$$P_i = C_i \int_0^m \delta(\mu) d\mu + R_i \delta(m_i) \quad (19)$$

(12) In particular, Schaefer (1981) follows this approach.

(13) As stated in Section 2 this is merely a simplifying assumption, and the results presented later in this section were derived using McCulloch's technique with discrete compounding. For clarity the description here follows the original.

where P_i , C_i , m_i and R_i are the price, coupon, maturity and redemption payment of the i^{th} bond.

To estimate the discount function, $\delta(m)$, it is defined to be a linear combination of a set of k (linearly independent) underlying *basis functions*:

$$\delta(m) = 1 + \sum_{j=1}^k a_j f_j(m) \quad (20)$$

where $f_j(m)$ is the j^{th} basis function, and a_j is the corresponding coefficient ($j=1, \dots, k$). There are a number of functional forms that the basis functions $f_j(m)$ can take to produce a sensible discount function, and this choice is discussed in detail in Section 3.2.

A system of n linear equations can be derived⁽¹⁴⁾ by combining equations (19) and (20), with the function weights a_j as the coefficients in each equation:

$$y_i = \sum_{j=1}^k a_j x_{ij} \quad (21)$$

where: $y_i = P_i - C_i m_i - R_i$

$$x_{ij} = C_i \int_0^{m_i} f_j(\mu) d\mu + R_i f_j(m_i)$$

The coefficients a_j ($j=1, \dots, k$) can be estimated from equation (21) using ordinary least squares, and the estimated discount function can then be calculated using equation (19).

(14) See Appendix A for the full derivation.

Having estimated the discount function, equations (13), (14) and (17) can be used to estimate the implied forward rate, zero coupon and par yield curves respectively. Given the assumption made by the Bank's model, the same inferences about the term structure of interest rates can be drawn from the estimated curves regardless of methodology.

The advantage of McCulloch's technique is that it makes explicit the assumption of an efficient market, ie one in equilibrium. Fitting through redemption yields can be regarded as simply fitting a curve through data and as such requires no assumption about the state of the market; however, the assumption is implicit as soon as such a curve is interpreted as a par yield curve.

3.2 Estimating functions

As described so far, both the McCulloch and Bank methods require a specification of one or more estimating function(s): when fitting through redemption yields, the functional form needs to be specified; whereas estimating a discount function using McCulloch's methodology requires the specification of basis functions ($f_j(m)$ in equation (20)). The choice of functions in both cases is crucial since it ultimately determines the trade-off between smoothness and flexibility discussed earlier, and therefore reflects prior beliefs about the shapes a yield curve should be able to take. This choice is unavoidably subjective but certain properties are essential; in particular an estimated discount function should be both positive and monotonic non-increasing (to avoid negative forward rates) and should equal unity at time $t=0$ (the present value of £1 receivable now is £1).

The simplest approach to fitting the discount function is that used by Carleton and Cooper (1976), who estimate the term structure of interest rates for the US government coupon securities (ie notes and bonds) market. They utilise the fact that the semi-annual interest payments made by nearly all securities in this market are made on only four days

of each year.⁽¹⁵⁾ This even-spacing of data points means that the discount factors can be estimated directly from equation (2) using ordinary least squares for maturities up to seven years,⁽¹⁶⁾ thus avoiding the need for approximating functions (and McCulloch's formulation) altogether. Although Carleton and Cooper did not constrain their estimates of the discount function, they apparently displayed the correct properties in most cases - ie were monotonic decreasing and non-negative. The main problem with this approach is the reliance on regularly spaced interest payment dates and as such it is not suitable for application to the UK market (or even to the US market beyond seven years). Furthermore, this method imposes no smoothness on the discount function, so the corresponding implied forward rate curve is jagged.

Polynomial splines

If data are not regularly spaced (as is the case in the UK and most other markets) the approach used by Carleton and Cooper is not feasible and instead an approach based on estimating or approximating functions is often used. McCulloch's (1971, 1975) implementation is given in equation (20) in which the discount function $\delta(m)$ is described as a linear combination of k approximating functions $f_j(m)$ ($j=1, \dots, k$) on which the coefficients ($a_j, j=1, \dots, k$) are estimated. One of the simplest implementations (discussed by McCulloch, 1971) is to let $f_j(m)=m^j$ for $j=1, \dots, k$. The discount function generated by this set of approximating functions will then be a simple k^{th} degree polynomial.⁽¹⁷⁾ However, unless observations are spaced equally through the maturity range, such a polynomial tends to fit well at the short end and badly at the

(15) Namely, 15 February, 15 May, 15 August and 15 November.

(16) Data beyond seven years could not be used due to the sparsity of observations in that maturity range.

(17) Chambers, Carleton and Waldman (1984) apply such a polynomial directly to the spot curve. They decide on k using a stepwise procedure.

long end or vice versa. To solve this problem it is possible to increase k , the order of the polynomial, but this can cause instability in the parameter estimates.

To solve these problems McCulloch suggested the use of piecewise polynomial functions or splines to approximate the discount function. Intuitively, a polynomial spline can be thought of as a number of separate polynomial functions, joined "smoothly" at a number of so-called *join, break* or *knot* points. The word "smooth" has a precise mathematical meaning, but in the context of a piecewise r -degree spline it is generally taken to mean that the $(r-1)^{\text{th}}$ derivatives of the functions either side of each knot point are continuous.⁽¹⁸⁾ Using this piecewise approach the polynomials can be of much lower order and generate a more stable curve.

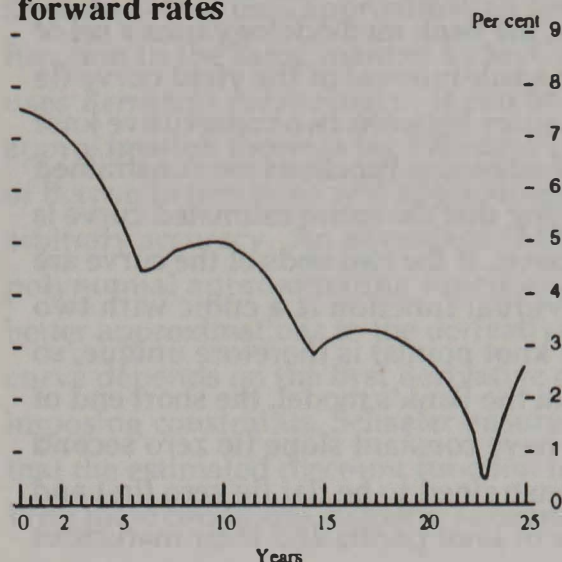
In his first paper McCulloch (1971) uses a quadratic spline to estimate the discount function. This has superior properties to that of the simple polynomial but also has several shortcomings. A major drawback is that use of a quadratic spline for the discount function can lead to what McCulloch terms "knuckles" in the corresponding forward rate curve. This effect is illustrated in Figure 3.1, and is caused by the fact that specifying the discount function by a piecewise quadratic function means it has a discontinuous second derivative, resulting in a forward rate curve with a discontinuous first derivative.⁽¹⁹⁾

(18) One consequence of this definition is that the r^{th} derivative of the spline is a step function.

(19) If $\delta(m)$ is the discount function and $\rho(m)$ is the forward rate curve it can be shown that $\rho'(m) = (\delta'(m)/\delta(m))^2 - (\delta''(m)/\delta(m))$.

Figure 3.1

McCulloch "Knuckle" Effect on forward rates



The obvious way to avoid this effect is to increase the order of the estimating functions and use (for example) a *cubic spline*. The simplest implementation of a cubic spline is that presented by McCulloch (1975). In this formulation the basis functions $f_j(m)$ in equation (20) are specified as a family of cubics that are constrained to be smooth around each knot point.⁽²⁰⁾ This specification is certainly flexible enough to model any reasonably-shaped discount function (and yield curve). It can in fact be too flexible, as it does not constrain the discount function to be non-increasing; so forward rates calculated using equation (13) may be negative.

The current Bank methodology uses a slightly more complicated technique to fit a cubic spline through observed redemption yields on

(20) The actual forms specified by McCulloch are reproduced in Appendix B, but in practice any set of cubic equations that are constrained to be smooth around the knot points could be used. Note also that the basis functions are defined along the length of the estimation space (ie from zero to the maturity of the longest bond) and combined using equation (20) (and the weights estimated by equation (21)) to obtain the estimated discount function.

stocks. Unlike McCulloch's methodology, where cubic functions define basis functions along the length of the discount function which are weighted and then added together, the Bank methodology uses a set of cubic functions each of which fits a sub-interval of the yield curve (ie each function fits the curve in the space between two consecutive knot points). The second derivatives of adjoining functions are constrained to be equal at the knot point, meaning that the entire estimated curve is "smooth" in the sense described above. If the two ends of the curve are also constrained, then each individual function is a cubic with two constraints and (for a set of fixed knot points) is therefore unique, so the entire fitted curve is unique. In the Bank's model, the short end of the yield curve is constrained to have constant slope (ie zero second derivative) and the long end is constrained to be flat (ie zero first and second derivatives). The number of knot points and their maturities are fixed,⁽²¹⁾ and the yields at each knot point are estimated such that the sum of squared residuals between observed and fitted yields is minimised.⁽²²⁾

(21) There are currently six knot points, equally spaced in transformed time (see Mastronikola 1991, page 8).

