The non-linear Phillips curve and inflation forecast targeting

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Abstract

This paper extends the Svensson (1997a) inflation forecast targeting framework with a convex Phillips curve. An asymmetric target rule is derived, which implies a higher level of nominal interest rates than the Svensson (1997a) forward-looking version of the reaction function popularised by Taylor (1993). Extending the analysis with uncertainty about the output gap, it is found that uncertainty induces a *further* upward bias in nominal interest rates.

Keywords: inflation targets, non-linearities, asymmetries, stochastic control

JEL Codes: E31, E42, E52, E58

1 Introduction[1]

The 1990s saw the introduction of explicit inflation targets for monetary policy in a number of countries: New Zealand, Canada, the United Kingdom, Sweden, Finland and Spain. Inflation targeting has been introduced as a way of further reducing inflation and to influence market expectations, after disappointment with monetary targeting (New Zealand and Canada) or fixed exchange rates (United Kingdom, Sweden and Finland).

The relation between inflation targets and central bank preferences has been thoroughly investigated. On the one hand there is a theoretical literature (Walsh (1995), Svensson (1997)) that concludes that inflation targets can be used as a way of overcoming credibility problems because they can mimic optimal performance incentive contracts.[2] On the other hand there is an empirical literature that tests whether inflation targets have been instrumental in reducing the policy-implied short-term trend rate of inflation (Leiderman and Svensson (1995)). Broadly speaking, the evidence is that inflation targets have indeed brought about a change in policymakers' inflation preferences.

Unlike the relation between inflation targets and central bank preferences, a relatively underexplored issue is how to translate inflation targets into short-term interest rates. This is the issue of how to map explicit *targets* for monetary policy into monetary policy *instruments*, or how to *implement* an inflation targeting framework. An exception is a recent and important contribution by Svensson (1997a). He shows that — because of lags in the transmission process of short-term interest rates to inflation — inflation targeting implies inflation *forecast* targeting. In his analysis the central bank's forecast becomes an explicit intermediate

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(1) This paper was written while Schaling was an Economist in the Monetary Assessment and Strategy Division of the Bank of England. The author is grateful for helpful comments by Marco Hoeberichts, Alison Stuart, Tony Yates, Andy Haldane, Mike Joyce, Douglas Laxton, Lavan Mahadeva, Peter Westaway, Jagjit Chadha, Paul Tucker, Alistair Milne, Peter Pauly and seminar participants at the Bank of England, the South African Reserve Bank, CentER, RAU, the University of the Witwatersrand and attendants at the third Econometrics Conference at the University of Pretoria. Bruce Devile and Martin Cleaves helped to prepare the paper.
(2) This literature is surveyed in Schaling (1995). Also, by increasing the accountability of monetary policy, inflation targeting may reduce the inflation bias of discretionary policy. See Svensson (1997), and Nolan and Schaling (1996).

target and its optimal reaction function has the same form as the Taylor rule (1993).(3) Recently, Clarida, Gali and Gertler (1997b) have shown that this type of reaction function does quite a good job of characterising monetary policy for the G3. The kind of rule that emerges is what they call 'soft-hearted' inflation targeting. In response to a rise in expected inflation relative to target, each central bank raises nominal interest rates sufficiently to push up real rates, but there is also a modest pure stabilisation component to each rule.

The 1990s have also seen the development of the literature on the so-called non-linear Phillips curve. (Chadha, Masson and Meredith (1992), Laxton, Meredith and Rose (1995), Clark, Laxton and Rose (1995,1996), and Bean (1996).) This recent literature puts the time-honoured inflation output trade-off debate in a fresh perspective by allowing for convexities in the transmission mechanism between the output gap and inflation. More specifically, according to this literature, positive deviations of aggregate demand from potential (the case of an upswing or 'boom') are *more inflationary* than negative deviations (downswings) are *disinflationary*.(4)

This paper marries both strands of the literature. The Svensson (1997a) inflation forecast targeting framework is extended with a convex Phillips curve. Using optimal control techniques, an asymmetric policy rule is derived that implies higher nominal interest rates than the Svensson (1997a) forward-looking version of the reaction function popularised by Taylor (1993). This means that, if the economy is characterised by asymmetries, the Svensson (1997a) linear target rule may underestimate the correct level of interest rates.

The rest of the paper is organised into five sections followed by an Appendix. The model is set out in Section 2. The asymmetric policy rule in the deterministic case is presented in Section 3. In Section 4 we extend the analysis with uncertainty about the output gap. Section 5 compares the implications of multiplicative parameter uncertainty for policy with those of

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(3) For an interesting recent study of the Taylor rule in a UK context, see Stuart (1996).
(4) There is also the view that the Phillips curve is *concave* (Stiglitz (1997)). It can be modelled by changing the sign of j in equation **(2.1)**. Obviously, all policy conclusions are reversed.

the classic Brainard (1967) analysis. Section 6 concludes, and the Appendix provides proofs behind key results.

2 A non-linear Phillips curve

As stated by Laxton *et al* (1995, pages 345-46) the broad acceptance of the expectations-augmented Phillips curve — and the associated 'natural rate' hypothesis — led to the important conclusion that a long-run trade-off between activity and inflation did not exist. Subsequent research on output-inflation linkages has focused on how expectations are formed and the reasons for price 'stickiness' that cause real variables to respond to nominal shocks. Almost all of this work, however, has been predicated on the assumption that the trade-off between activity and inflation is *linear*, that is the response of inflation to a positive gap between actual and potential output is identical to a negative gap of the same size. Though analytically convenient, the linear model ignores much of the historical context underlying the original split between classical and Keynesian economics: under conditions of full employment, inflation appeared to respond strongly to demand conditions, whereas in deep recessions, it was relatively insensitive to changes in activity.(5)

Many of the tests for non-linearity that have been performed have been uninformative because the filters that people have chosen have been fundamentally inconsistent with the existence of convexity. However, when properly tested, there is some evidence for asymmetries. Laxton *et al* (1995) find that by pooling data from the major seven OECD countries the Phillips curve is non-linear. Clark *et al* (1996) — using quarterly data from 1964–90 — find that the US inflation-output trade-off is non-linear. Debelle and Laxton (1997) find that the unemployment-inflation trade-off is non-linear in the United Kingdom, the United States and Canada. Finally, recent research at the Bank of England (Fisher *et al* (1997)) also finds that a Phillips curve that embodies a mild asymmetry is consistent with UK data.

(5) Indeed, as pointed out by Laxton *et al* (1995), the original article by Phillips emphasised such an asymmetry, with excess demand having had a much stronger effect in raising inflation than excess supply had in lowering it.

2.1 Non-linear output inflation dynamics

The main purpose of this section is to combine a convex Phillips curve along the lines of Laxton, Meredith, and Rose (1995) with the Svensson (1997a) model of inflation targeting, to allow for lags in the transmission process of short-term interest rates. We use this model to analyse the effects of delaying monetary policy measures on the future levels of inflation and nominal interest rates.

The functional form we employ to represent the non-linearity in the inflation-output relationship is

$$\Delta p_{t+1} = f(\bullet) = \frac{a_1 y_t}{1 - a_1 j y_t} \qquad \textbf{(2.1)}$$

where p is $p_t - p_{t-1}$, ie the inflation (rate) in year *t*, p_t is the (log) price level, *y* is an endogenous variable output, $a_1 > 0$ and $0 \leq j < 1$ are parameters, and Δ is the backward difference operator. We normalise the natural rate of output *in the absence of uncertainty* to zero.(6) This means that y is the (log) of output relative to potential, ie the output gap. Equation **(2.1)** is graphed in Figure 2.1. Its relevant properties can be derived by looking at the first derivative of $f(\bullet)$ — ie the slope of the output-inflation trade-off:

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(6) With uncertainty, the natural rate of output in the non-linear model will always be *below* that of the linear model. See, for instance, Clark *et al* (1995). The reason is that if output were maintained, on average, equal to the natural rate of the linear model, then the asymmetry in the response of inflation to demand shocks would make it impossible to maintain inflation at a constant inflation target. To see this formally, lead the Phillips curve one period and take expectations at time *t*, which yields $E_t \Delta p_{t+2} = E_t[a_1 y_{t+1} / 1 - a_1 j y_{t+1}]$. In a sustainable equilibrium with a constant rate of inflation equal to the inflation target, $E_t \Delta p_{t+2} = 0$.

