Averaging in a framework of zero reserve requirements: implications for the operation of monetary policy

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Abstract

If a central bank is unable to forecast accurately the banking system's demand for reserves, then the volatility of the money-market interest rate is likely to increase. Although reserve averaging is one possible means of dealing with this, positive reserve requirements may have undesirable properties. In this paper, we examine the operational implications of combining averaging with a zero reserve requirement. We then examine the optimum system of penalty charges for overnight overdrafts and for missing the averaging requirement, as well as the consequent behaviour of the money-market interest rate relative to the central bank's target rate.

1 Introduction

Central banks around the world employ a wide range of operational techniques to implement monetary policy. They supply or drain liquidity through different combinations of open market operations (OMOs) and standing facilities; some target a money-market interest rate directly, while others operate mainly as rate-takers, controlling interest rates indirectly via pressure on banks' reserves; and they operate at widely different maturities, from overnight to three months.

When designing an operational framework, one objective is likely to be to limit interest rate volatility arising from any inability to forecast accurately banks' demand for reserves. A more stable overnight rate helps to increase the transparency of monetary policy. As financial markets have become more international and capital has become more mobile, the sensitivity of the exchange rate and market interest rates to expectations of monetary policy has risen. In other words, expectations now play an increasingly important role in the transmission of monetary policy, and the costs of sending the wrong signals have therefore increased. Limiting the volatility of short-term market interest rates around its policy rate maximises a central bank's influence over market expectations, and minimises the chances for the market to misinterpret policy.

Operating more frequently, using standing facilities to set a floor and a ceiling for an interest rate corridor, and taking steps to improve the accuracy of the central bank's forecast of banks' demand for liquidity are possible ways of limiting the volatility of money-market interest rates. Reserve averaging is another, and one that has been quite widely employed, very often in conjunction with a minimum reserve requirement. Under such a system, banks must hold a given level of reserves at the central bank, on average, during the maintenance period; if they hold less than the requirement one day, they must offset this by holding more reserves than required before the end of the period.

Table A presents the main features of reserve requirements in the G7 countries. While many central banks have lowered the level of reserves that banks must maintain during any period,[1] only the United Kingdom and Canada have a zero reserve requirement. The main motivation for lowering requirements seems to have been to remove distortions within the market for financial services, and to discourage the relocation of business to other financial centres with lower, or even zero, minimum reserves. Moreover, other benefits sometimes associated with positive reserve requirements can be provided by other instruments that do not distort the market.[2]

Table A: Main features of reserve requirements in the G7

	United Kingdom	United States	Germany	France	Italy	Japan	Canada
Requirement (as % of liabilities)	Zero	3-10	1.5-2	0.5-1	15	0.05-1.3	Zero
Averaging	No	Yes	Yes	Yes	Yes	Yes	Yes
Length of period	1 day	2 weeks	1 month	1 month	1 month	1 month	4-5 weeks
Penalties	Repo rate +1-2%	Discount rate +2%	Lombard rate +3%	Average O/N rate +3%	Discount rate +10%	Discount rate +3.75%	Bank rate n/a
Remuneration	-	-	-	-	5.5%	-	-
Interest rate corridor	-	-	Yes	Yes	Yes	-	Yes
Policy rate can change during maintenance period	No	Yes	Yes	Yes	Yes	Yes	Yes

Source: Borio (1997).

As Table A shows, the United Kingdom is the only country in the G7 not to employ averaging; banks must balance their positions every day. Without averaging, the stream of shocks to banks' demand for liquidity each day could translate into sharp movements in short-term interest rates. To prevent this, the Bank of England relies on updating its forecast of the demand for reserves,

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(1) For example, the United States lowered reserve requirements in 1992, Germany in 1994 and 1995, France in 1990, 1991 and 1992.
(2) For example, central banks can use reserve requirements to increase the banking sector's demand for liquidity from open market operations. However, another means of creating a structural demand for central bank money, which the Bank of England employs, is the sale of central bank or government paper.

and operating several times a day to meet changes in demand, rather than employing averaging. However, the flows between the government and the private sector, which directly determine the liquidity of the banking system, are intrinsically volatile, and therefore difficult to forecast accurately. So updating the forecast of the demand for liquidity during the day may not significantly improve its accuracy. Consequently, the volatility of the overnight rate in the United Kingdom remains high by international standards.(3)

Table A also shows that several countries, especially in Europe, use standing facilities (deposit and lending) to create an interest rate corridor. By keeping market rates within this corridor, standing facilities could also reduce their volatility. But the narrower this corridor is, the more banks substitute central bank refinancing for borrowing in the money market. And this reduction in market liquidity may reduce the central bank's control over market interest rates. Most central banks that operate a corridor system operate a fairly wide one—about 2%—so that market rates can still fluctuate widely. Averaging, however, can fully stabilise the market rate of interest, by encouraging banks to postpone borrowing when the interest rate is temporarily high until later in the maintenance period, and to bring borrowing forward when the interest rate is low. So averaging can be a more effective instrument for reducing the volatility of market interest rates.

One reason for not operating averaging has been to avoid the implicit tax on the banking system associated with a positive reserve requirement. But it is now recognised that averaging does not need a positive reserve requirement, as long as overdrafts are made readily available. Table A shows that Canada operates a system of zero reserve requirements with averaging, as does Mexico. Furthermore, in determining how the European System of Central Banks might implement monetary policy in EMU, the European Monetary Institute considered a wide spectrum of possible frameworks, one of which was averaging around a zero reserve requirement. The model in this paper describes how such a system could operate.

However, a drawback to averaging, irrespective of whether the reserve requirement is zero or positive, is that it allows banks to speculate if they expect a change in the central bank's dealing rate during the maintenance period. For example, if banks expect a rise in the maintenance period, they will

(3) See Escriva and Fagan, (1996) and Borio (1997).

try to meet all of their expected liquidity needs early in the maintenance period, before the central bank raises its policy rate. If the central bank meets all of this demand, and does raise rates, it loses the revenue it would have earned later in the period on the quantity it would have lent at its higher rate. Alternatively, if the central bank does not meet this demand, the market bids up the overnight rate towards the level it expects after the rate rise. So speculation results in either a loss of control over the overnight rate, or a financial loss to the central bank if the market anticipates correctly.

One way to prevent speculation would be for central banks to change their policy rates only at the start of each maintenance period and rule out any further changes until the start of the next period. The results in Table A reveal that the maintenance period in most of the countries that operate averaging is one month, and that most of the central banks do change rates during the maintenance period. Given the low frequency with which central banks generally change their policy rate, doing so only at the start of a one-month maintenance period would not prove to be too much of a constraint. But speculation about changes in policy rates is likely to be one factor that keeps central banks from operating maintenance periods longer than one month.

Poole's seminal (1968) paper showed that in a world where shocks to banks' demand for liquidity were symmetrical, banks would aim to meet a daily reserve requirement exactly only if the penalty for not doing so were twice the market rate of interest. However, as Table A shows, the penalty for missing the reserve requirement in most countries is significantly less than twice the market rate, and so banks should aim below the reserve requirement. The fact that they do not could be a result of central banks employing additional non-price penalties.

Much of the research following Poole's paper centred on the calculation of reserve requirements, rather than on averaging. Spindt and Hoffmeister (1988), however, present a model where the closer to the end of the maintenance period a shock to liquidity occurs, the greater the proportion that banks seek to offset on each of the remaining days (ie the higher the 'work-off' rate). More recently, research has focused on the effect of lower reserve requirements, as central banks have lowered the reserves that they require banks to hold. Brunner (1993) extends Poole's model to the case where the central bank operates several times a day, and examines banks' behaviour and the overnight rate at each of these rounds of operations. Brunner concludes that lowering reserve requirements has an ambiguous effect on the volatility of the overnight rate. Feinman (1993) and Weiner (1992) argue that lower reserve requirements make it more difficult for central banks to forecast accurately the demand for reserves, and that they introduce excessive volatility in short-term interest rates, and thereby reduce the effectiveness of the central bank's monetary operations.

The use of sweep accounts by US banks to switch liabilities subject to reserve requirements into accounts not subject to reserve requirements has lowered banks' effective reserve requirements. This has provoked renewed interest in the link between the level of the reserve requirement and the effectiveness of OMOs. Sellon & Weiner (1996) review monetary policy in an environment of low reserve requirements, concluding that low requirements make it more difficult for central banks to control market rates of interest.

This paper extends Brunner's analysis to a three-day maintenance period, where banks must meet a zero reserve requirement on average—the sum of their daily positions must not be negative. The volatility of the overnight rate and the 'work-off' rate depend critically on the structure of the penalty for missing the reserve requirement and the cost of end-of-day overdrafts. Volatility is minimised when overdrafts are free. If overdrafts are not free, banks try to immediately adjust to liquidity shocks in the market, rather than passively absorbing them on their end-of-day balance. So central banks may contribute to the volatility of market interest rates through their choice of penalties for overdrafts. Moreover, the model confirms that Poole's 'two times' rule applies to a system with averaging.

Accordingly, the model shows that, under a system of zero reserve requirements with averaging, it is possible to implement monetary policy to keep interest rates around the desired level while meeting the banking system's liquidity needs, as long as (collateralised) overdrafts are readily available, and preferably free. It is not claimed that this system is unique in doing this; a wide variety of operational arrangements can be, and indeed are, in practice, employed to implement monetary policy. But the model does suggest that if averaging is adopted as a feature of the operational framework, then positive reserve requirements, with all their drawbacks, are not needed to confer its supposed benefits.

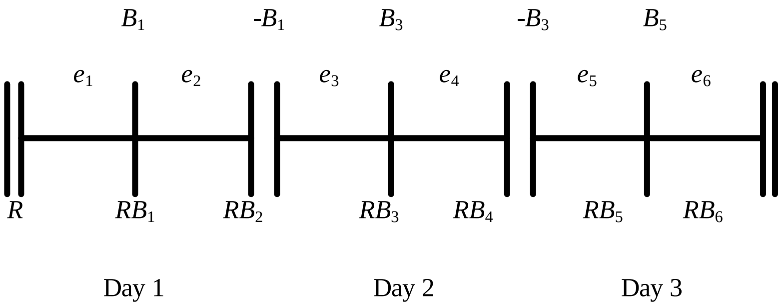
Moreover, averaging in a system of zero requirements with free overdrafts does not introduce any drawbacks additional to averaging with a positive reserve requirement. So for example, all overdrafts could be collateralised just as OMOs are collateralised; from the central bank's point of view, secured overdrafts are no more risky than secured OMOs. Furthermore, banks would manage their reserves no differently from under averaging with a positive requirement; under both systems, with the appropriate penalty structure, banks seek to meet the reserve requirement, and smooth their borrowing. And while this paper does not directly consider the implications for the central bank's operational income, any reduction in the central bank's income results from the lowering of the reserve requirement, not from providing overdrafts free of charge; any income that the central bank loses by extending a free overdraft, it recovers when banks borrow additional funds via OMOs later in the period to offset the overdraft.