(22) It could be argued that the Bank's model in fact uses a third order *exponential spline* (see below), since the time to maturity on each bond is transformed using equation (22) before estimation. However, since the motivation for using the transformation is different from that for using an exponential spline, it seems more useful to describe the Bank's model as using a cubic spline (in transformed time).

Bernstein polynomials

Schaefer (1981) uses approximating functions to estimate the discount function in the same manner as McCulloch, but instead of cubics he uses *Bernstein polynomials*. It can be shown using the Weierstrass approximation theorem (eg Williams 1991, page 74) that combinations of Bernstein functions will approximate any continuous function with arbitrary accuracy. An advantage of these functions over conventional polynomial approximating functions is that they give considerably better approximations to the derivatives; important since the forward curve depends on the first derivative of the discount function.⁽²³⁾ By imposing constraints, Schaefer ensures that the a_j 's are non-negative, that the estimated discount function is non-negative and that $\delta(0)=1$. With these conditions, negative forward rates are avoided.⁽²⁴⁾

(23) For a more detailed account of the use of Bernstein functions in this context see Schaefer (1982).

(24) If $\rho(m)$ is the forward rate curve and $\delta(m)$ the discount function it can be shown that $\rho(m) = -\delta'(m)/\delta(m)$ (equation (12)). Clearly, $\rho(m)$ will be negative - if either $\delta'(m)$ is positive or $\delta(m)$ is negative - Schaefer's constraints ensure that neither of these conditions arise. Since Schaefer's discount function is a linear combination of monotonic non-increasing approximating functions he ensures that it is monotonic by constraining the a_j s to be non-negative.

Exponential splines

One of the main criticisms levelled at both cubic and Bernstein polynomial functions as a choice of approximating functions is that these can lead to forward rate curves which exhibit undesirable (and unrealistic) properties for long maturities ie rise or fall steeply. Vasicek and Fong (1982) detail a method that can be used to produce asymptotically flat forward curves. Central to their approach is the characterisation of the discount function as essentially exponential in shape. They argue that splines, as piecewise polynomials, have a different curvature from exponentials and so will not provide a good local fit to the discount function.⁽²⁵⁾ Vasicek and Fong claim that this poor local fit will result in the spline "weaving" around the discount function, thus producing highly unstable forward rates. Also, polynomial splines cannot be forced to tail off in an exponential form as maturity increases.

Vasicek and Fong suggest applying a transform to the argument m of the discount function $\delta(m)$. This transform has the form:

$$m = - (1/\alpha) \ln(1-x), \text{ where } 0 \leq x < 1 \quad (22)$$

and has the affect of transforming the discount function from an approximately exponential function of m to an approximately linear function of x .⁽²⁶⁾ Polynomial splines can then be employed to estimate this transformed discount function. Using this transform it is easy to impose additional constraints on the discount function.⁽²⁷⁾ The

(25) This is refuted by Shea (1985) who insists that a piecewise polynomial function should be able to mimic well a piecewise exponential function.

(26) Here, x is referred to as transformed time.

(27) One such condition that they impose is the non-negative condition.

parameter α constitutes the limiting value of the forward rates, and can be fitted to the data as part of the estimation.

Vasicek and Fong use a cubic spline to estimate the transformed discount function. In terms of the original variable m this is equivalent to estimating the discount function by a third order exponential spline ie between each pair of knot points $\delta(m)$ takes the form:

$$\delta(m) = b_0 + b_1 e^{-\alpha m} + b_2 e^{-2\alpha m} + b_3 e^{-3\alpha m} \quad (23)$$

Although Vasicek and Fong claim to have tested exponential splines successfully, they provide no evidence. Consequently, Shea (1985) presents some empirical results on the suitability of exponential splines for yield curve modelling. He concludes that there is no evidence to support the claim that exponential splines produce more stable estimates of the term structure than polynomial splines - the discount function often deviating from the expected exponential decay. Shea found that the asymptotic property only constrained the forward curve to flatten at maturities beyond the longest observable bond and exhibited little influence over its shape or level at other maturities. An additional observation was that one of the factors driving the instability of the Vasicek and Fong model was the data-conditioning properties of the exponential transform, $x=1-e^{-\alpha m}$. For small $\hat{\alpha}$, this caused the observed x to become bunched so that substantial portions of the estimation interval $[0,1]$ contained no data, leading to particularly unstable and unrealistic asymptotic forward rates. In such circumstances Shea had to coax the nonlinear estimation program to converge to a solution. It is possible that this problem was caused by Shea's choice of knot points, which appears to be in line with McCulloch's convention of placing equal numbers of observations (if possible) between knots.⁽²⁸⁾

(28) This was certainly the rule used in Shea (1984).

Chambers, Carleton and Waldman (1984) have incorporated the exponential characteristic in a different manner. Here, a polynomial functional form is applied directly to the spot curve. The spot curve can then be related to observable bond prices by exponentiation of this functional form.

B-splines

An important observation made by Shea (1984) concerns the choice of basis functions when defining a spline function. He reports that some spline bases, such as that chosen by McCulloch (1971,1975) can generate a regressor matrix with columns that are nearly perfectly collinear, resulting in possible inaccuracies arising from the subtraction of large numbers. As a solution he advocates the use of a basis of "B-splines". These are functions which are identically zero over a large portion of the approximation space (unlike those used by McCulloch) and so prevent the loss of accuracy due to cancellation. By using a B-spline basis it is also easier to impose constraints on the spline function.

Steeley (1991) also recommends the use of B-splines for the same reason. He provides comprehensive details of how B-splines can be used to fit a discount function, and concludes that by their use spline functions can be viewed as a robust alternative to both cubic and Bernstein polynomials.

Problems using spline functions as estimation functions

Shea (1984) considers some of the pitfalls encountered when using splines to model the term structure. First, he demonstrates that the constraints implicit in the McCulloch cubic spline do not restrict the discount function to its desired negative slope, and can consequently produce an estimate for the discount function which starts to slope upward at the longest maturities. The forward rate curve generated by such a discount function will feature negative interest-rate estimates. Without the imposition of constraints (discussed earlier) the Schaefer

polynomial would display similar characteristics. Shea argues that Schaefer's constraint on the slope of the discount function to be everywhere negative, though serving to prevent negative forward rates does nothing for the general stability of the forward curve.

One alternative "fix" suggested by Shea on such occasions is the use of *ad hoc* constraint specification. In its more obvious form this might consist of changing the number or location of the knot points. However, Shea goes on to suggest the use of localised constraints to deal with specific problem areas. One such constraint suggested was a simple restriction of fixed proportions between the first derivatives of the discount function at different maturities. This is of particular use at the long end where it can be applied to ensure that the discount function remains negatively sloped. Although these manual adjustments to the term structure are acceptable in a research and development context, they will clearly be of limited use for practitioners in an operational environment, where yield curve updates may be required on a real time basis. Also, changes in the curve may be wrongly attributed to events in the market when in fact they are solely due to a change in the constraint specification.

Knot points

Another decision that needs to be made when using any kind of spline function is the appropriate number of knot points. If the number of knots is too low then the model will not fit the data closely when the term structure takes on difficult shapes, while if it is too high the estimated curve may conform too readily to unrepresentative outliers.

The Bank yield curve model currently uses six knots, which are spaced evenly in transformed time (see above). The approach adopted by McCulloch (1975) and several subsequent researchers is to set the number of knots to be equal to the square root of the number of bonds to be used in the estimation process. These knots are then spaced evenly amongst the number of observations (maturities). Given the

current number of bonds in the UK market, this approach also suggests the use of six knot points. One advantage of the McCulloch convention is that the positioning of the knots will automatically change with a shift in the structure of government debt - unlike the knot points in the Bank's model, which will remain fixed. On the negative side, allowing the knots to move on a day to day basis may give the false impression that the term structure has changed.