Taking account of Jensen's inequality we get $0 = f(E_t y_{t+1}) + j / 2 f''(E_t y_{t+1}) s_e^2$. This equality then (implicitly) defines $E_t y_{t+1}$, the average level of output in the presence of shocks. With the convexity parameter value used in this paper ($j = 0.5$) this level lies about 0.1 percent below the corresponding level of output in the absence of shocks. Since several empirical papers — see for instance Debelle and Laxton (1997) — suggest a larger gap between the stochastic and deterministic equilibrium.

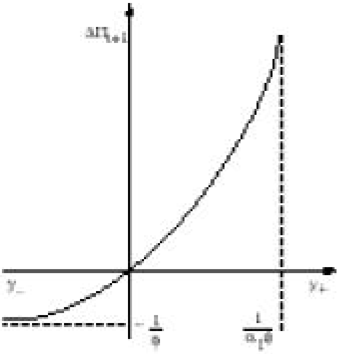
$$f'(\bullet) = \frac{a_1}{[1 - a_1 j y_t]^2} \qquad \textbf{(2.2)}$$

Following Laxton, Meredith and Rose (1995, pages 349-50), it is useful to consider the limiting values of $f(\bullet)$ and its derivative for some specific values of j and y, ie:

$$\lim_{j \to 0} f'(\bullet) = a_1 \qquad \textbf{(2.3a)}$$

$$\lim_{y \to \frac{1}{a_1 j}} f'(\bullet) = \infty, f(\bullet) = \infty \qquad \textbf{(2.3b)}$$

Figure 2.1 The Phillips Curve



$$\lim_{y \to -\infty} f'(\bullet) = 0, \; f(\bullet) = \frac{-1}{j} \qquad \textbf{(2.3c)}$$

$$f'(0) = a_1, \; f(0) = 0 \qquad \textbf{(2.3d)}$$

Equation **(2.3a)** shows that, as the parameter j becomes very small, the Phillips curve approaches a linear relationship, hence (as in Bean (1996)) the parameter j indexes the curvature.

Equation **(2.3b)** indicates that the effect on next year's inflation rises without bound as output approaches $1/a_1 j$. Hence, as in Chadha, Masson and Meredith (1992) — henceforth CMM — $1/a_1 j$ represents an *upper bound* (henceforth y_{max}) beyond which output cannot increase in the short run. Having described the Phillips curve it remains to specify the evolution of output. Following Svensson (1997a, page 1,115), we assume that output is serially correlated, decreasing in the short-term interest rate and increasing in an exogenous demand shock x:

$$y_{t+1} = b_1 y_t - (i_t - p_t) + x_{t+1} \qquad \textbf{(2.4)}$$

where $0 < b_1 < 1$. As can be seen from equations **(2.1)** and **(2.4)**, the real base rate affects output with a one-year lag, and hence inflation with a two-year lag, the control lag in the model.(7) The exogenous variable is also serially correlated and assumed to be subject to a random disturbance, e_{t+1}, not known at time *t*.

(7) With *rational* expectations of inflation in equation **(2.4)**, the following happens: through the Phillips curve **(2.1)** it can be seen that inflation at time $t+1$ depends on the $f(\bullet)$ function. This means that — with model-consistent expectations — expected inflation responds in a non-linear fashion to the output gap as well. More specifically, a positive output gap will increase *expected* inflation by more than a negative gap will reduce it. Of course, this implies that *ex ante* real rates now also respond asymmetrically. This add-on effect will thus reinforce the transmission effects of the asymmetry of the Phillips curve.

$$x_{t+1} = b_2 x_t + e_{t+1} \text{ (8)} \qquad \textbf{(2.5)}$$

2.2 Optimal monetary policy

As in Svensson (1997a), monetary policy is conducted by a central bank with an inflation target p^* (say 2.5% per year). We interpret inflation targeting as implying that the central bank's objective in period *t* is to choose a sequence of current and future interest rates $\{i_t\}_{t=t}$ such that

$$\underset{\{i_t\}}{Min} E_t \sum_{t=t}^{\infty} d^{t-t} \left[\frac{(p_t - p^*)^2}{2} \right] \qquad \textbf{(2.6)}$$

where the discount factor d fulfills $0 < d < 1$ and the expectation is conditional on the central bank's information set, Ω_t, which contains current (predetermined) output and inflation, its forecast of the demand shock and its perception of the asymmetry in the economy j.(9) Thus the central bank wishes to minimise the expected sum of discounted squared future deviations from the inflation target. This is consistent with the United Kingdom's new monetary framework, where the operational target for monetary policy is an underlying inflation rate (measured by the twelve-month increase in the RPI excluding mortgage interest payments) of 2.5%. For simplicity we focus on

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(8) It is not really necessary to specify a distribution as long as it is assumed that this has *finite* support. This is necessary because by inverting the Phillips curve it can be seen that output will hit the constraint if inflation goes to infinity. With inflation targeting (that serves as a natural brake on the expansion of output) and (appropriately specified) finite support of shocks, inflation will always be close enough to the target to prevent output hitting the capacity constraint.

(9) Note that here the central bank is conducting monetary policy from a clear *forward-looking* perspective. This means that — as elegantly stated by Greenspan in his Congressional testimony on 22 February 1995 — 'monetary policy will have a better chance of contributing to meeting the nation's macroeconomic objectives if we look forward as we act, however *indistinct* our view of the road ahead. Thus, over the past year [1994], we have firmed policy to head off inflation pressures not yet evident in the data.' An interesting parallel can be drawn. *If* policy takes account of the curvature, (as an information variable say) inflation will be closer to the target and similarly output will be closer to trend. This means that under optimal policy the observed (reduced-form) Phillips curve will almost certainly be either linear or non-existent. Thus, the more the central bank takes account of possible asymmetric (*ex ante*) inflation risks because of perceived nonlinearities in the inflation output relation, the less visible they will be in the data as a result. This problem has been studied formally by Laxton, Rose and Tambakis (1997).

the inflation objective and abstract from output stabilisation[10] and monitoring issues.[11]

Following Bean (1996), it is convenient to formulate this optimisation problem using dynamic programming. Let $V(p_t)$ be the minimised expected present value in **(2.6)** (the value function). Then:

$$V(p_t) = \underset{\{i_t\}}{Min}\{E_t[\frac{(p_t - p^*)^2}{2} + đE_t[V(p_{t+1})]\} \qquad \textbf{(2.7)}$$

Using **(2.1)** this can be written as

$$V(p_t) = \underset{\{i_t\}}{Min}\{E_t[\frac{(p_t - p^*)^2}{2} + đE_t[V(p_t + f(\bullet))]\} \qquad \textbf{(2.8)}$$

subject to **(2.4)** and **(2.5)**. Note that if $j = 0$ we obtain the Svensson (1997a) model exactly.

Since the interest rate affects inflation with a two-year lag, it is possible to express p_{t+2} in terms of year *t* and *t*+1 variables.
Leading the Phillips curve one period and substituting for output from **(2.1)** yields:

$$p_{t+2} = p_t + \frac{a_1 y_t}{1 - a_1 j y_t} + \frac{a_1 y_{t+1}}{1 - a_1 j y_{t+1}} \qquad \textbf{(2.9)}$$

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(10) Svensson (1997a, pages 1,130-34) shows that the weight on output stabilisation determines how *quickly* the inflation forecast is adjusted towards the inflation target. This is the most realistic case and is also relevant for the UK situation. The reason is that it is recognised by the Chancellor that sticking to the inflation target —in the case of external events or temporary difficulties — may cause undesirable volatility in output. However, in the more complicated case of multiplier uncertainty, Svensson (1997b) also focuses on strict inflation targeting. In order to keep our (already fairly complicated) analysis tractable, we focus on strict inflation targeting. Moreover, this facilitates comparison with the Svensson (1997b) results.
(11) Svensson (1997a, page 1,123) states that: 'Central banks have a strong tradition of secrecy mostly for no good reasons I believe'. For an alternative view where central bank secrecy may be beneficial because of a positive effect on output stabilisation see Eijffinger, Hoeberichts and Schaling (1997).

