

Band-pass filtering, cointegration, and business cycle analysis

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Contents

Abstract	5
Summary	7
1. Introduction	9
2. Frequency-domain analysis and linear filtering: a brief overview	13
3. Spurious cyclicalities from band-pass filtering	18
4. Macroeconomic examples	19
5. Conclusions	29
Appendix A: A simple expression for the ideal band-pass filter	31
Appendix B: Details on the macroeconomic models used in Section 4	32
Charts	35
References	49

Abstract

This paper critically assesses the practice of band-pass filtering—the non-structural, frequency-domain based decomposition of economic time series into trend and cyclical components—making two main points. First, it is shown that: (a) depending on the stochastic properties of the filtered process, the band-pass filtered cyclical component is entirely authentic, partly or mostly spurious, or even entirely spurious; and (b) as a simple consequence of the Lucas critique, the degree of authenticity of band-pass filtered cyclical components crucially depends on the monetary rule followed by the policy-maker.

Second, taking a number of macroeconomic models as data-generation processes it is shown that band-pass filtering: (a) may markedly distort key business cycle stylised facts, as captured by the cross-correlations and the cross-spectral statistics between the cyclical components of the variables of interest and the cyclical component of GDP; and (b) may well create entirely spurious stylised facts. For example: both productivity and the money supply may appear procyclical even when they follow random walks by construction; the real wage may appear procyclical when in fact it is countercyclical; in general, the Phillips correlation between inflation and the cyclical component of economic activity will appear weaker than it is in reality. Again, the degree of authenticity of business cycle stylised facts uncovered via band-pass filtering crucially depends on the monetary rule followed by the policy-maker.

Keywords: Band-pass filter; time series; frequency domain; unit roots; business cycles; Lucas critique; monetary policy.

JEL classification: E30, E32.

Summary

In recent years, band-pass filtering—the non-structural, frequency-domain based decomposition of economic time series into trend and cyclical components—has become more and more popular among macroeconomists, as a way of capturing and describing business cycle stylised facts. Compared with the Hodrick-Prescott filter, the band-pass filter offers the advantage of allowing the researcher to target a specific frequency band, thus extracting from the series of interest all the components associated with that band, while essentially discarding all the others. The growing interest of the macroeconomics profession in band-pass filtering techniques is demonstrated, first by the number of recent papers on business cycle stylised facts that make use of the band-pass filter, second, by the inclusion in the recent *Handbook of Macroeconomics* of a chapter on US post-World War II business cycle stylised facts entirely based on band-pass filtering, and third, by the continuing attempts to develop new and better approximations to the ideal band-pass filter.

This paper critically assesses the practice of band-pass filtering, making two main points. First, it is shown that, depending on the stochastic properties of the filtered process, the band-pass filtered cyclical component could be entirely authentic, partly or mostly spurious, or even entirely spurious. While, in general, there does not exist any universally valid measure of authenticity for band-pass filtered cyclical components, it is shown that for unobserved components (UCARIMA) processes there does indeed exist such a natural measure, based on the integral of the spectral density of the band-pass filtered process. Taking a simple sticky-price DSGE model as the data-generation process, it is shown that: (a) under a number of circumstances, band-pass filtered output may provide a surprisingly bad proxy for the structural output gap; and (b) as a technique for extracting a proxy for the output gap, band-pass filtering suffers from the distinct disadvantage that, as a simple consequence of the Lucas critique, the accuracy of the approximation is not invariant to the monetary rule followed by the policy-maker, and in fact crucially depends on it.

Second, taking some alternative macroeconomic models as data-generation processes, it is shown that band-pass filtering: (1) may markedly distort key business cycle stylised facts, as captured by the cross-correlations and the cross-spectral statistics (gain, phase angle, and coherence) between the cyclical components of the variables of interest and the cyclical component of GDP; and (2) may well create entirely spurious stylised facts. For example: (a) the Phillips correlation between inflation and the cyclical component of economic activity will in general appear weaker than it is in reality; (b) both money supply and productivity may appear procyclical even when they follow random walks by construction; (c) the real wage may appear procyclical when in fact it is countercyclical. These results are not peculiar to a particular class of model, but instead illustrate a general problem: the presence of stochastic trends, and possibly of cointegrating relationships among macroeconomic variables, may significantly alter the business cycle stylised facts as captured by the band-pass filter. Again, the degree of authenticity of business cycle stylised facts uncovered via band-pass filtering crucially depends on the monetary rule followed by the policy-maker.

The general conclusion emerging from the paper is that, far from being the neutral, atheoretical, and objective approach to the study of business cycle stylised facts that it is often claimed to be, band-pass filtering may markedly distort those very same stylised facts in unpredictable ways, simply because such distortions crucially depend on the unknown true structure of the economy that the researcher is investigating.

1. Introduction

In recent years, band-pass filtering—the non-structural, frequency-domain based decomposition of economic time series into trend and cyclical components—has become more and more popular among macroeconomists, as a way of capturing and describing business cycle stylised facts. Compared with the Hodrick-Prescott filter, the band-pass filter indeed offers the advantage of allowing the researcher to target a specific frequency band, thus extracting from the series of interest all the components associated with that band, while essentially discarding all the others. The growing interest of the macroeconomics profession in band-pass filtering techniques is demonstrated, first by the number of recent papers on business cycle stylised facts that make use of the band-pass filter,⁽¹⁾ second, by the inclusion in the recent *Handbook of Macroeconomics* of a chapter on US post-World War II business cycle stylised facts entirely based on band-pass filtering,⁽²⁾ third, by the continuing attempts to develop new and better approximations to the ideal band-pass filter.⁽³⁾

This paper critically assesses the practice of band-pass filtering, making two main points. First, it is shown that, depending on the stochastic properties of the filtered process, the band-pass filtered cyclical component could be entirely authentic, partly or mostly spurious, or even entirely spurious. While, in general, there does not exist any universally valid measure of authenticity for band-pass filtered cyclical components, it is shown that for unobserved components (UCARIMA) processes there does indeed exist such a natural measure, based on the integral of the spectral density of the band-pass filtered process. Taking a simple sticky-price DSGE model as the data-generation process, it is shown that: (a) under a number of circumstances, band-pass filtered output may provide a surprisingly bad proxy for the structural output gap; and (b) as a technique for extracting a proxy for the output gap, band-pass filtering suffers from the distinct disadvantage that, as a simple consequence of the Lucas critique, the accuracy of the approximation is not invariant to the monetary rule followed by the policy-maker, and in fact crucially depends on it.

Second, taking some alternative macroeconomic models as data-generation processes, it is shown that band-pass filtering: (1) may markedly distort key business cycle stylised facts, as captured by the cross-correlations and the cross-spectral statistics (gain, phase angle, and coherence) between the cyclical components of the variables of interest and the cyclical component of GDP; and (2) may well create entirely spurious stylised facts. For example: (a) the Phillips correlation between inflation and the cyclical component of economic activity will in general appear weaker than it is in reality; (b) both money supply and productivity may appear procyclical even when they follow random walks by construction; (c) the real wage may appear procyclical when in fact it is countercyclical. These results are not peculiar to a particular class of model, but instead illustrate a

⁽¹⁾ See Englund, Persson and Svensson (1992), Baxter (1994), Hassler *et al* (1994), Baxter and King (1999), King and Watson (1996), Bergman *et al* (1998), Basu and Taylor (1999) and Haldane and Quah (1999). For a critical perspective, see Murray (2000). See also the related work of Pollock (2000, 2001).

⁽²⁾ See Stock and Watson (1999).

⁽³⁾ See Christiano and Fitzgerald (1999).

general problem: the presence of stochastic trends, and possibly of cointegrating relationships among macroeconomic variables, may significantly alter the business cycle stylised facts as captured by the band-pass filter. Again, the degree of authenticity of business cycle stylised facts uncovered via band-pass filtering crucially depends on the monetary rule followed by the policy-maker.

The general conclusion emerging from the paper is that, far from being the neutral, atheoretical, and objective approach to the study of business cycle stylised facts that it is often claimed to be, band-pass filtering may markedly distort those very same stylised facts in unpredictable ways, simply because such distortions crucially depend on the unknown true structure of the economy that the researcher is investigating.

1.1. Structural and non-structural trend-cycle decompositions: an ‘econometric free lunch’?

What is a trend, and what is a cyclical component? This apparently simple question possesses a unique answer only within the context of a well-defined structural model. To fix ideas, let us assume that Christiano, Eichenbaum and Evans (2001) (dynamic stochastic general equilibrium) DSGE model represents the authentic structure of the economy. For all of the endogenous variables, the model uniquely defines a structural trend and a structural cyclical component, which, in what follows, will be defined as the *authentic* trend and cycle components. The attractiveness of such a trend-cycle decomposition is clear. First, the decomposition is uniquely defined. Second, it automatically and directly follows from the structure of the economy. This, unfortunately, creates an obvious problem: in order to implement such a decomposition, a researcher has to commit to a particular structural model.⁽⁴⁾

Since nobody knows exactly what the authentic structure of the economy really is, non-structural decompositions between trend and cyclical components—univariate or multivariate Beveridge-Nelson decompositions, one-sided or two-sided HP filters, band-pass filters, etc—may provide a reasonable second-best solution. One disadvantage of non-structural decompositions is clear: while a structural decomposition is uniquely defined, there are potentially infinite ways of decomposing a time series into a trend and a cyclical component based on non-structural methods. A second potential disadvantage of such an approach has been overlooked in the literature.⁽⁵⁾ The adoption of a particular non-structural decomposition may be justified if and only if such a decomposition may reasonably be expected to perform well under a wide range of possible alternative circumstances. As this paper shows, in the case of band-pass filtering such a presumption

⁽⁴⁾ If we knew, with absolute certainty, that a particular macroeconomic model represents the authentic structure of the economy, and we further knew the stochastic properties of all structural shocks and error terms, the optimal thing to do would be to estimate the model via maximum likelihood. (Typically, the model would involve unobserved variables like technology, but if we knew their stochastic properties, the problem could be trivially solved by casting the model in state-space form and applying the Kalman filter.) This would give us optimal estimates of trend and cyclical components, and would allow us to establish a set of stylised facts concerning growth and cyclical fluctuations. Finally, it would provide us with optimal estimates of a number of quantities of direct interest to policy-makers, like the output gap.

⁽⁵⁾ With the notable exception of Cogley (1997).

is in general incorrect, for the simple reason that the performance of the filter is not invariant with respect to the structure that is being filtered. On the contrary, the ability of the band-pass filter to extract reasonably accurate approximations to the authentic trend and cyclical components, as defined by the true model of the economy, crucially depends on the model's structure. It is shown that, depending on the particular structure that is being filtered, the performance of the band-pass filter may be extremely good, extremely bad, or anything in between, and since the true structure of the economy cannot be known, it is logically impossible to tell, *a priori*, whether the performance of the filter will be reasonably good or not. To put it differently, there is no such a thing as an 'econometric free lunch': if the researcher is not willing to commit to a particular structural model, and therefore decides to resort to a frequency domain based decomposition between trend and cyclical components, then a price has to be paid. And the price is that it will generally be impossible to tell the true meaning of such components.⁽⁶⁾

1.2. Spurious and 'contaminated' business cycle stylised facts from band-pass filtering

Band-pass filters are typically used to establish so-called 'business cycle stylised facts', in terms of the cross-correlations and/or the cross-spectral statistics between the cyclical components of the variables of interest and the cyclical component of a reference series (from now on, I will assume that real GDP is such a series). Ideally, we would like to be able to establish the relationships between the authentic cyclical components of the series of interest and the authentic cyclical component of GDP. Unfortunately the problem is that, by the very nature of band-pass filtering, this is technically impossible: band-pass filtering is based on the notion of extracting from a series all of the components lying within a pre-specified frequency band,⁽⁷⁾ irrespective of the fact that such components may come from filtering a stochastic trend, as opposed to filtering the authentic stationary component of the process. As a result, two types of problem will typically appear.

First, key business cycle stylised facts will end up being distorted and contaminated by the presence of the filtered stochastic trend(s). A typical example is the Phillips correlation between inflation and the cyclical component of economic activity, which, as we will see, will in general appear weaker than it is in reality. The problem will be particularly serious when a cointegrating relationship exists between the series of interest and GDP. In such a circumstance, indeed, the business cycle stylised facts captured by the band-pass filter will reflect both the relationship between the cyclical components of the two series, and the cointegrating relationship between the two stochastic trends. A typical example is the real wage. Assuming the presence of a unit root in technology, the real

⁽⁶⁾ In the recent work of Pedersen (2001), cyclical components are defined as the outcome of the ideal band-pass filter (for a definition of the ideal band-pass filter, see Section 2.4 below). Such a purely tautological approach clearly eliminates the problem at the root by 'defining it away', and possesses a number of distinctly unpalatable implications—for example, based on such a definition, a white noise process possesses both a 'trend' and a 'cyclical' component. The absurdity of such implications clearly casts doubts on the meaningfulness of such an approach.

⁽⁷⁾ With quarterly data, this is usually taken to be $[p/16, p/3]$, associated with fluctuations between 6 and 32 quarters (see, for example, the recent work of Baxter and King (1999) or Christiano and Fitzgerald (1999)).

wage and GDP share a common stochastic trend, and therefore tend to covary positively not only at very low frequencies, but also within the business cycle frequency band. As I will show in Section 4.4, the band-pass filtered real wage may appear to be procyclical even if the cyclical component of the real wage is negatively correlated with the cyclical component of GDP, simply because the authentic negative correlation may be ‘swamped’ by the positive cointegrating one.

Second, band-pass filtering may well create entirely spurious stylised facts. Technology, for example, may appear strongly procyclical even when it follows a random walk by construction, due, once again, to the cointegrating relationship with GDP.

1.3. Observational equivalence between alternative economic structures

A conceptually related problem is the observational equivalence between radically alternative economic structures. When seen through the lenses of the band-pass filter, economic structures with markedly different stochastic properties may indeed give rise to qualitatively similar business cycle stylised facts.

As a simple example, consider the following alternative and contrasting ‘visions of the world’. According to vision A, economic variables either contain deterministic time trends, or, in case they possess stochastic trends, the amount of power displayed by such trends within the business cycle frequency band is negligible compared with the amount of power displayed by the stationary components of the processes.⁽⁸⁾ According to vision B, on the other hand, the opposite is true: economic time series do contain large stochastic trends, while the stationary components of the processes are essentially negligible, so that most of the variance at the business cycle frequencies comes from movements in the stochastic trends. Consider now a researcher who, armed with the band-pass filter, wants to establish a number of basic business cycle stylised facts on the economy under investigation. What kind of picture would be obtained? In the case where vision A is the correct one, band-pass filtering the economy would generate a number of mostly authentic⁽⁹⁾ cyclical components, which, for plausible propagation mechanisms, would exhibit significant correlations with one another and, in particular, with GDP. What about the case in which vision B is, instead, the correct one? Unfortunately, a qualitatively similar picture would emerge. Indeed, since (a) in the reduced form of the model most (or all) variables are driven by the same common structural shocks, and (b) in many cases cointegrating relationships between various stochastic trends may plausibly be expected to exist, band-pass filtering the economy will produce a number of mostly spurious cyclical components, which, under plausible assumptions, would exhibit significant correlations with one another and with GDP. Once again, exclusive reliance on a frequency domain based decomposition between trend and cyclical components will not allow the researcher to say anything about the authentic meaning of the ‘business cycle stylised facts’ uncovered.

⁽⁸⁾ To put it differently, the standard deviations of the shocks to the stochastic trends are small, and most of the variation at the business cycle frequencies comes from the authentic stationary components of the series.

⁽⁹⁾ This is shown in Section 3.