The paper is organised as follows. The next section describes the model's key assumptions and results. The following section sets out how the central bank operates, how shocks affect the market's reserve position, and how the reserve requirement works. We derive the market's demand for reserves at the central bank's first operation of the maintenance period, and the next section describes the central bank's policy rule for supplying reserves at its operation. The paper then models the market's behaviour on the first day of the maintenance period, including the pattern of market interest rates. The other days of the maintenance period are then described, and the whole system is compared with a system without averaging. The final section relaxes the assumption that the central bank cannot change its official interest rate during the course of the maintenance period, and examines banks' speculative behaviour when they expect just such a change.

2 The model's assumptions and main results

In the basic model, the central bank operates a three-day maintenance period, using daily OMOs in overnight funds, to meet banks' demand for funds and to enforce its target for the money-market interest rate. Banks borrow via these OMOs each morning and afternoon to adjust their reserves following shocks to their liquidity. Figure 1 illustrates the flows in the maintenance period.

Figure 1: The three-day maintenance period



where	e_t	is a reserve shock, uniformly distributed over the range (-2,2), and e_t>0 is a positive shock to the market's reserves
	B_t	is the market's borrowing from the central bank
	RB_t	is the market's reserve balance at time t
	R	is the reserve balance with which the market begins the maintenance period

The first part of the paper assumes that the central bank's target rate and the sensitivity of supply to movements in the market interest rate remain constant throughout the maintenance period. In other words, the central bank changes its target rate only at the beginning of each maintenance period, in contrast with most central banks' practice, as Table A indicates. By operating in this manner, the central bank avoids any speculation about changes in monetary policy within the maintenance period, and ruling out speculation allows us to focus clearly on how the market behaves under averaging, and under different systems of penalty charges for missing the reserve requirement.

We assume that the central bank operates daily, to show that, contrary to what is sometimes claimed, averaging does not preclude the central bank from operating frequently within the averaging period. The frequency of the central bank's intervention, as we shall see later, affects the money market only after the central bank's last operation. Operating every other day would allow banks to speculate about interest rate changes, if the maturity of an operation were allowed to overlap the beginning of the next maintenance period, or alternatively would leave the money market to clear by itself on the last day of the maintenance period.

The market takes the form of a representative bank, so that the market is assumed to function competitively, without market power or asymmetric information. Therefore, the money-market interest rate will equate one-for-one with the central bank's dealing rate,(4) and the money market and the central bank's operations are perfect substitutes. Obviously, by assuming a representative bank, we ignore the interactions of the banks in the money market, which in practice may be very important.

We also assume that the central bank cannot forecast accurately the banking sector's demand for liquidity; if it could, averaging's stabilisation function would be less important. Every day, there are two reserve shocks, one in the morning, and one in the afternoon.(5) While the market learns the effect of these as soon as they happen, the central bank does not observe them until settlement, at the end of each day. So by noon, when the central bank conducts its daily operations, the market's reserve balance is likely to be different from the level it would want to maintain at the end of the day. To adjust towards its desired level, therefore, the market must borrow at the central bank's round of OMOs. The model implicitly assumes that there is a large money-market shortage, as is the case in the United Kingdom, so that even if banks wish to finish a day with a negative level of reserves, they would still need to roll over a large amount of past assistance. In other words, there is always a need to borrow reserves from the central bank.

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(4) The money-market interest rate and the central bank's dealing rate are identical, but because there is a representative bank, there is no money market as such in this model. However, we use the term money-market interest rate throughout the paper, to distinguish the central bank's dealing rate from its target rate.

(5) The model's results rest on the assumption that the central bank can observe any shocks at the end of the day, and adjust its forecast of demand for the remainder of the maintenance period accordingly. If shocks are serially correlated, then the central bank can include this in its forecast as well. Therefore, whatever the behaviour of shocks, banks expect the central bank to balance total liquidity demanded during the maintenance period with the total amount it supplies; shocks around the central bank's forecast always have a mean of zero. And as a result, banks do not try to compensate for shocks as they happen—they wait for the central bank to do so.

Since the central bank can learn the exact outcome of each day's shocks only at the close of business, when settlement occurs, it can only forecast demand for its noon operation; actual demand, *ex post*, will differ because of the morning's shock. Because the morning's shock is zero only on average, in practice, the central bank will make mistakes. As a result, on any one day there will tend to be too many, or too few, reserves on offer at the central bank's target rate. Unless the central bank is willing to supply an unlimited quantity of reserves at the target rate (the supply curve is perfectly elastic), this will result in money-market interest rates moving away from the central bank's target. The extent of this movement will depend on how willing the central bank is to adjust the supply of reserves in response.

To encourage the market to meet the zero cumulative reserve requirement at the close of the maintenance period, the central bank levies a penalty rate, p, on any deficiency. Some central banks that operate a system of zero reserve requirements, such as Canada, also make a charge for intra-period overdrafts. To accommodate this, the model allows the central bank to charge an overdraft rate, o, on end-of-day advances; positive balances are unremunerated. In practice, central banks may levy non-pecuniary penalties on overdrafts as a further discouragement. Later in the paper, we examine the effect of charging for end-of-day overdrafts, and derive the optimal overdraft charge. Intuitively, however, if overdrafts are expensive, banks will be less willing to incur them. So if banks have just experienced a negative reserve shock, they will continue to try to borrow more reserves on the same day, even if rates spike upwards. In other words, charging for overdrafts tends to dampen, at least in part, the smoothing effect of averaging.

This means that if the central bank wishes to stabilise money-market interest rates, it should allow free intra-period overdrafts. In addition, the paper shows that if the central bank is to induce banks to try to meet the zero reserve requirement precisely, it should combine free intra-period overdrafts with a penalty for failing to meet the reserve requirement of twice the market rate of interest on the final day of the maintenance period. This result is a generalisation of the conclusion of Poole (1968).

In this model, by offering free intra-period overdrafts, the central bank minimises the volatility of the money-market interest rate. At the other extreme, if the only charge is for intra-period overdrafts, and no penalty is imposed on missing the reserve requirement, the model is effectively a one-day maintenance period.[6] In general, as the central bank shifts a larger share of the total penalty for missing the reserve requirement (ie the sum of the end-of-day overdraft rate and the end-of-period penalty) to the end-of-period penalty, the average volatility of money-market interest rates falls. This is because under averaging, neither banks nor the central bank need to adjust to reserve shocks on the day that they occur, except on the final day of the averaging period. However, if the central bank charges for overdrafts, it imposes a 'tax' on switching demand; so, for example, a bank will not want to absorb a negative shock to its reserves by going overdrawn. Reserve shocks, therefore, will have a greater impact on the money market, if the central bank charges for intra-period overdrafts.

By the reasoning above, interest rates will be at their most volatile on the last day of the averaging period, since at other times shocks have less of an impact, because the central bank has an opportunity to rectify the mistake in its forecast later in the maintenance period. However, because we examine a three-day maintenance period, we also find that the average volatility of market rates rises on each day of the maintenance period, just as the work-off rate in response to a given shock to liquidity increases as the maintenance period wears on. It follows that the average volatility of money-market interest rates falls as the length of the maintenance period increases. But this conclusion depends upon the assumption that the central bank does not change official interest rates within the maintenance period. This is an assumption that becomes increasingly unrealistic the longer the maintenance period is.[7]

If the central bank is allowed to change its target rate within the maintenance period, speculation against the central bank could increase volatility substantially. If banks expect a rate rise, they will bring demand forward, and push it back when they expect a reduction. In other words, they attempt to do most of their borrowing when they expect that interest rates will be at their cheapest. As a result, the money-market interest rate early in the maintenance period moves towards the consensus level expected later in the period. But whatever the outcome for interest rates, the incentive to speculate disappears

(6) However, in practice there might be non-pecuniary penalties for overdrafts, which would make the two systems not exactly equivalent.
(7) This last assumption is relaxed in the last section, and the resultant speculative behaviour of banks examined.

once the level of official rates for the remainder of the period becomes known: the money-market interest rate remains at the central bank' s dealing rate. So the market interest rate diverges from the prevailing central bank target rate only at the start of the maintenance period, for as long as the future level remains uncertain.

3 The banking system's reserve target

The banking system has to manage its reserves taking into account that it will experience a shock to these reserves in the morning, and another in the afternoon. Both shocks each day are assumed to be uniformly distributed over the range (-2,2).[8] The central bank operates in overnight funds once a day, after the morning shock but before the afternoon one. From Figure 1, and remembering that central bank assistance must be repaid the next day, it can be seen that the market' s reserve balance at each point in time can be written as:

$$\begin{aligned}
RB_1 &= R + B_1 + e_1 \\
RB_2 &= RB_1 + e_2 \\
RB_3 &= RB_2 + B_3 - B_1 + e_3 \\
RB_4 &= RB_3 + e_4 \\
RB_5 &= RB_4 + B_5 - B_3 + e_5 \\
RB_6 &= RB_5 + e_6
\end{aligned} \qquad \textbf{(1)}$$

where R is the market' s reserve balance at the beginning of the maintenance period.

Since the reserve requirement is that the cumulative reserve balance is at least zero, the market fails to meet the requirement if by the close of the last day,

$$CRB_3 \equiv RB_2 + RB_4 + RB_6 < 0 \qquad \textbf{(2)}$$

Alternatively, using **(1)**, this condition can be rewritten in terms of the last shock of the maintenance period (ie the only shock for which the market is unable to correct before the end of the maintenance period) and the sum of

(8) The demand for reserves depends on how reserve shocks are distributed. By assuming a (linear) uniform distribution, we obtain a linear demand for reserves, and more analytically intuitive results.

reserve balances to date (the cumulative reserve balance to date):

$$\begin{aligned}&\left(RB_2 + RB_4 + RB_5 + e_6\right) < 0\\ &-\left(RB_2 + RB_4 + RB_5\right) > e_6\end{aligned} \qquad \textbf{(3)}$$

The market will not satisfy its requirement if the final shock is sufficiently negative to erode a positive cumulative reserve balance, or not sufficiently positive to reverse a negative cumulative balance.