As in Svensson (1997a), the interest rate in year *t* does not affect the inflation rate in year *t* and *t*+1, only in year *t*+2, *t*+3 etc; similarly the interest rate in year *t*+1 will only affect the inflation rate in year *t*+3, *t*+4 etc. Therefore we can solve the dynamic programming problem by assigning the interest rate in year *t* to the inflation target for year *t*+2, the interest rate in year *t*+1 to the inflation target for year *t*+3 etc. Thus, we can find the optimal interest rate in year *t* as the solution to the simple period-by-period problem:(12)

$$\underset{i_t}{Min}\, E_t\, \delta^2 [\frac{(\pi_{t+2} - \pi^*)^2}{2}] \qquad \textbf{(2.10)}$$

The first-order condition for minimising **(2.10)** with respect to i_t is:

$$\frac{\partial E_t\, \delta^2\, L(\pi_{t+2})}{\partial i_t} = \frac{\delta^2}{2}\, [2(E_t\pi_{t+2} - \pi^*)\, \frac{\partial E_t\pi_{t+2}}{\partial i_t}] = - \qquad \textbf{(2.11)}^{(13)}$$

$$\frac{\delta^2 \alpha_1}{(1 - \alpha_1 \varphi\, [\beta_1 y_t - (i_t - \pi_t) + \beta_2 x_t])^2} (E_t\pi_{t+2} - \pi^*) = 0$$

using (from **(2.9)**) that the effect of interest rate increments on expected inflation two years ahead is

$$\frac{\partial E_t \pi_{t+2}}{\partial i_t} = \frac{\partial E_t \pi_{t+2}}{\partial E_t\, y_{t+1}} \cdot \frac{\partial E_t\, y_{t+1}}{\partial i} = -$$

$$\frac{\alpha_1}{(1 - \alpha_1 \varphi\, [\beta_1 y_t - (i_t - \pi_t) + \beta_2 x_t])^2} = -f'(E_t y_{t+1}) \qquad \textbf{(2.11a)}$$

It follows that the first-order condition can be written as

$$E_t \pi_{t+2} = \pi^* \qquad \textbf{(2.12)}$$

(12) For a proof see Appendix A of Svensson (1997a).

(13) For analytical tractability in this section we do not analyse the implications of uncertainty about the output gap. This makes the analysis fairly complicated, as it implies solving a non-linear stochastic control problem that excludes closed form solutions for interest rates. We analyse this issue in Section 4.

Hence, as in Svensson (1997a, page 1,118), the interest rate in year *t* should be set so that the *inflation forecast* for p_{t+2} , the mean of inflation conditional upon information available in year *t*, equals the inflation target.

The one to two year inflation forecast is given by

$$E_t p_{t+2} = p_t + f(\bullet) + E_t f(\bullet)_{t+1} \qquad \textbf{(2.13)}$$

The last term is the forecast of the inflationary pressure implied by next year's output gap. Using **(2.1)** and **(2.4)** this forecast is

$$E_t f(\bullet)_{t+1} = E_t\left(\frac{a_1[b_1 y_t - r_t + x_{t+1}]}{1 - a_1 j\,[b_1 y_t - r_t + x_{t+1}]}\right)$$
$$= \frac{a_1[b_1 y_t - r_t + b_2 x_t]}{1 - a_1 j\,[b_1 y_t - r_t + b_2 x_t]} = f(E_t\, y_{t+1}) \qquad \textbf{(2.14)}^{(14)}$$

where $r = i - p$ is the real base rate.

Substituting **(2.1)** and **(2.14)** into **(2.13)** and setting the one to two year inflation forecast equal to the inflation target leads to the central bank's *optimal policy rule*:

$$r = \frac{(1 - a_1 j\,[b_1 y_t - r + b_2 x_t])}{a_1}(p_t - p^*)$$
$$+ \frac{(b_1 - 2a_1 j\, b_1 y_t - a_1 j\,[b_2 x_t - r_t])}{[1 - a_1 j\, y_t]} y_t + b_2 x_t \qquad \textbf{(2.15)}$$

where $b_1 = (1 + b_1)$

(14) Because we abstract from the implications of uncertainty about the output gap there is no Jensen's inequality effect in **(2.14)**. This extension is addressed in Section 4.

According to this equation, the optimal short-term interest rate is a *non-linear* function of the deviation from the inflation target $(p - p^*)$ on the one hand, and the output gap (y), on the other. This is in contrast to Bean (1996), who gets a *linear* policy rule. This is owing to the fact that he employs a specific functional form for the non-linear Phillips curve.(15)

An important limiting case of **(2.15)** is when j becomes very small. In the latter case the Phillips curve approaches the standard linear functional form and the policy rule collapses to:

$$r - r^* = a_1(p_t - p^*) + b_1 y_t \qquad \textbf{(2.16)}^{(16)}$$

where $a_1 = \frac{1}{a_1}$, $r^* = b_2 x_t$

which — as in Svensson (1997, page 1,119) — is essentially a *forward-looking* version of the simple backward-looking reaction function popularised by Taylor (1993). In what follows, for brevity's sake **(2.16)** is referred to as the Taylor rule.(17) The non-linear rule **(2.15)** will be analysed in detail in the next section.

3 A non-linear policy rule

In this section the focus is on the properties of the non-linear rule. It is shown that nominal interest rates according to this rule are higher than under the Svensson (1997a) forward-looking version of the Taylor rule. This means that

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(15) In fact his specification is probably the *only* specification that (together with standard quadratic preferences over inflation and output) implies a linear policy rule as the solution to the associated dynamic programming problem.

(16) Note that this solution *does* take account of uncertainty about the output gap. The reason is that, because of *certainty equivalence*, the optimal control trajectory for the stochastic problem is identical with the solution to the deterministic problem when the error terms take their (zero) expected values.

(17) Also, it should be emphasised that the original Taylor rule is an *instrument rule*: it directly specifies the reaction function for the instrument in terms of current information. In contrast a *target rule* results in an endogenous optimal reaction function expressing the instrument as a function of the available relevant information. For this distinction, see Svensson (1997a, page 1,136). We call **(2.16)** forward-looking because — although interest rates feed off current-dated variables only — the latter are *leading indicators* of future inflation. For more details, see Svensson (1997a).

— if the economy is characterised by asymmetries — the Svensson rule may underestimate the correct level of interest rates.

To recap we focus on our initial result, ie equation **(2.15)**.

Rearranging and using that $b_2 x_t = r^*$ we get:

$$r - r^* = \frac{1/a_1}{\{1 - j\,[(p - p^*) + f(\bullet)]\}}(p - p^*) + \frac{1/a_1 f(\bullet)}{\{1 - j\,[(p - p^*) + f(\bullet)]\}} + b_1 y_t$$

Equation **(3.1)** is the central result of this section and shows that the real interest rate penalty $r - r^*$ is a non-linear function of the deviation of the inflation rate from its target $p - p^*$ and the output gap y.

In order to make progress, it is useful to focus on the inflation argument in the rule. So for the moment we set $y = 0$ in **(3.1)**. This yields

$$r - r^* = \frac{1/a_1}{\{1 - j\,(p - p^*)\}}(p - p^*) \qquad \textbf{(3.1a)}$$

The most interesting feature of **(3.1a)** is that the elasticity of the interest rate penalty with respect to deviations from the inflation target is *state-contingent*, meaning that this elasticity depends on the *level of inflation*.

To give a numerical example, consider the effects of a +0.5% and a -0.51% deviation of inflation from target. We analyse the implications of these inflation gaps for short-term interest rates under the following parameter values: $a_1 = 0.5, j = 0.5$ and $r^* = 3.80$. Then the appropriate interest rate penalties are +1.33% and -0.80% respectively. In the linear case (Taylor rule) we get +1.00% and -1.00%. Hence the interest rate response is *asymmetric*; positive deviations from the inflation target imply higher (absolute values of) real interest rate penalties than negative deviations.[18]

The intuition behind this result is the following. If inflation is *above* target, short-term real interest rates will be *below* their equilibrium level. The result of this is that there are inflationary pressures in the economy that — if left to their own devices — will increase tomorrow's output gap. Since the Phillips curve is non-linear, this positive output gap at time $t+1$ will increase the inflation rate at time $t+2$ *by more* than if the world was linear. To offset this, the central bank needs to increase nominal interest rates at time t *further* than in the Svensson model. Of course, in case of a negative deviation from the inflation target, ie when real interest rates are *above* their equilibrium level, the reverse is true. The associated *disinflationary* pressures will depress tomorrow's output gap. However, this will cause *less* disinflation than in the linear case. Hence the central bank does not need to cut rates by as much.