The rest of the paper is organised as follows. Section 2 provides a brief introduction to frequency-domain analysis and linear filtering techniques. Section 3 illustrates the problem of spurious cyclicity in a univariate context. Section 4 analyses the problem in a multivariate context by means of a number of theoretical macroeconomic examples, focusing in particular on the problem of the spurious and contaminated business cycle stylised facts generated by band-pass filtering. Section 5 concludes.

2. Frequency-domain analysis and linear filtering: a brief overview⁽¹⁰⁾

Let y_t , $t = -\infty, \dots, +\infty$, be a covariance-stationary stochastic process, let

$$g_y(z) = \sum_{k=-\infty}^{+\infty} z^k \mathbf{g}_k^y \quad (1)$$

be its autocovariance-generating function, and let us further assume that y_t possesses absolutely summable autocovariances, namely

$$\sum_{k=-\infty}^{+\infty} |\mathbf{g}_k^y| < +\infty \quad (2)$$

A well-known result in time series analysis is that, under such conditions, y_t possesses a *Cramer representation*—that is, it can be decomposed into the infinite sum of either sine and cosine waves, or complex exponentials⁽¹¹⁾

$$y_t = \frac{1}{2\mathbf{p}} \int_{-\mathbf{p}}^{\mathbf{p}} c_w e^{i\mathbf{w}t} d\mathbf{w} = \frac{1}{2\mathbf{p}} \int_{-\mathbf{p}}^{\mathbf{p}} c_w [\cos(\mathbf{w}t) + i \sin(\mathbf{w}t)] d\mathbf{w} \quad (3)$$

for $\mathbf{w} = [-\mathbf{p}, \mathbf{p}]$, where the *Fourier coefficients*—the c_w 's—are given by

$$c_w = \sum_{t=-\infty}^{+\infty} y_t e^{-i\mathbf{w}t} \quad (4)$$

The Cramer representation of a covariance-stationary stochastic process provides a mathematically rigorous way of expressing the common sense notion that (economic) time series do contain components associated with different frequencies of oscillation—very slow-moving, low-frequency components intuitively associated with the notion of trend; medium-frequency components that can

⁽¹⁰⁾ An introduction to frequency-domain analysis can be found in Wei (1990). Excellent references are Sargent (1987) and Fuller (1996). MATLAB programs for spectral and cross-spectral analysis based on the formulas contained in Fuller (1996, chapter 7) are available from the author on request.

⁽¹¹⁾ Due to De Moivre's formula (see, for example, Hamilton (1994), page 716), $\exp(\pm i k \mathbf{w}) = \cos(k \mathbf{w}) \pm i \sin(k \mathbf{w})$, where i is the imaginary number, $i = (-1)^{1/2}$, the two expressions are perfectly equivalent.

be thought of as being associated with the notion of business cycle fluctuations; and fast-moving, high-frequency components associated with seasonal and irregular (or noise) factors.

2.1. Spectral analysis

Which frequencies of oscillation are dominant, and which are less important in explaining movements in y_t ? A simple answer to such a question is provided by the *spectral density*, or *spectrum*, of the process, defined as the Fourier transform⁽¹²⁾ of the autocovariance-generating function, namely as

$$S_y(\mathbf{w}) \equiv g_y(e^{-i\mathbf{w}})g_y(e^{i\mathbf{w}}) \equiv |g_y(e^{-i\mathbf{w}})|^2 = \sum_{k=-\infty}^{+\infty} \mathbf{g}_k^y e^{-i\mathbf{w}k} \quad (5)$$

The spectral density of a covariance-stationary process y_t decomposes the overall variance of the process frequency by frequency, thus highlighting which frequencies of oscillation are more important, and which are less important, in explaining its movements over time. By applying De Moivre's formula (see footnote 11) to (5) it can be easily shown that

$$S_y(\mathbf{w}) = \mathbf{g}_0^y + 2 \sum_{k=1}^{+\infty} \mathbf{g}_k^y \cos(\mathbf{w}k) \quad (6)$$

where \mathbf{g}_0^y is the variance of the process, which implies the following two important properties of the spectrum. First, the spectrum is always real. Second, it is symmetric with respect to $\mathbf{w} = 0$, which implies that, in analysing it, we can restrict our attention to the interval $[0, \mathbf{p}]$.

Chart 1 shows the spectral densities of four stationary processes, a white noise, an MA(1), an AR(1), and an AR(2). The broken vertical lines indicate the business cycle frequency band, $[\mathbf{p}/16, \mathbf{p}/3]$, associated with fluctuations between 6 and 32 quarters. The fact that the spectral density of a white noise process is perfectly flat implies that all frequencies are equally important in explaining its movements over time. In the other three cases, on the other hand, the shape of the spectral density clearly indicates which frequencies dominate, and which play instead a minor role. The MA(1) process $y_t = \mathbf{e}_t - 0.75\mathbf{e}_{t-1}$, for example, mostly contains rapid, high-frequency components, while in the case of the AR(1) process $y_t = 0.75y_{t-1} + \mathbf{e}_t$ the opposite is true, thus implying that the process is a highly persistent one. Finally, in the case of the AR(2) process in panel (d) most of the variance is concentrated within the business cycle frequency band.⁽¹³⁾

⁽¹²⁾ The Fourier transform of a generic polynomial $p(z)$ is obtained by replacing z with $\exp(-i\mathbf{w})$, for $\mathbf{w} = [-\mathbf{p}, \mathbf{p}]$, where i is the imaginary number.

⁽¹³⁾ As stressed for example by Sargent (1987, page 261), the fact that an AR(1) process has a peak in the spectrum at $\mathbf{w} = 0$, which means that most of its variance is concentrated at the very low frequencies, automatically implies that the business-cycle frequency component of economic activity must necessarily possess (at least) two autoregressive roots. As panel (d) clearly shows, an AR(2) process can indeed possess a peak in the spectrum within the business-cycle frequency band.

2.2. Cross-spectral analysis

Consider now two jointly covariance-stationary stochastic processes with absolutely summable autocovariances and cross-covariances, y_t and x_t , and, to fix ideas, think of x_t as the ‘input series’ and of y_t as the ‘output series’.⁽¹⁴⁾ It can be easily shown that the relationship existing between the two series at the various frequencies—in terms of leads/lags, strength of the correlation, etc—is entirely summarised⁽¹⁵⁾ by the two series’ *cross-spectrum*, defined as the Fourier transform of the cross-covariance generating function, namely as

$$S_{xy}(\mathbf{w}) \equiv g_x \left(e^{-i\mathbf{w}} \right) g_y \left(e^{i\mathbf{w}} \right) = \sum_{k=-\infty}^{+\infty} \mathbf{g}_k^{xy} e^{-i\mathbf{w}k} \quad (7)$$

where the \mathbf{g}_k^{xy} ’s are the cross-covariances. While spectral densities are always real, cross-spectra are, in general, complex quantities, and can therefore be written as

$$S_{xy}(\mathbf{w}) = \sum_{k=-\infty}^{+\infty} \mathbf{g}_k^{xy} \cos(\mathbf{w}k) + i \sum_{k=-\infty}^{+\infty} \mathbf{g}_k^{xy} \sin(\mathbf{w}k) \equiv c_{xy}(\mathbf{w}) + iq_{xy}(\mathbf{w}) \quad (8)$$

where $c_{xy}(\mathbf{w})$ and $q_{xy}(\mathbf{w})$ are, respectively, the *co-spectrum* and the *quadrature spectrum*. The following three quantities—respectively known as *gain*, *phase angle* (or simply *phase*), and *coherence*—are crucial in cross-spectral analysis:

$$G_{xy}(\mathbf{w}) = \frac{[c_{xy}^2(\mathbf{w}) + q_{xy}^2(\mathbf{w})]^{1/2}}{S_x(\mathbf{w})} \quad (9)$$

$$\mathbf{f}_{xy}(\mathbf{w}) = \tan^{-1} \left[\frac{q_{xy}(\mathbf{w})}{c_{xy}(\mathbf{w})} \right] \quad (10)$$

$$K_{xy}^2(\mathbf{w}) = \frac{c_{xy}^2(\mathbf{w}) + q_{xy}^2(\mathbf{w})}{S_x(\mathbf{w})S_y(\mathbf{w})} \quad (11)$$

and have the following interpretation. The gain is the absolute value of the \mathbf{b} coefficient in the OLS regression of y_t on x_t frequency by frequency, while the coherence is the R^2 in such a regression. Finally, the phase angle captures the lead-lag relationship between the two variables: a positive (negative) phase angle at frequency \mathbf{w} implies that x_t leads (lags) y_t at such frequency. As previously mentioned, cross-spectral statistics capture all aspects of the relationship between y_t and x_t at the various frequencies, with a single exception: the sign of the correlation between the two variables.

⁽¹⁴⁾ This is the terminology of the so-called transfer function models (see, for example, Wei (1990), Chapter 13).

⁽¹⁵⁾ With a single exception to be discussed shortly.

Since the gain is the absolute value of the \mathbf{b} coefficient in the OLS regression of y_t on x_t frequency by frequency, it clearly cannot shed any light on the sign of the correlation between the two series.

Chart 2 shows the estimated gain, phase, and coherence between the rate of capacity utilisation in the US manufacturing sector and the rate of unemployment, based on quarterly data for the period 1948 Q1-1998 Q1, together with the 90% confidence bands.⁽¹⁶⁾ Estimated cross-spectral statistics clearly show: (a) the existence of a strong and statistically significant relationship between the two variables, especially at the business cycle frequencies, and (b) how, within the business cycle frequency band, capacity utilisation leads the rate of unemployment.

2.3. Linear filtering

Let y_t , $t = -\infty, \dots, +\infty$, be once again a covariance-stationary process with absolutely summable autocovariances, and let z_t be given by

$$z_t = \sum_{j=-\infty}^{+\infty} b_j y_{t-j} = \sum_{j=-\infty}^{+\infty} b_j L^j y_t \equiv b(L)y_t \quad (12)$$

A crucial question is: what is the relationship between the original process y_t and the new process z_t we have obtained by passing y_t through the filter $b(L)$? Consider the transfer function of the filter, defined as the Fourier transform of $b(L)$. In general, the transfer function is a complex quantity, and can therefore be written in polar notation⁽¹⁷⁾ as

$$b(e^{-i\omega}) = \sum_{j=-\infty}^{+\infty} b_j e^{-i\omega j} = \mathbf{q}(\omega) e^{i\mathbf{y}(\omega)} \quad (13)$$

where $\mathbf{q}(\omega)$ and $\mathbf{y}(\omega)$ are, respectively, the *gain* and the *phase shift* of the filter. The squared gain and, respectively, the phase shift of $b(L)$ provide an answer to the following two questions: (a) what is the impact of the filter $b(L)$ on the stochastic properties of $z_t = b(L)y_t$, frequency by frequency?; and (b) what is the lead/lag relationship between z_t and y_t , frequency by frequency? For our purposes the phase shift of the filter is much less relevant, and in what follows I will therefore concentrate exclusively on the squared gain. A squared gain function equal to one over a frequency band \mathbf{W} implies that the filter $b(L)$ leaves the stochastic properties of y_t over such a band completely unaffected—to put it differently, over the band \mathbf{W} the stochastic properties of y_t and z_t are identical. On the other hand, a squared gain equal to zero over \mathbf{W} implies that the variance of y_t is completely erased, so that z_t does not possess any component within the band. Finally, a squared gain equal to, say, k implies that all of the components of y_t get amplified (dampened) by a factor k .

⁽¹⁶⁾ The rate of unemployment refers to the overall labour force, 16 aged and over. The data are from CITIBASE.

⁽¹⁷⁾ See, for example, Sargent (1987, Chapter XI).

2.4. The ideal band-pass filter

The ideal band-pass filter is defined as a filter whose squared gain function is equal to zero outside the band of interest, and equal to one inside the band. By construction, therefore, such a filter completely shuts off all the frequencies the researcher is not interested in, at the same time leaving the ones of interest completely unaffected. As shown, for example, in Sargent (1987, page 259), for a frequency band $[\mathbf{w}_L, \mathbf{w}_U]$, with $0 < \mathbf{w}_L < \mathbf{w}_U \leq \mathbf{p}$, the ideal band-pass filter is given by

$$\mathbf{b}(L) = \sum_{j=-\infty}^{\infty} b_j L^j \quad (14)$$

where the filter's weights are given by

$$b_0 = \frac{\mathbf{w}_U - \mathbf{w}_L}{\mathbf{p}} \quad b_j = \frac{\sin(\mathbf{w}_U j) - \sin(\mathbf{w}_L j)}{\mathbf{p}} \quad \text{for } j = \pm 1, \pm 2, \pm 3, \dots \quad (15)$$

As is well known, the ideal band-pass filter could be implemented if and only if the researcher had access to a series of infinite length. Given the data limitations of real-world situations it is necessary to resort to some kind of approximation. Three well-known approximations that have been proposed in recent years are the frequency-domain filter of Englund, Persson and Svensson (1992), and Hassler, Lundvik, Persson and Söderlind (1994), and the two approximated band-pass filters of Baxter and King (1999) and Christiano and Fitzgerald (1999).

Chart 3 plots the squared gain of the ideal band-pass filter, together with the squared gains of some popular filters. Panel (b) clearly shows how the widely used first-difference filter dramatically distorts the stochastic properties of the filtered process, erasing most of the variance at the business cycle frequencies, and strongly amplifying the high-frequency components of the data. Panel (c), on the other hand, illustrates how the Hodrick-Prescott (1997) filter operates essentially as a 'high-pass' filter, removing the very low frequencies, and leaving all other components of the data virtually unaffected. Finally, panel (d) plots the squared gain of the Baxter-King (1999) filter, currently the most popular approximated band-pass filter. As the chart clearly shows, the Baxter-King filter indeed represents an excellent approximation to the ideal filter.

2.5. The issue of stationarity

Up until now we have assumed that the stochastic processes we are dealing with are covariance-stationary. Since non-stationary processes do not possess a Cramer representation, applying frequency-domain logic to such a class of processes is indeed impossible. This, however, does not rule out the possibility of applying frequency-domain analysis to *filtered* processes. Indeed it can be easily shown that the ideal band-pass filter, and many popular linear filters, contain a number of differencing operations sufficient to induce stationarity both in I(1) and in I(2)

processes.⁽¹⁸⁾ To the extent that the highest order of integration of economic time series can reasonably be expected to be two,⁽¹⁹⁾ these filters will therefore induce stationarity in the data, thus making it possible to apply frequency-domain analysis to the filtered series.

3. Spurious cyclicalty from band-pass filtering

Let us define the log of the variable of interest as $y_t = \ln(Y_t)$, and let us start by assuming that it admits the following trend stationary representation

$$y_t = \mathbf{a} + \mathbf{g} t + c_t \quad (16)$$

where \mathbf{a} and \mathbf{g} are constants, t is a time trend,⁽²⁰⁾ and c_t is a stationary stochastic process on which we do not need to impose any particular structure. Let us now define a frequency band of interest, $\mathbf{W} = [\mathbf{w}_L, \mathbf{w}_U]$, with $0 < \mathbf{w}_L < \mathbf{w}_U \leq \mathbf{p}$, and let us suppose we want to extract from y_t all the components of c_t associated with such a band. By defining the ideal band-pass filtered x as $\mathbf{b}(x)$ we have⁽²¹⁾

$$\mathbf{b}(y_t) = \mathbf{b}(\mathbf{a} + \mathbf{g} t) + \mathbf{b}(c_t) = \mathbf{b}(c_t) \quad (17)$$

In the case of a trend stationary process, therefore, band-pass filtering allows us to extract from y_t all of the components of c_t associated with the frequency band \mathbf{W} , and only those components—in other words, the filtered component we obtain from band-pass filtering a trend-stationary process is always entirely authentic, and band-pass filtering does not introduce any spurious element.

