Productivity versus welfare:

or, GDP versus Weitzman's NDP

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Abstract

How should productivity and welfare be measured when the composition of the capital stock is shifting towards assets with shorter lives? What sort of adjustment, if any, should be made for depreciation? While GDP is still appropriate as a measure of output, I argue that Weitzman's NDP (WNDP) — nominal net domestic product deflated by the price index for consumption — is the appropriate measure of welfare. The rate at which the WNDP frontier is shifting out over time is analogous to the rate of growth of aggregate total factor productivity (TFP). Like the latter, it can be decomposed into the contributions made by TFP growth in individual industries, though with a different pattern of weights. The argument is illustrated by the experience of the United States in the 1990s. Here net investment increased more rapidly than gross investment and both grew faster than GDP, while the aggregate depreciation rate rose. Nevertheless the aggregate capital stock grew more slowly than GDP, and depreciation as a proportion of GDP was flat. Both official NDP and WNDP have been growing a little more slowly than GDP. But the acceleration of WNDP post 1995 was as great as that of GDP. Also, the rise in the growth rate of the WNDP frontier was equal to that of aggregate TFP.

Summary

Traditionally, productivity at the aggregate level has been measured using GDP, ie the measure of output is gross of depreciation. But suppose that the composition of the capital stock is shifting towards assets with shorter lives, so that the average depreciation rate is rising. This suggests that some part of what GDP measures as an increase in output may be illusory: some of the extra output is needed just to maintain the capital stock at its existing level. This question is given a sharper focus by the experience of the United States in the 1990s, where the growth rates of labour productivity and total factor productivity (TFP) rose, while investment shifted towards short-lived ICT assets. This raises the possibility that the US productivity improvement might be just a statistical illusion.

This paper has a theoretical and an empirical part. In the theoretical part, I compare measures of productivity with measures of welfare. I conclude that while GDP is satisfactory as a measure of *output*, it is outclassed as a measure of *welfare* by what I call Weitzman's NDP (WNDP). This is nominal net domestic product (consumption plus *net* investment) deflated by the price index for consumption. I extend the theory behind WNDP to the case where TFP growth can vary across sectors. This is the empirically relevant case for analysing recent US experience.

The aggregate TFP growth rate is the rate at which the GDP frontier is shifting out over time. This can be decomposed into a weighted average of the TFP growth rates in the various industries. Analogously, we can define the rate at which the WNDP frontier is shifting out over time. I call this the growth rate of total factor welfare (TFW). Like aggregate TFP growth, TFW growth can also be decomposed into a weighted average of TFP growth rates in the various industries, but the weights are not the same as for the GDP frontier. Hence the growth of welfare over time can be analysed using the same tools as have been developed for the analysis of the growth of output.

In the empirical part of the paper, I apply some of these ideas to the experience of the United States in the 1990s. In principle, one might expect WNDP to have grown more slowly than GDP over this period, for several reasons. First, the weight on consumption is higher in WNDP (or in NDP) than in GDP and consumption has been growing more slowly than investment. Second, the relative price of investment goods has been falling and this reduces WNDP growth. Third, one might have expected depreciation to have risen as a proportion of GDP, thus raising the share of consumption in WNDP still further.

In practice, WNDP has grown a bit more slowly than GDP. But the gap between the two growth rates was actually somewhat larger in the period 1973-90 than it was post 1990. And the acceleration of WNDP post 1995 was equal to that of GDP. The explanation is twofold. The ratio of depreciation to GDP has in fact been stable, despite the growing importance of short-lived assets. And net investment has grown more rapidly than gross investment. The growth rates of TFP and of TFW in the US non-farm business sector are also compared and found to be similar in the 1990s. Moreover, they display an almost identical increase after 1995.

GDP is a measure of output, not of welfare. So even if GDP had grown significantly faster than WNDP, this would not by itself suggest measurement error. In fact, the two have grown at similar rates in the 1990s and accelerated by the same amount. So it seems that, in practice, GDP has provided as reliable a measure of the improvement in US living standards over this period as WNDP, even though WNDP is conceptually superior as a welfare measure.

1. Introduction

There has been a long controversy in economics over when it is appropriate to measure output gross of depreciation and when it should be measured net. To get an idea of a nation's or a person's standard of living at a point in time, it clearly makes sense to subtract depreciation from *income* in current prices. But should *real output* be measured net of depreciation? In the present paper, I argue that gross output is the natural measure for productivity analysis, while a particular concept of net output due to Weitzman is appropriate for welfare analysis. But I also argue that the choice between net and gross output at the aggregate level is not an absolute one. One can apply growth accounting to both net and gross output: the same factors, such as TFP growth in individual industries, can help to explain both the growth of output and the growth of welfare.

The basic concept of productivity is output per unit of input. This is unproblematic if there is only one output (eg bread) and only one input (eg labour). But even for a single product there are normally many inputs, eg bread-making requires intermediate inputs like flour and energy and durable assets like bread ovens. And we frequently wish to make statements about productivity at the industry level or at the whole-economy level. For these purposes, we need (at the very least) measures of aggregate output.

At the whole-economy level, GDP is the most widely used measure of output. GDP in current prices can be calculated either as the sum of final expenditures on consumption and investment goods or as the sum of value added in different industries. The growth of real output can then be calculated in either one of two ways: *either* as a weighted average of the growth rates of real expenditure on consumption and investment goods *or* as a weighted average of the growth rates of real value added. When done consistently (for instance by using double deflation to calculate real value added), these two approaches yield in principle an identical answer. In both approaches, output of investment goods like bread ovens is included.(1)

When we wish to explain the level or growth rate of output, the natural organising concept is the production function, defined in terms of *gross* output. A production function defined in terms of net output does not seem to make much sense, even in a one-good, 'corn' economy. As Hulten (1992) points out, productivity growth in bread-making surely affects all the output produced, not just the proportion classified as net output. When we move outside a one-good economy, the difficulties mount. Bread output can be measured as the number of loaves produced per period, but what is bread output net of depreciation on ovens?