Next we focus on the output gap argument; hence we look at the opposite case to the one analysed above. Setting $p = p^*$ in **(3.1)** yields:

$$r - r^* = \frac{1/a_1 f(\bullet)}{\{1 - j f(\bullet)\}} + b_1 y_t \qquad \textbf{(3.1b)}$$

It can be shown that **(3.1b)** has characteristics similar to **(3.1a)**. In particular, the elasticity of the interest rate with respect to output depends on the *level of the output gap*. To give a numerical example, consider the effects of a +0.50% and -0.50% output gap on the real interest rate penalty. Using the same parameters as in the inflation example and setting $b_1 = 0.7$, we get +1.02% and -0.75% respectively. In the linear case (Taylor rule) we get +0.85% and -0.85% respectively.

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(18) Note that applying Svensson's distinction between 'official' versus *implicit* inflation targets — and for ease of exposition setting $y = 0$ — it is possible to reformulate the non-linear policy rule **(3.1a)** as a *linear* response to a non-linear (state-contingent) implicit inflation target p_t^b. After some algebra it can be shown that **(2.16)** can then be reformulated as

$$r - r^* = a_1(p_t - p_t^b) \text{ where } p_t^b \equiv \frac{p^* - jp(p - p^*)}{1 - j(p - p^*)}.$$

Thus, the interest rate response is also asymmetric with respect to the output gap. Positive output gaps imply higher (absolute values of) real interest rate penalties than negative output gaps.[19] The intuition is as follows. If output is *above* trend at time t, then because of serial correlation in output, tomorrow's output gap will be higher as well. Then, because of the asymmetry, the inflation rate at time $t+2$ will increase *by more* than if there were no asymmetries. In order to prevent this from happening, the central bank needs to put up nominal interest rates by more than suggested by the forward-looking version of the Taylor rule. Similarly, in the case of a negative output gap, the danger of *disinflation* is less severe, calling for a less substantial cut than according to the linear rule.

The above analysis sheds some light on the mechanics of our policy rule **(3.1)**. However, this was done by focusing on the inflation 'gap', given a zero output gap and *vice versa*. In the real world it is not very likely that those are the only relevant cases. So we now drop this restriction and allow both gaps to vary simultaneously. To get a feel for what happens in this case, an illustration is provided in Table 3.1.

(19) Clarida and Gertler (1997a) have found that it is possible to represent Bundesbank policy actions in terms of an interest rate reaction function that maps back into a Taylor-type rule. Their specification allows a modified Taylor rule with *linear* responses to expected inflation and *asymmetric* responses to the output gap.

Table 3.1 Implications of policy rules for short-term interest rates[20]

Inflation minus target	Output gap	Real rate penalty	Idem 'Taylor' rule	Nom interest rate 'Taylor' rule **(2.16)** [21]	Idem non-linear rule **(3.1)**	Idem with unc about output gap **(4.4)**	Interest rate bias in basis points[22]	'Brainard' effect in basis points
-0.50	-0.50	-1.41	-1.85	3.95	4.39	4.80	44 (41)	+ 30
-0.50	0.00	-0.80	-1.00	4.80	5.00	5.34	20 (34)	+ 23
0.00	-0.50	-0.75	-0.85	5.45	5.55	5.80	10 (25)	+ 15
0.00	0.00	0.00	0.00	6.30	6.30	6.50	0 (20)	+ 10
-0.50	0.50	-0.04	-0.15	5.65	5.76	6.02	11 (25)	+ 15
0.50	-0.50	0.30	0.15	6.95	7.10	7.24	15 (14)	+ 5
0.00	0.50	1.02	0.85	7.15	7.32	7.46	17 (14)	+ 5
0.50	0.00	1.33	1.00	7.80	8.13	8.24	33 (11)	+ 3
0.50	0.50	2.94	1.85	8.65	9.74	9.82	109 (8)	+ 1

This table maps output and inflation gaps into real interest rate *penalties* (columns 3 and 4), and into *nominal* interest rates (the shaded columns 5, 6 and 7). Please note that the table is *not* computed by stochastic simulations. All that is necessary to obtain the numbers in the table is to start with certain output and inflation gaps, and plug these into the policy rule **(3.1)** (and **(4.4)** for column 7), given the parameter values used earlier. Also note that our previous numerical examples are reported in rows 8 and 2, (for the inflation example) and rows 7 and 3 (for the output gap example).

Consider first the shaded row. This row corresponds with the case of *neutral monetary conditions*, meaning that the economy is operating at full potential (zero output gap) and inflation on course (equal to the inflation target). Thus both gaps are zero and real interest rates are at their equilibrium level. Note that, in this case, the linear and non-linear policy rules imply the same level of short-term interest rates.

(20) Note that, whereas Taylor prescribes coefficients of one half on both the inflation and output gaps under plausible parameter values, the 'Svensson' rule responds to inflation and output gaps with elasticities of 2 and 1.7 respectively. In this respect see Broadbent (1996), who finds numbers of 5 and 3.5. Also, as pointed out by Svensson (1997a, page 1,133), with a positive weight on output stabilisation, the coefficients in the optimal reaction function — and consequently the numbers in the table — will be smaller.

(21) Nominal interest rate = $r + p = (r - r^*) + r^* + p$, where $p = (p - p^*) + p^*$.

(22) First number = **(3.1) -/- (2.16)**. Bias due to uncertainty = **(4.4) -/- (3.1)** in brackets. The effects of uncertainty will be explained in Section 4.

However, by looking at the other rows in this table it becomes immediately clear that in all other cases, short-term interest rates are always higher under the non-linear rule. To see this, consider the first set of numbers in column 8. The difference in nominal rates is zero for neutral monetary conditions but ranges from about 40 to 100 basis points otherwise. Hence, the numbers suggest that interest rates are higher in a non-linear than in a linear world.

In order to investigate this conjecture formally consider the following equation:

$$r_{NL} - r_L = \frac{1/a_1[(p - p^*) + f(\bullet)]}{1 - j[(p - p^*) + f(\bullet)]} - 1/a_1(p - p^*) - y_t \qquad \textbf{(3.2)}$$

This is the algebraic equivalent of the first set of numbers in column 8 of Table 3.1. It is obtained by subtracting the level of interest rates according to the Taylor rule r_L (given by equation **(2.16)**) from that under the non-linear rule r_{NL} (equation **(3.1)**).

From equation **(3.2)** we conclude that the level of short-term interest rates as implied by the non-linear policy rules is *higher* than under the Taylor rule. For a proof see the appendix, where we show that **(3.2)** has a *local minimum* at $(p - p^*, y) = (0,0)$. Hence, under non-neutral monetary conditions, interest rates according to the non-linear rule are higher than under the Taylor rule.

The intuition is as follows. If the Phillips curve is non-linear, then positive shocks to demand — in the form of positive output and/or inflation gaps — are more dangerous for inflation then if the world is symmetric. This means that the central bank will need to raise rates by more than in the Svensson model. Similarly, negative gaps will be less disinflationary, urging the central bank to cut by less. Of course the net result is that nominal interest rates are higher on average.

Note that there is one interesting intermediate case that we did not investigate.[23] This is the scenario where the model is *non-linear*, but the policy rule remains *linear* (ie of the form given by **(2.16)**). Using stochastic simulations, it can then be shown that interest rates will be higher than under a *linear* Phillips curve with a *linear* policy rule. Moreover, it is then possible to analyse how much further interest rates need to rise under the *optimal* (asymmetric) policy rule compared with the linear rule. The level of interest rates in the non-linear model under the non-linear policy rule can then be decomposed into two parts: (i) the jump in rates caused by the change from a linear to a non-linear model (where the policy rule remains linear), and (ii) the further change in rates (in the non-linear model) caused by the switch from a linear to a non-linear policy rule. The stochastic simulations show that both the effects under (i) and (ii) are positive; the effect under (i) is quantitatively the most important.[24]

4 Uncertainty about the output gap

We analyse the effects of uncertainty about the output gap on the setting of short-term interest rates. This uncertainty takes the form of random shocks to the output gap. This effect is captured in the model by the term e_{t+1} in equation **(2.5)**. Thus, from the perspective of the central bank, the inflation rate becomes a random variable that can only be *imperfectly* controlled.[25] More specifically, because of the non-linearity of the economy, uncertainty about the true value of next year's output gap implies that the *slope* of the Phillips curve — and hence the *effect* of interest rate increments on inflation two years ahead — also becomes a random variable. Hence, the combination of *additive* uncertainty about the economy combined with a non-linear structure gives rise to issues of *multiplier* or *model* uncertainty. However, the implications for optimal policy are quite different here from either the standard Brainard (1967) analysis, or from Svensson's (1997b) extension of his inflation forecast targeting framework with model uncertainty.