Figures 3.2-3.5 illustrate the kinds of effect that changing the number or location of the knots in the Bank model can have on the forward rate curve.⁽²⁹⁾

Figure 3.2

Forward curve for different numbers of knot points

cob 30/3/92

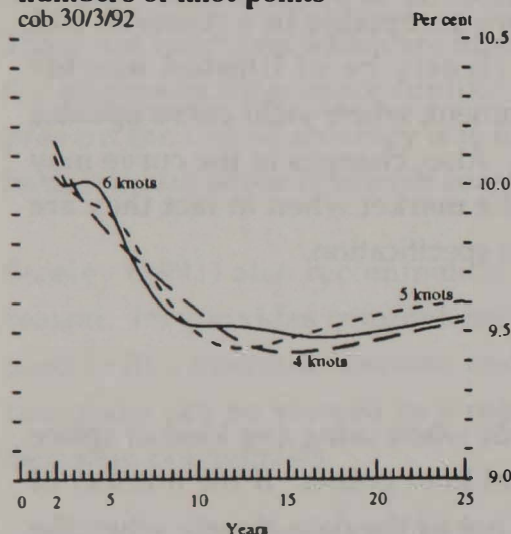
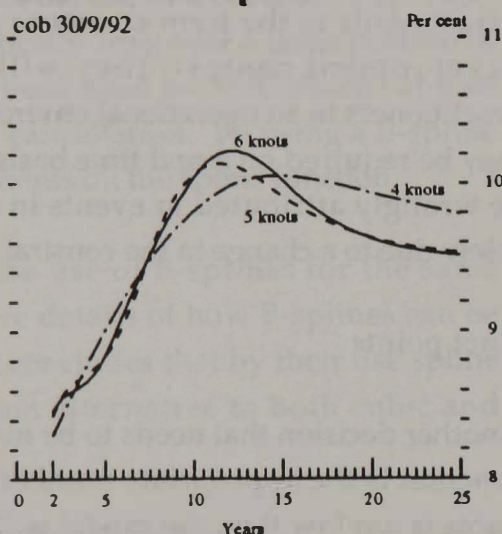


Figure 3.3

Forward curve for different numbers of knot points

cob 30/9/92



Figures 3.2 and 3.3 show the effect of reducing the number of knots to four or five, but still spacing these points evenly in transformed time. In the case of 30 September 1992, reducing the number of knots from six to four raises the forward curve by over 30 basis points in places. This smoothing also removes the point of inflection at the 3 year horizon.

(29) Such effects also occur when considering a par or zero coupon yield curve, but are less significant.

Figure 3.4

Forward curve for different positioning of knots

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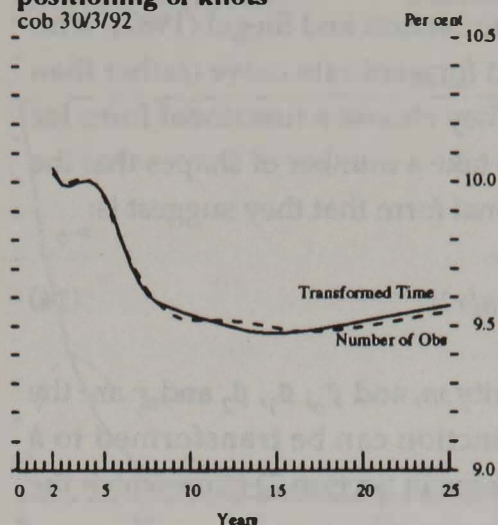
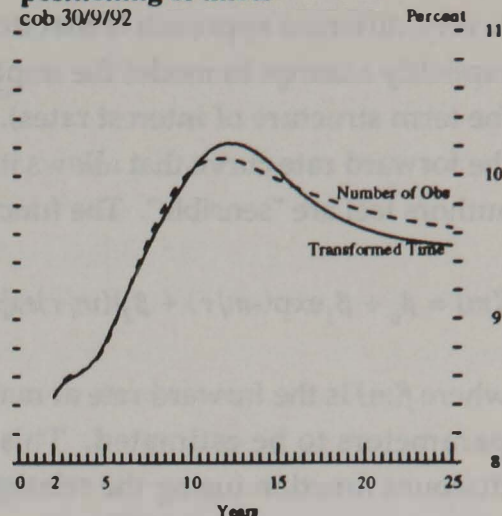


Figure 3.5

Forward curve for different positioning of knots

cob 30/9/92



Figures 3.4 and 3.5 compare the effect of switching from knots spaced evenly in transformed time to knots spaced evenly by number of observations. In the example of 30 September this produces a shift in the forward curve of up to 13 basis points.

Surprisingly, aside from Steeley (*op cit*), there seems to have been little effort in the literature devoted to testing sophisticated techniques for specifying the optimal number and location of knot points. That such techniques already exist (eg de Boor, 1978) makes this all the more surprising.

Nelson and Siegel (1987)

A very different approach is that due to Nelson and Siegel (1987), who explicitly attempt to model the implied forward rate curve (rather than the term structure of interest rates). They choose a functional form for the forward rate curve that allows it to take a number of shapes that the authors feel are "sensible". The functional form that they suggest is:

$$f(m) = \beta_0 + \beta_1 \exp(-m/\tau) + \beta_2 [(m/\tau) \exp(-m/\tau)] \quad (24)$$

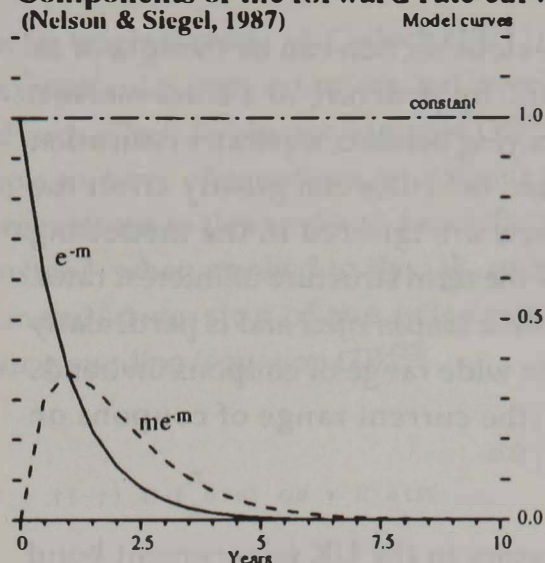
where $f(m)$ is the forward rate at maturity m , and β_0 , β_1 , β_2 and τ are the parameters to be estimated. This function can be transformed to a discount function (using the relationships in Section 2) from which the parameters are estimated.⁽³⁰⁾

By considering the three components that make up this function (see Figure 3.6) it is clear how, with appropriate choices of weights, it can be used to generate forward rate curves of a variety of shapes, including monotonic and "humped". An important property of this model is that β_0 specifies the long rate to which the forward rate asymptotes horizontally. Furthermore, this approach avoids the problem in spline-based models of choosing the "best" knot point specification.

(30) Equation (24) can also be transformed to a spot rate curve, to which Nelson and Siegel fit US Treasury Bill data (because Treasury Bills are zero coupon instruments).

Figure 3.6

Components of the forward rate curve
(Nelson & Siegel, 1987)



From the Nelson and Siegel forward rate equation it is possible to derive algebraic expressions for the spot curve and the discount function, though not unfortunately for the par yield curve.⁽³¹⁾

It is interesting to note that Svensson (1993) estimates spot and forward rate curves using McCulloch's (1971,1975) approach of fitting a discount function to bond price data, but uses the Nelson and Siegel functional form instead of a spline function. Svensson argues that for monetary policy applications a simplistic functional form of this nature is perfectly acceptable. In his paper on estimating Swedish forward rates (Svensson 1994) he increases the flexibility of the original Nelson and Siegel model by adding a fourth term to the forward rate equation (equation (24)). This term takes the form $\beta_3(m/\tau_2)\exp(-m/\tau_2)$ and provides two extra parameters for estimation. However, Svensson concludes that the original Nelson and Siegel model produced a satisfactory fit on most occasions.

(31) To obtain a par yield curve numerical methods must be applied.

4 Modelling the effect of tax (the "Coupon Effect")

The techniques outlined in the previous section can be thought of as methods to estimate the yield-maturity structure of a bond market. However, the existence of coupon paying bonds complicates estimation of the term structure. In particular, tax rules can greatly affect the prices of bonds and, if their effects are ignored in the modelling process, can distort any estimate of the term structure of interest rates. This is what is commonly known as the *coupon effect* and is particularly important in the UK because of the wide range of coupons on bonds currently trading in the market (the current range of coupons on conventionals being 3% to 15 1/2%).⁽³²⁾

A substantial proportion of investors in the UK government bond market are taxed at their marginal rate of tax on any coupon income they receive, but are exempt from taxation on capital gain. Bonds with high coupons clearly provide more of their return in the form of coupon income than do bonds with low coupons. Therefore investors facing a non-zero marginal income tax rate but no tax on capital gain will *ceteris paribus* prefer low coupon to high coupon bonds, whereas those paying no income or capital gains tax will be indifferent between the two types. The preference of tax-paying investors for low coupon bonds will increase their price relative to high coupon bonds, a distortion that needs to be removed when attempting to measure the underlying term structure. This section outlines and compares three methodologies for taking account of the coupon effect.

(32) Another possible effect caused by bonds paying coupons is what might be termed a 'duration effect', since two bonds of the same maturity but with different coupons will have different durations and different exposure to interest rate risk. Such effects are not considered by any of the three models described here, presumably since the tax effect is considered to dominate.