It seems clear that there must be a connection between productivity and welfare or at least between productivity and potential economic welfare.(2) A sustained rise in economic welfare requires a sustained rise in productivity. But productivity and welfare are not identical. From a productivity viewpoint, it makes no difference whether a good is for consumption or investment. But from a welfare point of view only consumption matters; investment goods are desired only insofar as they can produce more consumption goods in future. In particular, it is *net* investment

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(1) Real GDP is not just an arbitrary empirical concept, but can be given an interpretation in terms of economic theory. In this theory, aggregate production frontiers play the role that utility functions have in the theory of the consumer price index: see Caves *et al* (1982), Diewert (1987) and Fisher and Shell (1998). Suppose there are n industries producing final goods in a closed economy with outputs $Q_1, \ \ldots \ , Q_n$. Then the production possibility frontier can be written as $f(Q_1, \ \ldots \ , Q_n, t) = 0$ where t is time. This shows the maximum amount of any one output that can be produced given all the other outputs. The rate at which this frontier is shifting over time is precisely what an output index is designed to measure and is the theoretical counterpart of GDP growth.

(2) Welfare has many dimensions, eg human rights, equality, health, personal security and autonomy. These may or may not be correlated with economic welfare, that is, the welfare derivable from the goods and services measured in the national accounts.

in investment goods that matters for welfare, since net investment increases future consumption. The part of gross investment that simply maintains the productive capacity of the existing capital stock at its current level does not add to welfare. This suggests that to measure welfare we should subtract depreciation (capital consumption) from GDP to obtain net domestic product (NDP).

It turns out, as will be shown below, that NDP as conventionally measured by statistical agencies is not quite the right concept for welfare measurement either. The reason is that the growth of conventional NDP is a weighted average of the growth rates of consumption and net investment, where both consumption goods and investment goods are measured in their own units. A better measure of welfare is Weitzman's net domestic product (WNDP): consumption plus *net* investment, measured in *consumption* units, ie deflated by the price index for consumption (Weitzman (1976, 1997)). This paper discusses the difference between these two concepts, WNDP and GDP. It analyses how their growth rates are related.(3)

The growth rate of total factor productivity (TFP) is the rate at which the GDP frontier is shifting out over time. This paper shows how an analogous concept, the rate at which the WNDP frontier is shifting over time, can be defined. The rate at which the WNDP frontier is shifting is the rate at which welfare would change, holding constant the inputs. This new concept, which I call the growth rate of total factor welfare (TFW), is characterised and related to the more familiar TFP growth rate.

Preferences as expressed in the pattern of expenditure obviously matter in measuring welfare. But the same is true of productivity at the aggregate level. At the single-product or single-process level, productivity is a technical concept. But once we move beyond a single product to the industry or whole-economy level, matters grow a little more complicated. Aggregate output or productivity growth is an expenditure-weighted average. Expenditures depend on preferences, so preferences and not just technology influence the aggregate measure. But one contribution of this paper is to show how the growth of welfare, like the growth of aggregate productivity, depends on productivity growth in individual industries. It turns out that the growth of TFW, like that of aggregate TFP, is a weighted average of the growth rates of TFP in individual industries, though the weights are different.

The context of the present study is the rise in the growth rate of GDP per hour worked in the United States in the second half of the 1990s. This productivity acceleration was accompanied by an investment boom in ICT goods like computers and software. The boom was fuelled by a massive fall in the relative price of these goods. The share of ICT investment in total investment rose (Jorgenson and Stiroh (2000); Oliner and Sichel (2000)). But since computers and software have relatively short economic lives, one might expect that aggregate depreciation was rising too. Since welfare is related to net, not gross, investment, it is far from obvious that welfare, even potential welfare, was rising at the same rate as productivity.(4)

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(3) Hulten (1992) is a general discussion of the net output/gross output controversy with references to the earlier literature.

(4) In a recent speech, Mervyn King has drawn attention to the difficulties of interpretation which may arise when net and gross output are growing at different rates and has called for more research (King (2001)). In a wide-ranging attack on the notion of a new economy, Kay (2001a) has claimed that US productivity statistics are seriously distorted by the use of GDP. He follows up this claim in Kay (2001b). As will be clear from what follows, this paper reaches a different conclusion to Kay's. Both of us however agree on the usefulness of what is called here Weitzman's NDP.

Plan of the paper

Section 2 introduces and motivates Weitzman's concept of net domestic product. It discusses why the growth of WNDP may differ from that of GDP. It defines the concept of total factor welfare (TFW) growth, the rate at which the WNDP frontier is shifting out over time. It explains how this concept differs from the more familiar concept of aggregate TFP growth, which is the rate at which the GDP frontier is shifting out over time. Both TFP and TFW growth are shown to be weighted averages of TFP growth in individual industries, but the weights are not the same. Section 3 applies these various measures to the United States in the 1990s. The paths followed by gross and net investment, by depreciation and by capital stocks are analysed. Estimates of WNDP are presented and compared with GDP. The rates at which the two frontiers are shifting are compared. Section 4 concludes.

2. GDP, NDP and WNDP

Weitzman's net domestic product, denoted by Y, is defined as net domestic product,[(5)] *measured in consumption units*. Alternatively, it is nominal NDP deflated by the price index for consumption. In symbols,

$$Y(t) = C(t) + p(t)J(t) \qquad \textbf{(1)}$$

where C is real consumption, J is real *net* investment ($J = \dot{K}$ where K is the capital stock), and p is the relative price of investment goods in terms of consumption goods. For simplicity of exposition, I assume a single consumption good and a single investment good. The single consumption good could be thought of as a chain index of consumption goods. It is straightforward to extend the analysis to many investment goods: see below. At this point we can think of I and J as chain indices of many investment goods and p as the corresponding price index (relative to the consumption price index).[(6)]

The intuition behind WNDP is that only consumption matters for welfare. So the current level of consumption must obviously be part of the measure. In addition, net investment increases future consumption. The second term in **(1)** can be interpreted as the present value of the future stream of consumption generated by adding to the capital stock.