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(23) I owe this suggestion to Peter Westaway.
(24) The results are available from the author upon request.
(25) This is also true in the linear stochastic model but there the forecast error does not depend on the interest rate.

We now extend the analysis of Section 3. As stated in Section 2, we can find the optimal interest rate in year *t* as the solution to the problem:

$$\underset{i_t}{Min}\, E_t\, d^2[\frac{(p_{t+2}-p^*)^2}{2}] = \underset{i_t}{Min}\, E_t\, d^2 L(p_{t+2}) \qquad \textbf{(2.10)}$$

subject to **(2.1)**, **(2.4)** and **(2.5)**.

The expected value of the discounted loss can be written as:(26)

$$E_t\, d^2 L(p_{t+2}) = E_t\, d^2[\frac{(p_{t+2}-E_t p_{t+2})+(E_t p_{t+2}-p^*)}{2}]^2 = \qquad \textbf{(4.1a)}$$
$$\frac{d^2}{2}[Var_t p_{t+2} + (E_t p_{t+2} - p^*)^2]$$

and we can define:

$E_t p_{t+2} \equiv \overline{p_{t+2}} + (E_t p_{t+2} - \overline{p_{t+2}}) = \overline{p_{t+2}} + E_t d_{t+2}$, ie the one to two year inflation forecast equals the *deterministic* (or certainty equivalent) inflation forecast $\overline{p_{t+2}} = p_t + f(\bullet) + f(E_t y_{t+1})$

where: $$f(E_t y_{t+1}) = \frac{a_1[b_1 y_t - r_t + b_2 x_t]}{1 - a_1 j\,[b_1 y_t - r_t + b_2 x_t]} \qquad \textbf{(4.1b)}$$

plus the *expected deviation* $E_t d_{t+2}$ of the one to two year inflation forecast from the certainty equivalent forecast:

$$E_t d_{t+2} = E_t p_{t+2} - \overline{p_{t+2}} = (p_t + f(\bullet) + E_t f(\bullet)_{t+1}) - (p_t + f(\bullet) + f(E_t y_{t+1})) \qquad \textbf{(4.1c)}$$
$$= E_t f(\bullet)_{t+1} - f(E_t y_{t+1})$$

This split is important because it will enable us to identify one of the two channels through which the uncertainty affects *inflation forecast targeting*.

---

(26) Using $p_{t+2} = E_t p_{t+2} + (p_{t+2} - E_t p_{t+2})$.

Substituting the decomposition of the one to two year inflation forecast into **(4.1a)** gives:

$$E_t d^2 L(p_{t+2}) = \frac{d^2}{2}[Var_t p_{t+2} + (\overline{p_{t+2}} + E_t d_{t+2})^2 + (p^*)^2 - 2p^*(\overline{p_{t+2}} + E_t d_{t+2})] \qquad \textbf{(4.2)}$$

The advantage of **(4.2)** over **(2.10)** is that the stochastic elements of the solution have been isolated in the terms $E_t d_{t+2}$ and $Var_t p_{t+2}$. It is precisely through these two terms that the uncertainty about the output gap affects inflation forecast targeting.

We will now derive the policy rule in the presence both of asymmetries and uncertainty. Because the rule is highly non-linear, unlike the previous section it is not possible to derive an *explicit* function that maps output and inflation gaps into the appropriate level of interest rates. Instead we resort to numerical methods. However, we are able to derive robust qualitative analytical results. The punchline is that, no matter which parameter values are chosen, nominal interest rates will be *higher* the greater the uncertainty about the output gap.

The first-order condition is:

$$(\overline{p_{t+2}} - p^*) + E_t d_{t+2} + \frac{\P Var_t p_{t+2} / \P i_t}{2[\P \overline{p_{t+2}} / \P i_t + \P E_t d_{t+2} / \P i_t]} = 0 \qquad \textbf{(4.3)}^{(27)}$$

where the first term is the difference between the certainty equivalent inflation forecast **(4.1b)** and the inflation target; the second term is the expected deviation of the one to two year inflation forecast from its certainty equivalent value **(4.1c)**; and the last term captures the effect of

---

(27) Note that in the *deterministic* case $E_t d_{t+2} = \P Var_t p_{t+2} / \P i_t = 0$ and we get $\overline{p_{t+2}} = p^*$, which is the first-order condition in the *certainty equivalence case* as in Svensson (1997a, page 1,118).

nominal interest rates on the conditional variance of inflation, ie on the *variability* or ‘risks’ surrounding the central forecast.

Substituting **(4.1b)** and **(4.1c)** into **(4.3)** and rearranging leads to the central bank’s *optimal policy rule*:

$$
\begin{aligned}
r - r^{*} = {} & \frac{1/a_1}{\{1 - j\,[(p - p^{*}) + f(\bullet) + E_t\, d_{t+2} + \dfrac{\P Var_t\, p_{t+2} / \P i_t}{2[\P\, \overline{p_{t+2}} / \P i_t + \P E_t\, d_{t+2} / \P_{\,t}]}]\}}(p - p^{*}) \\
& + \frac{1/a_1 f(\bullet)}{\{1 - j\,[(p - p^{*}) + f(\bullet) + E_t d_{t+2} + \dfrac{\P Var_t\, p_{t+2} / \P_{\,t}}{2[\P \overline{p_{t+2}} / \P_t + \P E_t\, d_{t+2} / \P_t]}]\}} \qquad \textbf{(4.4)} \\
& + \frac{1/a_1 (E_t d_{t+2} + \dfrac{\P Var_t\, p_{t+2} / \P_t}{2[\P \overline{p_{t+2}} / \P_t + \P E_t\, d_{t+2} / \P_t]})}{\{1 - j\,[(p - p^{*}) + f(\bullet) + E_t d_{t+2} + \dfrac{\P Var_t\, p_{t+2} / \P_{\,t}}{2[\P \overline{p_{t+2}} / \P i_t + \P E_t\, d_{t+2} / \P i_t]}]\}} + b_1 y_t
\end{aligned}
$$

where

$$
\begin{aligned}
& E_t d_{t+2} = (j\ /2) f''(E_t y_{t+1}) s_e^2 > 0 \\
& \P Var_t p_{t+2} / \P i_t = -2 f'(E_t y_{t+1}) f''(E_t y_{t+1}) s_e^2 < 0 \qquad \textbf{(4.5)} \\
& \P \overline{p_{t+2}} / \P i_t = -f'(E_t y_{t+1}) < 0 \\
& \P E_t d_{t+2} / \P i_t = -(j\ /2) f'''(E_t y_{t+1}) s_e^2 < 0
\end{aligned}
$$

According to equation **(4.4)**, the optimal short-term interest rate is determined by the deviation from the inflation target $(p - p^{*})$ on the one hand, and the output gap y (through the terms $b_1 y_t$ and $f(\bullet)$) on the other.

An important limiting case of **(4.4)** is where s_e^2 becomes very small. In this case the stochastic elements of the rule, $E_t d_{t+2}$, $\partial E_t d_{t+2} / \partial i_t$ and $\P Var_t p_{t+2} / \P i_t$ become very small as well, and the policy rule collapses to:

$$r - r^* = \frac{1/a_1}{\{1 - j\,[(p - p^*) + f(\bullet)]\}}(p - p^*) + \frac{1/a_1 f(\bullet)}{\{1 - j\,[(p - p^*) + f(\bullet)]\}} + b_1 y_t \qquad \textbf{(3.1)}$$

which is the asymmetric policy rule for the case where $j > 0$, ie the *certainty equivalent rule* in the non-linear model. Of course, if we set $j = 0$ in **(3.1)**, the asymmetric certainty equivalent rule collapses to the *symmetric certainty equivalent rule* **(2.16)**. This means that it in turn collapses to the Svensson result. Table 4.1 summarises the cases discussed above.