McCulloch (1975)

In his original work, McCulloch (1971) overlooked the possible effect of taxation rules on bond prices, but developed the model to take account of such effects in his second paper (1975). In this paper, McCulloch sets up a number of equations for various types of bonds not all of which are relevant to this study as they reflect US tax laws in the early 1970s. Instead, when applied to the UK, only one equation is required - an amended version of the price equation assuming continuous compounding (equation (7)):⁽³³⁾

$$P = (1-\tau) C \int_0^m \delta(\mu) d\mu + R \delta(m) \quad (25)$$

where: τ = effective income tax rate
(P , m , C , R and δ as defined before)

From equation (25) a least squares estimation procedure analogous to equation (21) can be formulated.⁽³⁴⁾ The *effective* income tax rate τ is the rate that minimises the sum of squared residuals (between actual and fitted prices) produced by the model, and therefore requires some form of nonlinear search to find the optimal value of τ . It is not clear how τ should be interpreted (see below), but McCulloch describes it as "the approximate rate at which the Treasury recaptures its interest payments when it floats new debt".

(33) Note that the formulation of equation (25) implies that there is no tax on capital gains. While any capital gain made is indeed exempt from Capital Gains Tax, under certain circumstances market-makers have to treat capital gain as profit for tax purposes and therefore it may be taxed at their (corporation) rate of tax. In this sense equation (25) may be an over-simplification. Also, the continuous compounding approximation is again not necessary but we follow the original.

(34) Once the post-tax discount function has been estimated (using equation (20)) the equivalent pre-tax implied forward rate, zero coupon and par yield curves can be obtained by using equations (13), (14) and (17) respectively with τ as a scaling factor. See Appendix A for details.

Schaefer (1981)

Schaefer begins from a slightly different perspective, using a simple example to highlight the fact that, given a set of bond prices it is impossible to derive a unique term structure using the bond price equation (1) (which he calls the "no-arbitrage" condition) if there exists more than one category of taxpayer in the market and the tax treatment of long and short positions is symmetric. His example is as follows:

Suppose there are two investors in the market - one tax-exempt and the other facing an income tax rate of 50% - and two one-period bonds with coupons 4% and 10%. Both bonds make payments only at maturity, when each pays a coupon payment and repays the principal (£100, say). Using the bond price equation (1) and the after-tax cashflows we get the following price equations for the tax-exempt investor:

$$p_1 = \frac{104}{(1+r_1)} \quad \text{and} \quad p_2 = \frac{110}{(1+r_1)} \quad (26a)$$

whereas, for the tax-paying investor:

$$p_1 = \frac{102}{(1+r'_1)} \quad \text{and} \quad p_2 = \frac{105}{(1+r'_1)} \quad (26b)$$

The two pairs of equations are evidently inconsistent: if p_1 and p_2 are fixed then r_1 and r'_1 cannot be equal, implying that for each bond one class of investor values it higher than the other. In turn this implies that costless arbitrage would be possible between the different categories of investors (to the cost of the tax authorities). Since this is inconsistent with equilibrium as well as being unrealistic, Schaefer

assumes that short-sales are banned.⁽³⁵⁾ This assumption implies (amongst other things) that no arbitrage is possible and hence that no bond can be underpriced. Therefore, for any given tax rate τ , the price equation (1) becomes:

$$\text{Price} \geq \frac{C}{(1+r_1)} + \frac{C}{(1+r_2)^2} + \dots + \frac{R+C}{(1+r_n)^n} \quad (27)$$

where all cashflows are post-tax and the term structure r_i is *specific* to the tax rate τ . For an investor facing an income tax rate τ each bond is either efficient (if its market price exactly equates to the investor's valuation of that bond) or inefficient (with a market price greater than the value of the bond to the investor). Thus, Schaefer argues, there is no unique term structure of interest rates but rather a series of tax-specific term structures, each of which should be estimated using only those bonds which are efficiently held by investors in that tax bracket. The estimation involves one further (and essentially arbitrary) assumption about the series of cashflows required by investors, and requires the solution of a linear program to select the group of bonds that minimises the cost of providing these cashflows, subject to price constraints on each gilt (based on equation (27)).

Schaefer's specification of the problem highlights a number of difficulties with McCulloch's approach. First, by definition, McCulloch's methodology will calculate the term structure for only one category of taxpayer (facing the effective tax rate) and thus ignores the problem outlined above caused by the existence of more than one category. Also, the effective tax rate calculated using equation (25) will be some kind of "average" of all income tax rates faced by investors,

(35) In practice this is not the case - gilt-edged market makers are allowed to short sell - but Derry and Pradhan (1993) suggest some reasons why the market might behave in a manner that is analytically equivalent to the simplifying assumption of no short sales.

rather than the marginal rate of the investor determining prices of bonds. Second, this tax rate is (implicitly) assumed to apply to all bonds along the length of the curve, which is unrealistic if any of the categories of investors have preferences for the maturity of debt they want to hold.

The model specified by Schaefer is well suited to an individual institution making decisions on which bonds to hold, since the applicable tax rate is clear and the profile of cashflows required should also be known (or at least reasonably well approximated). However, problems arise when using this model to identify a (single) "market" term structure. There are two possible ways forward:

- (a) Simply select one of the various tax specific term structures and use this as a "representative" term structure, for example the term structure for 0% taxpayers. The problem with this approach is that it ignores information from all bonds other than those used to determine the particular term structure.⁽³⁶⁾ This data shortage problem can be alleviated a little by also including "near-efficient bonds" - those within a tolerance limit of being efficient bonds - and including these in a McCulloch-type regression,⁽³⁷⁾ but a lot of information is still being ignored and the term structure cannot be described as being representative of the whole market without further assumptions being made.
- (b) Another method derives from a suggestion in Schaefer (1981) outlining how a representative par yield curve can be estimated from a set of tax specific term structures. Using equation (16) and scaling by τ (see Appendix A), the (pre-tax) coupon $C_{\tau,m}$ required

(36) Typically around ten bonds (from a total of approximately 45) are selected as "efficient" by Schaefer's criterion and therefore eligible for use in the estimation.

(37) This was a suggestion made to the authors by Professor Schaefer and his colleague, Roger Brown.

by an investor facing tax rate τ to value an m period bond at its face value R is:

$$C_{\tau, m} = \frac{R(1-d)^m}{(1-\tau) \sum_{i=1}^m d^i}$$

Since the market price of a bond is determined by the investors who give it the highest value, the "market par yield" $y(m)$ is given by the lowest coupon stock with maturity m that at least one investor will price at par:

$$y(m) = \min_{\tau} (C_{\tau, m}) \quad (28)$$

from which a representative zero coupon curve and forward rate curve can be calculated using the relationships detailed in Section 2.

Approach (b) has the advantage over approach (a) that it does represent the whole market, rather than a specific category of taxpayer. However, to obtain an accurate term structure in this way requires the identification of all distinct tax categories (not an easy task) and the estimation of all their separate term structures.

Both approaches suffer from two drawbacks when used to estimate market representative curves. First, both require a function specifying the cashflows required by at least one category of investor in all periods. This is essentially an arbitrary selection and it is not clear what effect different functional forms may have on resulting term structures. Second, the estimation method depends crucially on the assumption that no bonds are underpriced and so, as Schaefer states (page 429):

"To the extent that...underpricing does occur, our estimates [of term structures] may be upward biased."

The Bank of England model (Mastronikola, 1991)

The Bank of England yield curve model is primarily used to provide advice to the Treasury on the level at which to set PWLB and NLF lending rates. Essentially, these are the respective rates at which local authorities and nationalised industries can borrow funds from the Government, and are calculated by adding a margin to the yield curve in order to ensure they are close to market rates.

The current yield curve model tackles the problem highlighted by Schaefer by noting that this tax effect manifests itself entirely through the bond coupons⁽³⁸⁾ and attempts to correct for it by modelling the relationship between yield and coupon as well as that between yield and maturity explicitly. The Bank's model therefore estimates a yield *surface* (yield as a function of coupon and maturity), thus allowing the size of the coupon effect to vary with maturity.⁽³⁹⁾ The par yield curve can be obtained from such a surface by noting that the yield of a bond trading at par must equal its coupon (the same condition used to derive equations (16) and (17) above); so the par yield curve can be thought of as the intersection between the yield surface and the "yield = coupon" plane (Mastronikola 1991, Diagram A).

The Bank models the yield-coupon relationship for a given maturity using *Capital-Income curves* that describe the trade-off between capital gain (assuming the bond is held to maturity) and income. For a bond with coupon C and redemption payment R (equal to £1, say) trading at price P , capital gain and income (the latter being defined as the bond's running yield - see Section 2.4) are given by:

(38) Assuming that investors are exempt from paying tax on capital gains.

(39) Unlike McCulloch (1975).

$$\text{Capital Gain} = \frac{1}{P} - 1 \quad \text{Income} = \frac{C}{P} \quad (29)$$

Note that, for a given maturity, describing the relationship between capital gain and income is equivalent to describing the relationship between yield and coupon since the only variables in (29) are coupon and price (which, if maturity is fixed, is a function of yield and coupon only). So having modelled the relationship between capital gain and income, this relationship can be transformed to provide the relationship between yield and coupon.