Weitzman (1976) gave a more formal justification. He showed that WNDP is proportional to the yield on wealth (assuming perfect competition and no externalities). That is, it is equivalent to permanent income. His argument was as follows. Consider an economy which behaves as if it is governed by a social planner who seeks to maximise wealth (W), defined as the present value of the stream of real consumption:

$$\max\ W(t) = \int_t^\infty C(s) \cdot e^{-r(s-t)} \cdot ds \qquad \textbf{(2)}$$

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(5) Net national product is Weitzman's term, though net national *income* might be more appropriate. I use the term 'net domestic product' to retain the link with the national accounts and because I do not adjust for net income from abroad. See Sefton and Weale (1996a) for extensions of Weitzman's analysis to an open economy and to trade in natural resources. Sefton and Weale (1996b) present unofficial estimates of WNDP for the United Kingdom.

(6) It might seem that measuring WNDP is easier than measuring GDP, since the only price we need to know is the price of consumption. But this would be an illusion. WNDP requires us to measure depreciation and to do this properly, we need to know the quality-adjusted prices of investment goods and ideally second-hand asset prices too (the latter in order to estimate depreciation rates).

subject to the production set and given initial capital stocks; here r is the rate of discount (real rate of interest). Along the optimal path, Weitzman showed that

$$Y(t) = r \cdot \int_t^{\infty} C(s) \cdot e^{-r(s-t)} \cdot ds = r \cdot W(t) \qquad \textbf{(3)}$$

So WNDP (= Y) is the yield on wealth (the present value of consumption), ie it is equivalent to permanent income (see Theorem 2 and Remark 2 in the Annex). WNDP can therefore be considered a cardinal measure of welfare. This is important because WNDP is directly observable: only current prices and quantities enter into WNDP, while measuring wealth directly would require forecasting an infinite stream of future consumption.(7) Note that WNDP as defined in equation **(1)** is the Hamiltonian associated with the problem of equation **(2)**, provided that we can equate actual market prices with shadow prices.

Despite its theoretical importance, Weitzman's NDP is not generally calculated by statistical agencies. In the United Kingdom for example, it is possible to derive net domestic product at 1995 basic prices from published series.(8) And the ONS publishes 'net national disposable income at 1995 market prices'. This concept equals GDP at 1995 market prices adjusted for fixed capital consumption, net income from abroad, and for gains or losses on the terms of trade. But in both these cases, investment, imports and government expenditure are valued in their own base year prices, not in consumption units.

Two possible qualifications to the use of WNDP as a measure of welfare should be noted. First, a constant real interest rate is used in equation **(2)** which is only appropriate in a steady state. However, it is possible to derive an analogue of Weitzman's result, equation **(3)**, when the real interest rate varies over time (see equation **(44)** and Remark 2 in the Annex). It turns out that WNDP can still be interpreted as permanent income.

The second qualification is more substantial. Weitzman's original result was derived on the assumption that TFP growth is zero.(9) Weitzman (1997) derives a more general result, when TFP growth is allowed to be non-zero (see also Löfgren (1992) for a related result). In a steady state, he shows that

$$Y(t) = \left[\frac{r-g}{r-g+\lambda} \right] rW(t) \qquad \textbf{(4)}$$

where λ is the (constant) rate at which the WNDP frontier is shifting out, holding the inputs constant (more on this below), and g is the overall steady-state growth rate of WNDP; see Theorem 2 and the Corollary to this theorem in the Annex for a proof.(10)

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(7) Others, eg Usher (1980) and Scott (1990), have also suggested that, when measuring real income, investment should be deflated by the price of consumption goods. The original idea goes back to Hicks (1939, chapter XIV) and (1940). But Weitzman was the first to put this suggestion on a clear theoretical footing. See also Heal (1998, chapters 11 and 12) for a discussion of the connections between Hicksian income and welfare in the context of depletable or renewable natural resources.

(8) This is 'gross value added at 1995 basic prices' [ABMM] *minus* 'fixed capital consumption at 1995 prices' [YBFX].

(9) Alternatively, it can be interpreted as applying when national income accounting is 'complete': all possible sources of growth are accounted for, including knowledge stocks and 'atmospheric' factors which influence productivity (Weitzman (2002)). The disadvantage of this interpretation is that it cannot be applied empirically.

(10) That is, overall growth of WNDP is a result of input growth and also of the shift in the WNDP frontier with inputs held constant; the latter factor is symbolised by λ.

It follows from **(4)** that (if λ is positive), the *level* of WNDP will underestimate the yield on wealth. But the *growth rate* of WNDP will still correctly measure the growth rate of wealth, provided that the real interest rate and the rate at which the WNDP frontier is shifting are constant.

To illustrate, consider the problem of measuring the growth rate of wealth in the United States over the period 1990-95. With the benefit of hindsight, we now know that TFP growth rose in the subsequent period 1995-2000 (see eg Oliner and Sichel (2000)). Hence over 1990-95 wealth was rising more rapidly than WNDP. This is because, in 1990-95 and relative to the 1980s, the era of higher TFP growth is now getting closer and so looms larger in present value terms.[(11)] But if we assume (heroically) that TFP growth will be constant from now on at its 1995-2000 level, then the growth of wealth over 1995-2000 is measured exactly by the growth of WNDP.

So far WNDP has been presented as equivalent to permanent income. It is natural to ask too, what is the connection between permanent income and some utility-based measure of welfare? It might be objected that Weitzman's main result rests on the rather restrictive assumption that the social planner maximises the present value of consumption. Surely a more general assumption is that the social planner maximises the present value of the *utility* from consumption?[(12)] This criticism is incorrect. As the Annex shows, maximising the present value of utility is equivalent to maximising wealth (the present value of consumption), provided that consumption is discounted at the interest rate (or rates) which maximises utility.[(13)] However the Annex also shows that WNDP and the present value of utility will not necessarily produce the same ranking of alternative situations. Ranking by WNDP assumes in effect that the marginal utility of income is the same in the situations being compared. To the extent that this is not the case, the rankings may differ. This difficulty is not of course unique to WNDP; it affects all money-based indicators of welfare (including GDP).

Comparing the growth rates of GDP and WNDP

A Divisia index of the growth of WNDP is

$$\hat{Y}(t) = v(t)\hat{C}(t) + [1 - v(t)]\hat{J}(t) + [1 - v(t)]\hat{p}(t) \qquad \textbf{(5)}$$

where $v(t) = C(t)/[C(t) + p(t)J(t)]$ is the nominal share of consumption in WNDP, and a hat (^) denotes a growth rate (a total logarithmic derivative with respect to time), eg $\hat{Y} = d\ln Y/dt$.