Table 4.1 Classification of policy rules

Phillips curve	Uncertainty about the output gap	
	No uncertainty $s_e^2 = 0$	Uncertainty $s_e^2 > 0$
Linear $j = 0$	Svensson result **(2.16)**	Svensson result **(2.16)**
Non-linear $j > 0$	Non-linear certainty equivalent rule **(3.1)**	Non-linear rule **(4.4)**

Turning to the case where both j and s_e^2 are positive, from equations **(4.4)** and **(4.5)** it can be seen that the stochastic elements of the rule $E_t d_{t+2}$, $\P E_t p_{t+2} / \P i_t$ and $\P Var_t p_{t+2} / \P i_t$ depend on the level of the interest rate. Thus, *both* the left-hand side and the right-hand side of equation **(4.4)** depend on the interest rate. Therefore, it is not possible to derive an *explicit* function that maps output and inflation gaps into the appropriate level of interest rates. Instead, we have to resort to numerical methods to find the level of the real interest rate that is implicitly determined by equation **(4.4)**.

Setting s_e^2 at 0.925[(28)] and keeping the real interest rate at the *certainty equivalent* level according to rule **(3.1)**, we can compute the effect of the uncertainty on the inflation forecast and on the *risks* surrounding the forecast.

We find that the inflation forecast is adjusted upwards. This forecast now *overshoots* the 2.5% target level that would be attained in two years time with interest rates according to **(3.1)** and no uncertainty. Moreover, the same is true for the conditional variance of inflation. At the level of interest rates implied by the certainty equivalent rule **(3.1)**, we get a variance of up to 86% of the variance of the shock to the output gap. This means that only a very small amount of the demand shock is dampened before it passes through and causes significant inflation risks. Clearly, in the presence of uncertainty, interest rates according to **(3.1)** are at a sub-optimal level.

To find the appropriate level we numerically compute the real interest rate that solves the first-order condition. The results can be found in column 7 of Table 3.1. It immediately becomes clear that short-term interest rates according to rule **(4.4)** are *higher* than under the certainty equivalent non-linear rule **(3.1)**. To see this, consider the numbers in brackets in column 9. The difference owing to the uncertainty is about 25 basis points for neutral monetary conditions and ranges from about 10 to 40 basis points otherwise.[(29)] This means that uncertainty induces a

---

(28) This is the MSE of ONS revisions to real GDP in the late 1980s. For more details see Dicks (1997). Obviously this is a crude way of parameterising the model, but in the linear case there is a one-to-one correspondence between the conditional variance of the output gap at time *t* and the variance of shocks s_e^2 . Also, this highlights another attractive feature of the model. We have a natural mapping of noisy data (which is very much a real-life problem) into issues of multiplier uncertainty.

(29) Note that, strictly speaking, the definition of neutral monetary conditions needs to be changed in the non-linear model. The reason is that the natural rate of output now lies below the natural rate of output in the linear model. With the parameter values in the paper this difference amounts to about -0.1% of GDP. Therefore neutral monetary conditions now mean inflation at target and output at the adjusted natural rate. Indeed it can be shown that with inflation on target and output at -0.13 the interest rate bias disappears and the appropriate level of the real interest rate (as defined by the policy rule **(4.4)**) is equal to $r^* = 3.8$.

further upward bias in nominal interest rates on top of the effect of the non-linearity *per se* as analysed in Section 3.

In order to investigate these results more formally consider **(4.4)**. In this equation the stochastic elements of the solution have been isolated in the terms $E_t\delta_{t+2}$, $\partial E_t\delta_{t+2} / \partial i_t$ and $\partial Var_t\pi_{t+2} / \partial i_t$. The sign of $E_t\delta_{t+2}$ in **(4.1c)** will always be positive, implying that the one to two year inflation forecast will be *higher* than the certainty equivalent inflation forecast as derived in Section 3. The reason is that positive shocks to the output gap are *more inflationary* then negative shocks are *disinflationary*, hence with equal probabilities of positive and negative shocks, the inflation forecast will be adjusted upwards, and the more so the higher the variance of shocks hitting the output gap σ_ε^2.

This can be restated in a more technical way by noting that the forecast of tomorrow's inflationary pressure, $E_t f(\bullet)_{t+1}$, involves the expectation of a convex function which will always be higher then the value of the f function at the expected value, $f(E_t y_{t+1})$. Hence, the first channel through which the uncertainty affects inflation forecast targeting is the *Jensen's inequality effect*. Note that from **(4.5)** this effect becomes smaller the higher the interest rate, ie $\partial E_t\pi_{t+2} / \partial i_t < 0$.

The *second* channel through which the uncertainty affects inflation forecast targeting is its effects on the *conditional variance* of the one to two year inflation forecast $Var_t\pi_{t+2}$. This is important because it implies that in the case of imperfect control of the inflation rate the policymaker should also take account of the *risks* surrounding the central inflation projection. It can be shown that this variance is:

$$Var_t\pi_{t+2} = [f'(E_t y_{t+1})]^2 \sigma_\varepsilon^2 \qquad \textbf{(4.6)}$$

From **(4.5)** it can be seen that by increasing interest rates this variance can be reduced. The reason is that by putting up rates, today's forecast of tomorrow's output gap goes down. This means that next year's Phillips curve will be *flatter,* which in turn implies that the effects of demand shocks at time t+1 on inflation in two year's time will be smaller. Hence, the variability of inflation around the central projection can be reduced by increasing short-term interest rates. For instance, returning to our earlier numerical example, by putting up rates to their appropriate level, the conditional variance of inflation is reduced from 86% to about 51% of the initial variance of demand shocks.

The implication for policy is that with uncertainty about the output gap (and asymmetries in the output inflation trade-off), cautious policymaking implies a more activist (more aggressive) rather than a less activist (more passive) interest rate policy.

To recap, the intuition is that a higher variance of shocks hitting the output gap implies a higher inflation forecast (through Jensen's inequality effect) and a higher conditional variance of inflation. Both can be reduced by increasing nominal interest rates above their certainty equivalent level.

To see the benefits of this policy from a different perspective, consider the implications of stabilisation for the *level* of output. With a convex Phillips curve, the mean level of output is inversely related to the *variability* of inflation around the central projection. Therefore, a monetary strategy that reduce this variability (by responding correctly to the multiplier uncertainty issue) does not only keep the inflation rate closer to the target, but also has the important added bonus of pushing up the level of output.(30)

5 Brainard uncertainty and non-linearities

Note that the results with respect to the conditional variance of inflation are the opposite of those assumed in Brainard's (1967) multiplier

(30) I owe this insight to Clark *et al* (1995, page 8). They in turn quote Mankiw (1988, page 483). The result can be verified by inverting the Phillips curve **(2.1)**. This yields

$y_t = \dfrac{\Delta p_{t+1}}{a_1(1 + j\Delta p_{t+1})}$. Leading this equation one period and taking expectations at time *t* of

the resulting *concave* function yields the result that expected output, $E_t y_{t+1}$, is inversely

related to the conditional variance of inflation $Var_t p_{t+2}$.

uncertainty analysis.[31] The reason is that in Brainard's analysis the variance of the target variable is a linear function of the variance of the policy multiplier. As a result, uncertainty about the effects of policy calls for a *less activist* policy. Moreover, the policy multiplier is positively related to the level of the instrument. It follows that policies that are 'too activist' increase the variance of the target variable, thereby reducing the success of stabilisation policy. In this section we show that, in the non-linear stochastic model, uncertainty about the effects of policy does not make the monetary authorities less activist in the Brainard sense. This is because the model has the property that the variance of the target is inversely related to the instrument, and it thus provides a counter-example to the Brainard case.

In his (1967) paper, Brainard identified two types of uncertainty that a policymaker may face. First, at the time he must make a policy decision he is uncertain about the impact of the exogenous variables that affect the target variable. This may reflect the policymaker's inability to forecast perfectly either the value of exogenous variables or the response of the target variable to them. Second, the policymaker is uncertain about the response of the target variable to any given policy action. He may have a central estimate of the expected value of the response coefficient, but he is aware that the actual response of the target variable to policy action may differ substantially from the expected value.