For a particular (fixed) maturity m , a capital-income diagram describes the trade-off between capital gain and income received on a bond. For a fixed yield (y), capital gain (CG in what follows) can be shown⁽⁴⁰⁾ to be a linear function of income (r):

$$\frac{1}{P} - 1 = A (y-r) \quad (30)$$

where:

$$A = \frac{\left[1 + \frac{1}{2} y (1-\tau) \right]^m - 1}{y} \quad (31)$$

This relationship is shown graphically in Figure 4.1(a) below. It can be interpreted as an indifference curve describing the various balances between capital gain and income to which an investor (facing tax rate τ - see below) is indifferent.

(40) See Mastronikola (1991), equations (3) and (4) on page 11. Note that r is the running yield here and has no connection with the spot rates r_t defined previously.

Figure 4.1(a)

Capital Gain v Income: Constant Yield

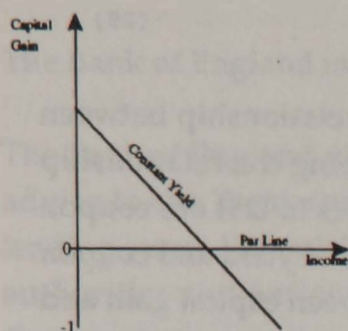
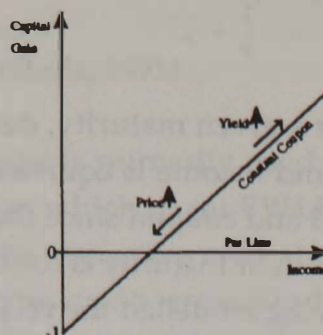


Figure 4.1(b)

Capital Gain v Income: Constant Coupon



Note in Figure 4.1 the line $CG = 0$ is called the *par line* since a bond bought at par and held to maturity yield no capital gain. Figure 4.1(b) shows the relationship between capital gain and income if the coupon C is fixed; clearly (from equation (29)) as the price rises both income and capital gain fall. The relationship is linear with slope $(1/C)$ since, from the definition of income (r):

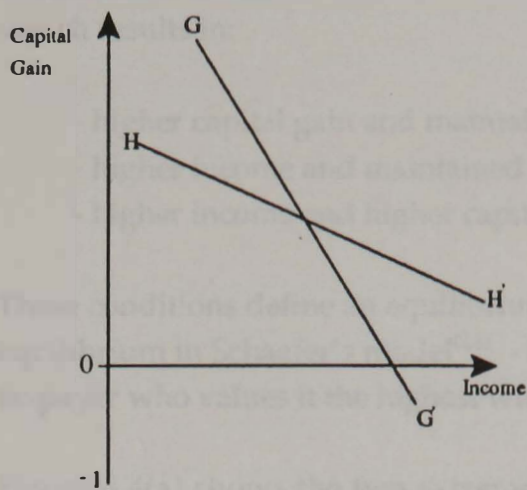
$$r = \frac{C}{P}$$

$$\therefore \frac{1}{P} - 1 = \frac{r}{C} - 1 \quad (32)$$

For a fixed maturity and yield, the constant term A in equation (30) depends only on the tax rate τ and therefore it is this income tax rate alone that determines the slope of the indifference line in Figure 4.1(a). Figure 4.2 demonstrates this by displaying the indifference lines for two categories of investors: gross investors (who are exempt from paying income tax, ie $\tau=0$) represented by the line GG' and a tax-paying investor represented by the line HH' (defined by $\tau>0$).

Figure 4.2

**Capital Gain v Income :
The effect of tax**

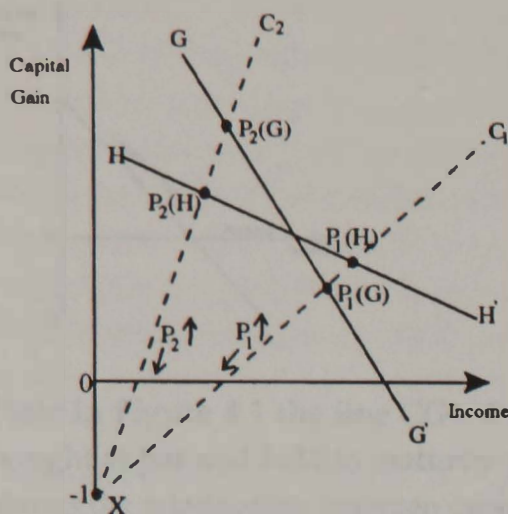


Since tax-paying investors pay tax on income but not on capital gain, they require a larger increase in income than gross investors to offset a unit decrease in capital gain - hence HH' is less steeply sloped than GG' in Figure 4.2.

Figure 4.3 illustrates how two bonds (1 and 2) with the same maturity m but different coupons (C_1 and C_2) are priced in a market with these two categories of investors.

Figure 4.3

Capital Gain v Income



The slopes of the constant coupon lines representing the bonds (denoted C_1 and C_2 in Figure 4.3) are $1/C_1$ and $1/C_2$ (from equation (32)) so, since the gradient of C_2 is greater than the gradient of C_1 in Figure 4.3, $1/C_2 > 1/C_1$; ie the coupon C_1 must be larger than the coupon C_2 . This illustrates the general property of such diagrams that constant coupon lines representing high coupon bonds are less steep than those representing low coupon bonds.

The intersections of C_1 and C_2 with GG' (denoted $P_1(G)$ and $P_2(G)$) indicate how gross investors will value the stream of cashflows from bonds 1 and 2 respectively. Likewise, $P_1(H)$ and $P_2(H)$ represent the valuations of the same two bonds made by taxpayers. It has already been noted that an increase in a bond's price moves it along its constant coupon line towards the origin on a capital-income diagram (see Figure 4.1(b)), and so it is clear from Figure 3.3 that the gross investors will value bond 1 higher than the tax-paying investors (since $P_1(G)$ is closer to the origin than $P_1(H)$), whereas bond 2 will be priced higher by the taxpayers. So, if investors are rational, the higher coupon bond's price will be set by the tax-exempt investor, whereas the lower coupon bond's price will be determined by the tax-paying investor. Such a

specification models a market assumed to be in "equilibrium under switching" (Mastronikola 1991, page 10), in which no investor can switch from one stock to any combination of other stocks if such a switch results in:

- higher capital gain and maintained income, or
- higher income and maintained capital gain, or
- higher income and higher capital gain.

These conditions define an equilibrium equivalent to the "no arbitrage" equilibrium in Schaefer's model⁽⁴¹⁾ - for each bond it is the category of taxpayer who values it the highest who determines its price.

Figure 4.4(a) shows the two extreme indifference lines, for the gross investors and 100% taxpayers. The line for an investor facing a 100% income tax rate is horizontal, since such an investor will only invest in bonds providing a pure capital gain.

Figure 4.4(a)

Capital Gain v Income :
Extreme Cases

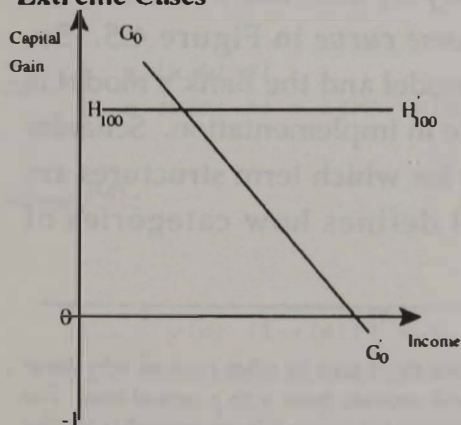
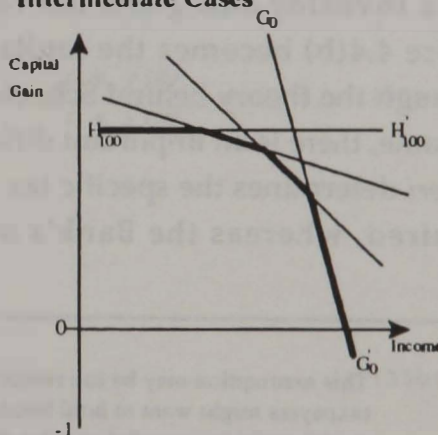


Figure 4.4(b)

Capital Gain v Income :
Intermediate Cases



(41) Therefore this model also depends on an assumption that short sales are restricted.