Now consider a Divisia index of the growth of real GDP (Z). In nominal terms GDP is the sum of consumption and *gross* investment.[(14)] So the growth of real GDP is:

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(11) This assumes that the effect of any changes in the real interest rate can be disregarded.
(12) More general still would be to take account of leisure. I do not make this extension here since I wish to relate the theoretical measures to the national accounts. Of course, the latter do not adjust for changes in leisure either.
(13) This conclusion goes back to Fisher (1930), as Hulten (1992) points out. However, there is a qualification. It is reasonable to discount utility at a constant rate. But the real interest rate generated by a utility-maximising program will not be constant over time, unless the economy is in steady state: see the Annex.
(14) The difference between gross investment and net investment is called capital consumption by national income statisticians and replacement investment by economists. It is also often called depreciation and for brevity I follow this usage here. But it should be noted that replacement and depreciation are not the same concepts. Depreciation is the loss in value experienced by an asset as it ages, while replacement is the amount which must be invested to restore the stock of some asset to its previous level of productive capacity. The difference is clearest in the case where the productive capacity of an asset follows the 'light bulb' pattern, ie constant up till the moment of failure. In this case, the asset falls in value with age, since there are progressively fewer years over which profits can be earned. But replacement is zero up till the moment of failure. Hence the difference between replacement investment and

$$\hat{Z}(t) = w(t)\hat{C}(t) + [1 - w(t)]\hat{I}(t) \qquad \textbf{(6)}$$

where *I* is *gross* investment and $w(t) = C(t)/[C(t) + p(t)I(t)]$ is the nominal share of consumption in GDP.

There are three differences between the equation for WNDP growth, equation **(5)**, and that for GDP growth, equation **(6)**. First, the weight for consumption is lower in the GDP equation: $w < v$ since gross investment exceeds net investment ($I > J$). Second, the WNDP equation contains the growth rate of net investment while the GDP one contains that of gross investment. And third, there is a relative price term in **(5)** which is not present in **(6)**.

In a one-good, Solow model, in steady state, none of these differences would matter. Gross investment, net investment and consumption would all grow at the same rate and the relative price term would be zero by assumption. Hence GDP and WNDP would grow at the same rate. Out of steady state, gross investment and net investment may grow at different rates, so GDP and WNDP growth may diverge even in this simple model.

In practice, there is a long-run tendency in the United States for the relative price of investment goods to fall and for investment to grow more rapidly than consumption (Greenwood *et al* (1997); Whelan (2001)). This suggests that the Solow model may be a misleading guide even to the long run. This tendency has been exacerbated as ICT investment has grown in importance. So the relative price term in equation **(5)** is usually negative, and increasingly so in recent years.

The difference between the growth rates of GDP and WNDP is found by subtracting equation **(6)** from equation **(5)**:

$$\hat{Z} - \hat{Y} = [(w - v)\cdot(\hat{C} - \hat{I}) + [(1 - v)\cdot(\hat{I} - \hat{J})] - (1 - v)\cdot\hat{p} \qquad \textbf{(7)}$$

Now $w < v$ and under current circumstances $\hat{C} < \hat{I}$ and $\hat{p} < 0$. So the first and third terms on the right-hand side of equation **(7)** are positive, which tends to make GDP grow more rapidly than WNDP. However, as we shall see, in the United States net investment has been growing more rapidly than gross investment $(\hat{J} > \hat{I})$.[(15)]

The WNDP frontier

Aggregate TFP growth is the rate at which the GDP frontier is shifting out over time. It is natural therefore to ask whether we can define an analogous concept, the rate at which the WNDP frontier is shifting. The answer is yes. This section shows how to do this and how the two concepts are related to each other.[(16)]

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depreciation for the stock of such an asset as a whole depends on the age structure of the stock. However, in the theoretical part of this paper I assume that the depreciation rate is geometric and in this case depreciation and replacement are identical (Jorgenson (1989); Oulton (2001b)). Also, the empirical part of the paper uses the US NIPA and these assume by and large geometric depreciation (U.S. Department of Commerce (1999)).

(15) If we wanted to explain the difference between GDP growth and that of official NDP, we could use the right-hand side of equation **(7)**, but with the term $(1 - v)\hat{p}$ omitted.

(16) Both Löfgren (1992) and Weitzman (1997) employ the concept of a shifting WNDP frontier. But to the best of my knowledge its properties have not been described before. Hulten (1992) argues that the growth of net output cannot be decomposed into factor input growth and a residual in the same way as gross output can be. But his objection is really to a net output production function at the level of a single product. As the argument will show, at the aggregate level we can analyse the shift in the WNDP frontier using concepts derived from standard production theory.

With m capital goods, WNDP can be written

$$Y = C + \sum_{i=1}^{m} p_i \dot{K}_i = C + \sum_{i=1}^{m} p_i I_i - \sum_{i=1}^{m} \delta_i p_i K_i \qquad \textbf{(8)}$$

Here δ_i is the depreciation rate of the ith capital good and p_i is its relative price: $p_i = P_i / P_c$ where (P_i, P_c) are the nominal prices of consumption and capital good i respectively.

We assume the existence of neo-classical production functions with constant returns to scale in the m + 1 industries:

$$\begin{aligned} C &= A \cdot L_c \cdot f^c(K_{1c}/L_c, K_{2c}/L_c, ..., K_{mc}/L_c) \\ I_i &= A \cdot B_i \cdot L_i \cdot f^i(K_{1i}/L_i, K_{2i}/L_i, ..., K_{mi}/L_i), \quad i = 1, ..., m \end{aligned} \qquad \textbf{(9)}$$

Here output of (and gross investment in) the ith investment good is I_i. The capital input levels in the jth industry are $(K_{1j}, ..., K_{mj})$, $j = c, 1, 2, ..., m$, and the labour inputs in the m + 1 industries are $(L_c, L_1, ..., L_m)$. The level of TFP in the consumption good industry is A; the level of TFP in the ith investment good industry is $A \cdot B_i$. The growth rates of TFP are $\hat{A}$ in the consumption sector and $\hat{A} + \hat{B}_i$ in the ith investment goods industry.[17]

Given these production functions, there exist also dual price (unit cost) functions with the following properties:

$$\begin{aligned} P_c &= A^{-1} \cdot P_L \cdot g^c(P_{K1}/P_L, P_{K2}/P_L, ..., P_{Km}/P_L) \\ P_i &= A^{-1} \cdot B_i^{-1} \cdot P_L \cdot g^i(P_{K1}/P_L, P_{K2}/P_L, ..., P_{Km}/P_L), \quad i = 1, ...m \end{aligned} \qquad \textbf{(10)}$$

where P_L is the wage and $(P_{K1}, P_{K2}, ..., P_{Km})$ are the rental prices of the m types of capital.