We can substitute out the expressions for reserve balances in **(3)**, using **(1)**, to obtain an alternative expression for the condition under which the market fails to meet its reserve requirement:

$$-\left(3(R + e_1 + e_2) + 2(e_3 + e_4) + e_5 + B_1 + B_3 + B_5\right) > e_6 \qquad \textbf{(4)}$$

An important feature of this expression are the weights. Any shocks that arise on the first day have three times the impact on the cumulative reserve balance of shocks on the final day of the maintenance period. In the same way, the second day' s shocks have twice the impact. This is because shocks are assumed to be independent, so that they do not necessarily cancel each other out. In other words, a shock has a permanent effect on the level of reserve balances.

By the time settlement occurs at the end of the day, the central bank will know the full size of that day' s reserve shocks. On the next day, therefore, the central bank can incorporate the net effect of the previous day' s reserve shocks into its forecast of demand. Thus a net positive shock on the first day would reduce demand over the maintenance period as a whole by an amount equal to three times its net value. But supply on the last two days could similarly rise by three times the net value of any shocks. Thus, although demand will shift, the central bank can ensure that supply does too; aggregate supply over the whole maintenance period equals aggregate demand, at the central bank' s dealing rate. Hence, in the absence of any further reserve shocks, the central bank should be able to maintain the market rate of interest at its target level. So the multiple impact of shocks arising early in the maintenance period should not, in practice, affect future market interest rates. But if overdrafts are not free, a reserve shock will push the current market rate of interest away from the central bank' s target rate, because banks will be unwilling to maintain excessively positive or negative reserve balances.

The market will have an end-of-day overdraft if the final reserve shock on any day within the maintenance period is greater than its reserve balance at noon the same day. Thus, the market will go overdrawn if:

$$\begin{aligned}
-RB_2 &= (RB_1 + e_2) < 0 \quad \text{or} \quad -RB_1 > e_2 \quad \text{on the first day} \\
-RB_4 &= (RB_3 + e_4) < 0 \quad \text{or} \quad -RB_3 > e_4 \quad \text{on the second day} \\
-RB_6 &= (RB_5 + e_6) < 0 \quad \text{or} \quad -RB_5 > e_6 \quad \text{on the third day}
\end{aligned} \tag{5}$$

The size of the market's end-of-day overdraft each day will be:

$$\begin{aligned}
&-RB_1 - e_2 \quad \text{on the first day} \\
&-RB_3 - e_4 \quad \text{on the second day} \\
&-RB_5 - e_6 \quad \text{on the third day}
\end{aligned} \tag{6}$$

4 The market's demand for reserves

At noon each day, the market has to decide how to manage its reserves, via its borrowings, to minimise the cost of meeting the central bank's reserve requirement. We can use conditions **(3)–(6)**, which describe when the market goes overdrawn and when it misses its reserve requirement, to specify the expression for the market's expected costs at the time of the central bank's first operation of the maintenance period:[9]

(9) Following Brunner (1993).

$$E(C_1) = r_1^e B_1 + r_3^e B_3 + r_5^e B_5 + o \int_{-2}^{-RB_1} (-RB_1 - e_2) f(e_2) de_2$$
$$+ o \int_{-2}^{-RB_3} (-RB_3 - e_4) f(e_4) de_4$$
$$+ o \int_{-2}^{-RB_5} (-RB_5 - e_6) f(e_6) de_6 \qquad \textbf{(7)}$$
$$+ p \int_{-2}^{-RB_2 - RB_4 - RB_5} (-RB_5 - RB_4 - RB_2 - e_6) f(e_6) de_6$$

The first three terms of **(7)** represent the cost of borrowing in the central bank's OMO (or, equivalently, the market) on any day (where r_t^e is the central bank's dealing rate expected at time t, and B_t is the quantity borrowed). The next three terms capture the expected cost of going overdrawn, where o is the overdraft rate and the integrals represent the expected size of the overdraft: the probability that the afternoon shock on the first, second or third days is sufficient to erode fully the market's noon reserve balance, multiplied by the size of the overdraft. Since positive end-of-day balances are assumed not to earn interest, charging for overdrafts introduces an asymmetric cost, the effects of which, we see later, are to make borrowing (and hence interest rates) more volatile. The final term in **(7)** is the expected cost of missing the cumulative reserve requirement, where p is the penalty rate and the integral is the amount by which the market expects to miss the requirement: the probability that the final shock of the maintenance period is sufficient to turn the market's cumulative reserve balance negative, multiplied by the extent to which the market is short.

The market's objective is to minimise the cost of managing its reserves, which it can do by altering the pattern of its borrowing from the central bank. Altering its borrowing in this way leads to a simultaneous change of equal size in its reserve balance; that is, borrowing fewer funds means that its reserve balance will be lower by a corresponding amount. However, borrowing slightly less today may mean that the market will have to borrow more later on in the maintenance period, perhaps when it will be more expensive to do so. Therefore, the market has simultaneously to choose all its remaining borrowings to minimise its expected cost given in **(7)**. The first-order conditions are:[10]

$$
\begin{aligned}
r_1 &= o \int_{-2}^{-RB_1} f(e_2)de_2 + p \int_{-2}^{-RB_2-RB_4-RB_5} f(e_6)de_6 \\
r_3^e &= o \int_{-2}^{-RB_3} f(e_4)de_4 + p \int_{-2}^{-RB_2-RB_4-RB_5} f(e_6)de_6 \\
r_5^e &= o \int_{-2}^{-RB_5} f(e_6)de_6 + p \int_{-2}^{-RB_2-RB_4-RB_5} f(e_6)de_6
\end{aligned}
\qquad \textbf{(8)}
$$

Using the assumption that the shocks are uniformly distributed, it is possible to calculate explicit values for the probabilities defined by the integrals. But to solve for the equilibrium of the system, we first need to complete the model's description with how the central bank supplies reserves.

5 The central bank's supply of reserves

The central bank is assumed to target a fixed money-market rate and to supply reserves with the aim of meeting that target. This supply function takes a very general form, allowing for the case where the central bank pegs its interest rate as well as when it targets the supply of the money-base, or indeed where it operates somewhere between the two. We assume that the central bank aims to supply the quantity of reserves that it forecasts the market will require. However, if the central bank makes a mistake in its forecast, so that the market rate of interest moves away from its interest rate target, r^T, it will adjust supply in order to prevent interest rates moving too much;[11] the sensitivity of supply to movements of the market interest rate is measured by the parameter g A value of zero for g means that the market rate of interest is pegged at the central bank's target rate, and a value of infinity indicates that the supply of reserves is fixed at the central bank's forecast. For example, if g is high, even a very large negative spread between the central bank's target rate and the market rate of interest will not persuade the central bank to supply many more reserves. But if by chance demand is equal to the forecast, the market rate of

(10) Using Leibniz's rule.

(11) So for example, if the central bank, in conducting its operations, finds that demand is high, it may not fully scale bids back to the level of the forecast.

interest will exactly equal the central bank's target rate. In addition, we assume that during the whole maintenance period, the central bank maintains both the same target rate and the same sensitivity of supply to changes in the interest rate. The supply of reserves on the first day of the maintenance period is given by:

$$B_1 = B_1^F + \frac{r_1 - r^T}{g}$$

$$r_1 = a_1 + gB_1 = r^T + g(B_1 - B_1^F) \qquad \textbf{(9)}$$

where $a_1 = r^T - gB_1^F$, r^T is the central bank's target rate of interest, and B_t^F is the central bank's forecast of demand at the start of the day.

Supply on the second and third days therefore follows similar rules:

$$\begin{aligned} r_3 &= r^T + g(B_3 - B_3^F) \\ &= a_3 + gB_3 \end{aligned} \qquad \textbf{(10)}$$

$$\begin{aligned} r_5 &= r^T + g(B_5 - B_5^F) \\ &= a_5 + gB_5 \end{aligned} \qquad \textbf{(11)}$$

The expected value of each shock is zero. Thus, as on the first day, the market interest rate on each day of the maintenance period will, on average, equal the central bank's target rate. In practice, though, there will be mistakes in the central bank's forecast of demand, which will feed through to the money-market interest rate. As a result, a spread will open up between the market rate and the central bank's target rate, the size of which will depend on the forecast error, as well as the sensitivity of supply to movements in market rates, g The smaller the parameter g is, the closer the market interest rate will be to the target rate, for any given shock.

At noon of the maintenance period's first day, the market's expectation of how much it will need to borrow on the remaining two days will differ from the central bank's forecast of future demand, because of the reserve shock that morning. However, at the end of the day, the central bank learns the extent of the day's shocks, and can update its forecast of the market's demand for reserves over the whole of the maintenance period. So the market's expectation of its future demand and the central bank's forecast will again be identical. As a consequence, the market should expect the central bank to be able to keep the corresponding market interest rate at its target rate. Algebraically, this is written as:

$$
\begin{aligned}
r_3 &= r^T + g(B_3 - B_3^F) \\
B_3^F &\equiv E_2(B_3)
\end{aligned}
\qquad \textbf{(12)}
$$

where $E_2(B_3)$ is the expectation, at the end of the first day, of the market's demand for reserves at noon the next day; and

$$
\begin{aligned}
E_1(r_3) &= E_1(r^T) + g\left(E_1(B_3) - E_1(B_3^F)\right) \\
E_1(r_3) &= r^T + g\left(E_1(B_3) - E_1(B_3)\right) \\
E_1(r_3) &= r^T
\end{aligned}
\qquad \textbf{(13)}
$$

where $E_1(B_3)$ is the expectation, at noon of the first day, of the market's demand for reserves at noon the next day.