Let us now assume, instead, that y_t admits the following representation, suggested by the work of Beveridge and Nelson (1981)

$$y_t = \mathbf{t}_t + c_t \quad (18)$$

where \mathbf{t}_t is a stochastic trend, and c_t is a stationary component.⁽²²⁾ Passing y_t through the ideal band-pass filter we now have

$$\mathbf{b}(y_t) = \mathbf{b}(\mathbf{t}_t) + \mathbf{b}(c_t) \quad (19)$$

⁽¹⁸⁾ King and Rebelo (1993) show that the Hodrick-Prescott filter contains four differencing operations. Baxter and King (1999) show how their approximated band-pass filter contains at least two differencing operations. In Appendix A, I show that the ideal band-pass filter, too, contains at least two differencing operations.

⁽¹⁹⁾ See, for example, the discussion in Baxter and King (1999).

⁽²⁰⁾ I consider a linear time trend just for convenience.

⁽²¹⁾ Indeed, by applying the expression for the ideal band-pass filter developed in Appendix A—namely, expression **(A1)**—it can be trivially shown that $\mathbf{b}(\mathbf{a} + \mathbf{g} t) = 0$.

⁽²²⁾ Once again, we do not need to impose any particular structure on **(18)**—either on \mathbf{t}_t , or on c_t , or on the correlation between the two components.

and $\mathbf{b}(y_t)$ is therefore equal to the sum of two components, an authentic component, $\mathbf{b}(c_t)$, and a spurious one, $\mathbf{b}(t_t)$. It is important to stress how in filtering non-stationary processes with the time-series representation (18) this will *always* be the case, simply because the stochastic trend t_t possesses a certain amount of power within the frequency band of interest, which the band-pass filter will retain.⁽²³⁾ While I have not been able to find any universally valid measure of authenticity for band-pass filtered cyclical components, for one particular class of stochastic processes—UCARIMA, or unobserved components⁽²⁴⁾ ones—there does indeed exist such a natural metric:

$$\Gamma = \frac{\int_{\mathbf{w} \in \Omega} S_{\mathbf{b}(c)}(\mathbf{w}) d\mathbf{w}}{\int_{\mathbf{w} \in \Omega} S_{\mathbf{b}(y)}(\mathbf{w}) d\mathbf{w}} \quad (20)$$

where $S_{\mathbf{b}(c)}(\mathbf{w})$ is the spectral density of $\mathbf{b}(c_t)$, while $S_{\mathbf{b}(y)}(\mathbf{w})$ is the spectral density of $\mathbf{b}(y_t)$. The denominator therefore measures the variance of the filtered process, while the numerator measures the variance of the filtered stationary component. The key identifying assumption of UCARIMA models is that all of the structural shocks are uncorrelated at all leads and lags, which amounts to assuming that $\mathbf{b}(c_t)$ and $\mathbf{b}(y_t)$ are uncorrelated at all frequencies. Under such an assumption Γ varies between 0 and 1, and for Γ which tends to 1 $\mathbf{b}(y_t)$ is to be regarded as mostly authentic, while for Γ which tends to 0 it is to be regarded as mostly spurious. The intuition is that for values close to 1 most of the variance of the filtered process comes from filtering the stationary component, while for values close to 0 it comes from filtering the stochastic trend.

4. Macroeconomic examples

4.1. Is band-pass filtered output a reasonably good proxy for the structural output gap?

Does the band-pass filtered cyclical component of GDP represent a reasonably good proxy for the structural output gap? This question is addressed both via the natural metric proposed in the previous section, and via theoretical cross-spectral statistics. Once again, the focus is on the ideal band-pass filter.⁽²⁵⁾

⁽²³⁾ Such a result is therefore independent of the specific assumptions we make about the stochastic properties of t_t and c_t , and about the correlation between the two components, and only depends on y_t possessing a stochastic trend, as opposed to a deterministic one.

⁽²⁴⁾ Unobserved components models have been advocated by Harvey (see, for example, Harvey (1985, 1989) and Maravall (see, for example, Maravall (1995)). Applications can be found in Watson (1986) and Clark (1987, 1989).

⁽²⁵⁾ Focusing on the theoretical case of the ideal band-pass filter clearly highlights how the problems analysed in this paper have nothing to do with the purely practical issue of designing better and better approximations to the ideal band-pass filter.

Let us start by considering a simplified version of Erceg *et al*'s (1998) DSGE model.⁽²⁶⁾ Specifically, and in contrast to the original model, I assume full wage flexibility, I set the shocks to the marginal utility of consumption, leisure, and real money holdings equal to zero, and I assume the central bank sets the nominal interest rate according to

$$i_t = \mathbf{f}p_t + \mathbf{y}_t \quad (21)$$

where p_t is the rate of inflation, $\mathbf{f} > 1$, and \mathbf{y}_t is a shock to the policy rule, which is assumed to follow the stationary AR(1) process $\mathbf{y}_t = \mathbf{r}\mathbf{y}_{t-1} + \mathbf{e}_t$. Finally, the log of technology, x_t , is assumed to follow a random walk with innovation v_t (the two structural innovations are assumed to be completely uncorrelated at all leads and lags). It can easily be shown that such a model possesses a UCARIMA reduced form—in other words, stochastic trends and cyclical components are driven by completely unrelated processes—which makes it possible to apply the indicator illustrated in the previous section. Conditional on the same set of parameters chosen by Erceg *et al*,⁽²⁷⁾ Chart 4a shows the value⁽²⁸⁾ taken by Γ for different values of \mathbf{r} (which, without any loss of generality, has been restricted between 0 and 1) and of the ratio between the standard deviations of the two structural shocks, $k = \mathbf{s}_v / \mathbf{s}_e$. As the chart makes clear, unless \mathbf{s}_v happens to be sufficiently small compared to \mathbf{s}_e , the band-pass filtered cyclical component of GDP may provide a surprisingly bad approximation to the structural output gap. The intuition for such a result is simple: the larger the standard deviation of the technology shock with respect to the standard deviation of the shock to the policy rule, the larger the fraction of the variance of GDP at the business cycle frequencies originating from movements in potential output, as opposed to movements in the output gap, and, as a result, the worse the approximation provided by band-pass filtered GDP to the structural output gap.

This simple example also illustrates a second shortcoming of band-pass filtering as a technique for extracting a proxy for the structural output gap: as a simple consequence of the Lucas critique, the quality of the approximation is not invariant to the monetary rule followed by the policy-maker, and on the contrary crucially depends on it. Chart 4b clearly illustrates such a problem, by focusing on the relationship between Γ and the key parameter of the monetary policy rule, \mathbf{f} , for different values of $k = \mathbf{s}_v / \mathbf{s}_e$. Without any loss of generality, \mathbf{f} has been restricted between 1 and 10. As the chart makes clear, (a) once again, *ceteris paribus*, Γ is a decreasing function of k ; and (b) the stronger the central bank's reaction to inflation, the less authentic band-pass filtered output becomes. The intuition for such a result is straightforward: the stronger the central bank's reaction to inflation, the closer the economy gravitates around a steady-state equilibrium with zero inflation and a zero output

⁽²⁶⁾ I focus on the original 1998 working paper version, instead of the one published in the *Journal of Monetary Economics*, 2000, for reasons of personal convenience.

⁽²⁷⁾ See Erceg *et al* (1998, Table 3).

⁽²⁸⁾ The two spectral densities in (20) have been computed based on the formula for the ideal band-pass filter derived in Appendix A. The infinite sum has been approximated with a sum with a finite but very large number of terms, 50,000. Results are insensitive to the number of terms as long as such number is sufficiently large.

gap,⁽²⁹⁾ and, as a result, the larger the fraction of the variance of band-pass filtered output originating from filtering potential output, as opposed to filtering the output gap.⁽³⁰⁾

Let us now suppose that log GDP does indeed possess a UCARIMA representation: specifically, following Watson (1986), I assume it can be represented as the sum of two orthogonal components, a natural one following a random walk with drift, $\mathbf{t}_t = \mathbf{t}_{t-1} + \mathbf{d} + u_t$, and a cyclical one following a stationary AR(2), $c_t = \mathbf{f}_1 c_{t-1} + \mathbf{f}_2 c_{t-2} + v_t$. Maximum likelihood estimation of such a process based on US data⁽³¹⁾ produces the following results: $\mathbf{d} = 0.008$ (0.0007); $\mathbf{s}_u = 0.007$ (0.002); $\mathbf{s}_v = 0.007$ (0.002); $\mathbf{f}_1 = 1.472$ (0.152); $\mathbf{f}_2 = -0.518$ (0.155). Based on these estimates and on (20), it is then easy to compute a MLE estimate⁽³²⁾ of Γ for US GDP conditional on the stochastic process we postulated. The value we obtain, $\Gamma = 0.785$, suggests that, conditional on the assumed time series representation, more than one fifth of the variance of band-pass filtered output is spurious, in the sense of coming from filtering potential output, as opposed to filtering the output gap.

Finally, following Kiley (1996), consider a Calvo sticky-price framework in which the logs of both potential output and the money stock follow random walks, with orthogonal innovations respectively given by u_t (demand shock) and v_t (supply shock). It can be easily shown that, in contrast to the simplified version of Erceg *et al*'s (1998) model that we previously analysed, the output gap is here driven by both demand and supply shocks, so that the natural and cyclical components of GDP are not orthogonal to each other. Under such circumstances, Γ is no longer bounded between 0 and 1, and does not provide an indication of the authenticity of band-pass filtered output. In order to investigate the accuracy of the approximation provided by band-pass filtered output to the structural output gap, the theoretical cross-spectral statistics between the two variables are analysed. Focusing on the business cycle frequency band, and for four different values of $k = \mathbf{s}_u / \mathbf{s}_v$,⁽³³⁾ Chart 5 shows the phase angle, gain, and coherence between the structural output gap and band-pass filtered output.

⁽²⁹⁾ Quite obviously, the same result obtains when the central bank reacts either to the output gap, or to both inflation and the output gap.

⁽³⁰⁾ Over the most recent years, a number of papers (see, for example, Wynne and Koo (2000)) have resorted to band-pass filtering techniques in order to assess the degree of business cycle synchronisation within the euro zone. Now, assume that (a) the degree of business cycle synchronisation within the euro zone has increased over time, and is now greater than it was (say) during the 1970s; and (b) the 'quality' of monetary policy has steadily improved over time, in the specific sense that, over the most recent years, output gaps have been much smaller than they were during the 1970s. Based on the results contained in this paragraph, it immediately follows that the use of band-pass filtering techniques will tend to systematically underestimate not only the *current* degree of business cycle synchronisation among countries, but also the *average* degree of synchronisation over the sample period, for the simple reason that filtered output over the most recent years, during which synchronisation has been stronger, provides a noisier and worse approximation to the structural output gap.

⁽³¹⁾ Data are from Table 1.2 of the National Income and Product Accounts, and the sample period is 1947 Q1-1998 Q4. Estimation has been performed via exact maximum likelihood. The likelihood function has been computed via the Kalman filter and has been maximised numerically with respect to unknown parameters via the EM algorithm for state-space models derived in Shumway and Stoffer (1983).

⁽³²⁾ Indeed, since Γ is a continuous function of the vector of parameters of interest, plugging MLE parameters estimates into (20) we obtain the MLE estimate of Γ , conditional on the chosen data-generation process.

⁽³³⁾ \mathbf{a} , the key parameter in the Calvo model (representing the fraction of firms which are allowed to reset their prices in each single period) has been set equal to 0.2, a standard choice in the literature.

Ideally, we would like the phase shift to be equal to zero, and both the gain and the coherence to be equal to unity across the whole band. Unfortunately, as the charts make clear, unless the standard deviation of the technology shock is relatively small compared with the standard deviation of the money shock, this is not the case. In particular, for k between 0.5 and 2, band-pass filtered output displays (a) a phase shift with respect to the structural output gap, and (b) both a gain and a coherence significantly lower than one.

4.2. *The Phillips correlation*⁽³⁴⁾

Let us now consider the Phillips correlation between inflation and the cyclical component of economic activity. Having just found that, under a number of circumstances, band-pass filtered output may represent a poor proxy for the output gap, it should come as no surprise that the Phillips correlation, too, may end up being distorted by band-pass filtering. Chart 6, based, once again, on Calvo's version of Kiley's (1996) model, shows that this is indeed the case. For three values of the ratio between the standard deviations of the two structural shocks, the chart shows the 'true' and 'false' theoretical cross-spectral statistics for the Phillips correlation, where the true statistics refer to the relationship between inflation, which is $I(0)$, and the output gap, while the false ones refer to the relationship between ideal band-pass filtered inflation and ideal band-pass filtered output.⁽³⁵⁾ As the graphs make clear, once again, the greater the variance of shocks to potential output compared with the variance of money shocks, the more band-pass filtering distorts the authentic, structural Phillips correlation, (a) decreasing both the gain and the coherence—eg causing the correlation to appear weaker than it is in reality—and (b) introducing a spurious phase shift between output and inflation, thus giving the illusion that inflation leads the cycle when, in fact, inflation and the cyclical component of output are perfectly synchronised.

In their investigation of the Phillips correlation via the Baxter-King filter, King and Watson (1994, footnote 9) state:

One concern about exploration of filtered data [...] is that one is uncovering 'spurious relations' that arise from the filtering rather than from the original series. [...] However, the features that we stress are unlikely to be spurious. We are not concerned with the periodic nature of the univariate series, which is an artifact of the filtering. Rather, we are interested in the comovements of the two series [...]. *Note also that we have applied the same symmetric linear filter to each series, so that no phase shifts have been induced.* (emphasis added)

Conditional on taking Kiley's (1996) model as the data generation process, the Phillips correlation uncovered via band-pass filtering is clearly not spurious but, at the same time, it may get significantly distorted, and King and Watson's presumption that the application of the same

⁽³⁴⁾ Recent investigations of the Phillips correlation via band-pass filtering techniques include King and Watson (1994, 1996), Stock and Watson (1999), Baxter and King (1999), and Christiano and Fitzgerald (1999).

⁽³⁵⁾ All the expressions for cross-spectral statistics are based on the expression for the ideal band-pass filter derived in Appendix A. Once again, I approximate the infinite sums with finite sums with a large number of terms, 50,000.

symmetric linear filter to both variables does not induce any phase shift⁽³⁶⁾ in the relationship under investigation is mistaken: as we have seen, band-pass filtering does indeed introduce a spurious phase shift between output and inflation.

Finally, on strictly logical grounds, and based on the same reasoning as in Section 4.1, it is clear that the degree of authenticity of the Phillips correlation uncovered via band-pass filtering will crucially depend on the monetary rule followed by the policy-maker. The more aggressively the policy-maker reacts to deviations of output from potential, and/or to deviations of inflation from a target value, and the smaller the output gap, the larger the fraction of the variance of band-pass filtered output coming from filtering potential output, and therefore the noisier the Phillips correlation uncovered via band-pass filtering will be. It is quite reasonable, for example, to expect that the degree of authenticity of the Phillips correlation uncovered via band-pass filtering was much greater during the 1970s than in recent years.

4.3. Productivity

The procyclical behavior of measured productivity is one of the best-known stylised facts in the whole field of macroeconomics. A currently popular explanation for such a phenomenon, proposed for example by Basu and his co-authors,⁽³⁷⁾ and by Burnside, Eichenbaum and Rebelo,⁽³⁸⁾ ascribes it to unobserved input variation, namely to the unobserved cyclical utilisation of the factors of production, both capital and labour. In their recent investigation of US post-World War II business cycle stylised facts via the Baxter-King filter, Stock and Watson (1999) state:

‘[b]oth total factor productivity and labour productivity are procyclical and slightly lead the cycle [...]’.

As I will now show, on strictly logical grounds, the procyclical behaviour of productivity cannot possibly be established via band-pass filtering, for the simple reason that, due to the presence of a cointegrating relationship with GDP, band-pass filtered productivity appears procyclical even when productivity follows a random walk by construction. If technology and output share a common stochastic trend, they tend to move together not only at the frequencies near zero, but also within the business cycle frequency band. As a result, analysing the relationship between band-pass filtered productivity and band-pass filtered output gives the illusion of procyclicality, even though technology follows a random walk by construction.