By duality,

$$\begin{aligned} -\partial \ln P_c / \partial t &= \hat{A} = \partial \ln C / \partial t \\ -\partial \ln P_i / \partial t &= \hat{A} + \hat{B}_i = \partial \ln I_i / \partial t, \quad i = 1, ..., m \end{aligned} \qquad \textbf{(11)}$$

Hence

$$\partial \ln p_i / \partial t = -\hat{B}_i, \quad i = 1, ..., m \qquad \textbf{(12)}$$

The rate at which the WNDP frontier is shifting, denoted by λ, is the rate at which Y grows over time, if the inputs (labour and the capital stocks in every industry) are held constant.[18] By analogy with TFP, I call this the growth rate of total factor welfare (TFW). From **(8)**:

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(17) Models of this type have been studied by Bakhshi and Larsen (2001).

(18) Note that this does *not* imply that the $\dot{K}_i$ are constant. In fact, $\partial \dot{K}_i / \partial t = \partial I_i / \partial t$.

$$\lambda = \frac{\partial \ln Y}{\partial t} = \left(\frac{C}{Y}\right) \cdot \left(\frac{\partial \ln C}{\partial t}\right) + \sum_{i=1}^{m} \left(\frac{p_i I_i}{Y}\right) \cdot \left(\frac{\partial \ln I_i}{\partial t}\right) + \sum_{i=1}^{m} \left(\frac{p_i \dot{K}_i}{Y}\right) \cdot \left(\frac{\partial \ln p_i}{\partial t}\right) \quad \textbf{(13)}$$

Now define d_i as the ratio of depreciation on capital of type i to NDP in current prices:

$$d_i = \frac{\delta_i p_i K_i}{Y} \quad \textbf{(14)}$$

and define the aggregate depreciation ratio as

$$d = \sum_{i=1}^{m} d_i \quad \textbf{(15)}$$

Then we can rewrite the equation for λ in simpler form as

$$\lambda = (1 + d) \cdot \hat{A} + \sum_{i=1}^{m} d_i \cdot \hat{B}_i = \hat{A} + \sum_{i=1}^{m} d_i \cdot (\hat{A} + \hat{B}_i) \quad \textbf{(16)}$$

Two comments on this result are in order.[19] First, the rate at which the WNDP frontier shifts depends, via d, on the capital-output ratio, which depends in turn on preferences as well as technology: more patient consumers will want to hold a higher level of capital in relation to income. Second, if we want to *analyse* the growth of welfare, and not just to measure it, we need to know TFP growth rates in the different industries. So, though WNDP excludes depreciation, to analyse its growth we need the industry production functions, and the production function is a gross output concept.

Next, we want to compare λ, the growth rate of TFW, with the more familiar aggregate TFP growth rate, which we denote by μ. The latter is the rate at which the GDP frontier is shifting, ie it is the growth rate of GDP holding the inputs constant:

$$\mu = \frac{\partial \ln Z}{\partial t} = \left(\frac{C}{p_z Z}\right) \cdot \hat{A} + \sum_{i=1}^{m} \left(\frac{p_i I_i}{p_z Z}\right) \cdot (\hat{A} + \hat{B}_i) \quad \textbf{(17)}$$

In other words, aggregate TFP growth is a weighted average of TFP growth in the consumption and investment goods industries, where the weights are the output shares of each industry in nominal GDP.[20] Now define $s_i = p_i I_i / p_z Z$, the ratio of gross investment in capital good i to GDP in current prices and $s = \sum_{i=1}^{m} s_i$, the aggregate gross investment/GDP ratio. Then equation **(17)** can be written more compactly as

$$\mu = \hat{A} + \sum_{i=1}^{m} s_i \cdot \hat{B}_i = (1 - s) \cdot \hat{A} + \sum_{i=1}^{m} s_i \cdot (\hat{A} + \hat{B}_i) \quad \textbf{(18)}$$

When we compare the rate of growth of the WNDP frontier **(16)** with that of the GDP frontier **(18)**, we can see that in general there is no presumption that one will necessarily grow faster than the other. The coefficient on $\hat{A}$ is larger in the WNDP frontier, so this tends to make WNDP

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[19] Analogously, we could derive an expression for the rate at which the 'official NDP' frontier is shifting. The difficulty with this is that official NDP is a hybrid concept. It is neither a measure of output nor one of welfare.
[20] This is an illustration of the principle of Domar aggregation (Oulton (2001a)).

shift more rapidly than GDP. In fact, if TFP grew at the same rate in all industries $(\hat{B}_i = 0, \text{ all } i)$, then the growth rate of the two frontiers would be related as

$$\lambda = (1+d)\cdot\hat{A} = (1+d)\cdot\mu > \mu \qquad \textbf{(19)}$$

On the other hand, if TFP growth is faster in investment goods industries, then it is possible, though by no means certain, that the GDP frontier is shifting more rapidly.