This model can easily be generalised to any number of categories of taxpayers, as illustrated in Figure 4.4(b). All the intersections between indifference curves are assumed to occur above the par line, since all bonds trading below the par line are priced above par and therefore cause a capital loss if held to redemption (this loss being balanced by above par coupon payments). Such bonds should therefore be held only by gross investors, and only bonds lying above the par line will be held by tax-paying investors of any kind.^{(42) (43)}

If the market is in equilibrium under switching, the prices of bonds in this diagram will be set along the heavy boundary (corresponding to an "efficient frontier"). However, as mentioned earlier (with reference to Schaefer's model), it is difficult to specify a number of distinct categories of taxpayers since, apart from the four current personal rates in the UK (0%, 20%, 25% and 40%), there are a number of institutions that have exemptions (including pension funds and some foreign investors) whilst others pay at their corporation tax rate (currently 25% or 33% in the UK) and may be able to offset some income against other losses for tax purposes. For this reason, therefore, the Bank model allows for a continuous spectrum of income taxpayers between the gross investor and 100% tax rate payer, and the boundary in Figure 4.4(b) becomes the *capital-income curve* in Figure 4.5. So, although the theory behind Schaefer's model and the Bank's model is the same, there is an important difference in implementation. Schaefer *a priori* determines the specific tax rates for which term structures are required, whereas the Bank's model defines how categories of

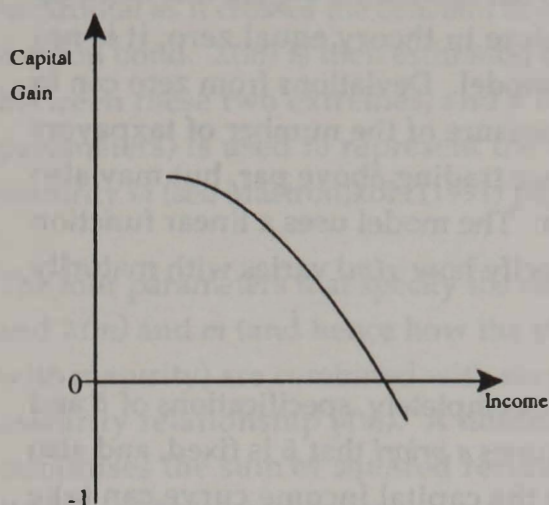
(42) This assumption may be too restrictive since there may be other reasons why some taxpayers might want to hold bonds that will provide them with a capital loss. The model could be amended to relax this assumption if it was felt unreasonable by (for example) restricting all intersections to occur above the constant coupon line representing the highest coupon bond in the market.

(43) The Bank model does not constrain the gross investor's tax rate to be 0% but instead allows it to vary with maturity. The estimated value of this parameter at each maturity perhaps gives an indication of whether or not bonds with that maturity and prices above par are in fact held by 0% taxpayers.

taxpayers interact and thereby estimates a single term structure representative of the market as a whole.

Figure 4.5

Capital Income Curve



The capital-income curve is defined in the Bank's model by the following equation:

$$CG = \begin{cases} \alpha (y(m) - r) & r \geq y(m) \\ \alpha (y(m) - r) + \lambda(m) (y(m) - r)^\delta & r < y(m) \end{cases} \quad (33a)$$

where⁽⁴⁴⁾:

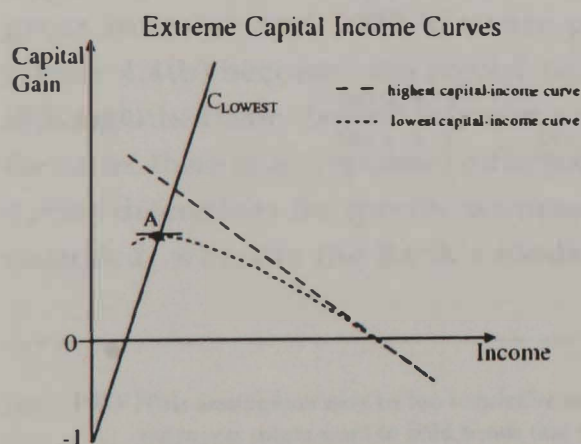
$$\alpha = \frac{\left[1 + \frac{1}{2} y(m) (1 - \tau(m)) \right]^m - 1}{y(m)} \quad (33b)$$

(44) Compare equation (33b) with equation (31). Note also that δ here is not related to the discount function in Section 2.

and $y(m)$ is the (given) par yield at maturity m . Note from these equations that the straight line segment of the capital-income curve (below the par line) is, for a given maturity m and associated par yield $y(m)$, dependent on the value of $\tau(m)$ (the so-called *effective tax rate at par* at maturity m), whilst the segment above the par line is dependent on $\tau(m)$, $\lambda(m)$ and δ .⁽⁴⁵⁾ Although $\tau(m)$ represents the tax rate faced by gross investors and should therefore in theory equal zero, it is not constrained to be so in the Bank's model. Deviations from zero can to some extent be interpreted as a measure of the number of taxpayers holding high coupon bonds that are trading above par, but may also reflect underpricing in the market. The model uses a linear function (defined by two parameters) to specify how $\tau(m)$ varies with maturity m .

To specify the capital income curve completely, specifications of δ and $\lambda(m)$ are required. The model assumes *a priori* that δ is fixed, and also assumes two extreme forms that the capital income curve can take (Figure 4.6):

Figure 4.6



(45) δ can be thought of as the "degree" of curvature of the capital-income curve above the par line (eg quadratic if $\delta=2$, etc.) whilst $\lambda(m)$ can be thought of as a "weight" that specifies how much this curvature comes into play at maturity m .

The upper extreme curve corresponds to the case where all bonds above the par line are held by gross investors, and can be represented by $\lambda(m) = 0$. Conversely, the lower extreme curve represents the largest tax effects. It is constructed by assuming a 100% taxpayer is holding the lowest coupon bond in the market, so the capital income curve becomes horizontal as it crosses the constant coupon line representing the lowest coupon bond. $\lambda(m)$ is then estimated to represent the true curve lying between these two extremes, and a linear function (defined by two parameters) is used to represent the relationship between $\lambda(m)$ and maturity m (see Mastronikola (1991) pages 9-18 for the full derivation).

The four parameters that specify the relationships between $\tau(m)$ and m , and $\lambda(m)$ and m (and hence how the yield-coupon relationship varies with maturity) are combined with six parameters to specify the yield-maturity relationship $y(m)$. A nonlinear estimation technique that minimises the sum of squared residuals between the observed and fitted yields is used to estimate the values of these ten parameters (along with two others)⁽⁴⁶⁾ simultaneously. The Bank's model therefore uses a curve fitting technique to estimate the tax rate faced by the category of taxpayers who determine the price of each bond. In a sense this is the reverse of Schaefer's approach, which involves determining the optimal set of bonds that each category of taxpayer should hold then, from the prices of bonds in this subset, calculating the tax-specific term structure.

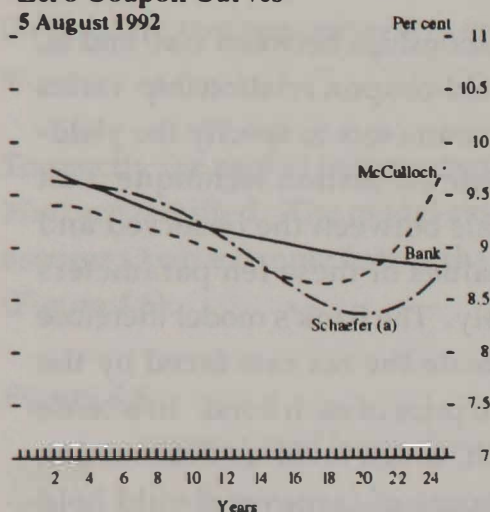
(46) Two other effects are modelled using dummy variables to represent whether or not a bond is trading ex-dividend (XD) and/or Free of Tax to Residents Abroad (FOTRA). See Mastronikola (1991), pages 18-9.

5 A comparison of the three models

Figures 5.1 and 5.2 illustrate zero coupon curves for the Bank, McCulloch and Schaefer methodologies. The Schaefer curve is for a 0% income taxpayer and includes near efficient bonds. Although the three curves are of broadly similar shape there are differences between them of up to 100 basis points. This is primarily due to the lack of constraints on long rates in the McCulloch/Schaefer methodology.

Figure 5.1

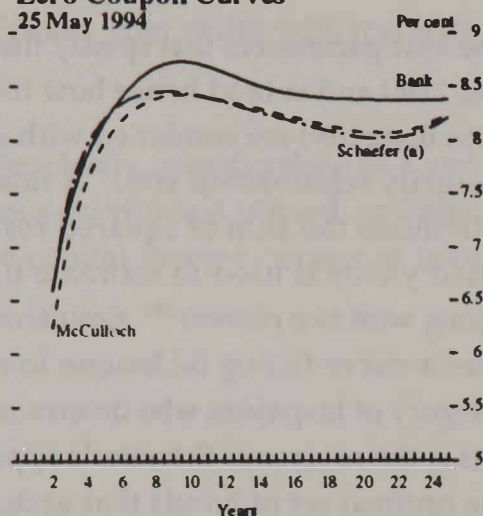
Zero Coupon Curves
5 August 1992



(a) Schaefer yield curve generated for 0% taxpayer
(including near-efficient bonds)

Figure 5.2

Zero Coupon Curves
25 May 1994



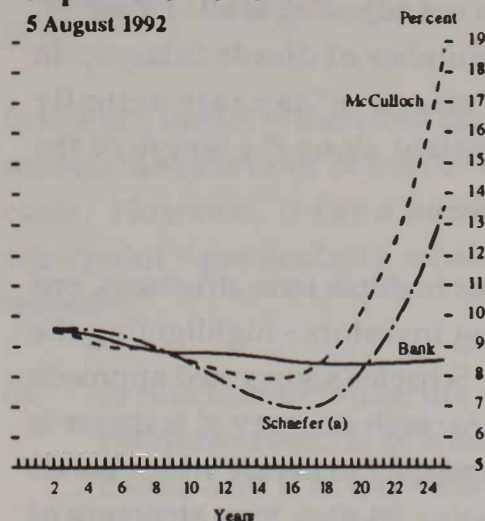
(a) Schaefer yield curve generated for 0% taxpayer
(including near-efficient bonds)

Figures 5.3 and 5.4 show the forward rate curves corresponding to the zero coupon curves in Figures 5.1 and 5.2. These graphically illustrate the sensitivity of the forward rate curve - in particular to long end constraints and to the number and location of knot points.