By the same reasoning it can be shown that:

$$E_1(r_5) = r^T \qquad \textbf{(14)}$$

Similarly, on the second day of the maintenance period, the market will expect the interest rate on the final day of the period to equal the central bank's target rate. So throughout the maintenance period, the market expects the central bank to supply reserves in order to keep future interest rates at its target level, whatever the current level of the money-market interest rate.(12)

6 The first day of the maintenance period

First, to simplify the later analysis, we shall define the sum of the end-of-day overdraft rate and the end-of-period penalty rate as the 'cumulative penalty', C. Thus:

---

(12) Because the market knows that the central bank will behave like this, and that its actions today have no effect on the price it must pay tomorrow and *vice versa*, there is no need to use backward induction when solving the model—we can, more intuitively, solve the model forward.

$$C \equiv o + p$$
$$o = C - p \tag{15}$$

We first solve for the end-of-day balances and the end-of-period balance that the market wishes to maintain, as at the very start of the maintenance period, before it experiences any reserve shocks or undertakes any borrowing. So first thing in the morning, at the very start of the maintenance period, the market intends to manage its borrowing to finish each day with the following balances:

$$RB_2 = RB_4^* = RB_6^* = \frac{2C - 4E_0(r_5)}{C + 2p} = \frac{2C - 4r^T}{C + 2p} \tag{16}$$

$$CRB_3^* = RB_2 + RB_4^* + RB_6^* = 3\left[\frac{2C - 4r^T}{C + 2p}\right]$$
$$= 0 \ \ iff\, C = 2r^T \ \text{ ie } \ o + p = 2E(r_5) \tag{17}$$

The market plans to finish each day with the same reserve balance. In addition, the second line of **(17)** indicates that the market aims to finish the maintenance period with a zero cumulative reserve balance only if the sum of the overdraft rate and the end-of-period penalty rate equals twice the last day's market rate of interest.

This is a modification of the rule first highlighted by Poole (1968), where in a one-day maintenance period, a bank would aim to meet the reserve requirement exactly only if the penalty for not doing so was fixed at twice the money-market interest rate. Since banks lend out their reserves at the market rate of interest, the cost of lending out too many reserves is the interest income earned minus the penalty for failing to meet the reserve requirement. Similarly, the cost of holding more than enough reserves is the interest income foregone. In essence, banks must balance these costs, allowing for the likelihood of each occurring. Poole showed that in a world where shocks are symmetrical around zero, the costs are equalised only when the penalty is twice the market rate of interest. If the market is to aim to meet the requirement exactly, it must be indifferent between meeting and missing the reserve requirement. Ignoring overdrafts, it must balance the cost of missing the requirement (the end-of-period penalty weighted by the probability that it will miss) against the cost of acquiring more reserves to meet it, the market rate of interest. Because shocks are symmetrically distributed around zero, if the market aims to meet the requirement exactly, half the time it will miss, so that the expected cost of missing the reserve requirement is half the penalty for doing so. This equals the cost of borrowing more reserves only if the penalty for not doing so is twice the market rate of interest.

Clearly, the higher the cost of missing the reserve requirement, the less frequently the bank will want do so. Therefore, a higher end-of-period penalty induces the market to miss the requirement less frequently, which requires it to hold a larger buffer of reserves to insulate it from most reserve shocks. So a penalty higher than twice the market rate induces the market to aim to finish the maintenance period with a positive level of reserves.

Under averaging, the only modification is that banks must weigh the cumulative cost of missing the requirement (the end-of-period penalty plus the overdraft rate) against the cost of borrowing more reserves. After all, to miss a zero requirement and hence incur a penalty, a bank must go overdrawn at least once. The market will adjust both its borrowing and its planned end-of-day balances in response to reserve shocks, and hence also the cumulative reserve balance it wants to hold at the close of the maintenance period. Therefore, by noon, the market may be aiming for a different pattern of end-of-day balances from at the start of the day. It is easiest to examine how the market manages its borrowings and target balances together by looking at two separate cases: first when overdrafts are free, and second when the central bank charges for end-of-day overdrafts.

First-day borrowing and target balances combined

i) Overdrafts are free

Let us first examine the case where overdrafts are free, but the penalty for missing the requirement is twice the last day's market rate of interest. Since the latter has still to be determined, the market has to plan its behaviour around its expectation that future rates in the maintenance period will equal the target rate. By using **(1)**,**(9)** and **(12)**–**(14)** to solve the first-order conditions **(8)**, we arrive at the following expressions describing the market's behaviour at noon on the first day:

$$RB_2^* = \frac{2p - 4r^T}{3p} + e_1 \quad \textbf{(18)}$$

$$RB_4^* = RB_5^* = \frac{2p - 4r^T}{3p} - \frac{e_1}{2} \quad \textbf{(19)}$$

$$CRB_3^* = 3\left(\frac{2p - 4r^T}{3p}\right) \quad \textbf{(20)}$$

$$B_1 = B_1^F \quad \textbf{(21)}$$

$$B_3^* = B_5^* = B_1^F - e_1 \frac{3}{2} \quad \textbf{(22)}$$

$$r_1 = r^T = E_1(r_3) = E_1(r_5) \quad \textbf{(23)}$$

This behaviour, which is summarised in Table B, is illustrated in Figure 2 for the case in which, earlier in the morning, the market is subjected to a positive shock to its reserves. The shock should reduce the quantity of reserves that the market will need to borrow at noon in order to reach a given cumulative reserve balance by the end of the period. As a result, the market can react in one of two ways: it can either borrow less today, or it can plan to wait, and borrow less on the two remaining days of the period. Borrowing less today will depress the money-market interest rate: we saw above that the central bank supplies funds in such a way that the interest rate falls if demand drops below the forecast.

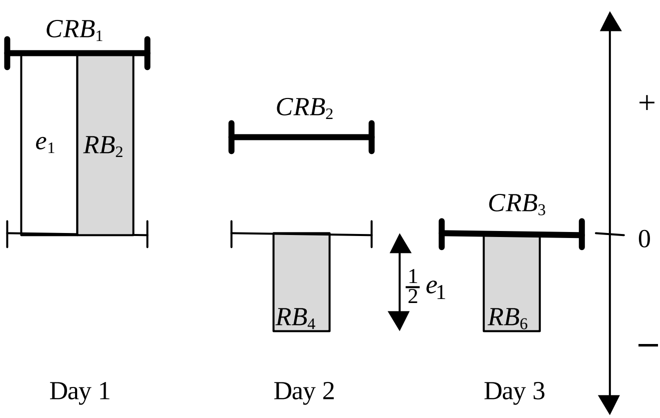
The fall in the overnight interest rate encourages the market to bring forward its demand for reserves, which stops only when the price incentive disappears, ie it is no cheaper to acquire more funds today than during the remainder of the maintenance period. Since future interest rates are expected to remain at the central bank's target rate, this must be when the first day's interest rate has returned to the target rate **(23)**. But given the central bank's supply policy, this also means that the market must borrow exactly the amount that was forecast, as given in **(21)**. According to **(18)**, because the market does not borrow any less on the first day than was forecast, and has benefited from a positive reserve shock, the market's reserve balance will rise by the full value of the reserve shock. This is also illustrated in Figure 2. Assuming that there are no more shocks during the remainder of the maintenance period, this is how the market will finish the first day.

Table B: Borrowing and end-of-day balances

	Free overdrafts	Overdrafts not free
First day's borrowing	B_1^F	$B_1^F - e_1(1-d-b)$
Future borrowing	$B_1^F - e_1 3/2$	$B_1^F - e_1(1+d)$
First day's interest rate	r^T	$r^T - e_1(1-d-b)$
Future interest rate	r^T	r^T
First day's end-of-day balance	e_1	$e_1(d+b)$
Future end-of-day balances	$-\frac{e_1}{2}$	$-e_1 d$
End-of-period cumulative reserve balance	0	$e_1(b-d)$

where $b \geq d$, $a + d \leq 1$.

Figure 2: Reserve balances when overdrafts are free



The market opens the second day carrying more reserves than it needs to, in order to arrive exactly at its desired cumulative reserve balance at the close of the period; so it needs to shed reserves before the end of the period. Given a sufficiently large structural shortage at the start of the maintenance period, this means that the market must borrow less than was originally forecast, by rolling over less assistance than it did the day before, as given in **(22)**. If the structural position of the money market is broadly level, the central bank will have to drain reserves.

In either case, because of the triple impact of a shock on the first day of the maintenance period, the market can afford to reduce its total borrowing by three times the value of the morning's shock, and still finish the maintenance period exactly satisfying the reserve requirement. Since it spreads this reduction equally over the remaining two days of the maintenance period, the market reduces each day's borrowing by one and a half times the value of the morning's shock. The net result of this is that, as **(20)** shows, the market plans to finish the maintenance period with a level cumulative balance, exactly satisfying the reserve requirement, only if the penalty for missing the requirement, p, is set at twice the market rate of interest expected on the last day of the maintenance period (r^T). If the central bank sets the penalty higher than this, the market never fully offsets the value of the reserve shock on the following two days, and plans to finish the period with a cumulative balance in credit.

ii) Charging for overdrafts

The story is different when the central bank charges for overdrafts. Figure 3 illustrates the mechanics, while the expressions describing the market's behaviour, corresponding to **(18)–(23)** are:

$$RB_2 = \frac{2C - 4r^T}{C + 2p} + e_1 \left[\frac{4g(C+p)}{4g(C+p) + (C-p)(C+2p)} \right] \qquad \textbf{(24)}$$

$$RB_4^* = RB_6^* = \frac{2C - 4r^T}{C + 2p} - e_1 \frac{4gp}{4g(C+p) + (C-p)(C+2p)} \qquad \textbf{(25)}$$

$$CRB_3^* = RB_2 + RB_4^* + RB_6^* = 3\left(\frac{2C - 4r^T}{C + 2p}\right) + e_1 \frac{4g(C-p)}{4g(C+p) + (C-p)(C+2p)} \quad \textbf{(26)}$$

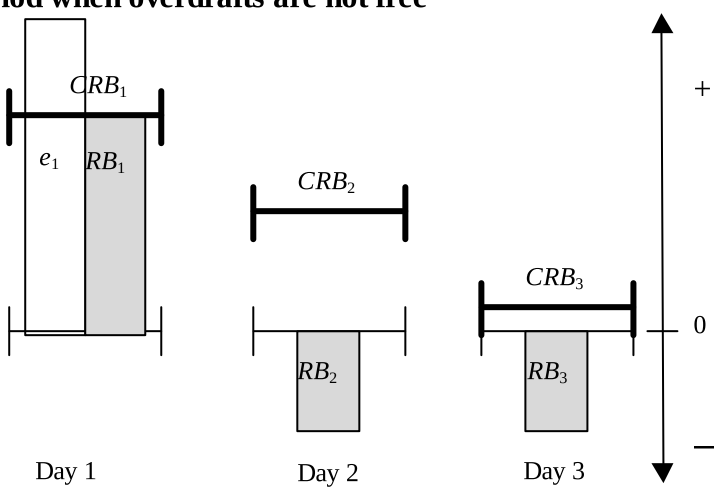
$$B_1 = B_1^F - e_1 \frac{(C-p)(C+2p)}{4g(C+p) + (C-p)(C+2p)} \quad \textbf{(27)}$$

$$B_3^* = B_5^* = B_1^F - e_1 \frac{(4g + C - p)(C+2p)}{4g(C+p) + (C-p)(C+2p)} \quad \textbf{(28)}$$

$$r_1 = r^T - ge_1 \frac{(C-p)(C+2p)}{4g(C+p) + (C-p)(C+2p)} \quad \textbf{(29)}$$

As before, each such shock will push the market's demand away from the central bank's forecast. So according to expression **(27)**, if the reserve shock is positive, the market will borrow less than was forecast, and more than was forecast if the shock is negative. As a result, a wedge is driven between the money-market interest rate and the central bank's target rate, as in **(29)**, the size of which depends on the magnitude of the shock, the penalty for missing the requirement and the interest elasticity of supply.