Depreciation plays a dual role in the growth rate of the WNDP frontier. First, the larger is depreciation as a proportion of NDP, the larger the coefficient on $\hat{A}$ in equation **(16)**. This tends to make the WNDP frontier grow faster than the GDP one. But this is simply due to the fact that nominal NDP is smaller than nominal GDP. A 1% rise in GDP due to TFP growth will raise WNDP by more than 1%, since the derivation of λ assumes in effect that the whole of the rise in GDP is available to raise NDP. If we assume a one-good model with a constant growth rate of labour input, ie $(\hat{B}_i = 0, \text{ all } i)$, then the long-run, steady-state, growth rate of GDP (and of NDP) per unit of labour is μ / (labour's share in GDP). Since $\text{GDP/NDP} = 1+d$, it is easy to see from **(19)** that this long-run growth rate also equals λ / (labour's share in NDP); that is, the faster growth of λ is exactly balanced by the higher share of labour in NDP.[21]

The second effect of depreciation on the WNDP frontier arises when the rate of technical progress in the investment goods industries differs from that in the consumption goods industry. This effect is rather more interesting economically. If technical progress is faster in investment goods industries $(\hat{B}_i > 0, \text{ all } i)$, then the larger is depreciation as a proportion of NDP, the larger is the second term on the right-hand side of equation **(16)** and this tends again to raise λ in relation to μ. Technical progress in investment goods industries lowers the cost in foregone consumption of maintaining the capital stock at a given level. The larger is depreciation and the faster is TFP growth in the investment goods industries, the faster does the WNDP frontier shift. Paradoxically, therefore, though higher depreciation lowers the *level* of WNDP, it tends to raise the *growth rate* of the WNDP frontier.[22]

Top down versus bottom up measures of the growth rates of TFP and TFW

Up to this point we have employed a bottom up measure of aggregate TFP growth, equation **(17)**. We can of course also measure TFP growth in a top down way by

$$\mu = \hat{Z} - \alpha\hat{K} - (1-\alpha)\hat{L} \qquad \textbf{(20)}$$

where K is an index of aggregate capital and α is the share of gross profits in GDP. If any given input is paid the same price in all industries, then the top down and bottom up measures must produce exactly the same answer (Jorgenson *et al* (1987, chapter 2)). Is it possible to produce an equivalent top down measure of TFW growth? In general the answer is no. But there is a special case in which it can be done. This is where the production functions (or equivalently the unit cost

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(21) Hulten (1979) has suggested that the contribution of TFP growth to output growth should be measured not by TFP growth alone but by the steady-state growth rate to which TFP gives rise: that is, not by μ but by μ / (labour's share in GDP). This amounts to assigning to TFP as a source of growth the benefits of the capital accumulation induced by rising TFP.

(22) Since the relative prices of investment goods are falling when these goods enjoy faster technical progress, the weights on the $\hat{B}_i$ in equation **(16)** may be changing too. However in a steady state we would expect them to be constant.

functions) are all the same up to a scalar multiple (TFP). In other words, TFP growth rates differ between industries but the share of any given input in total cost is the same across industries. In this special case, the Annex proves as Theorem 3 that

$$\lambda = \hat{Y} - \beta\hat{K} - (1-\beta)\hat{L} \qquad \textbf{(21)}$$

where β is the current price share of *net* profits in *net* domestic product.

It is simple to generalise this result to the case of many investment goods. It is also useful to rearrange the result, putting the growth of WNDP per unit of labour on the left-hand side. The result is:

$$\hat{Y} - \hat{L} = \sum_{i=1}^{m} \beta_i (\hat{K}_i - \hat{L}) + \lambda \qquad \textbf{(22)}$$

where β_i is the *net* profit generated by the *i*th capital stock[23] as a proportion of aggregate *net* output (WNDP) and $\sum_{i=1}^{m} \beta_i = \beta$. In other words the growth of WNDP per unit of labour can be decomposed into capital deepening (the growth of each type of capital per unit of labour, weighted by the net profit shares) plus the growth of TFW. A similar generalisation and rearrangement for TFP growth, equation **(20)**, yields:

$$\hat{Z} - \hat{L} = \sum_{i=1}^{m} \alpha_i (\hat{K}_i - \hat{L}) + \mu \qquad \textbf{(23)}$$

where α_i is the *gross* profit generated by the *i*th capital stock as a proportion of GDP and $\sum_{i=1}^{m} \alpha_i = \alpha$. Now, necessarily, the net profit share is less than the gross profit share ($\beta_i < \alpha_i$) and the difference is larger for short-lived assets like computers. So the contribution of capital accumulation in computers to WNDP growth is likely to be smaller than its contribution to GDP growth, a point we shall return to below.[24]

3. Productivity versus welfare in the United States

In this section we quantify some of the concepts just discussed, employing US data from the 1990s. We start by asking what has happened to investment, depreciation and capital stocks. We then go on to measure the differences between GDP, official NDP and Weitzmans's NDP. Finally, we estimate the rate at which the WNDP frontier has been growing and compare it with the growth of aggregate TFP.

We saw in the previous section that the growth rate of TFW depends in part on the ratio of depreciation to NDP (*d*) or equivalently, the ratio to GDP. The latter can be decomposed as follows:

$$\frac{\text{Depreciation}}{\text{GDP}} = \frac{\text{Depreciation}}{\text{Capital stock}} \cdot \frac{\text{Capital stock}}{\text{GDP}} \qquad \textbf{(24)}$$

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[23] The net profit generated by an asset is the gross profit less depreciation. Gross profit is the rental price times the stock of the asset. The rental price is given by the Hall-Jorgenson formula.
[24] This point, illustrated with UK data, has been made by Martin (2002).

In what follows we measure the two ratios on the right-hand side.

Investment and depreciation in the United States in the 1990s

Table A shows the growth rates[25] of GDP and its main expenditure components in the 1990s. All components show an acceleration when the second half of the 1990s (1995-2000) is compared with the first half (1990-95). Gross private investment has been growing much more rapidly than personal consumption and shows a greater acceleration. In the second half of the 1990s, the growth rates of both gross investment and net investment accelerated. This was true for both private and government investment. But the acceleration was greater for net investment. The growth of capital consumption also accelerated: see Table B.

Table A
Average annual growth rates of US GDP and components, % p.a.

	1990-2000	*1990-95*	*1995-2000*	*Acceleration (1995-2000 over 1990-95)*
GDP	3.19	2.35	4.02	1.67
Personal consumption	3.35	2.52	4.19	1.67
Gross private domestic investment	6.70	4.58	8.82	4.24
Exports of goods and services	6.77	6.78	6.76	-0.02
Imports of goods and services	8.85	6.76	10.94	4.18
Government	1.25	0.27	2.23	1.96
Government consumption	*0.93*	*0.30*	*1.57*	*1.27*
Gross government investment	*2.37*	*0.16*	*4.58*	*4.42*

Source: US NIPA.[26] Government consumption calculated as Fisher index of federal, state and local consumption.