Figure 5.3

Implied Forward Rates

5 August 1992

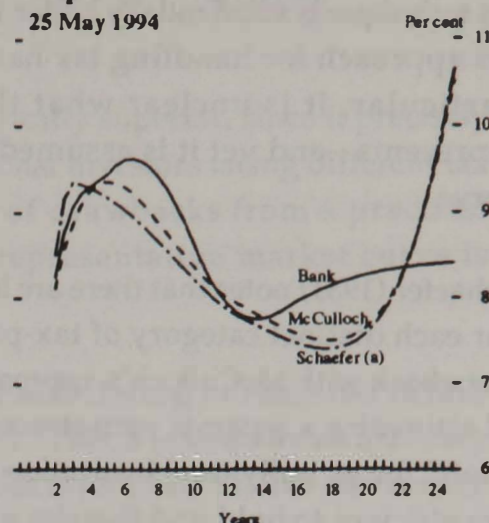


(a) Schaefer yield curve generated for 0% taxpayer (including near-efficient bonds)

Figure 5.4

Implied Forward Rates

25 May 1994



(a) Schaefer yield curve generated for 0% taxpayer (including near-efficient bonds)

The choice between fitting a par yield curve (the Bank model) and fitting a discount function (McCulloch 1971, 1975 and Schaefer 1981) is to some extent a matter of taste and prior beliefs about market behaviour. The discount function approach is explicitly consistent with economic theory but can be very difficult to estimate, leading to the sensitivity of the forward rate curve to small changes in the discount function. The approach of fitting through yields, whilst theoretically less attractive, appears more robust in practice (particularly when producing implied forward rate curves), and can be justified if it is believed that it better reflects market practice. This choice is inextricably linked with the choice of basis functions - the properties of the estimated term structure depend to a large degree on the properties of the chosen underlying basis functions.

The choice of model for the tax effects is a different matter, and is to some degree independent of the choice between fitting through yields or fitting a discount function. For example, there is no reason why the Bank's method for estimating the coupon effect cannot be used in conjunction with a model that fits a discount function. McCulloch

(1975) introduced the methodology for adjusting yield curves for taxation, and he demonstrates in his paper that adjusting for tax using his technique is substantially better than not adjusting at all. However, his approach for handling tax has a number of disadvantages; in particular, it is unclear what the "effective" tax rate actually represents - and yet it is assumed constant along the length of the curve.

Schaefer (1981) notes that there are in fact multiple term structures, one for each distinct category of tax-paying investors - highlighting the drawback with McCulloch's approach. Schaefer's suggested approach of estimating a separate term structure for each category of taxpayer is useful for an individual institution attempting to decide which bonds are efficient to hold and thereby assessing its own term structure of interest rates, but causes problems when trying to estimate a single "market" yield curve. It is necessary either to identify all distinct categories of taxpayer, something that could easily change on a daily basis, or to assume that one particular term structure is somehow representative of the market and, in the process, discard information from all bonds that are inefficient for that particular category of investor. The assumption that no bond is underpriced also leaves open the possibility that the estimated term structure is biased.

The current Bank model (Mastronikola, 1991) attempts to model both the yield-maturity and the yield-coupon relationships of the bond market and, in a way different from that adopted by Schaefer, also models which bonds are held by which category of investor. Despite the (possibly restrictive) assumption that all bonds trading above par are held by gross investors⁽⁴⁷⁾ the Bank's model effectively estimates the gross investors' par yield curve using information from all bonds in the market, rather than just the efficient subset. Although the Bank's

(47) Although, as explained previously, this assumption is not as restrictive as it first appears since a "gross" investor as defined by the Bank's model can face an (estimated) tax rate $\tau > 0$.

model will not produce tax-specific term structures, it is possible for each bond to measure the tax rate of the investor determining its price.⁽⁴⁸⁾

Schaefer's model is therefore theoretically superior, since it precisely models the behaviour of a set of rational investors facing different tax rates. However, it has a number of drawbacks from a practical viewpoint - particularly when a representative market curve is required:

- (i) A function defining the size and timing of required future cashflows needs to be specified. This is essentially an arbitrary choice and it is not clear what effect this choice has on the derived term structures.
- (ii) The assumption that no bond is underpriced may lead to bias in the estimated term structures.
- (iii) If a representative market curve is required, then either all distinct categories of taxpayers need to be identified, or the assumption that one category is "representative" needs to be made.

The Bank model avoids these drawbacks, but at the expense of theoretical precision. For these reasons, Schaefer's model seems preferable when attempting to estimate the term structure for a particular category of investor but is less suitable for estimating a term structure that is intended to be representative of the market as a whole.

(48) This is done by calculating the slope of the capital-income curve (for the bond's maturity) where it crosses the appropriate constant coupon line, and using equation (31) to determine τ .

6 Conclusion

This study has investigated the properties of three models that estimate the term structure of interest rates - two prominent models from the literature due to McCulloch (1975) and Schaefer (1981), and the model used currently by the Bank. The three models were compared (on theoretical grounds: their methodologies for handling tax effects in particular. Examples of curves produced using each of the methodologies were also presented.

There are (at least) three aspects to the estimation problem that are more or less distinct. These are:

- (a) Which curve should be estimated first? McCulloch developed the methodology for fitting the discount function (and Schaefer also uses this approach), which is theoretically attractive but can be difficult to estimate in practice. In particular it can prove difficult to derive sensible forward curves from an estimated discount function. Alternatively, the approach of simply fitting a curve through redemption yields to obtain a par yield curve (as the Bank's model does) can be used. This is less attractive theoretically (although it may be justifiable if it is common market practice), but more robust in practice. Until satisfactory curves can be derived from an estimated discount function it seems sensible to continue fitting a par yield curve.⁽⁴⁹⁾
- (b) What basis functions should be used to define the shape of the estimated discount function/par yield curve? This is a separate choice to (a), although many applications use variations on the cubic spline for estimation, and depends to a large extent on the required flexibility of the derived curves. The choice of basis functions may be critical to the shape of the curves produced by

(49) The approach due to Nelson and Siegel (1987) - and augmented by Svensson (1994) - may prove useful in this respect.

either method in (a), but appears to be a more important consideration when the discount function approach is used. The consensus view in the literature appears to be a choice between using B-splines (most recently endorsed by Steeley, 1991) or a more restrictive functional form of the kind suggested by Nelson and Siegel.

- (c) How should tax effects be accounted for? McCulloch produced the first solution to this problem, which is probably a little too restrictive. Schaefer highlighted that there are in fact as many separate term structures as there are distinct categories of tax-paying investors, and his approach is well suited for an individual or institution to estimate the tax-specific term structure that they face. However, there are drawbacks to this methodology when attempting to estimate a single "market" term structure of interest rates, an area where the current Bank tax model has practical advantages over Schaefer's model, but at the expense of theoretical rigour.

Finally, it is worth reiterating that the choice of tax model is a separate issue from the choice of methodology for estimating the term structure. For example, an approach based on fitting the discount function and modelling tax effects using the Bank's technique might be desirable.

Appendix A: Deriving the McCulloch Equations

Using continuous compounding

To estimate the discount function $\delta(m)$ from observed prices of n bonds, the discount function is written as a linear combination of basis functions:

$$\delta(m) = 1 + \sum_{j=1}^k a_j f_j(m) \quad (\text{A1})$$

where $f_j(m)$ is the j^{th} basis function, and a_j is the corresponding coefficient ($j=1, \dots, k$).

The price of the i^{th} bond is given by:

$$P_i = C_i \int_0^{m_i} \delta(\mu) d\mu + R_i \delta(m_i) \quad (\text{A2})$$

where P_i , C_i , R_i and m_i are the price, coupon, redemption payment and maturity of the i^{th} bond.