Let us now rephrase the above in the context of inflation forecast targeting. To make things comparable with Brainard, for the moment we focus on the *linear* version ($j = 0$) of the stochastic model presented earlier. Type 1 uncertainty means that when the central bank sets its instrument variable, the nominal interest rate at time t, it is uncertain about the realisation of the exogenous shock to the output gap at time $t+1$. Here the central bank's inability to forecast next year's output gap perfectly implies that it is also unable to forecast inflation perfectly. As a consequence, inflation in two years time will differ from its forecast at time t (which is the basis for its interest rate policy). More specifically, if

(31) Throughout the paper if we refer to the Brainard result, we mean Brainard's result for the one instrument and one target case where the random response coefficient is uncorrelated with the exogenous disturbances.

the output gap is higher than expected, inflation overshoots its target and *vice versa*.

The second type of uncertainty means that the central bank may have a central estimate of the expected value of the response coefficient of inflation in two years time with respect to the nominal interest rate at time t, but that it is aware that this central estimate is subject to error. More specifically, assume that the central estimate is $-a_1$ — being the product of the interest elasticity of output (which is -1) and the slope of the Phillips curve (which is a_1) in the linear stochastic model — and that the variance of this central estimate is s_t^2.

Brainard shows that both types of uncertainty imply that the policymaker cannot guarantee that the target variable will assume its target value. But they have quite different implications for policy action. The first type of uncertainty, if present by itself, has nothing to do with the actions of the policymaker; it is, as Brainard (1967, page 413) describes it, 'in the system' *independent* of any action he takes. He then states that if all of the uncertainties are of this type, optimal policy behaviour is *certainty equivalence* behaviour. That is, the policymaker should act on the basis of expected values as if he were certain that they would actually occur. Moreover, since in this case the variance and higher moments of the distribution of the target variable do not depend on the policy action taken, the policymaker's actions only shift the location of the target variable's distribution. In the presence of the second type of uncertainty, however, the shape as well as the location of the distribution of the target variable depends on the policy action. In this case, the policymaker should take into account his influence on the *variability* of the target variable. In his analysis(32) Brainard assumes that the variance of the target variable is a linear and increasing function of the level of the policy instrument. It follows that policies that are 'too activist' increase the variance of the target variable, thereby worsening the performance of economic policy. Brainard thus shows that uncertainty about the

(32) Here we focus on Brainard's most simple case; ie the one instrument and one target case where the random response coefficient is uncorrelated with the exogenous disturbances. The reason for doing this is that this case has the closest correspondence to inflation forecast targeting. There we also have one target, inflation, and one instrument, the nominal interest rate.

response coefficient, ie about the policy *multiplier*, leads to an optimal policy that is less active. As the variance of the multiplier rises, the policy of trying to minimise the variance of the target variable tends towards lowering the optimal amount of policy.

Let us now rephrase the above in the context of inflation forecast targeting. An example of *certainty equivalence behaviour* is the Svensson **(2.16)** forward-looking policy rule. This rule is optimal in the *linear* stochastic model. Because shocks to the output gap have a zero expected value at time t, it is optimal for the central bank to act as if these zero values would actually occur.

An example of *uncertainty about response coefficients* is Svensson’s (1997b) extension of his inflation forecast targeting framework with multiplier uncertainty. Indeed, he finds that multiplier uncertainty calls for a more gradual adjustment of the conditional inflation forecast toward the inflation target. This means that — similar to Brainard — optimal monetary policy will be less activist in the sense that the response coefficients in the optimal policy rule for short-term interest rates decline with the uncertainty.(33)

Let us now focus on inflation forecast targeting in the *non-linear* model and relate the effects of uncertainty about the output gap to the Brainard paper. Here — following Brainard’s terminology — it appears that we

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(33) The above can be derived by resorting to the linear model (setting j equal to zero) and modifying equation **(2.4)** as $y_{t+1} = b_1 y_t - t_t(i_t - p_t) + x_{t+1}$ with $E_t(t_1) = 1$

$E_t(t_1^2) = s_t^2$ and $E_t(t_1 e) = 0$ **(2.4')**

This means that the effects of interest rate changes on tomorrow’s output gap are now uncertain because the interest elasticity of output is a random variable. If $s_t^2 \rightarrow 0$ the central estimate is not subject to error and the equation reduces to **(2.4)**. It can be shown that

$Var_t p_{t+2} = a_1^2 (r^2 s_t^2 + s_e^2)$ so that $¶Var_t p_{t+2} / ¶_t > 0$ and we obtain the standard Brainard (1967) result. It can be shown that the optimal (linear) policy rule then becomes:

$$r = \frac{r^*}{(1+s_t^2)} + \frac{a_1}{(1+s_t^2)}(p_t - p^*) + \frac{b_1}{(1+s_t^2)} y_t \quad \textbf{(2.16')}$$

So, as in Svensson (1997b), the response coefficients decline with uncertainty, calling for more cautious policymaking. If $s_t^2 \rightarrow 0$ this rule reduces to **(2.16)**.

only have type 1 uncertainty. That is, because of an *additive* (white noise) shock to tomorrow's output gap, the central bank is unable to forecast inflation perfectly. If the model were linear, certainty equivalence would hold and that would be the end of the story. However in a non-linear model this uncertainty has very different implications.

Similar to the linear model the uncertainty enters the story through additive shocks to the output gap at time $t+1$. Suppose now that the output gap is higher than expected. In the non-linear model the slope of the Phillips curve, $\partial \Delta p_{t+1} / \partial y_t$, depends on the level of the output gap. Because the Phillips curve is convex, its *slope* is increasing in the *level* of the output gap (see equation **(2.1)** and Figure 2.1). Thus, if the output gap turns out to be higher than expected (because of a positive shock), the slope of the Phillips curve is also higher than expected. Similarly, if we have a negative shock the slope of the Phillips curve will be lower than expected.

Interestingly, the above implies that the central bank becomes uncertain about the response of inflation to any given policy action. This response coefficient is equal to the product of the interest elasticity of output (which is -1) and the slope of the Phillips curve (which now depends on the realisation of the additive shock to output).(34) Thus if the slope of the Phillips curve is *higher* than expected (because of a positive realisation of the demand shock), the response coefficient of inflation in two years' time with respect to the nominal interest rate at time t is *lower* than expected. Because the response coefficient is negative (increasing the nominal interest rate reduces inflation), in this case monetary policy turns out to be *more effective* than expected. Similarly, if the slope of the Phillips curve is lower than expected the response coefficient is higher (less negative) than expected. In this case monetary policy is less effective than expected. To conclude, in the non-linear model *additive* shocks to the output gap generate uncertainty about the policy *multiplier*; ie type 1 uncertainty has type 2 implications.

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(34) This can be seen by adapting **(2.11a)**. The algebraic expression for the response coefficient of inflation in two year's time with respect to the nominal interest rate at time t is:

$$\frac{\partial p_{t+2}}{\partial i_t} = \frac{\partial y_{t+1}}{\partial i_t} . \frac{\partial p_{t+2}}{\partial y_{t+1}} = -1 . f'(y_{t+1}) = -\frac{a_1}{(1 - a_1 j\,[b_1 y_t - (i_t - p_t) + b_2 x_t + e_{t+1}])^2} .$$

From the previous paragraph we learnt that the slope of the Phillips curve depends on the realisation of the shock to the output gap. With a positive realisation monetary policy was shown to be more effective than expected at time t and *vice versa*. Meaning that the dampening effect of a *given* nominal interest rate at time t on inflation in two years time is proportional to the realisation of the shock. However, in this paper we are concerned with *optimal* policy and it is, therefore, of some interest to relax the assumption of a given nominal interest rate.

Suppose the central bank decides to increase the nominal interest rate. From equation **(2.4)** it follows that a higher nominal interest rate — *ceteris paribus* — lowers the level of tomorrow's output gap. Moreover, in the non-linear model the *slope* of the Phillips curve is increasing in the level of the output gap. Thus, a higher nominal interest rate lowers the slope of the Phillips curve. This in turn implies that any positive output shock that may hit the economy at time $t+1$ will be less inflationary. Similarly, by lowering the slope of the Phillips curve, a higher interest rate will also dampen the disinflationary effects of negative shocks. Thus, in the non-linear model a higher nominal interest rate causes positive demand shocks to induce *less inflation* and negative shocks to cause *less disinflation*. Of course, if the central bank decides to cut the nominal interest rate, the reverse applies. By increasing the slope of the Phillips curve, a lower interest rate amplifies the inflationary effects of positive output shocks and enhances the disinflationary effects of negative shocks. Thus, nominal interest rates can dampen or amplify the second-round effects of output shocks on inflation.