Figure 3: Reserve balances over the maintenance period when overdrafts are not free



If the market experiences a negative shock to its reserves, it has two choices. First, it can borrow more the next day, when it will be able to borrow the funds at the central bank’ s target rate. The cost of this postponement, though, is having to pay the overdraft rate. Alternatively, the market can borrow all the funds needed on the same day as it suffered the reserve shock, and completely avoid being charged for an overdraft. But the drawback is that in so doing, the market drives up the interest rate, the price it must pay for reserves. Typically, the least costly route is a compromise: borrow some funds on the day of the shock, and borrow some more on the following days. Thus, in **(27)**, we can see that the market never quite offsets its demand by the full value of the shock, even when overdrafts are very expensive: the factor on e_1 is always positive but less than unity. Similarly, if borrowing more today would quickly drive up the interest rate (a large value for g), the market would rather postpone more borrowing until the following day, when the central bank will offer funds at its cheaper target rate.

If we first define the following:

$$d = \frac{4gp}{4g(C+p)+(C-p)(C+2p)} \qquad \textbf{(30)}$$

$$b = \frac{4gC}{4g(C+p)+(C-p)(C+2p)} \qquad \textbf{(31)}$$

where $C = o + p$, so $C > p$;

therefore $b \geq d$, $d + b \leq 1$,

we can simplify **(24)**–**(29)**, as is done in Table B.

Take the example again of a positive reserve shock on the morning of the first day of the maintenance period. Just as before, the market must choose whether to adjust borrowing today or tomorrow. From expression **(27)** and Table B, we see that unlike in the free-overdraft case, the market decides to do part of the adjusting today. Looking at the table, and from **(30)** and **(31)**, the market reduces its borrowing on the first day, below the amount forecast, but by less than the full value of the shock (since $a + b \leq 1$). The market rate of interest falls proportionately. Thus the market does adjust its borrowing on the same day that the shock occurs, but not by enough to offset its effect fully. Consequently, the market must finish the day with a positive balance, as in Figure 3, and in the fifth row of Table B.

As expression **(26)**, describing the cumulative reserve balance with which the market intends to finish the period, and the table make clear, the market plans to hold more reserves than necessary to meet the requirement. Nevertheless, because this balance is less than the first day's closing balance, the market has to shed reserves, by running a negative balance on the remaining two days of the maintenance period, as Figure 3 shows. To achieve this, the market has to borrow far fewer reserves than it did on the first day—see the second row of Table B. In addition, it spreads this borrowing equally over each remaining day.

The reason that the market does not bring more borrowing forward to take advantage of the low interest rate is that doing so would push up today's reserve balance. Although this does not incur a cost immediately, it does imply a cost tomorrow. In order to avoid the opportunity cost of holding too many reserves idle during the maintenance period, the market would have to offset a large positive balance on the first day with large negative positions on the other two days. In other words, it would have to run sizable overdrafts, which would, of course, be expensive under a regime where overdrafts were not free. Consequently, the market is not prepared to finish the first day with as large a positive reserve balance as it would if overdrafts were free.

As indicated in the table, the market still decides to miss the requirement, by finishing the maintenance period with a positive reserve balance. To put it another way, the market decides to bring some demand forward, and to shed reserves afterwards—but not enough to completely reverse the effect of the first morning's reserve shock on the cumulative reserve balance. So in Figure 3, the cumulative reserve balance falls over the maintenance period, after its initial spike, but never quite reaches zero. As **(26)** states, the extent to which the market misses the requirement depends on the size of the overdraft rate ($C - p$). Too expensive a rate, and it is simply not worth running large negative reserve balances, to meet the requirement exactly. After all, in such a world, most of the cost of missing the requirement would be the overdraft rate. The same story applies to the case of a negative shock, except that all the signs are reversed; so, for example, the market would aim not to hold enough reserves to meet the reserve requirement.(13)

---

(13) If the cumulative penalty is not sufficient to induce banks to aim to at least meet the requirement, Figure 3 would simply have to be rescaled; instead of the horizontal axis crossing the vertical at zero, it would do so at the cumulative reserve balances banks would be aiming for.

This reasoning indicates that it is no longer sufficient for the central bank to set the sum of the overdraft rate and the penalty rate for missing the reserve requirement at twice the final day's market rate of interest. To give banks an incentive to meet the reserve requirement exactly, the central bank should make no charge for intra-period overdrafts, and instead fix the penalty for missing the requirement at twice the last day's interest rate. Hence, Poole's 'two-times' rule generalises to a system of reserve requirements with averaging, and with a zero reserve requirement.

7 The second day of the maintenance period

On the second day of the maintenance period, the market has only two days remaining to satisfy its reserve requirement. So if the market experiences a reserve shock, it has only two more opportunities left to use the central bank's operations to help it to arrive at its desired position by the end of the period. Intuitively, therefore, we would expect the reserve shock on the morning of the second day to have more of an impact than the first morning's. In effect, the market begins a new maintenance period, but with only two days, rather than three. So as before, the market must manage its reserves to minimise its expected costs, which are given by an expression similar to **(7)**:

$$
\begin{aligned}
E(C_3) = r_3^e B_3 + r_5^e B_5 + o \int_{-2}^{-RB_3} (-RB_3 - e_4) f(e_4) de_4 \\
+ o \int_{-2}^{-RB_5} (-RB_5 - e_6) f(e_6) de_6 \\
+ p \int_{-2}^{-RB_2 - RB_4 - RB_5} (-RB_5 - RB_4 - RB_2 - e_6) f(e_6) de_6
\end{aligned}
\qquad \textbf{(32)}
$$

Once again, the market's expected costs over the remainder of the maintenance period depend on: the expected cost of borrowing at the central bank's market operations; the expected cost of going overdrawn each day; and the expected cost of missing the reserve requirement. And of course, the market wants to minimise these costs. So in the same way as before, we can obtain a set of expressions similar to **(18)–(23)**, describing how the market plans to allocate its borrowing, and plans its reserve balances for the remainder of the maintenance period:

$$B_3 = B_3^F - e_3 \frac{(C-p)(C+p)}{\left(C^2 - p^2 + 4\,gC\right)} \qquad \textbf{(33)}$$

$$B_5^* = B_3^F - e_3 \left[1 + \frac{4\,gp}{C^2 - p^2 + 4\,gC} \right] \qquad \textbf{(34)}$$

$$\text{where } B_3^F = \frac{\left[2(o+p) - 4a - RB_2(o+2p)\right]}{C + p + 4\,g} \qquad \textbf{(35)}$$

$$r_3 = r^T - ge_3 \frac{(C-p)(C+p)}{\left(C^2 - p^2 + 4\,gC\right)} \qquad \textbf{(36)}$$

$$r_5 = r^T \qquad \textbf{(37)}$$

$$RB_3 = RB_4^* = \left[\frac{2C - 4r^T}{C+p} \right] + e_3 \left[\frac{4\,gC}{4\,gC + C^2 - p^2} \right] - \frac{pRB_2}{C+p} \qquad \textbf{(38)}$$

$$RB_6^* = \left[\frac{2C - 4r^T}{C+p} \right] - e_3 \left[\frac{4\,gp}{4\,gC + C^2 - p^2} \right] - \frac{pRB_2}{C+p} \qquad \textbf{(39)}$$

$$CRB_3^* = \overline{RB}_2 + RB_4^* + RB_6^* = 2\left[\frac{2C - 4r^T}{C+p} \right] + \frac{4\,ge_3(C-p)}{4\,gC + C^2 - p^2} + \frac{(C-p)RB_2}{C+p} \qquad \textbf{(40)}$$

The market behaves in the same way as on the first day of the maintenance period; each expression takes the same general form as in Table B. So for example, if overdrafts are free ($C - p$ is zero), the market demands exactly the amount forecast at the start of the day, planning to reduce demand correspondingly on the next day. Consequently, the money-market interest rate remains pegged at the central bank's target rate for the whole of the maintenance period. But a shock on the second day of the maintenance period has a double impact on the cumulative reserve balance, as we saw in **(4)**; it feeds one-for-one into the daily balance on the day that it occurs, and the following day as well. So for example, in order to meet the reserve requirement exactly, after having experienced a positive shock, the market can lower its borrowing on the following day by twice the value of the shock.

The market's desired closing reserve position follows directly from how it manages its borrowing, the aim being to minimise the costs of holding too few or too many reserves over the period as a whole. But because of the reserve shock the previous afternoon (the first day of the period), it is likely that the market's closing balance turned out either higher, or lower, than the market would have wished. Moreover, the market probably still has not fully made up for the previous morning's shock either; when focusing on the first day of the period, we saw that the market planned to use the whole of the maintenance period to compensate for any shock. So in deciding how large a balance it would wish to hold, not only at the end of this day, but the next as well, the market will take account of the effect of yesterday's shocks on the end-of-day balance, and the latest shock, which transpired earlier that same morning. And as before, banks aim to finish the period with a zero cumulative reserve balance (**(40)** equals zero) only if overdrafts are free and the penalty for missing the requirement is twice the market rate of interest on the last day of the maintenance period.