Substituting the expression for the discount function (A1) into the i^{th} price equation gives:

$$P_i = C_i \int_0^{m_i} \left[1 + \sum_{j=1}^k a_j f_j(\mu) \right] d\mu + R_i \left[1 + \sum_{j=1}^k a_j f_j(m_i) \right]$$

$$\therefore P_i = C_i \left[m_i + \int_0^{m_i} \sum_{j=1}^k a_j f_j(\mu) d\mu \right] + R_i + R_i \sum_{j=1}^k a_j f_j(m_i)$$

$$\therefore P_i = C_i m_i + R_i + \sum_{j=1}^k a_j C_i \int_0^{m_i} f_j(\mu) d\mu + \sum_{j=1}^k R_i a_j f_j(m_i)$$

$$\therefore P_i - C_i m_i - R_i = \sum_{j=1}^k a_j \left[C_i \int_0^{m_i} f_j(\mu) d\mu + R_i f_j(m_i) \right]$$

which can be written:

$$y_i = \sum_{j=1}^k a_j x_{ij} \quad (\text{A3})$$

where:

$$y_i = P_i - C_i m_i - R_i$$

$$x_{ij} = C_i \int_0^{m_i} f_j(\mu) d\mu + R_i f_j(m_i)$$

Equation (A3) can then be used to obtain least-squares estimates \hat{a}_j , and the estimate of the discount function $\delta(m)$ is then given by:

$$\hat{\delta}(m) = 1 + \sum_{j=1}^k \hat{a}_j f_j(m) \quad (\text{A4})$$

Using discrete compounding

The analogy of (A2) using discrete compounding is:

$$P_i + ai_i = \frac{c_i}{2} \sum_{l=1}^n \delta(l) + R_i \delta(n) \quad (A5)$$

where P_i , ai_i and n are the clean price, accrued interest and the number of outstanding (semi-annual) coupon payments of size $c_i/2$ of the i^{th} bond.

Substituting the expression for the discount function (A1) into the i^{th} price equation gives:

$$P_i + ai_i = \frac{c_i}{2} \sum_{l=1}^n [1 + \sum_{j=1}^k a_j f_j(l)] + R_i [1 + \sum_{j=1}^k a_j f_j(n)]$$

$$\therefore P_i + ai_i = n \frac{c_i}{2} + R_i + \sum_{j=1}^k a_j \frac{c_i}{2} \sum_{l=1}^n f_j(l) + \sum_{j=1}^k a_j R_i f_j(n)$$

$$\therefore P_i + ai_i - n \frac{c_i}{2} - R_i = \sum_{j=1}^k a_j \frac{c_i}{2} \sum_{l=1}^n f_j(l) + R_i f_j(n)$$

which can be written:

$$y_i = \sum_{j=1}^k a_j x_{ij} \quad (\text{A6})$$

where:

$$y_i = P_i + ai_i - \frac{c_i}{2} - R_i$$

$$x_{ij} = \frac{c_i}{2} \sum_{l=1}^n f_j(l) + R_i f_j(n)$$

As with the case of continuous compounding, (A6) can then be used to obtain least-squares estimates \hat{a}_j , and the estimate of the discount function δ_m is then given by:

$$\hat{\delta}(m) = 1 + \sum_{j=1}^k \hat{a}_j f_j(m)$$

Tax-adjusted discrete compounding formulae

Using the formulae in the previous section ignores any effects caused by taxation of coupon payments. In this section we list the analogous formulae allowing for taxation of income at some rate τ . Incorporating the Accrued Interest Taxation Scheme the relationship between the price of the i^{th} bond and the discount function is now given by:

$$P_i + ai_i = ai_i \tau \delta(\tau_1) + \frac{c_i}{2} (1-\tau) \sum_{l=1}^n \delta(l) + R_i \delta(n)$$

Following the same process as in the previous section leads to the result that:

$$y_i = \sum_{j=1}^k a_j x_{ij}$$

where:

$$y_i = P_i + (ai_i - \frac{nc_i}{2}) (1-r) - R_i$$

$$x_{ij} = ai_i r f_j(t_1) + \frac{c_i}{2} (1-r) \sum_{l=1}^n f_j(l) + R_i f_j(n)$$

It is also possible to derive analogous equations for the relationships between the discount function and the par yield, zero coupon yield and forward rate curves derived in Section 2.

The equation relating the forward rate curve to the discount function [the analogy of equation (11)] is:

$$f_i = \frac{-\Delta d_i}{(1-r) d_i}$$

The zero coupon curve can be derived from the discount function [analogous to equation (15)] using the equation:

$$r_i = \frac{1}{1-r} \left[\left(\frac{1}{d_i} \right)^{1/i} - 1 \right]$$

Finally, the equation linking the par yield curve to the discount function [analogous to equation (16)] is:

$$y_m = \frac{R(1-d_m)}{(1-\tau) \sum_{i=1}^m d_i}$$

Appendix B: McCulloch's cubic spline specification

Assume k knots $\kappa_1, \dots, \kappa_k$ where $\kappa_1 = 0$ and $\kappa_k =$ maturity of the longest existing bond, the other knots positioned so that there is approximately the same number of bonds between each pair of knots.

The functions used are (for $j < k$)

for $m < \kappa_{j-1}$

$$f_j(m) = 0$$

for $\kappa_{j-1} \leq m < \kappa_j$

$$f_j(m) = \frac{(m - \kappa_{j-1})^3}{6(\kappa_j - \kappa_{j-1})}$$

for $\kappa_j \leq m < \kappa_{j+1}$

$$f_j(m) = \frac{c^2}{6} + \frac{ce}{2} + \frac{e^2}{2} - \frac{e^3}{6(\kappa_{j+1} - \kappa_j)}$$

where: $c = \kappa_j - \kappa_{j-1}$
 $e = m - \kappa_j$

for $\kappa_{j+1} \leq m$

$$f_j(m) = (\kappa_{j+1} - \kappa_{j-1}) \left[\frac{2\kappa_{j+1} - \kappa_j - \kappa_{j-1}}{6} + \frac{m - \kappa_{j+1}}{2} \right]$$

for $j = k$

And: $f_k(m) = m$ for all m .

Appendix C: Deriving an Implied Forward Rate Curve from a Par Yield Curve

The price equation for an n -period is:

$$P = \frac{C}{(1+r_1)} + \frac{C}{(1+r_2)^2} + \dots + \frac{C}{(1+r_n)^n} + \frac{1}{(1+r_n)^n}$$

where:

P = price of bond (per £1 nominal)

C = coupon (per £1 nominal)

r_i = i year spot/zero coupon rate

Let:

y_i = i year par yield

f_i = 1 year forward rate from period $i-1$ to period i

$$\therefore (1+r_i)^i = (1+f_1)(1+f_2)\dots(1+f_i) \quad (C1)$$

d_i = i th discount factor

$$= (1+r_i)^{-i}$$

Note: $y_1 = r_1 = f_1$

$$\text{so } P = d_1 C + d_2 C + \dots + d_n C + d_n$$

$$\therefore P = C \sum_{i=1}^n d_i + d_n$$

The par yield curve is constructed from notional bonds selling at par. Hence $P=1$ and the n year par yield $y_n = C$, the coupon on the notional n year par bond:

$$1 = y_n \sum_{i=1}^n d_i + d_n \quad (C2)$$

$$\text{ie } y_n = (1 - d_n) / \sum_{i=1}^n d_i$$

Now get an expression for $\sum_{i=1}^n d_i$ in terms of the par yields (y_i 's):

$$\text{Since } d_n = \sum_{i=1}^n d_i - \sum_{i=1}^{n-1} d_i$$

it follows that

$$y_n = \left[1 - \left(\sum_{i=1}^n d_i - \sum_{i=1}^{n-1} d_i \right) \right] / \sum_{i=1}^n d_i$$

$$\therefore y_n \sum_{i=1}^n d_i + \sum_{i=1}^{n-1} d_i = 1 + \sum_{i=1}^{n-1} d_i$$

$$\therefore \sum_{i=1}^n d_i = (1 + \sum_{i=1}^{n-1} d_i) / (1 + y_n) \quad (C3)$$

From equation (C3):

$$n=1: \sum_{i=1}^1 d_i = \frac{1}{(1+y_1)}$$

$$n=2: \sum_{i=1}^2 d_i = \frac{1}{(1+y_2)} + \frac{1}{(1+y_1)(1+y_2)} = \sum_{j=1}^2 \frac{1}{\prod_{i=j}^2 (1+y_i)}$$

$$n=3: \sum_{i=1}^3 d_i = \frac{1}{(1+y_1)} + \frac{1}{(1+y_2)(1+y_3)} + \frac{1}{(1+y_1)(1+y_2)(1+y_3)} = \sum_{j=1}^3 \frac{1}{\prod_{i=j}^3 (1+y_i)}$$

⋮

$$\sum_{i=1}^n d_i = \sum_{j=1}^n \frac{1}{\prod_{i=j}^n (1+y_i)}$$

But $d_n = 1 - y_n \sum_{i=1}^n d_i$ (from equation (C2))

$$\therefore d_n = 1 - y_n \sum_{j=1}^n \frac{1}{\prod_{i=j}^n (1+y_i)}$$

Since $d_n = (1 + r_n)^{-n}$:

$$r_n = \left[1 - y_n \sum_{j=1}^n \frac{1}{\prod_{i=j}^n (1+y_i)} \right]^{-1/n} - 1$$

The implied forward rates can then be calculated from equation (C1).

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