[25] Here and below growth rates are calculated as log differences multiplied by 100.
[26] In this and subsequent tables, the NIPA data post-date the major revision published in Summer 2001.

Table B
Average annual growth rates of gross and net investment in the United States, % p.a.

	1990-2000	*1990-95*	*1995-2000*	*Acceleration (1995-2000 over 1990-95)*
Private				
Gross investment	6.70	4.58	8.82	4.24
Consumption of fixed capital	5.26	3.85	6.66	2.81
Net investment	9.16	6.01	12.31	6.30
Government				
Gross investment	2.37	0.16	4.58	4.42
Consumption of fixed capital	3.21	2.83	3.59	0.76
Net government investment	0.85	-5.07	6.76	11.83
Total				
Gross investment	5.95	3.76	8.14	4.38
Net investment	7.76	3.96	11.56	7.60

Source: US NIPA. Total investment, gross and net, calculated as Fisher indices of private and government investment.

In the US national accounts, the stock of any asset is assumed to evolve (approximately) according to the simple accumulation equation:

$$K_{it} = I_{it} + (1-\delta_i) \cdot K_{i,t-1} \qquad \textbf{(25)}$$

where K_{it} is the stock of the *i*th asset at the *end* of period *t*, I_{it} is gross investment in period *t* and δ_i is the depreciation rate (see U.S. Department of Commerce (1999)).(27) With a few exceptions, the individual δ_i are not assumed to change over time (Fraumeni (1997)).(28) So any change in the ratio of aggregate depreciation to the aggregate capital stock (in current prices) indicates a change in the asset composition of the capital stock. This ratio can therefore be used to measure the aggregate depreciation rate. That is, define the aggregate depreciation rate at time *t*, δ_t, implicitly as:

$$\delta_t \cdot \sum_{i=1}^{m} P_{i,t-1} K_{i,t-1} = \sum_{i=1}^{m} \delta_i P_{i,t-1} K_{i,t-1}$$

Then

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(27) Though this equation holds at the asset level, it does not hold exactly for any larger aggregate, since the official estimates of net stocks are chained Fisher indices of the underlying assets. See Whelan (2000) for some of the pitfalls of dealing with chain indices.

(28) As Fraumeni (1997) explains, the depreciation rates are derived mostly from evidence on asset lives plus studies of second-hand asset prices. The latter tend to support the assumption of geometric depreciation. That assets depreciate in a value sense is not really controversial. After all, virtually all assets are eventually discarded or destroyed, and this alone would lead to a decline in asset value with age. Whether depreciation is due to physical deterioration or obsolescence is not the issue here: only the fact of depreciation is relevant. In the US NIPA, depreciation covers '… wear and tear, obsolescence, accidental damage, and aging …'. Obsolescence is however at a 'normal' or 'planned' rate. Obsolescence that is '… unusually or unexpectedly larger than the amounts built into the depreciation schedules …' is not included. Possible examples of the latter are '… sudden changes in energy prices, and increased foreign competition since the early 1970s …' (U.S. Department of Commerce (1999, page M-6)).

$$\delta_t = \sum_{i=1}^{m} \left(\frac{P_{i,t-1} K_{i,t-1}}{\sum_{i=1}^{m} P_{i,t-1} K_{i,t-1}} \right) \cdot \delta_i \qquad \textbf{(26)}$$

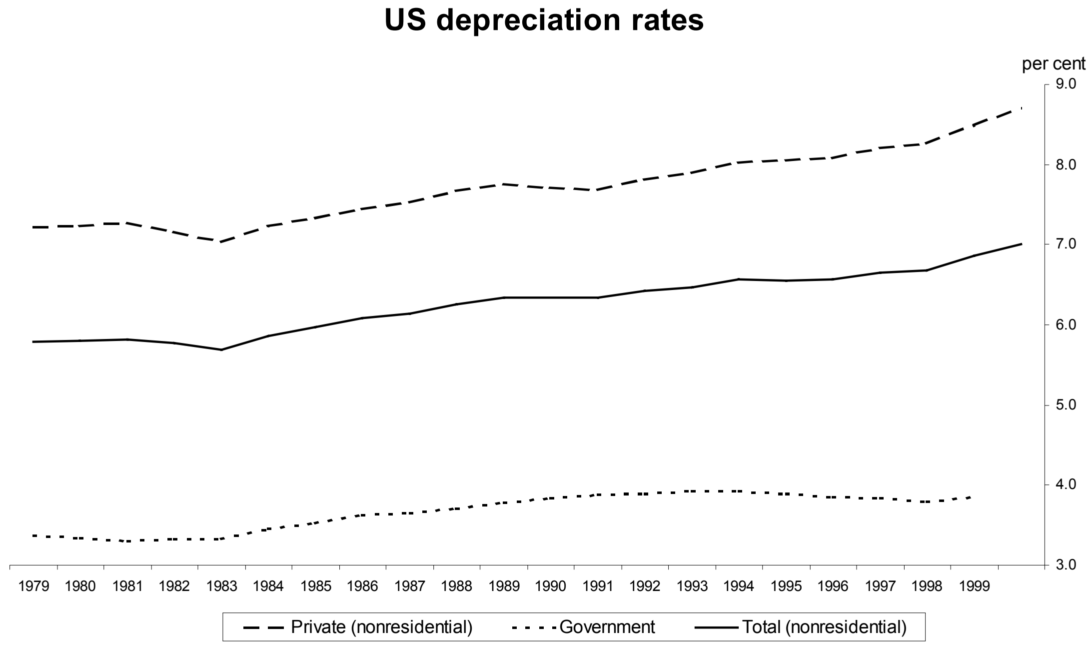
Chart 1 now shows that the aggregate depreciation rate has indeed been trending upwards from the early 1980s onwards. However, despite this, the ratio of depreciation to GDP in current prices has been remarkably flat in recent years, both for the business sector and for the whole economy: see Chart 2. After a spike in the early 1980s, the ratio has shown very little change. Arithmetically, the reason is that the increase in the depreciation rate has been outweighed by a fall in the capital-output ratio. As Table C shows, most asset stocks have been growing less rapidly in real terms than GDP; the major exception is equipment and software. Physical asset prices have been growing a bit faster than the price of GDP, again with the major exception of equipment and software. In consequence, the capital-output ratio in current prices has been falling. This is true for the whole economy, for the business sector, and even for the ratio of the equipment and software stock to business sector GDP.