The market manages its end-of-day balances to reach its desired cumulative position by the end of the period, following the same pattern as the day before. So for example, the second terms of **(33)** and **(34)** indicate that the market would respond to a positive shock the same morning by initially running a higher balance, but compensating the next day by running a lower position. In addition, the market follows the same pattern in continuing its adjustment to the previous day's shocks. So if the previous day's shocks together push the market's balance higher than it had wished, the market will offset this today and tomorrow by holding a lower balance, the last terms in **(38)** and **(39)**. But when we studied how the market managed its daily borrowing, we saw that it would not adjust its borrowing by the full impact of the shock. Thus, although the market will respond to a positive shock, for example, by planning to run first a positive balance, and then a negative reserve balance on the following day, the negative balance is never sufficient to fully offset the positive balance; the market finishes the maintenance period with a positive cumulative reserve balance, exceeding the reserve requirement.

On the last day of the maintenance period, there will not be any further opportunities for the bank to adjust its borrowing in order to work off a shock to its liquidity position. But on the two earlier days in the period, there is still at least one further open market operation through which the bank can work off any shock. With a three-day maintenance period, therefore, it is possible to examine how this work-off rate varies over the maintenance period. The work-off rate is defined to be the proportion of any reserve shortfall or surplus that a bank plans to make up on each of the remaining days of the maintenance period. Therefore, the planned work-off rate after the first morning's shock is the factor on e_1 in equation **(25)**, and the work-off rate after the second morning's shock is the factor on e_3 in equation **(39)**:

$$l_1 = \frac{4\gamma p}{4\gamma(C+p)+(C-p)(C+2p)}$$
$$l_2 = \frac{4\gamma p}{4\gamma C+(C-p)(C+p)} \qquad \textbf{(41)}$$

The difference between the two rates is:

$$l_2 - l_1 = \frac{4\gamma p}{p\left(4\gamma+(C-p)\right)} > 0 \qquad \textbf{(42)}$$

Equation **(42)** states that the work-off rate is higher on the second day of the period than on the first. In other words, the work-off rate rises during the course of the maintenance period, as the bank runs out of time to adjust towards its desired cumulative reserve balance, as in Spindt and Hoffmeister (1988). However, as expressions **(41)** and **(42)** make clear, this work-off rate does not depend only on the time remaining in the maintenance period, but also on the structure of penalties for end-of-day overdrafts and for missing the reserve requirement; the work-off rate falls the higher the overdraft rate is relative to the penalty for missing the reserve requirement at the end of the maintenance period.

8 The last day of the maintenance period

To all intents and purposes, the last day of the maintenance period is equivalent to a one-day maintenance period without averaging. It is true that in a maintenance period longer than one day, the accumulated shocks and past borrowing may mean that demand on the last day is very different from that in a typical one-day maintenance period. But in both cases, the central bank supplies funds based on its forecast of demand, so that if there is no shock that morning, the money-market interest rate remains at its target level.

So the behaviour of the money market will depend on the shock, the sensitivity of the central bank's supply of reserves to changes in the interest rate, and the cost to the market of finishing the day with a negative balance—all of which are the same whatever the length of the maintenance period.

The expression for the market's expected cost at noon of the last day, is again a simplified version of **(7)**:

$$E(C_5) = r_5 B_5 + o \int_{-2}^{-RB_5} (-RB_5 - e_6) f(e_6) de_6 + p \int_{-2}^{-RB_2 - RB_4 - RB_5} (-RB_5 - RB_4 - RB_2 - e_6) f(e_6) de_6 \quad \textbf{(43)}$$

The market minimises its costs, as before, choosing to borrow the quantity of reserves given by:

$$B_5 = B_5^F - e_5 \left[\frac{C}{4g + C} \right] \quad \textbf{(44)}$$

$$r_5 = r^T - e_5 \left[\frac{C}{4g + C} \right] \quad \textbf{(45)}$$

$$RB_6^* = 2 - \frac{4r_5}{C} - \frac{p}{C}(\overline{RB}_2 + \overline{RB}_4) \quad \textbf{(46)}$$

where a bar above indicates a predetermined variable

$$CRB_3^* = \overline{RB}_2 + \overline{RB}_4 + RB_6^* = 2 - \frac{4r_5}{C} + \left(\frac{C-p}{C} \right)(\overline{RB}_2 + \overline{RB}_4) \quad \textbf{(47)}$$

The main feature distinguishing **(44)** and **(46)** from their counterparts from earlier in the maintenance period, **(24)–(28)** and **(33)–(39)**, is that the wedge between the market's demand and the central bank's forecast does not depend upon the cost of going overdrawn. Increasing the cost of an overdraft has an effect only if it raises the cumulative penalty, C; if this is the case, the forecast error will tend to be larger. On the other hand, if a rise in the overdraft rate is matched by a reduction in the end-of-period penalty rate for missing the reserve requirement, so that C is kept constant, the typical forecast error will remain unchanged. Intuitively, on the last day of the maintenance period, the end-of-day overdraft rate and the end-of-period penalty amount to the same thing: a penalty for finishing the day with a negative balance. Thus it should come as no surprise that the split between a charge for overdrafts and for missing the requirement is irrelevant. And therefore, as was argued above, **(44)–(46)** would also describe the market's behaviour in a one-day maintenance period.

The level of reserves the market would wish to hold on the last day will depend on how many reserves it held at the end of each of the two preceding days, and how many it wants to hold over the maintenance period as a whole. The last two terms in **(47)**, the previous two days' closing balances, disappear only if overdrafts are free ($C - p = 0$). So even if the sum of the overdraft rate and the penalty for missing the reserve requirement is equal to twice the market rate of interest, unless overdrafts are free, the market will not aim to finish the maintenance period with a zero cumulative level of reserves. In a one-day maintenance period, earlier end-of-day balances do not directly affect the likelihood of hitting the reserve requirement on any given day. Therefore, the last two terms in **(47)** would automatically drop out; the split between the overdraft rate and penalty rate does not matter, because they amount to the same thing. Finally, assuming that overdrafts are free, we see that **(47)** equals zero only if the penalty for missing the reserve requirement is twice the money-market interest rate on the last day of the maintenance period. If the penalty is instead set at twice the rate on any of the earlier days, then **(47)** may not be zero, depending on the ratio of that day's interest rate to the final day's.

In a maintenance period longer than one day, the cost of missing the requirement is the penalty plus the cost of overdrafts. On the last day of the maintenance period, a negative cumulative reserve balance to date ($RB_2+RB_4<0$) means that the market, in incurring an overdraft, has already paid some of the cost of missing the requirement. So if the sum of the overdraft rate and the penalty rate is twice the market interest rate, the remaining cost must be less than twice the market interest rate. But in a one-day maintenance period, we know that a penalty rate of twice the market rate of interest is necessary to persuade banks to manage their remaining borrowings to hit the reserve requirement exactly, whatever the current level of their reserves. The same will be true in a longer maintenance period. To put it another way, the market chooses the level of reserves with which it finishes the day based on the future cost that it will have to pay for not meeting the requirement, not any costs that it has already paid.

In our example, the sum of the overdraft rate and the penalty rate is twice the market interest rate. But once the market has been subject to a negative shock, the price of doing nothing is less than twice the market rate of interest. As a result, the market would target a cumulative reserve balance below that necessary to meet the zero reserve requirement,[14] just as it would in a one-day maintenance period. Only if overdrafts are free is the cost of missing the zero cumulative requirement twice the market rate of interest both before and after any shocks. So only if overdrafts are free will the market target the same cumulative reserve balance throughout the period. And we know from the one-day maintenance period that this cost—the penalty for missing the requirement—must be twice the market rate of interest for banks to aim to meet the reserve requirement exactly.

9 A comparison of money-market behaviour during the maintenance period

The actual demand for central bank reserves on each day of the maintenance period, restated, is:

$$B_1 = B_1^F - e_1 \frac{(C-p)(C+2p)}{4g(C+p)+(C-p)(C+2p)} \qquad \textbf{(24)}$$

$$B_3 = B_3^F - e_3 \frac{(C-p)(C+p)}{\left(C^2 - p^2 + 4gC\right)} \qquad \textbf{(33)}$$

$$B_5 = B_5^F - e_5 \left[\frac{C}{4g+C}\right] \qquad \textbf{(44)}$$

$$\left(B_5 - B_5^F\right) > \left(B_3 - B_3^F\right) > \left(B_1 - B_1^F\right) \qquad \textbf{(48)}$$
$$\forall e_i = e_j \neq 0,\ \forall C\text{-}p > p$$

(14) Similarly, if the penalty for missing the requirement was set sufficiently high, the market would target a cumulative reserve balance above that necessary to meet the reserve requirement.

$$Var(r_t - r^T) = g^2 (B_t - B_t^F)^2 s^2 \qquad \forall t \tag{49}$$

where $s^2 = Var(e_i)$

From **(46)** and **(47)** we can write:

$$\begin{gathered} Var(r_5 - r^T) > Var(r_3 - r^T) > Var(r_1 - r^T) \\ \forall e_i = e_j \neq 0,\ \forall C\text{-}p > p \end{gathered} \tag{50}$$

This states simply that a shock of a given size tends to push that day's demand further away from the central bank forecast as the maintenance period gets shorter, with the result that the variance of the money-market interest rate rises as well.

The variance of the interest rate on the second day of the period is equal to that on the first only if overdrafts are free; if this is not so, the variance will be higher on the second day. And on the final day of the period, the variance of interest rates will definitely be higher than on any earlier day in the maintenance period, because the market has no time left to work off any reserve shock. Therefore, we can see that the average volatility of market interest rates increases each day of the maintenance period; so that volatility is not higher only on the last day of the maintenance period. So average volatility increases over each day of the maintenance period and, therefore, a longer period results in a lower average volatility of money-market interest rates. Therefore, in the absence of any changes in official interest rates, we can say with certainty that the average volatility of money-market rates over any maintenance period will be less than under any shorter maintenance period. So, for example, a one-day maintenance period can be improved upon by a two-day averaging period, which in turn can be improved upon by a three-day period, and so on. Thus, to minimise the average volatility of money-market interest rates, the central bank should operate as long a maintenance period as possible. However, the longer the maintenance period is, the more unrealistic our assumption becomes that the central bank does not change official interest rates. Hence, speculation may increase about changes in interest rates during the maintenance period, which would be reflected by more volatile policy-adjusted market interest rates. We examine this in the next section.