Chart 7 shows the theoretical cross-spectral statistics between ideal band-pass filtered output and ideal band-pass filtered productivity based on a simple aggregate demand-aggregate supply model

⁽³⁶⁾ It can be trivially shown, indeed, that in the case of symmetric filters—eg of filters for which $b_j=b_{-j}$ in (12)— $y(\omega)$ in (13) is equal to zero, so that no phase shift between y_t and z_t is induced.

⁽³⁷⁾ See Basu (1996, 1998), and Basu, Fernald and Kimball (1998).

⁽³⁸⁾ See for example Burnside, Eichenbaum and Rebelo (1993, 1995, 1996).

with Calvo sticky prices,⁽³⁹⁾ in which unobserved input variation is absent by construction (in other words, all of the relevant variables are perfectly observed). The logs of both driving processes, money and technology, are modeled as random walks, with uncorrelated structural shocks \mathbf{e}_t (money) and \mathbf{y}_t (technology). The charts clearly show how independent of the value of $k = \mathbf{s}_y / \mathbf{s}_e$, band-pass filtered productivity displays an insignificant phase shift with respect to band-pass filtered output, and a gain approximately equal to one; and while the specific value taken by the coherence crucially depends on k , it is generally quite significant.

Chart 8 reports results from 10,000 stochastic simulations of the model. Specifically, the three panels report the average cross-correlations at leads and lags between Baxter-King-filtered⁽⁴⁰⁾ technology and Baxter-King-filtered output, together with the 95% confidence bands, computed based on the 10,000 replications, for three different values of the ratio between the standard deviations of the two structural shocks. As the chart shows, technology clearly appears as procyclical, and the extent of procyclicality crucially depends on the ratio between the standard deviations of the two structural shocks—specifically, as expected, the strength of the spurious correlation is increasing in k : the larger the variance of the technology shock compared with the variance of the money shock, the more procyclical productivity appears.

4.4. *Is the real wage procyclical?*

A commonly heard argument⁽⁴¹⁾ against sticky-wage models, and in favour of sticky-price ones, is that models featuring sticky nominal wages automatically imply countercyclical real wages, a prediction that is not supported by the data. The pro or acyclicity of real wages may well be an entirely authentic business cycle stylised fact. As will be shown, however, once again on strictly logical grounds it cannot possibly be established via band-pass filtering for the simple reason that, assuming the presence of a unit root in technology, the cointegrating relationship between output and the real wage may cause the band-pass filtered real wage to appear procyclical even if the cyclical component of the real wage is negatively correlated with the cyclical component of output.

Let us consider once again the simple AD-AS model used in Section 4.3, and replace Calvo sticky prices with Calvo sticky wages. Chart 9, based, once again, on 10,000 replications of the model for three different values of the ratio between the standard deviations of the two structural shocks, shows the average cross-correlations at leads and lags between the Baxter-King filtered real wage and

⁽³⁹⁾ Specifically, the model features a constant capital stock and a Cobb-Douglas production function for output, and a labour supply curve according to which the amount of labour supplied is a linear increasing function of the real wage. Details about the model, as well as MATLAB programs for both stochastic simulations and the computation of theoretical cross-spectral statistics, are available on request.

⁽⁴⁰⁾ Since the Baxter-King filter is a symmetric linear filter, it does not introduce any spurious phase shift in the relationship under study (see footnote 37).

⁽⁴¹⁾ See, for example, Kimball (1995, page 1,244). To be entirely fair, however, Kimball also discusses a second, much stronger argument against models with sticky nominal wages, namely the (most likely) non-allocative nature of wage payments.

Baxter-King-filtered output, together with the average cross-correlations between the structural cyclical components of the two variables (plus the 95% confidence bands computed based on the replications). As the charts make clear, while the ‘true’ business cycle stylised facts are independent of the ratio between the standard deviations of the two structural shocks, the ‘false’ ones crucially depend on such a ratio, and while the structural cyclical component of the real wage is *negatively* correlated with the structural cyclical component of output, the band-pass filtered real wage is, depending on the relative magnitude of the standard deviations of the structural shocks, weakly or strongly *positively* correlated with band-pass filtered output.

Finally, once again, as a matter of pure logic, the correlation uncovered via band-pass filtering crucially depends on the monetary rule followed by the policy-maker. A policy-maker who reacted with infinite strength to deviations of output from potential, thus causing the output gap to be equal to zero in each period, would cause the relationship between output and the real wage uncovered via band-pass filtering *uniquely* to reflect the cointegrating relationship between the two variables. In less extreme cases, the stylised fact uncovered via band-pass filtering will *partly* reflect the authentic relationship between the two structural cyclical components, and *partly* the cointegrating relationship between the two stochastic trends—the degree of authenticity being uniquely determined by the particular monetary rule followed by the policy-maker.

4.5. Are monetary aggregates procyclical?

Together with the procyclical behaviour of measured productivity, the procyclicality of monetary aggregates is one of the best-known stylised facts about business cycle frequency economic fluctuations. In their recent investigation of US post-World War II business cycle stylised facts based on the Baxter-King filter, for example, Stock and Watson (1999) state:

‘Over the full sample, the log level of nominal M2 is procyclical with a lead of two quarters, and the nominal monetary base is weakly procyclical and leading.’

The procyclicality of monetary aggregates may well be an entirely authentic stylised fact. Once again, however, on strictly logical grounds it cannot be possibly established via band-pass filtering, for the simple reason that, as will be shown within the context of Kiley’s (1996) model⁽⁴²⁾—the same one used in Section 4.1—band-pass filtered money may appear as procyclical even when the money stock follows a random walk by construction.

Chart 10 reports the theoretical cross-spectral statistics between ideal band-pass filtered money, $\mathbf{b}(m_t)$, and ideal band-pass filtered output, $\mathbf{b}(y_t)$, for three possible values of the ratio between the standard deviations of the two structural shocks, u_t (money shock) and v_t (potential output shock).

⁽⁴²⁾ Qualitatively identical results can be obtained both within the simple AD-AS model of Section 4.3 and within the Erceg *et al* (1998) model.

Once again, the business cycle frequency band has been taken to be $[p/16, p/3]$. The charts clearly show, first, how the ‘cyclical’ component of money, $\mathbf{b}(m_t)$, slightly lags the cycle, and displays a significant gain with respect to band-pass filtered output at the business cycle frequencies (both the gain and the phase angle are independent of the relative magnitude of the two structural shocks). Second, the coherence between $\mathbf{b}(y_t)$ and $\mathbf{b}(m_t)$ depends on the relative magnitude of the two structural shocks, and is generally strong.

Chart 11 illustrates the same point, reporting the results from a numerical simulation of Kiley’s model. The model has been simulated 10,000 times for three different values of the ratio between the standard deviations of the two structural shocks, 0.5, 1, and 2. Each simulation comprises 200 observations, roughly equal to the length of the typical sample available for post-World War II US quarterly series. The three panels report the average cross-correlations at leads and lags between Baxter-King filtered money and Baxter-King filtered output, together with the 95% confidence bands (confidence bands have been computed based on the numerical simulations). From the charts we see that money displays a significant positive correlation with output at the business cycle frequencies, and tends to slightly lag output.

As the above simple example shows, the fact that $\mathbf{b}(m_t)$ is procyclical with respect to $\mathbf{b}(y_t)$ does not necessarily imply that money truly is procyclical. This is obviously not to say that the business cycle stylised facts about money established via band-pass filtering are an artifact of the filter. In fact, they might well be entirely authentic. However, the mere fact that cross-correlations similar to those found in the data can be generated by a model in which money is a random walk should induce one to be sceptical about the authenticity of the stylised facts uncovered via band-pass filtering.

4.6. *The cyclical behaviour of prices*

Focusing again on Calvo’s version of Kiley’s (1996) model,⁽⁴³⁾ Chart 12 shows the theoretical cross-spectral statistics between the structural output gap and the structural cyclical component of prices, and ideal band-pass filtered output, $\mathbf{b}(y_t)$, and ideal band-pass filtered prices, $\mathbf{b}(p_t)$, for three possible values of $k = \mathbf{s}_u / \mathbf{s}_v$. It is possible to see how the authentic relationship between the two structural components is independent of k , while the ‘false’ relationship between $\mathbf{b}(y_t)$ and $\mathbf{b}(p_t)$ crucially depends on the ratio between the standard deviations of the two structural shocks, and is, in general, quite significantly distorted. In particular, panel (c) clearly shows how, depending on the particular value taken by k , the extent of countercyclicality⁽⁴⁴⁾ may get either enhanced or attenuated.

⁽⁴³⁾ Once again, analogous results can be obtained based on other models—like Erceg *et al*’s (1998), and the simple AD-AS model I previously used—and are available on request.

⁽⁴⁴⁾ It can be trivially shown, indeed, that in this model prices are countercyclical.

4.7. Spot and forward exchange rates

Hai, Mark and Wu (1997) propose an unobserved components model for the spot and forward exchange rate motivated by Mussa's (1982) stochastic sticky-price model. The log spot and log forward rates, s_t and f_t , share a common stochastic trend, z_t (modelled as a driftless random walk), while their transitory components, s_t^T and f_t^T , are assumed to evolve according to a vector ARMA(1,1) process. Their maximum likelihood estimates indicate that exchange rate dynamics are almost entirely driven by the dynamics of the permanent component, which suggests that the cyclical components of both spot and forward exchange rates extracted via band-pass filtering will be largely spurious. Given that the model possesses a UCARIMA form, it is once again possible to apply the natural metric I proposed in Section 3 in order to evaluate, conditional on Hai *et al*'s MLE estimates, how authentic the band-pass filtered cyclical components of spot and forward exchange rates are. Focusing on the case of the ideal band-pass filter, and for a frequency band of interest $[p/16, p/3]$, we obtain 0.1979 for log spot rates, and 0.1657 for log forward rates.

Once again, although it is technically possible to band-pass filter any economic time series, the economic meaning of the components obtained based on such a methodology crucially depends on the structure being filtered. Since, in general, such a structure is unknown to the researcher, it is logically impossible to tell, *a priori*, the authentic meaning of the band-pass filtered cyclical components obtained.

4.8. Real exchange rates and real interest rates differentials

An early application of band-pass filtering techniques is Baxter's (1994) investigation of the relationship between real exchange rates and real interest rate differentials at the low-to-medium frequencies. Previous empirical studies, most of the times based on first-differenced data, had failed to uncover any relationship between the two variables, despite the fact that a host of theoretical sticky-price models predicts such a relationship to exist.⁽⁴⁵⁾ As stressed by Baxter, and as previously shown in Section 2.4, the application of the first-difference filter has the effect of destroying most of the variance at the low-to-medium frequencies, at the same time dramatically amplifying the high-frequency components of the data. Assuming that the relationship between real exchange rates and real interest rate differentials pertains to the low-to-medium frequencies, it is therefore not surprising that previous studies had been incapable of uncovering it. By extracting low, medium, and high frequency⁽⁴⁶⁾ components from both variables, Baxter identifies a significant relationship between them at the low-to-medium frequencies, at the same time rejecting the notion of any relationship at the high frequencies.

⁽⁴⁵⁾ See the extensive discussion in Baxter (1994, pages 17-26), both of the theory and of previous empirical studies.

⁽⁴⁶⁾ As usual, the three components are the ones associated with fluctuations, respectively, beyond 32 quarters, between 6 and 32 quarters, and below 6 quarters.

In a previous study based on an unobserved components model, Campbell and Clarida (1987) postulate a structure in which, first, the *ex ante* one-period real interest rate differential, d_t , follows the process

$$d_t = \frac{u_{1,t} + \mathbf{I}u_{2,t-1}}{(1 - \mathbf{r}L)} \quad (22)$$

where $u_{1,t}$ and $u_{2,t-1}$ are, respectively, a zero-mean error, and the (zero-mean) inflation surprise at time $t-1$; and second, the log real exchange rate, q_t , is given by

$$q_t = q_{t+1|t} + d_t + k_t \quad (23)$$

where $q_{t+1|t}$ is the rational expectation of q_{t+1} based on information available at time t , and k_t is a risk premium, which, for the sake of simplicity, is assumed to follow the process $k_t = kd_t$, with k constant. Assuming that

$$\lim_{j \rightarrow +\infty} q_{t+j|t} = w_t = \frac{u_{3,t}}{1 - L} \quad (24)$$

is the stochastic trend in the real exchange rate, it can be easily shown that (23) reduces to

$$q_t = \frac{u_{3,t}}{1 - L} + \frac{1+k}{1-\mathbf{r}} \left[\mathbf{I}u_{2,t} + \frac{u_{1,t} + \mathbf{I}u_{2,t-1}}{1 - \mathbf{r}L} \right] \quad (25)$$

Finally, Campbell and Clarida assume the three shocks are uncorrelated at all leads and lags, with the only exception of $u_{1,t}$ and $u_{3,t}$, for which they postulate $E[u_{1,t}u_{3,t}] = \mathbf{s}_{13}$. Based on their maximum likelihood estimates for four currencies⁽⁴⁷⁾ *vis-à-vis* the US dollar, they conclude that real exchange rate dynamics are almost entirely driven by the dynamics in the random walk component, while transitory stationary components are essentially negligible.

Let us assume, just for the sake of argument, that the Campbell-Clarida model represents the authentic structure of the world, as far as the particular relationship under study is concerned. Based on the results illustrated in the previous sections, the Campbell-Clarida estimates suggest, once again, that band-pass filtering the data will dramatically distort the authentic relationship between the structural stationary components of the real exchange rate and of the real interest rate differential. Based on (22) and (25), the theoretical cross-spectral statistics for the Campbell-Clarida model can be trivially computed, and conditional on their maximum likelihood estimates it is possible to evaluate how band-pass filtering the data would distort the authentic structural relationship between the two variables. Chart 13 reports results for the US dollar-Canadian dollar exchange rate, but

⁽⁴⁷⁾ German Mark, British pound, Canadian dollar, and the Japanese yen.

qualitatively similar results hold for all of the other three currencies, and are available from the author on request. As the graphs clearly show, although the induced phase shift is negligible, band-pass filtering significantly amplifies the gain between the two variables, and dramatically reduces the coherence. Once again, this is obviously not to say that the relationship uncovered by Baxter (1994) is to a large extent spurious. It is important however to stress how, as we have seen in this paper again and again, for plausible economic structures band-pass filtering has been shown to perform badly under a wide array of circumstances, which, again, casts doubts on its ability to extract reasonably accurate approximations to the authentic, structural trend and cycle components.

4.9. Does band-pass filtering necessarily distort business cycle stylised facts?

The following example, based once again on Kiley's (1996) model, illustrates a case in which band-pass filtering leaves the authentic stylised facts completely unaffected even in the presence of stochastic trends. As Chart 13 clearly shows,⁽⁴⁸⁾ in the case of real balances there is absolutely no difference between the 'true' and 'false' phase, gain, and coherence. Given the results reported so far, however, I tend to regard this case as peculiar and unrepresentative.

5. Conclusions

In recent years, band-pass filtering—the non-structural, frequency-domain based decomposition of economic time series into trend and cyclical components—has become more and more popular among macroeconomists, as a way of capturing and describing business cycle stylised facts. This approach is based on the notion of defining a business cycle frequency band; using frequency-domain techniques in order to extract from the series of interest all of the components associated with such a band; and analysing the relationship (in terms of cross-correlations, coherence, etc) between the band-pass filtered series of interest and a band-pass filtered reference series, usually GDP.