To be more precise, it can be shown that the *conditional variance* of the one to two year inflation forecast, $Var_t p_{t+2}$, is a *decreasing* function of the nominal interest rate. This can be seen from equations **(4.5)** and **(4.6)**. As explained above, the reason is that by putting up rates, today's forecast of tomorrow's output gap goes down. This means that next year's Phillips curve will be *flatter*, which in turn implies that the effects of demand shocks at time t+1 on inflation in two year's time will be smaller. Hence the variability of inflation around the central projection can be reduced by increasing short-term interest rates.

At this stage it is useful to summarise the results so far. First, we have shown that in the non-linear model *additive* shocks to the output gap imply *uncertainty about policy*; ie type 1 uncertainty has type 2 implications. Second, in the non-linear model the variance of the target variable (inflation) is a *decreasing* function of the level of the policy instrument (the nominal interest rate). Note that the second result is the *opposite* of the assumption in Brainard's (1967) analysis that the variance of the target variable is a linear and *increasing* function of the level of the policy instrument. It follows that policies that are 'too activist' increase the variance of the target variable, thereby worsening the performance of economic policy.

In contrast here the variance of the target variable is a non-linear and *decreasing* function of the level of the policy instrument (this can be seen from equations **(4.5)** and **(4.6)**). It follows that policies that are 'too activist' from a Brainard perspective may actually *decrease* the variance of the target variable, thereby *improving* the performance of policy. Thus, in the non-linear model uncertainty about the policy multiplier leads to an optimal policy that is more active. To be more precise, as the variance of the multiplier rises, the policy of trying to minimise the variance of the target variable tends towards *increasing* the optimal amount of policy, which here means a higher level of interest rates.

To see this, we focus on the central bank's optimal policy rule **(4.4)**. As stated above, the stochastic elements of this rule are isolated in the terms $E_t d_{t+2}$, $\P E_t p_{t+2} / \P i_t$ and $\P Var_t p_{t+2} / \P i_t$. Here the first two terms relate to the effects of the uncertainty on the inflation forecast, and hence capture the *Jensen's inequality effect*. The second channel through which the uncertainty affects inflation forecast targeting is through its effects on the conditional variance of the one to two year inflation forecast $Var_t p_{t+2}$.

We can isolate the implications of the second channel for the amount of optimal policy by abstracting from the Jensen's inequality effect. This can be done by setting $E_t d_{t+2}$ and $\partial E_t p_{t+2} / \partial i_t$ equal to zero in the central bank's optimal rule **(4.4)**. This yields

$$r - r^{*} = \frac{1/a_1}{\{1 - j\,[(p - p^{*}) + f(\bullet) + \frac{\P Var_t p_{t+2} / \P i_t}{2[\P \overline{p}_{t+2} / \P i_t]}]\}}(p - p^{*})$$

$$+ \frac{1/a_1 f(\bullet)}{\{1 - j\,[(p - p^{*}) + f(\bullet) + \frac{\P Var_t p_{t+2} / \P i_t}{2[\P \overline{p}_{t+2} / \P i_t]}]\}} \qquad \textbf{(5.1)}$$

$$+ \frac{1/a_1 \left(\frac{\P Var_t p_{t+2} / \P i_t}{2[\P \overline{p}_{t+2} / \P i_t]}\right)}{\{1 - j\,[(p - p^{*}) + f(\bullet) + \frac{\P Var_t p_{t+2} / \P i_t}{2[\P \overline{p}_{t+2} / \P i_t]}]\}} + b_1 y_t$$

Equation **(5.1)** implicitly defines the optimal level of the nominal interest rate in the non-linear stochastic model, where the uncertainty is only allowed to affect the variance of the target variable. This is as close as we can get to the linear-quadratic Brainard framework. Since both the left-hand side and the right-hand side depend on the nominal interest rate, again we have to resort to numerical methods to find the optimal level of the central bank's policy instrument. The results can be found in column 9 of Table 3.1. This column gives the *difference* between the level of nominal rates implied by the rule **(5.1)** and the non-linear certainty equivalent rule **(3.1)**. As can be seen from the numbers in the table the difference is *positive*, implying that in the non-linear model, uncertainty about policy calls for a *higher* rather than a *lower* optimal amount of policy.

6 Summary and concluding remarks

In this paper we extended the Svensson (1997a) inflation forecast targeting framework with a convex Phillips curve. Using optimal control techniques we derived an asymmetric policy rule. We found that nominal interest rates according to this rule were higher than under the Svensson forward-looking version of the Taylor rule.

Extending the analysis with uncertainty about the output gap we found that our earlier results became even stronger. We found that the uncertainty induced a *further* upward bias in nominal interest rates on top of the effect of the non-linearity *per se*. Also we found that the implications of uncertainty for optimal policy are quite different from either the standard Brainard (1967) analysis, or Svensson's (1997b) extension of his inflation forecast targeting framework with model uncertainty. More specifically, we find that the variability of inflation around the central projection can be reduced by increasing short-term interest rates. The implication for policy is that with uncertainty about the output gap (and asymmetries in the output inflation trade-off), cautious policymaking implies a higher interest rate on average.

The analysis can be extended in a number of ways. One is to investigate robustness of results with respect to alternative assumptions about inflation expectations. It would be interesting to see whether the same results are obtained with purely model consistent expectations, or a backward and forward-looking components model, or a multiple-regime model with credibility and learning.[35]

Another is to extend the objective function of the authorities to include an intrinsic weight on output stabilisation. Results can then be contrasted with pure inflation targeting. We leave those issues for further research.

(35) For an interesting analysis that builds on a trade-off between caution and learning (by experimentation) in policy, see Wieland (1998).

Appendix: the minimum of equation (3.2)

In this appendix we prove that the interest rate differential **(3.2)** has a local minimum at $(p - p^{*}, y) = (0{,}0)$.
The partial derivatives are given by

$$\frac{\P(r_{NL} - r_{L})}{\P(p - p^{*})} = \frac{1}{a_1}\left[\frac{1}{(1 - j\Gamma)^2} - 1\right] \qquad \textbf{(A.1a)}$$

$$\frac{\P(r_{NL} - r_{L})}{\P y} = \frac{1/a_1 f'(\bullet)}{(1 - j\Gamma)^2} - 1 \qquad \textbf{(A.1b)}$$

where $\Gamma \equiv (p - p^{*}) + f(\bullet) < \frac{1}{j}$

Hence, it can easily be seen that if $p = p^{*}$ and $y = 0$, $f(\bullet) = \Gamma = 0$ and $f'(\bullet) = a_1$ so that **(A.1a)** = **(A.1b)** = 0, and $(p - p^{*}, y) = (0{,}0)$ is a *stationary point*.

The second derivatives and the cross partials are

$$\frac{\P^2(r_{NL} - r_{L})}{\P(p - p^{*})^2} = \frac{2j}{a_1(1 - j\Gamma)^3} \qquad \textbf{(A.2a)}$$

$$\frac{\P^2(r_{NL} - r_{L})}{\P y^2} = \frac{1/a_1[f''(\bullet)(1 - j\Gamma) + (2j / a_1)[f'(\bullet)]^2]}{(1 - j\Gamma)^3} \qquad \textbf{(A.2b)}$$

$$\frac{\P^2(r_{NL} - r_{L})}{\P(p - p^{*})\P y} = \frac{\P^2(r_{NL} - r_{L})}{\P y\,\P(p - p^{*})} = \frac{2jf'(\bullet)}{a_1(1 - j\Gamma)^3} \qquad \textbf{(A.2c)}$$

Now because **(A.2a)** is positive, and the determinant

$$\left| \begin{array}{cc} \dfrac{\P^2 (r_{NL} - r_L)}{\P(p - p^*)^2} & \dfrac{\P^2 (r_{NL} - r_L)}{\P(p - p^*)\,\P y} \\ \dfrac{\P^2 (r_{NL} - r_L)}{\P(p - p^*)\,\P y} & \dfrac{\P^2 (r_{NL} - r_L)}{\P(y)^2} \end{array} \right|_{\substack{p - p^* = 0 \\ y = 0}} = \left| \begin{array}{cc} 2j \;/\; a_1 & 2j \\ 2j & 2j\,(1 + a_1) \end{array} \right| = \frac{4j^{\,2}}{a_1} \qquad \textbf{(A.3)}$$

evaluated at the stationary point is positive, the surface near (0,0) is in the shape of a ‘bowl’ and we have a local minimum.

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