10 Allowing the central bank to change interest rates within the maintenance period

Up to now, it has been assumed that the central bank can change interest rates only at the start of the maintenance period. In this section of the paper, we relax this assumption, and allow the central bank to change its target interest rate on the second day of the maintenance period.(15) The analysis focuses on the first day of the maintenance period, because by the second day of the maintenance period, the market has learnt whether or not interest rates will be changed. And once it knows for certain the level of interest rates for the remainder of the maintenance period, there will be no further speculation, and the analysis will be as in the earlier part of this paper. Moreover, as before, the central bank can adjust the supply of reserves to meet the market's remaining demand for liquidity exactly, in order to meet the reserve requirement. Therefore, without any reserve shocks, demand on the last two days of the maintenance period should equal the central bank's forecast, and hence market interest rates should remain in line with the central bank's target rate.

The incentive to speculate acts separately from the market's response to any reserve shocks. Therefore, we can assume that all shocks to date have been zero, without affecting the model's results in any way. And we are then able to focus more clearly on the effect of speculation. Of course, at any one time, the market will be uncertain as to the precise timing and magnitude of a change in the central bank's target rate. Nevertheless, there will be a market consensus as to the target rate expected to be set the next day. So, on the first day of the maintenance period, we assume that the market consensus is that the central bank will change rates on the next day by an amount s (which can be positive or negative), which corresponds to the mean expectation of the rate change. So if r_1^T is the central bank's target rate on the first day of the period, the market expects r_2^T to be:

$$E(r_2^T) = E(r_3^T) = r_1^T + s \qquad \textbf{(51)}$$

(15) Allowing the central bank to change interest rates on the final day of the maintenance period is a straightforward extension.

The central bank supplies reserves throughout the maintenance period according to the same schedule as before:

$$
\begin{aligned}
r_1 &= r_1^T + g(B_1 - B_1^F) \\
r_3 &= r_2^T + g(B_3 - B_3^F) \\
r_5 &= r_2^T + g(B_5 - B_5^F)
\end{aligned}
\qquad \textbf{(52)}
$$

So if demand does not differ from the level forecast by the central bank, the market rate of interest that day is pegged at the official target rate. If there is a mistake in the forecast, then, as before, the money-market rate deviates from the central bank's target rate by an amount dependent on the mistake, and the parameter g if g is zero, the central bank operates an interest rate peg, and the central bank always adjusts supply fully, to keep money-market rates at its target rate.

We can again derive expressions for the market's planned demand for reserves, by substituting **(52)** into **(10)** and taking expectations, using (**49**). Doing so, we obtain the following equations:

$$B_1 = B_1^F + s\frac{8p}{\left(4g(C+p)+C^2+pC-2p^2)\right)} \qquad \textbf{(53)}$$

$$B_3 = B_5 = B_1^F - s\frac{4(4g+C)}{\left(4g(C+p)+C^2+pC-2p^2)\right)} \qquad \textbf{(54)}$$

$$r_1 = r_1^T + s\frac{8pg}{\left(4g(C+p)+C^2+pC-2p^2)\right)} \qquad \geq r_1^T \qquad \textbf{(55)}$$

$$E(r_2) = E(r_3) = r_2^T \qquad \textbf{(56)}$$

From **(53)**, we see that demand does differ from the central bank's forecast if banks expect a change in interest rates (s is not zero). For example, if interest rates are expected to rise later in the period (s is positive), the market brings forward more of its expected borrowing, to benefit from cheap rates. From expression **(54)**, we see that the market compensates by reducing demand on the other two days of the maintenance period. The effect of this is to put upward pressure on the first day's money-market interest rates, so that they rise towards the mean level expected the next day, as **(55)** shows. In the expression describing the first day's interest rate, **(55)**, the factor on *s*, the mean change in official interest rates expected by the market, is less than or equal to one. So although the market interest rate moves towards the official rate that the market consensus expects will be set the next day, it may not fully adjust. Full adjustment occurs only if the central bank makes no charge for intra-period overdrafts ($C = p$), in which case, **(53)** and **(55)** become:

$$B_1 = B_1^F + \frac{s}{g} \tag{53a}$$

$$r_1 = r_1^T + s = r_2^T \tag{55a}$$

In our example, the market expects a rise in rates later in the maintenance period. So if overdrafts are free, the market brings its demand for reserves forward until money-market interest rates reach the level expected later in the maintenance period. At that point, the market cannot bring any more demand forward without pushing interest rates above the expected future cost. If overdrafts are not free, however, the market continues to bring forward some demand, but never enough to drive interest rates fully up to the level expected to be set by the central bank the next day.

To take advantage of cheap interest rates early in the maintenance period, the market must bring demand forward, finishing the day with a large positive balance. But the market will then only meet the reserve requirement exactly if it offsets this balance with correspondingly negative balances over the following days. A prohibitively high overdraft rate makes this strategy very expensive, so that banks cannot speculate profitably. As a result, the first day's interest rate remains near the central bank's target rate for that day. In other words, high overdraft rates, as would be expected, discourage speculation in the same way that they shift banks away from averaging their reserve requirement, and encourage them to try to meet the requirement every day.

If the central bank operates an interest rate peg, ($g = 0$), and overdrafts are free, we see from **(53)** and **(54)** that speculation becomes infinite. Again, assuming the market expected a rise in interest rates later in the maintenance period, from **(53)**, it borrows an infinite amount (or the maximum amount it can borrow) at the current low rate, and puts it back to the central bank on the next two days, earning an expected spread of s on the amount with which it was able to speculate, at the central bank's expense. The first day of the period's money-market interest rate, however, remains at the central bank's target interest rate.

11 Summary

In the model developed in this paper, an operational framework containing a zero reserve requirement and averaging was found to stabilise money-market interest rates and, because reserve demand is more predictable, to make the central bank's forecast of demand more accurate. It is not claimed that this system is unique in doing this, but the model does suggest that positive reserve requirements are not needed to confer the supposed benefits of averaging. The model also suggests that as reserve requirements in various countries have steadily been lowered, the availability and cost of overdrafts may have come to restrict the efficient operation of averaging systems. This is an argument for making overdrafts freely available, rather than an argument for positive reserve requirements. The model shows that the central bank can maximise the accuracy of its forecast of demand, and minimise the volatility of money-market interest rates, by allowing banks to obtain intra-period overdrafts without charge.

The benefits of averaging stem from the fact that on all but the last day of the maintenance period, the central bank can always accommodate shifts in demand for reserves at some point in the future: if the market bids up the interest rate, it ends up obtaining funds that it would expect to obtain more cheaply in the future; if it allows rates to soften, it misses an opportunity to fund itself cheaply. By postponing any change in its funding requirements, the market can always avoid disturbing the current interest rate, and hence minimise the expense of its reserve management. Thus, averaging not only smoothes the overnight money-market interest rate, but also reduces banks' costs.

Another feature of the model set out here is that Poole's (1968) well-known 'two times' rule for inducing banks to attempt to meet their reserve requirement exactly was also found to generalise to a system with averaging: the central bank should fix the penalty for missing the requirement at twice the (expectation of the) last day's market rate of interest.

The model also suggests that, contrary to what is sometimes claimed, there is no reason why averaging precludes the central bank from operating on a daily basis, although, of course, it does not have to do so. Even if overdrafts are free, money-market interest rates will become volatile as soon as the central bank has conducted the last operation of the maintenance period in which it can take account of any reserve shocks. Thus, in a system in which the central bank operates daily, volatility spikes only on the last day of the period. If instead, the central bank operated only periodically, with the last operation several days before the end of the period, interest rates would tend to spike on all of these days.

We also extended the model to the case where the central bank can alter its official target rate over the course of the maintenance period, and where the market attaches some probability to just such a move. In that case, the market will tend to bring demand forward if it expects a rise in interest rates, and postpone demand if it expects a reduction in rates, in an attempt to borrow most when interest rates are relatively cheap. Consequently, the market rate of interest early in the maintenance period will move away from the current level of the central bank's dealing rate, and towards the level expected later in the maintenance period. But just as charging for overdrafts prevents banks from averaging their reserve requirement, it also discourages banks from speculating about interest rate changes.

Mathematical appendix

Derivation of the model as at the first day of the maintenance period

The first-order conditions are,

$$\begin{aligned}
r_1 &= o\int_{-2}^{-RB_1} f(e_2)\,de_2 + p\int_{-2}^{-RB_2-RB_4-RB_5} f(e_6)\,de_6 \\
r_3 &= o\int_{-2}^{-RB_3} f(e_4)\,de_4 + p\int_{-2}^{-RB_2-RB_4-RB_5} f(e_6)\,de_6 \qquad \textbf{(i)} \\
r_5 &= o\int_{-2}^{-RB_5} f(e_6)\,de_6 + p\int_{-2}^{-RB_2-RB_4-RB_5} f(e_6)\,de_6
\end{aligned}$$

We can solve these for the market's desired borrowing, by substituting **(1)** from the main text into expression **(i)**, which relates borrowing to desired reserve balances and calculating the integral, given that we know $f(e_i)=1/4$:

$$\begin{aligned}
B_1(o+p) &= 2(o+p) - 4r_1 - (o+3p)(R+e_1) - pB_3 - pB_5 \\
B_3(o+p) &= 2(o+p) - 4r_3 - (o+3p)(R+e_1) - pB_1 - pB_5 \qquad \textbf{(ii)} \\
B_5(o+p) &= 2(o+p) - 4r_5 - (o+3p)(R+e_1) - pB_1 - pB_3
\end{aligned}$$

Supply takes the form:

$$\begin{aligned}
r_1 &= r^T + g(B_1 - B_1^F) \\
r_3 &= r^T + g(B_3 - B_3^F) \qquad \textbf{(iii)} \\
r_5 &= r^T + g(B_5 - B_5^F)
\end{aligned}$$

The central bank's forecast of demand on each day is:

$$
\begin{aligned}
B_1^F &= E_0(B_1) \\
B_3^F &= E_2(B_3) \qquad \textbf{(iv)} \\
B_5^F &= E_4(B_5)
\end{aligned}
$$

Inserting this into supply, we obtain:

$$
\begin{aligned}
r_1 &= r^T + g(B_1 - E_0(B_1)) \\
r_3 &= r^T + g(B_3 - E_2(B_3)) \qquad \textbf{(v)} \\
r_5 &= r^T + g(B_5 - E_4(B_5))
\end{aligned}
$$