This paper has critically assessed the practice of band-pass filtering, making three main points. First, I have shown that, depending on the stochastic properties of the filtered process, the band-pass filtered cyclical component is entirely authentic, partly or mostly spurious, or even entirely spurious. While, in general, there does not exist any universally valid measure of authenticity for band-pass filtered cyclical components, it has been shown that for unobserved-components (UCARIMA) processes there does indeed exist such a natural measure, based on the integral of the spectral density of the band-pass filtered process. Taking a sticky-price DSGE model as data generation process, it has been shown that under a number of circumstances, band-pass filtered output may provide a surprisingly bad proxy for the structural output gap; and as a technique for extracting a proxy for the output gap, band-pass filtering suffers from the distinct disadvantage that the accuracy of the

⁽⁴⁸⁾ Analogous results based on numerical simulations are not reported, but are available on request.

approximation is not invariant with respect to the monetary rule followed by the policy-maker, and on the contrary crucially depends on it.

Second, taking a number of macroeconomic models as data generation processes, it has been shown that band-pass filtering may markedly distort key business cycle stylised facts, as captured by the correlations and the cross-spectral statistics (gain, phase angle, and coherence) between the cyclical components of the variables of interest and the cyclical component of GDP; and may well create entirely spurious stylised facts. For example, the Phillips correlation between inflation and the cyclical component of economic activity may appear weaker than it is in reality; both money supply and productivity may appear procyclical even when they follow random walks by construction; the real wage may appear procyclical when in fact it is countercyclical. Again, the degree of authenticity of business cycle stylised facts uncovered via band-pass filtering crucially depends on the monetary rule followed by the policy-maker. The general conclusion emerging from the paper is that, far from being the neutral, atheoretical, and objective approach to the study of business cycle stylised facts that it is often claimed to be, band-pass filtering may markedly distort those very same stylised facts in ways that are completely unpredictable, simply because such distortions crucially depend on the unknown true structure the researcher is investigating.

Finally, although this paper has focused on band-pass filtering, similar problems may be reasonably expected to plague other detrending methods. Two directions of research therefore seem worth pursuing. First, check how severe the distortions induced by alternative detrending methods are, conditional on taking a number of macroeconomic models as data generation processes. Second, and possibly even more important, investigate whether it is possible to identify a detrending method whose performance is reasonably robust across a number of alternative macroeconomic structures. Conditional on taking the Christiano-Eichenbaum (1992) real business cycle model as data generation process, for example, Cogley (1997) has shown that the consumption-based detrending methodology proposed by Cochrane (1994) performs significantly better than any of the alternatives he considers (HP and band-pass filters, and a Beveridge-Nelson decomposition). It would be interesting to ascertain whether such a result is peculiar to the Christiano-Eichenbaum model, or is instead robust across a number of alternative macroeconomic structures.

Appendix A: A simple expression for the ideal band-pass filter

By definition, the squared gain function of $\mathbf{b}(L)$ is equal to 1 for $\mathbf{w} \in \mathbf{W}$ and zero otherwise, thus implying that the filter wipes out all the power outside the band \mathbf{W} , leaving everything inside the band completely unaffected. For our purposes, two are the key characteristics of the ideal band-pass filter. First, the filter's weights, given by (2) in the text, are symmetric, namely $b_j = b_{-j}$ for any $j = \pm 1, \pm 2, \pm 3, \dots$ (The proof is trivial). Second, the sum of the filter's weights is equal to zero. This is an immediate consequence of the trend reduction property of the ideal band-pass filter—namely the fact that, since the filter wipes out all the power outside the band \mathbf{W} , its squared gain function takes the value 0 for $\mathbf{w} = 0$ —and can be proved simply by noticing that, for $\mathbf{w} = 0$, the squared gain function of the filter is given by:

$$\left[\sum_{j=-\infty}^{+\infty} b_j e^{-i \cdot 0 \cdot j} \right] \left[\sum_{j=-\infty}^{+\infty} b_j e^{i \cdot 0 \cdot j} \right] = \left[\sum_{j=-\infty}^{+\infty} b_j \right]^2 = 0$$

which implies

$$\sum_{j=-\infty}^{+\infty} b_j = 0$$

Now, following the same logic of Baxter and King (1999, Appendix A), we have

$$\begin{aligned} \sum_{j=-\infty}^{+\infty} b_j L^j &= \sum_{j=-\infty}^{+\infty} b_j L^j - \sum_{j=-\infty}^{+\infty} b_j = \sum_{j=-\infty}^{+\infty} b_j (L^j - 1) = \sum_{j=1}^{+\infty} b_j (L^j + L^{-j} - 2) = - \sum_{j=1}^{+\infty} b_j (1 - L^j)(1 - L^{-j}) = \\ &= -(1 - L)(1 - L^{-1}) \sum_{j=1}^{\infty} b_j (1 + L + L^2 + \dots + L^{j-1})(1 + L^{-1} + L^{-2} + \dots + L^{-(j-1)}) \end{aligned}$$

which implies that the ideal band-pass filter can be rewritten as

$$\sum_{j=-\infty}^{\infty} b_j L^j = -(1 - L)(1 - L^{-1}) \sum_{j=1}^{\infty} b_j \sum_{h=(j-1)}^{(j-1)} (j - |h|) L^h \quad (\mathbf{A1})$$

In particular, the previous manipulations use both the symmetry property of the ideal band-pass filter's weights, and the fact that the sum of the filter's weight is equal to zero.

Appendix B: Details on the macroeconomic models used in Section 4

This appendix provides some details on the macroeconomic models used in Section 4.

B.1. Kiley's (1996) sticky-price model

In Kiley's (1996) sticky-price model the log of natural output, x_t , and the log of the money supply, m_t , are assumed to evolve according to random walk processes, $x_t = x_{t-1} + v_t$, and $m_t = m_{t-1} + e_t$, with v_t and e_t normally distributed shocks (assumed to be uncorrelated both contemporaneously and at all leads and lags), with mean zero and variances respectively equal to \mathbf{s}_v^2 and \mathbf{s}_e^2 . The actual level of output is given by $y_t = m_t - p_t$, where p_t is the log of the price level. The instantaneously optimal price for the single firm (the price the firm would like to charge in each period were it able to continuously reset its price) is given by $p_t^* = m_t - x_t$. Firms are assumed to set prices Calvo-style. The Calvo sticky-price model is described by the following two equations:

$$p_t = (1 - \mathbf{a})p_{t-1} + \mathbf{a}z_t \quad (\text{B1})$$

$$z_t = \mathbf{a}p_t^* + (1 - \mathbf{a})z_{t+1|t} \quad (\text{B2})$$

where \mathbf{a} , between 0 and 1, is the probability for the single firm of getting the chance to reset its own price (the probability is assumed to be the same for all firms); z_t is the optimal reset price at time t —namely, the price charged at time t by all firms which get the chance to reset their price; and $s_{t+j|t} \equiv E[s_{t+j}|t]$ is the expectation of s_{t+j} conditional on information available at time t .

It can be easily shown that the reduced-form expressions for output and the price level are given by

$$y_t = \frac{\mathbf{a}v_t + (1 - \mathbf{a})(1 - L)e_t}{(1 - L)[1 - (1 - \mathbf{a})L]} \quad p_t = \mathbf{a} \frac{e_t - v_t}{(1 - L)[1 - (1 - \mathbf{a})L]} \quad (\text{B3})$$

where L is the lag operator, from which the expressions for inflation, Dp_t , the output gap, $y_t^C = y_t - x_t$, the cyclical component of prices, $p_t^C = p_t - p_t^*$, and all of the other variables of interest can easily be derived.

B.2. Erceg et al's (1998) DSGE model

The simplified version of the DSGE model proposed by Erceg *et al* (1998) used in Section 4 is described by the following equations. The aggregate demand side of the economy is described by the following two log-linearised equations:

$$(\text{intertemporal IS curve}) \quad g_t = g_{t+1|t} - \frac{1}{\mathbf{s}} [i_t - \mathbf{p}_{t+1|t}] \quad (\text{B4})$$

$$(\text{monetary policy rule}) \quad i_t = \mathbf{f}_p \mathbf{p}_t + \mathbf{y}_t \quad (\text{B5})$$

where g_t is the structural output gap, i_t is the nominal interest rate (the monetary policy instrument of the central bank), p_t is the rate of inflation prevailing between periods $t-1$ and t , s is the inverse of the elasticity of intertemporal substitution in consumption, and y_t is a shock to the nominal interest rate, which is assumed to follow the stationary AR(1) process $y_t = r y_{t-1} + e_t$, with $0 < r < 1$, and e_t white noise $(0, \mathbf{s}_e^2)$. With sticky prices and fully flexible wages, the aggregate supply side of the model is described by the Phillips curve:

$$p_t = b p_{t+1} + \frac{k_p \Delta}{(1-a)} g_t \quad (\text{B6})$$

where b is the intertemporal discount factor of the representative individual, a is the capital share parameter in the Cobb-Douglas production function, k_p and D are defined as $k_p = [(1-x_p)(1-bx_p)]/x_p$, and $D = a + c l_L + (1-a)s$.⁽⁴⁹⁾ The real wage, z_t , evolves according to $z_t = z_t^* + g_t(D-a)/(1-a)$, with z_t^* being the natural level of the real wage (namely, the stochastic trend) which evolves according to $z_t^* = x_t(s + c l_L)/D$, where $x_t = x_{t-1} + v_t$ is the log of technology. Finally, the stochastic trend for output is given by $y_t^* = x_t(1 + c l_L)/D$, and overall output is therefore equal to $y_t = y_t^* + g_t$. It can be easily shown that inflation obeys the following expectational difference equation

$$\left\{ 1 - \frac{s(1-a)(1+b) + k_p \Delta}{s(1-a) + f_p k_p \Delta} B^{-1} + \frac{bs(1-a)}{s(1-a) + f_p k_p \Delta} B^{-2} \right\} p_t = \frac{-k_p \Delta y_{t|t}}{s(1-a) + f_p k_p \Delta} \quad (\text{B7})$$

where B is the backshift operator,⁽⁵⁰⁾ defined by the property $B^1 s_{t+j|t} = E[s_{t+j+1}|t]$. Assuming that the condition for determinacy, $f_p > 1$, is satisfied, the polynomial in the backshift operator in equation (B.7) has two stable roots, I_1 and I_2 , with $0 < I_1 < I_2 < 1$ (the roots are not reported here). Inflation is therefore given by

$$p_t = - \frac{k_p \Delta}{(1-I_1 r)(1-I_2 r)[s(1-a) + f_p k_p \Delta]} y_t \quad (\text{B8})$$

while the expressions for the output gap, overall output, the cyclical component of the real wage, and the overall real wage, follow immediately, and are not reported here, but are available from the author upon request.

B.3. A sticky-wage (sticky-price) aggregate demand-aggregate supply model

The sticky-wage AD-AS model used in Section 4.4 has the following structure.⁽⁵¹⁾ The log of output, y_t , is given by $y_t = a l_t + u_t$, where l_t and u_t are, respectively, the logarithms of employment and

⁽⁴⁹⁾ Where l_L is the ratio between labour and leisure in the steady-state, c is the inverse of the elasticity of intertemporal substitution for leisure, and x_p is the fraction of firms which are not allowed to reset their prices in each period.

⁽⁵⁰⁾ A discussion of the properties of the backshift operator can be found in Sargent (1987, ch. XIV).

⁽⁵¹⁾ Its sticky-price counterpart, used in Section 4.3, can be derived along the same lines, and is not described here. Details, however, are available upon request.

technology, and \mathbf{a} is the Cobb-Douglas parameter for labour (capital is assumed to be fixed). Labour demand is given by $l_t^D = [u_t - (w_t p_t)] / (1 - \mathbf{a})$, which comes directly from the profit-maximising condition for the single firm, with $(w_t p_t)$ being the log real wage. Log labour supply is given by $l_t^S = \mathbf{q}(w_t p_t)$, with $\mathbf{q} > 0$. The two driving processes for the logs of technology and money, respectively u_t and m_t , are $u_t = u_{t-1} + \mathbf{y}_t$ and $m_t = m_{t-1} + \mathbf{e}_t$, where \mathbf{y}_t and \mathbf{e}_t are the two structural shocks (assumed to be uncorrelated at all leads and lags). Households set nominal wages Calvo-style, with \mathbf{g} being the probability for the single household of getting the chance to reset its own wage, and the instantaneously optimal nominal wage being given by $w_t^* = p_t + u_t / [1 + \mathbf{q}(1 - \mathbf{a})]$. Households supply any amount of labour demanded at the posted wages, and the level of employment is therefore uniquely determined by labour demand. Substituting labour demand within the production function, $y_t = \mathbf{a}l_t + u_t$, we therefore get the expression for aggregate supply. Combining aggregate demand, $y_t^D = m_t p_t$, with aggregate supply we get the expression for the equilibrium price level, $p_t = \mathbf{a}w_t + (1 - \mathbf{a})m_t u_t$. It can be easily shown that the log nominal wage obeys the following expectational difference equation:

$$\left[1 - \frac{1 + (1 - \mathbf{g})^2 - \mathbf{a}\mathbf{g}^2}{(1 - \mathbf{g})} B^{-1} + B^{-2} \right] w_{t-1|t} = - \frac{(1 - \mathbf{a})\mathbf{g}^2}{(1 - \mathbf{g})} \left\{ m_{t|t} - u_{t|t} \frac{\mathbf{q}}{[1 + \mathbf{q}(1 - \mathbf{a})]} \right\} \quad (\mathbf{B9})$$

whose roots I_1 and I_2 (with $0 < I_1 < 1 < I_2 = 1/I_1$) are not reported here. Solving **(B.9)** we get the expression for w_t as a function of the structural shocks, $w_t = [H\mathbf{e}_t - K\mathbf{y}_t] / [(1 - L)(1 - I_1 L)]$ —where $H = [(1 - \mathbf{a})\mathbf{g}^2 I_1] / [(1 - \mathbf{g})(1 - I_1)]$, and $K = H\mathbf{q} / [1 + \mathbf{q}(1 - \mathbf{a})]$ —from which it is then easy to obtain the reduced-form expressions for all of the other variables of interest, output, the price level, inflation, etc. Finally, reworking the model under the assumption of full flexibility—namely, $w_t = w_t^*$ —it is possible to compute for each single variable (with the only exception of the rate of inflation) the stochastic trends, which allows us to compute the cyclical components.