Taking expectations as of the morning of the first day of the maintenance period:

$$
\begin{aligned}
r_1 &= r^T + g(B_1 - B_1^F) \\
r_3 &= r^T + g(B_3 - B_3^F) \\
E_1(r_3) &= r^T + g(E_1(B_3) - E_1(B_3^F)) \\
E_1(r_3) &= r^T + g(E_1(B_3) - E_1(E_2(B_3))) \qquad \textbf{(vi)} \\
E_1(r_3) &= r^T + g(E_1(B_3) - E_1(B_3)) \\
E_1(r_3) &= r^T
\end{aligned}
$$

Therefore:

$$r_5 = r^T + g(B_5 - B_5^F)$$
$$E_1(r_5) = r^T + g(E_1(B_5) - E_1(B_5^F))$$
$$E_1(r_5) = r^T + g(E_1(B_5) - E_1(E_4(B_5)))$$
$$E_1(r_5) = r^T + g(E_1(B_5) - E_1(B_5)) \qquad \textbf{(vii)}$$
$$E_1(r_5) = r^T$$

Substituting these expressions into the FOCs **(ii)**, we obtain the set of simultaneous equations:

$$B_1(o+p) = 2(o+p) - 4a - 4gB_1 - (o+3p)(R+e_1) - pB_3 - pB_5$$
$$B_3(o+p) = 2(o+p) - 4r^T - (o+3p)(R+e_1) - pB_1 - pB_5 \qquad \textbf{(viii)}$$
$$B_5(o+p) = 2(o+p) - 4r^T - (o+3p)(R+e_1) - pB_1 - pB_3$$

By symmetry:

$$B_3 = B_5 \qquad \textbf{(ix)}$$

Therefore, substituting out B_5 we obtain:

$$B_1(4g+C) = 2C - (C+2p)(R+e_1) - 2pB_3 - 4a$$
$$B_3(C+p) = 2C - (C+2p)(R+e_1) - 4r^T - pB_1 \qquad \textbf{(x)}$$

where $C = o + p$

Solving these simultaneously for B_1:

$$B_1\left[4\gamma(C+p)+C^2+pC-2p^2)\right]$$
$$=(C-p)(2C-(C+2p)(R+e_1))-4\alpha(C+p)+8pr^T$$
$$B_1=\frac{1}{4\gamma(C+p)+(C-p)(C+2p)} \qquad \textbf{(xi)}$$
$$\left[(C-p)\big(2C-(C+2p)(R+e_1)\big)-4\alpha(C+p)+8pr^T\right]$$

All elements of this except e_1 are forecastable, so that we can write **(xi)** as:

$$B_1=B_1^F-e_1\frac{(C-p)(C+2p)}{4\gamma(C+p)+(C-p)(C+2p)} \qquad \textbf{(xii)}$$
$$=B_1^F-e_1X$$
$$B_1-B_1^F=-e_1X$$

From supply, we know that:

$$r_1=\alpha+\gamma B_1=r^T+\gamma(B_1-B_1^F)$$
$$\alpha+\gamma B_1=r^T-\gamma Xe_1 \qquad \textbf{(xiii)}$$
$$r^T=\alpha+\gamma Xe_1+\gamma B_1$$

Substituting this expression for the central bank’ s target rate into **(xi)** we obtain:

$$
\begin{aligned}
&B_1\left[4g(C+p)+(C-p)(C+2p)\right]= \\
&(C-p)(2C-(C+2p)(R+e_1))-4a(C+p) \qquad \textbf{(xiv)} \\
&+8pa+8pgXe_1+8pgB_1
\end{aligned}
$$

Rearranging for B_1:

$$
\begin{aligned}
B_1 &= \frac{\left[2C-(C+2p)R-4a\right]}{4g+C+2p}+e_1\left[\frac{8pgX-(C+2p)(C-p)}{(C-p)(4g+C+2p)}\right] \\
&= \frac{\left[2C-(C+2p)R-4a\right]}{4g+C+2p} \qquad \textbf{(xv)} \\
&\quad +e_1\frac{(C+2p)}{4g+C+2p}\left[\frac{8pg}{4g(C+p)+(C-p)(C+2p)}-1\right]
\end{aligned}
$$

$$
B_1=\frac{\left[2C-(C+2p)R-4a\right]}{4g+C+2p}-e_1\left[\frac{(C+2p)(C-p)}{4g(C+p)+(C-p)(C+2p)}\right] \qquad \textbf{(xvi)}
$$

confirming **(xii)**. Defining $B_1^F = E_0(B_1)$:

$$
E_0(B_1)\equiv B_1^F=\frac{\left[2C-(C+2p)R-4a\right]}{4g+C+2p} \qquad \textbf{(xvii)}
$$

Substituting in **(xii)** and **(xiii)**:

$$B_1 = \frac{\left[2C-(C+2p)R-4r^T+4gB_1+4gXe_1\right]}{4g+C+2p} - e_1\left[\frac{(C+2p)(C-p)}{4g(C+p)+(C-p)(C+2p)}\right] \qquad \textbf{(xviii)}$$

Taking expectations:

$$E_0(B_1) \equiv B_1^F = \frac{\left[2C-(C+2p)R-4r^T+4gB_1^F\right]}{4g+C+2p}$$

$$B_1^F = \frac{\left[2C-(C+2p)R-4r^T\right]}{C+2p} \qquad \textbf{(xix)}$$

If we let $C = p$, we can see that:

$$B_1 = B_1^F \qquad \textbf{(xx)}$$

We can now substitute the expression for B_1, **(xii)**, into the expression for B_3 in **(x)**:

$$B_3(C+p) = 2C-(C+2p)(R+e_1)-4r^T-pB_1 \qquad \textbf{(xxi)}$$

$$B_3(C+p) = 2C-(C+2p)R-4r^T - pB_1 -(C+2p)e_1$$
$$= (C+p)B_1^F - pB_1 -(C+2p)e_1 \qquad \textbf{(xxii)}$$
$$= (C+p)B_1^F - pB_1^F + e_1(pX-(C+2p))$$

$$B_3 = B_1^F + e_1 \frac{(C+2p)}{(C+p)}\left[\frac{p(C-p)}{4g(C+p)+(C-p)(C+2p)} - 1\right]$$
$$= B_1^F - e_1 \frac{(4g+C-p)(C+2p)}{4g(C+p)+(C-p)(C+2p)} \qquad \textbf{(xxiii)}$$

If $C = p$:

$$B_3 = B_1^F - e_1 \frac{3}{2}$$
$$= B_1 - e_1 \frac{3}{2} \qquad \textbf{(xxiv)}$$

We have seen that:

$$B_1 = B_1^F - e_1 \frac{(C-p)(C+2p)}{4g(C+p)+(C-p)(C+2p)} \qquad \textbf{(xii)}$$

From supply **(iii)**, and from **(xii)** it can be seen that:

$$r_1 = r^T - ge_1 \frac{(C-p)(C+2p)}{4g(C+p)+(C-p)(C+2p)} \qquad \textbf{(xxv)}$$

$$Var(r_1 - r^T) = s^2 g^2 \left(\frac{(C-p)(C+2p)}{4g(C+p)+(C-p)(C+2p)}\right)^2 \qquad \textbf{(xxvi)}$$

$$\left.\frac{dVar(r_1 - r^T)}{dp}\right|_{C=\overline{C}} = 2s^2 g^2 .$$

$$\left(\frac{-\left(8\,g p C + (C-p)(2pC + 4p(C+p)\right)}{\left(4\,g(C+p) + (C-p)(C+2p)\right)^2}\right) \qquad \textbf{(xxvii)}$$

$$\left(\frac{(C-p)(C+2p)}{4\,g(C+p)+(C-p)(C+2p)}\right) < 0$$

If $C = p$:

$$\left.\frac{dVar(r_1 - r^T)}{dp}\right|_{C=\overline{C}} = 0 \qquad \textbf{(xxviii)}$$

Instead of solving **(i)** for borrowing, we can solve for the market's desired reserve balance at the end of each day:

$$\begin{aligned} RB_2(o+p) &= 2(o+p) - 4r_1 - pRB_4 - pRB_6 \\ RB_4(o+p) &= 2(o+p) - 4r_3 - pRB_2 - pRB_6 \\ RB_6(o+p) &= 2(o+p) - 4r_5 - pRB_2 - pRB_4 \end{aligned} \qquad \textbf{(xxix)}$$

Taking expectations, and substituting **(vi)**, **(vii)** and **(xiii)**:

$$\begin{aligned} RB_2(o+p) &= 2(o+p) - 4r^T + 4\,g e_1 X - pRB_4 - pRB_6 \\ RB_4(o+p) &= 2(o+p) - 4r^T - pRB_2 - pRB_6 \\ RB_6(o+p) &= 2(o+p) - 4r^T - pRB_2 - pRB_4 \end{aligned} \qquad \textbf{(xxx)}$$

By symmetry, $RB_4 = RB_5$, which gives us:

$$RB_2 C = 2C - 4r^T + 4gXe_1 - 2pRB_4$$
$$RB_4 (C+p) = 2C - 4r^T - pRB_2 \qquad \textbf{(xxxi)}$$

Thus, solving simultaneously:

$$RB_2 (C-p)(C+2p) = (C-p)(2C-4r^T) + 4gXe_1(C+p)$$
$$RB_2 = \frac{2C-4r^T}{C+2p} + e_1\left[\frac{4g(C+p)}{4g(C+p)+(C-p)(C+2p)}\right] \qquad \textbf{(xxxii)}$$

If $C = p$:

$$RB_2 = \frac{2p-4r^T}{3p} + e_1 \qquad \textbf{(xxxiii)}$$

$$RB_4 (C+p) = 2C - 4r^T - pRB_2$$
$$= 2C - 4r^T - p\frac{2C-4r^T}{C+2p} - pe_1 \frac{4g(C+p)}{4g(C+p)+(C-p)(C+2p)} \qquad \textbf{(xxxiv)}$$

$$RB_4 = \frac{2C-4r^T}{C+2p} - e_1 \frac{4gp}{4g(C+p)+(C-p)(C+2p)} \qquad \textbf{(xxxv)}$$

Thus if $C = 2r^T$, the market will aim for a zero end-of-day balance, and since the cumulative reserve balance is simply the sum of the end-of-day balances, the market will aim to finish the maintenance period with a zero reserve balance, or in other words, to meet the reserve requirement exactly.

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