Charts

Chart 1: Spectral densities for four covariance-stationary processes

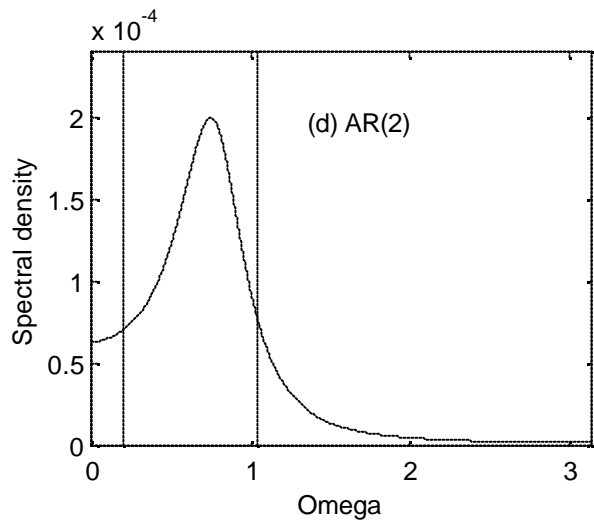
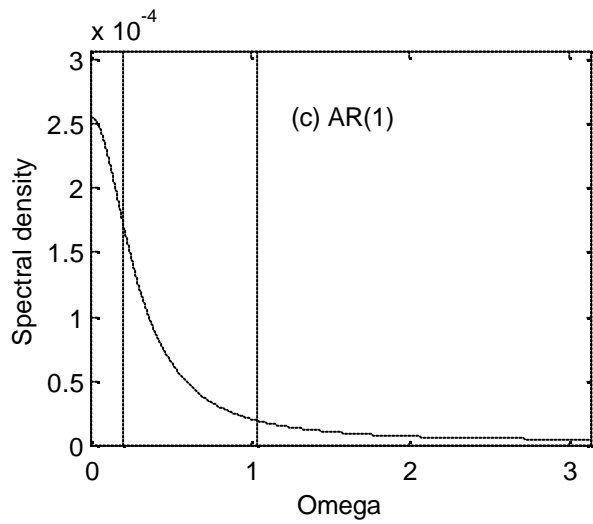
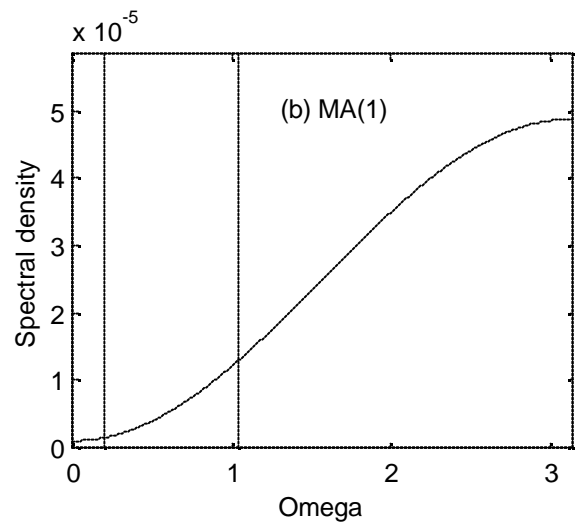
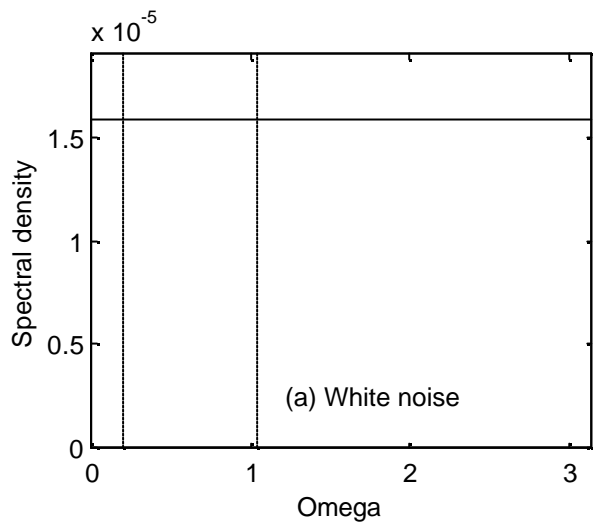
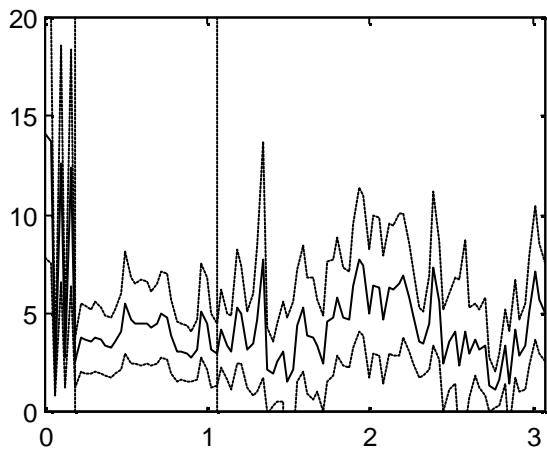
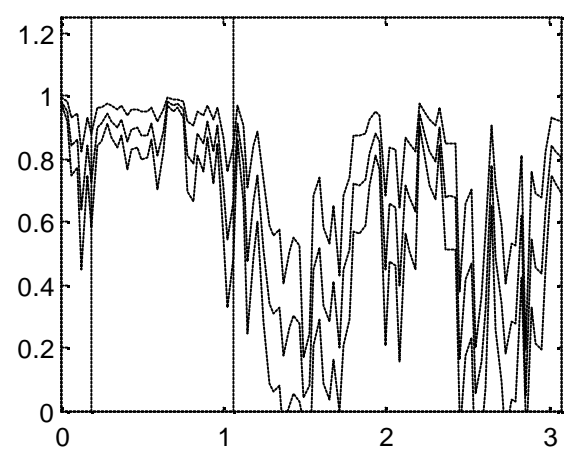


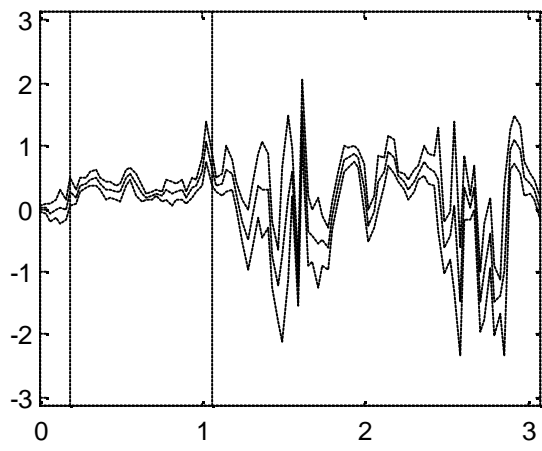
Chart 2: Estimated cross-spectral statistics for the rate of capacity utilisation in the US manufacturing sector and the rate of unemployment (overall labour force, 16 years of age and over)



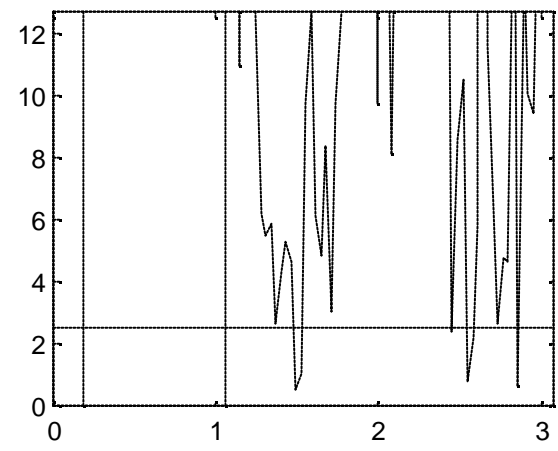
(a) Gain and 90% confidence bands



(b) Coherence and 90% confidence bands



(c) Phase angle and 90% confidence bands



(d) Testing: Coherence=0 (At 90%)

Chart 3: Squared gains of some popular filters, compared with the squared gain of the ideal band-pass filter

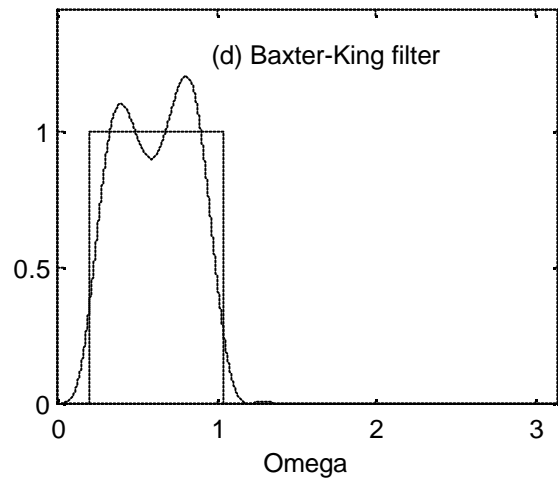
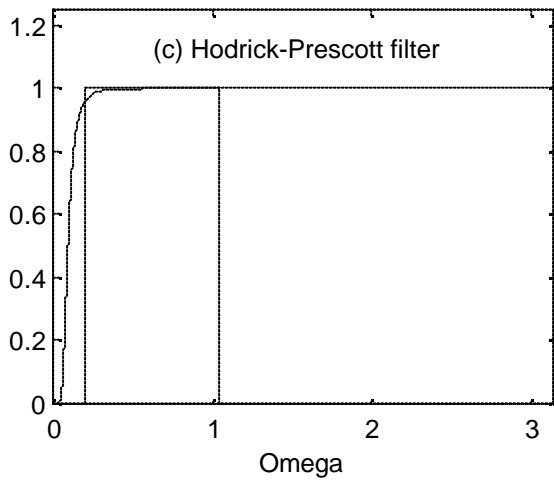
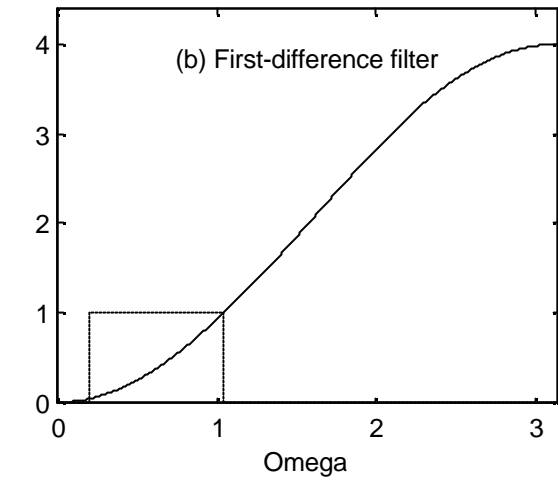
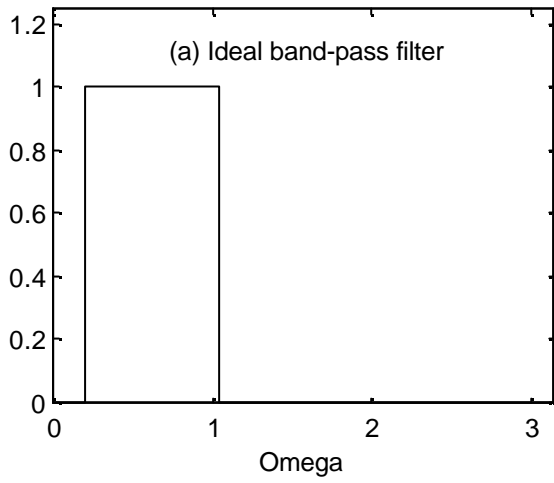


Chart 4: An exact measure of ‘how authentic’ the band-pass filtered cyclical component of GDP is in the Erceg, Henderson and Levin (1998) model, for different values of r , f , and k ($s_v = ks_e$)

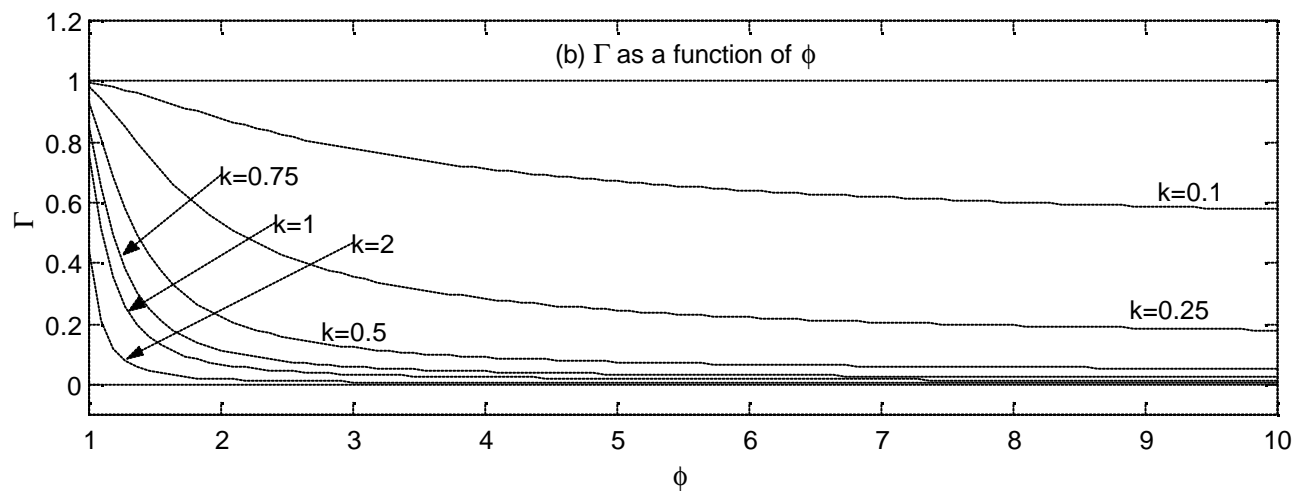
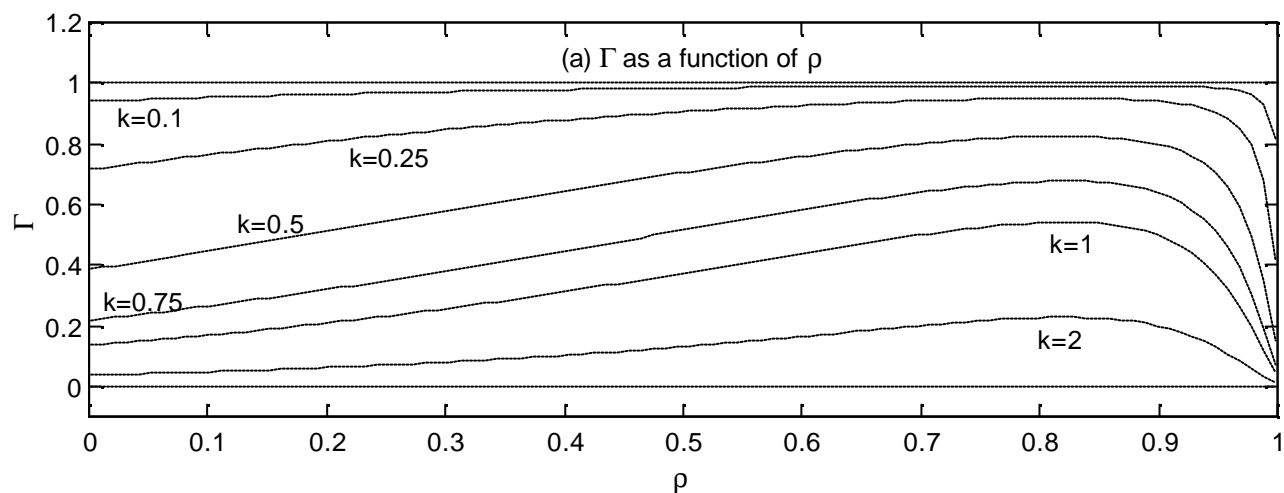


Chart 5: Theoretical cross-spectral statistics between the structural output gap and band-pass filtered output for Kiley's (1996) model, for different values of $k=S_u/S_v$

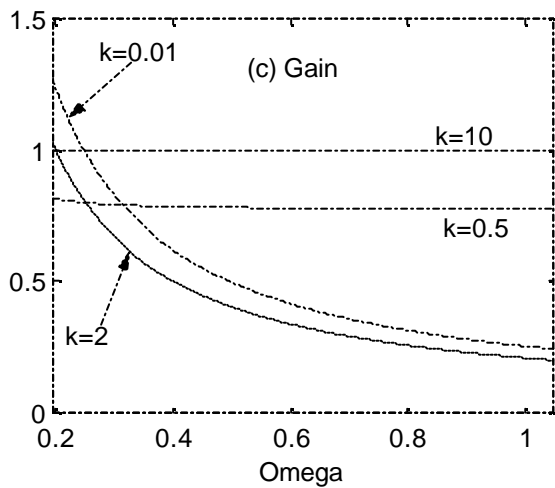
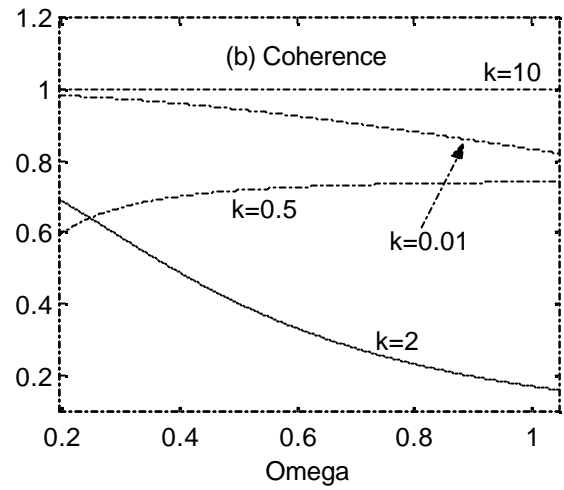
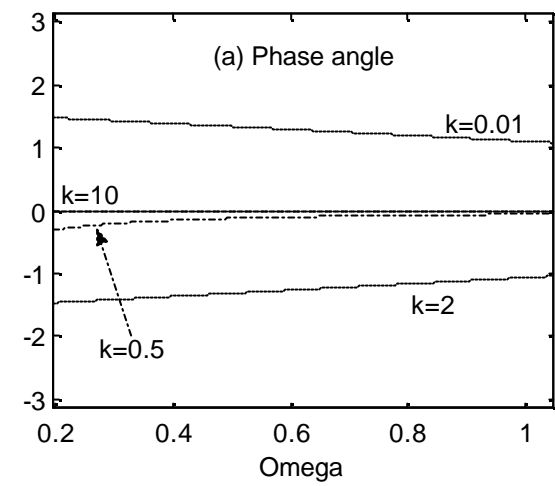


Chart 6: 'True' and 'false' business cycle stylised facts: the Phillips correlation between inflation and the output gap, based on Kiley's (1996) model, for different values of $k=S_u/S_v$, (theoretical cross-spectral statistics)

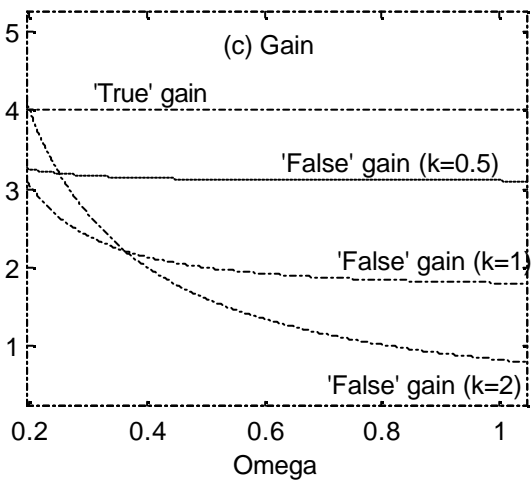
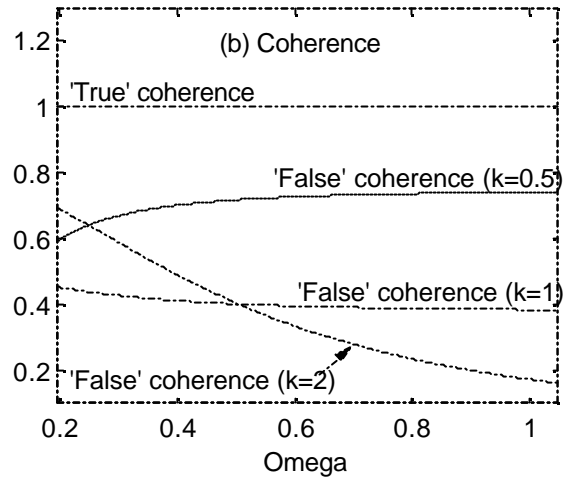
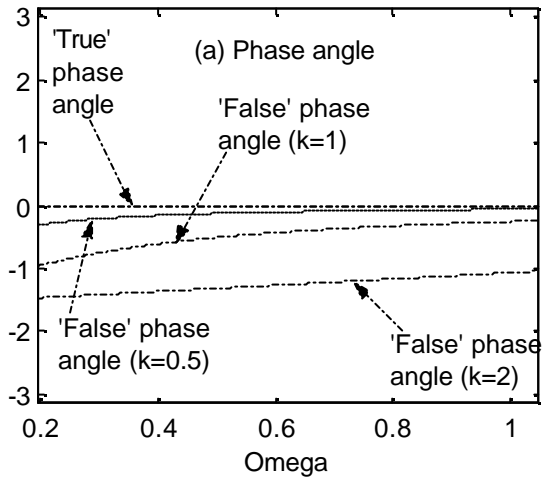


Chart 7: A false business cycle stylised fact within a simple AD-AS model: the ‘cyclical’ behaviour of productivity (theoretical cross-spectral statistics)

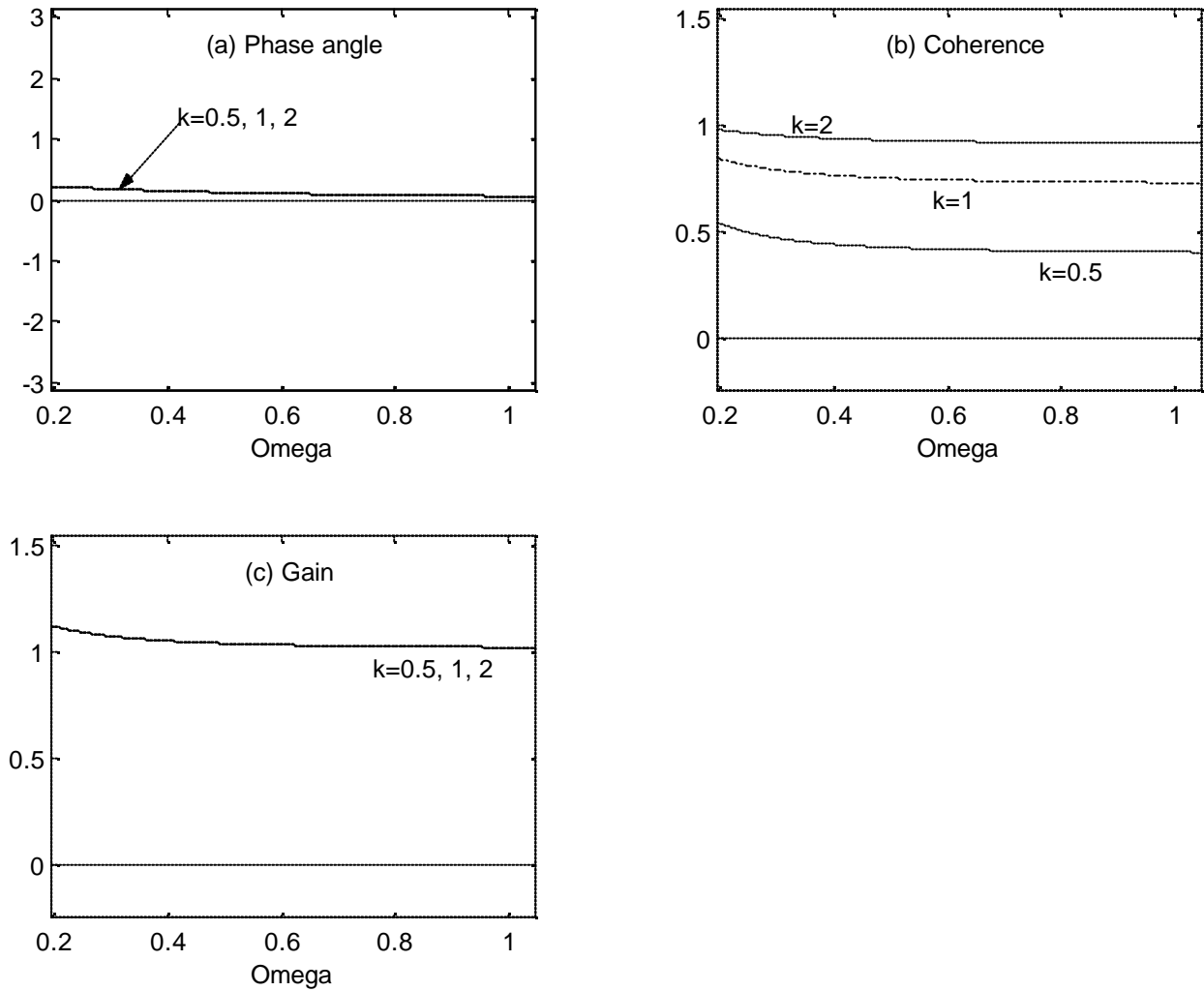


Chart 8: A false business cycle stylised fact within a simple AD-AS model: the ‘cyclical’ behaviour of productivity (cross-correlations and 95% confidence bands)

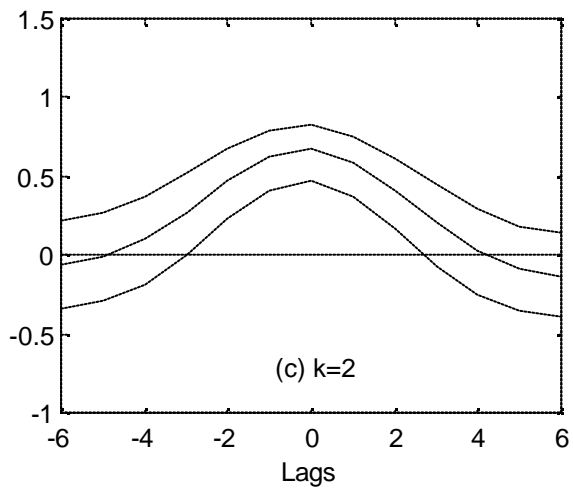
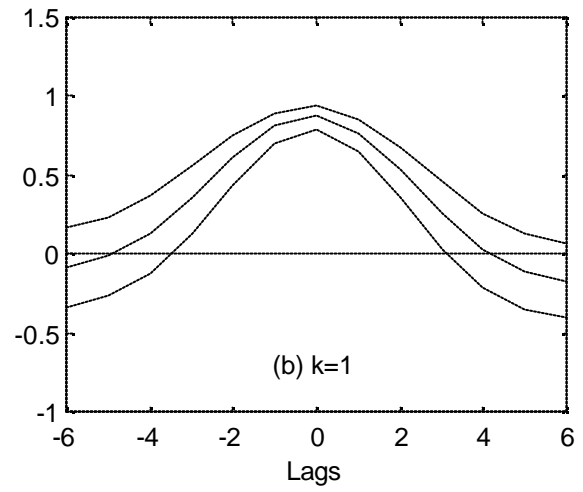
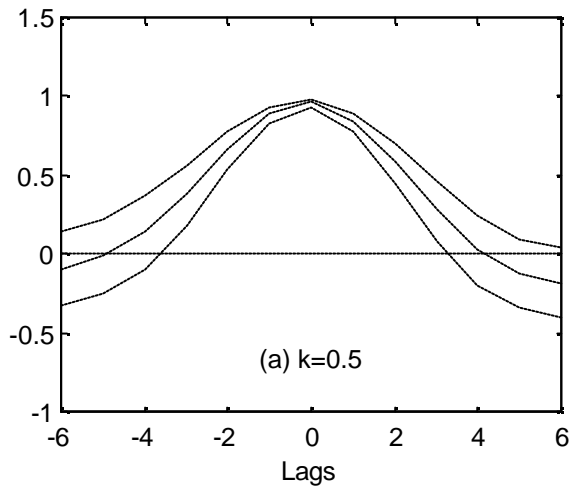


Chart 9: 'True' and 'false' business cycle stylised facts within a simple sticky-wage AD-AS model: the cyclical behaviour of the real wage

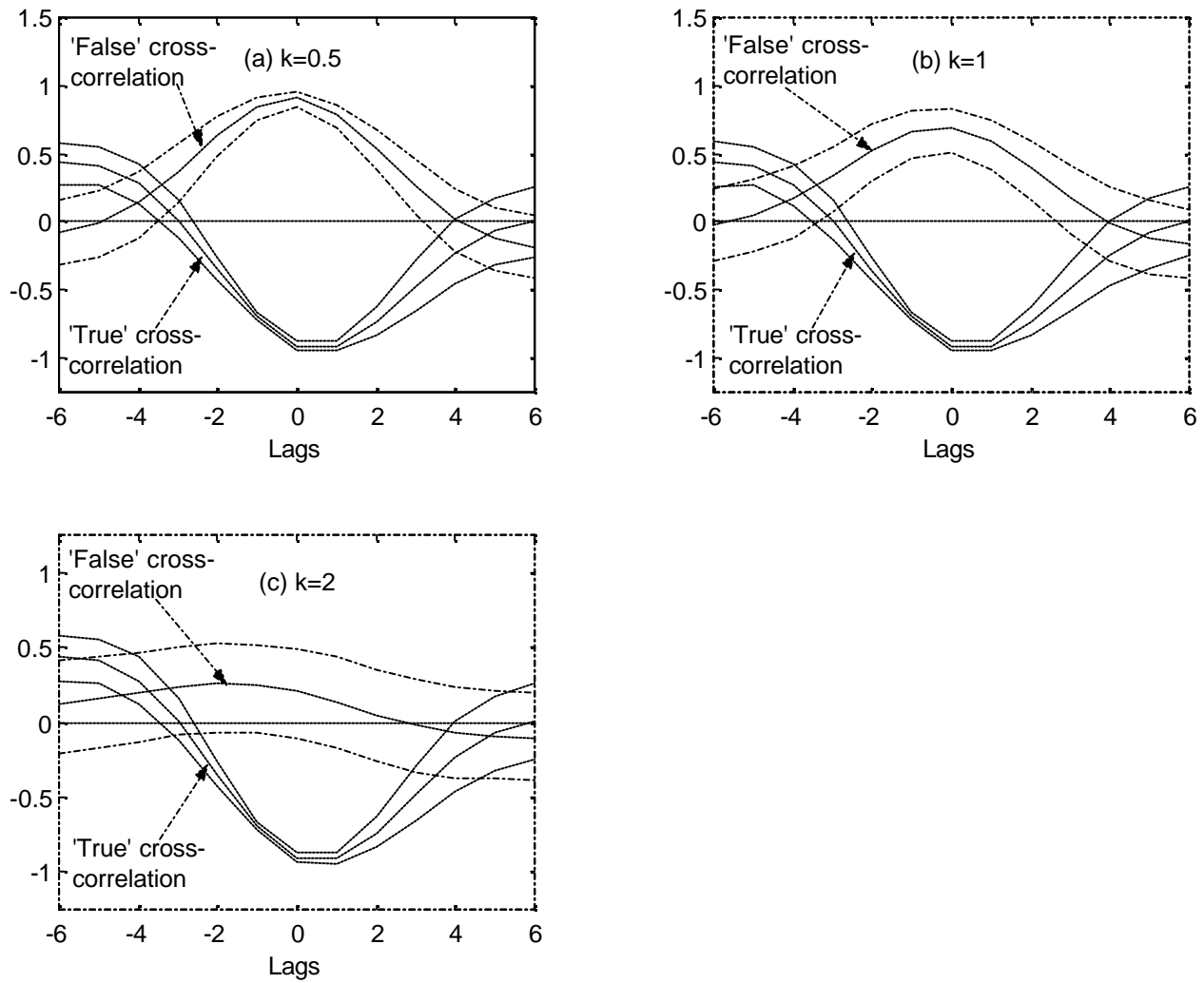


Chart 10: A false business cycle stylised fact within Kiley's (1996) model: the 'cyclical' behaviour of the money supply, for different values of $k=S_u/S_v$ (theoretical cross-spectral statistics)

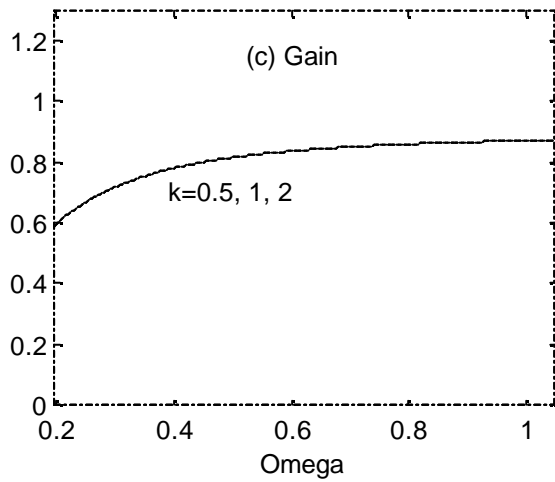
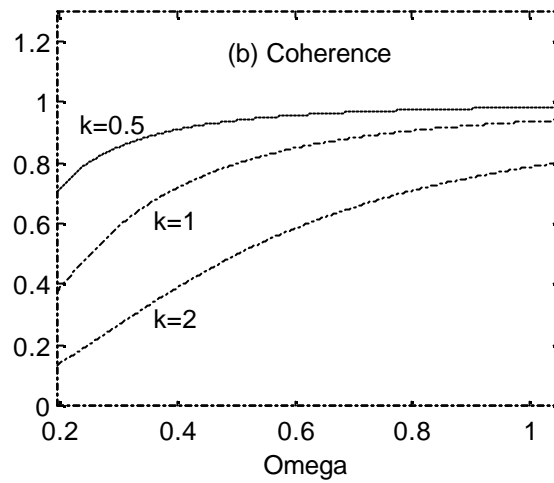
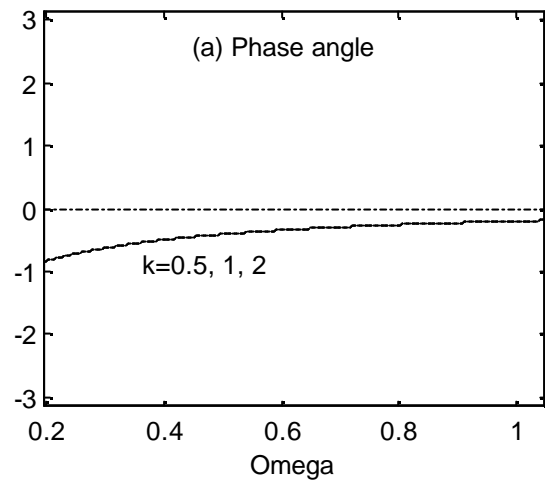


Chart 11: A false business cycle stylised fact within Kiley's (1996) model: the 'cyclical' behaviour of the money supply, for different values of $k=S_u/S_v$ (cross-correlations and 95% confidence bands)

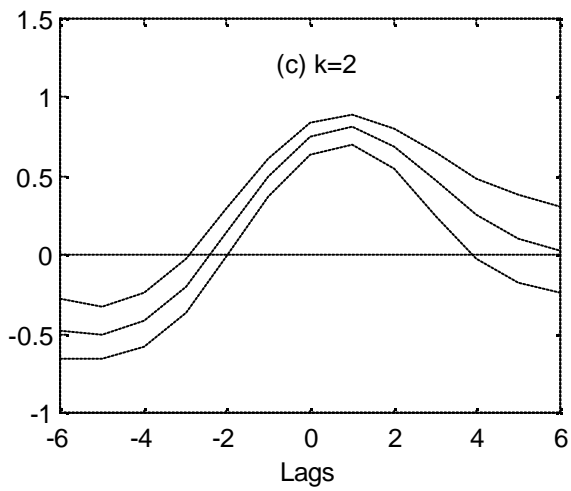
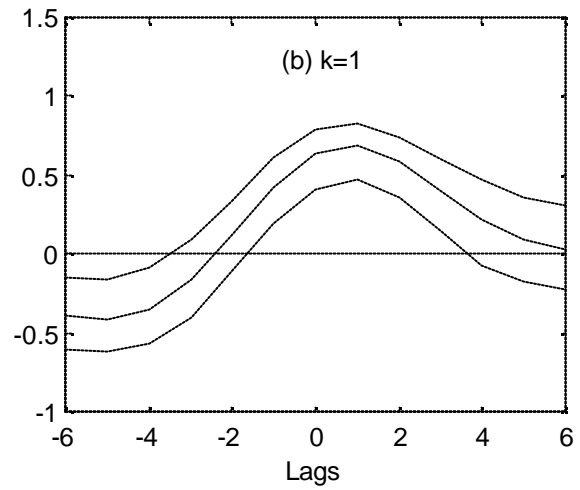
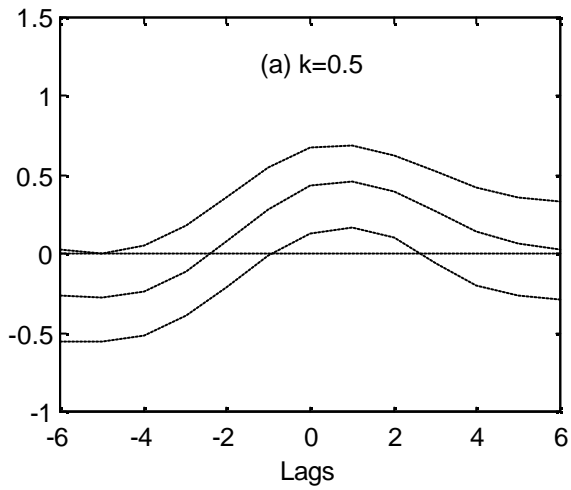


Chart 12: ‘True’ and ‘false’ business cycle stylised facts: the cyclical behaviour of prices, based on Kiley’s (1996) model, for different values of $k=S_u/S_v$ (theoretical cross-spectral statistics)

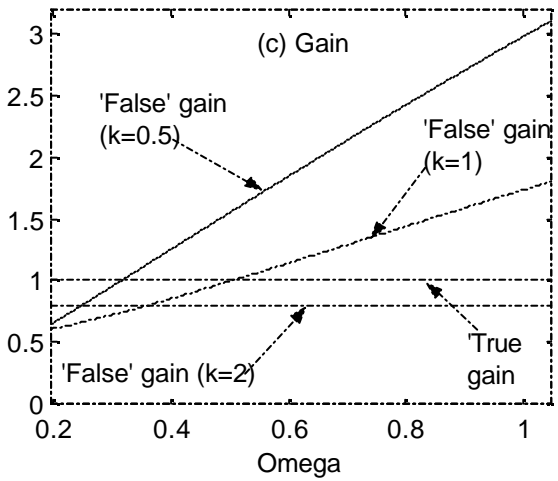
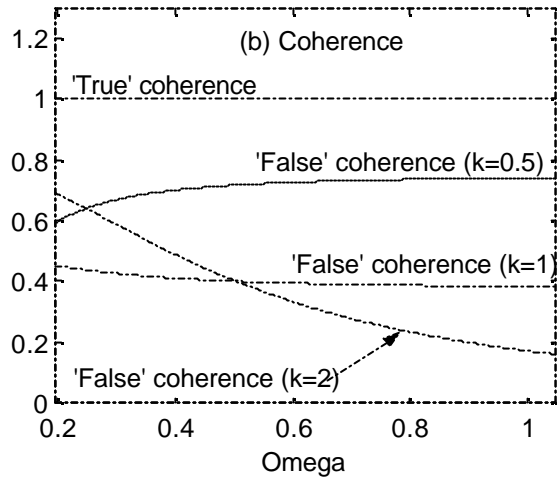
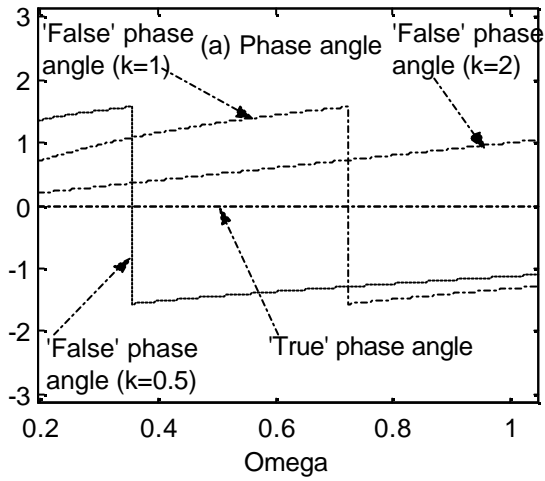


Chart 13: Real exchange rates and real interest rate differentials, true and false cross-spectral statistics, based on Campbell and Clarida (1987); (US dollar/Canadian dollar, 1979:10-1986:3)

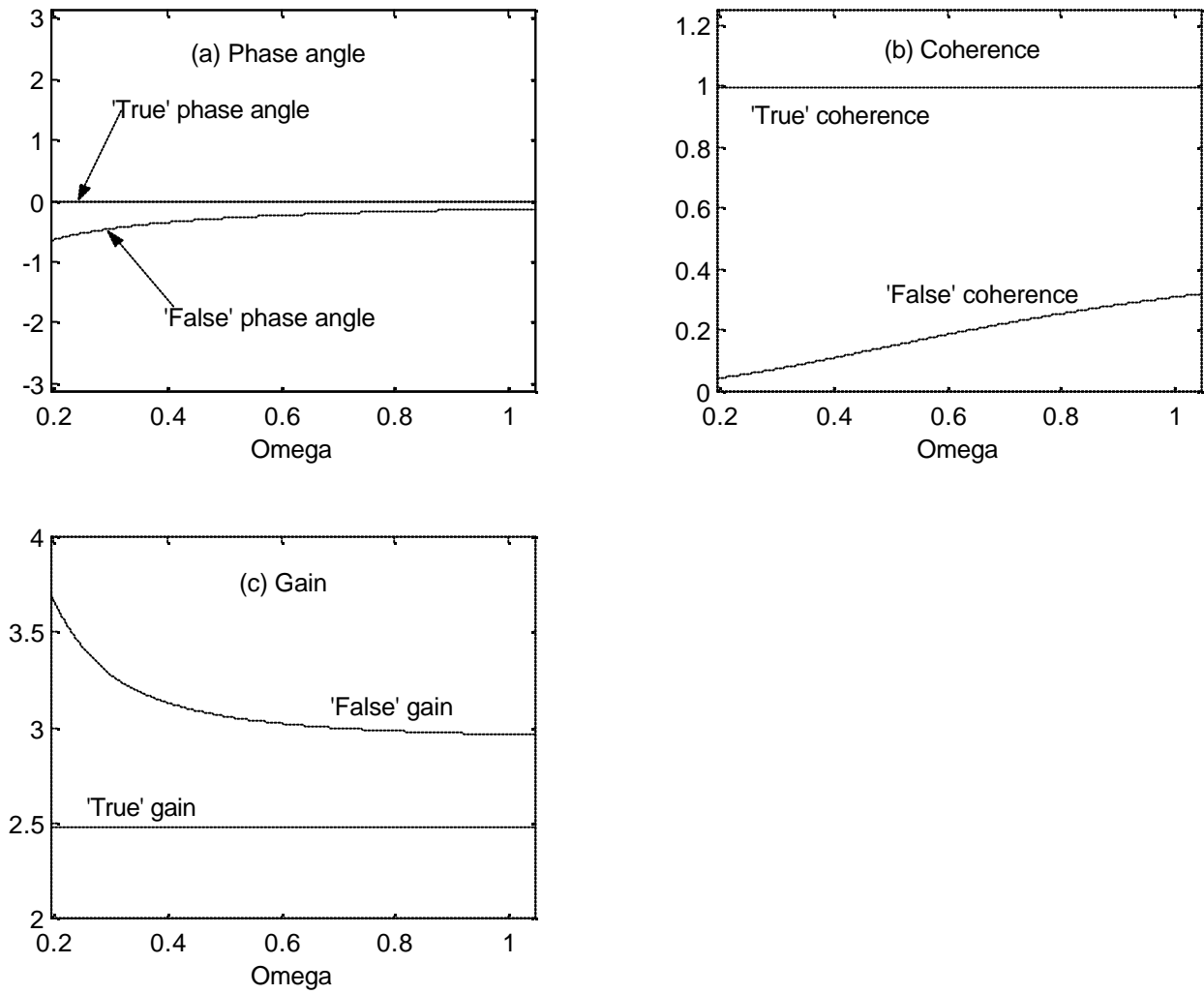
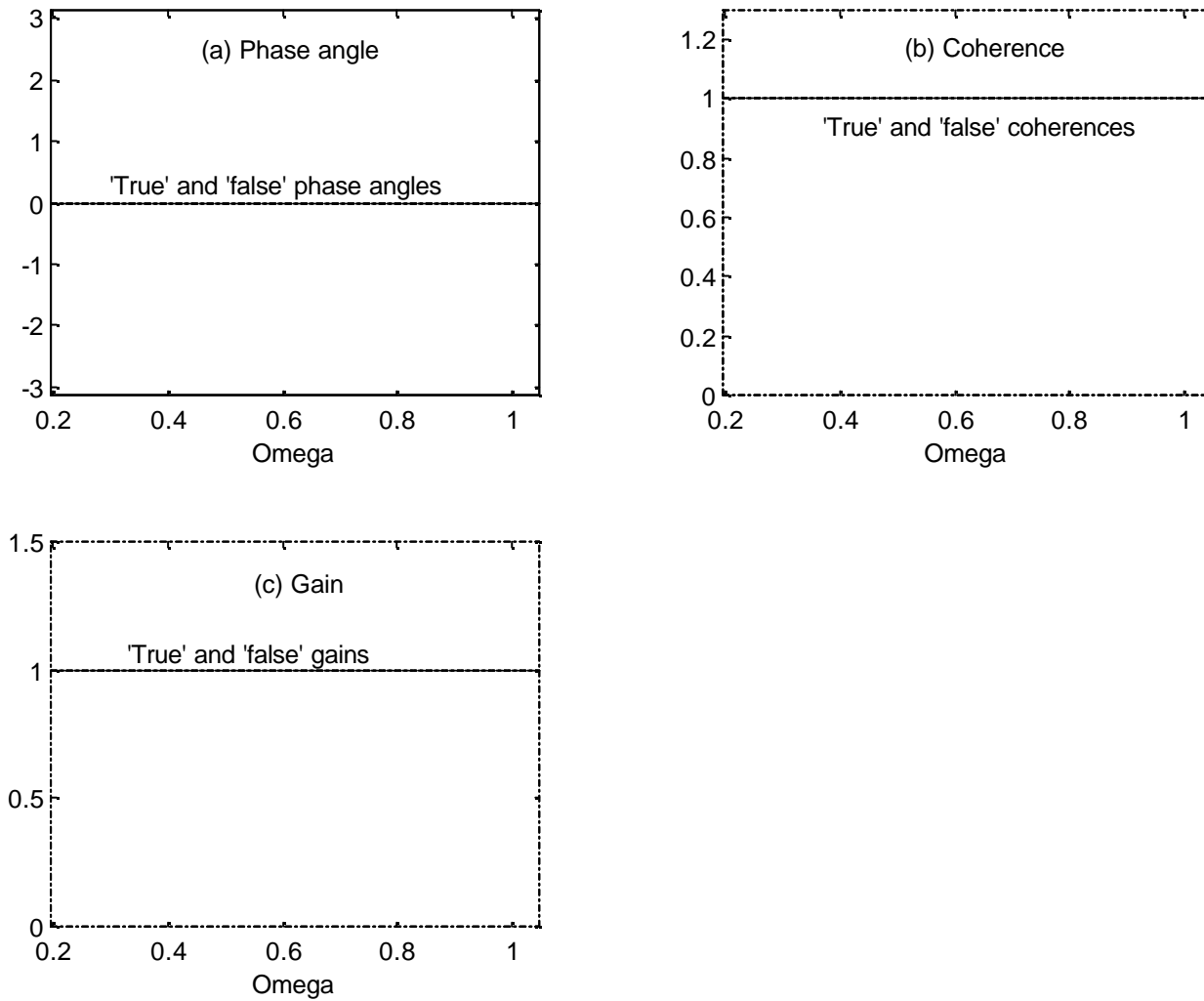


Chart 14: A case in which ‘true’ and ‘false’ business cycle stylised facts coincide within Kiley’s (1996) model: the cyclical behaviour of real balances, for different values of $k=S_u/S_v$ (theoretical cross-spectral statistics)



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