Table C
Growth rates of prices and quantities of fixed assets in the United States, 1990-2000

	All fixed assets	*Private fixed assets*	*Private non-resid-ential*	*Private equipment and software*	*Private structures*	*Private resid-ential*	*Govern-ment*	Memo item: GDP, whole economy	Memo item: GDP, business sector
1990-2000									
Quantities	2.39	2.54	2.84	4.71	1.64	2.24	1.81	*3.19*	*3.71*
Prices	2.58	2.52	1.95	0.42	2.94	3.11	2.82	*2.13*	*1.88*
Values	4.97	5.06	4.79	5.13	4.58	5.34	4.64	*5.31*	*5.58*
1990-95									
Quantities	1.94	1.98	1.98	3.13	1.25	1.98	1.79	*2.35*	*2.70*
Prices	2.50	2.43	2.11	1.37	2.57	2.78	2.75	*2.51*	*2.31*
Values	4.44	4.41	4.09	4.50	3.83	4.75	4.54	*4.86*	*5.01*
1995-2000									
Quantities	2.84	3.11	3.71	6.29	2.03	2.49	1.84	*4.02*	*4.71*
Prices	2.66	2.60	1.79	-0.53	3.30	3.44	2.89	*1.74*	*1.45*
Values	5.50	5.71	5.50	5.76	5.32	5.93	4.73	*5.77*	*6.15*

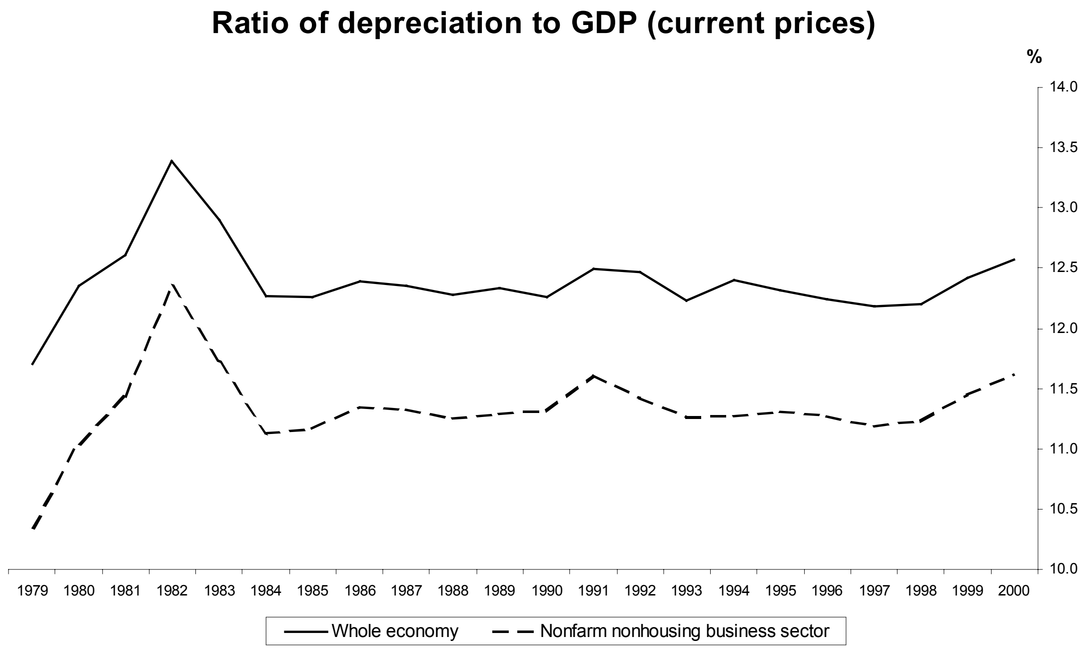
Source: US NIPA, fixed asset tables 1.1 and 1.2, for asset values and quantities. Price growth derived as value growth minus quantity growth. Business sector is non-farm and non-housing.

Chart 1



Source: US NIPA. Depreciation rate calculated as capital consumption as a proportion of the value of the stock at the end of the previous year, both in current prices. Residential housing excluded.

Chart 2



Source: US NIPA. 'Whole-economy' rate: aggregate capital consumption as a ratio to GDP. Business sector rate: non-residential capital consumption as a ratio to GDP in the non-farm non-housing business sector.

From Tables B and C, we can see that gross investment has been growing faster than GDP and net investment faster still. So it may seem surprising that asset stocks have been growing more slowly than GDP. The explanation for this apparent paradox is that gross investment and the capital stocks are chain indices which use different weights. Gross investment in equipment and software and the net stock of these assets have been growing very rapidly. But gross investment in structures and the net stock of structures have grown rather slowly. So it is quite possible for gross investment to grow more rapidly than GDP while the capital stock grows less rapidly.(29)

The slow growth of asset stocks may also seem surprising in the light of growth accounting studies, which suggest that capital deepening has played a large part in the productivity acceleration after 1995 (Oliner and Sichel (2000); Jorgenson and Stiroh (2000)). But these studies (rightly) seek to measure the growth of capital *services*, not that of the capital stock, as here. In a capital services measure, the rapid growth of equipment and software will have a bigger weight than in a wealth or stock measure, since these assets have high rates of depreciation and, in the case of ICT, falling prices. Hence capital services grow more rapidly than the capital stock (Oulton (2001b)).(30)

Are the BEA's depreciation rates too high as measures of economic depreciation?

The depreciation rates used by the BEA are based on panel data studies of new and second-hand asset prices, particularly those of Hulten and Wyckoff (1981a) and (1981b), which did not control fully for quality change. As Oliner (1993) and (1994) has shown, in the absence of full control for quality change, hedonic regressions will overestimate *economic* depreciation rates when quality is rising over time (see also Cummins and Violante (2002)). Currently, the depreciation rate in the NIPA is around 40% per annum for computers. Academic researchers have often assumed a lower rate, eg Jorgenson and Stiroh (2000) assume 31.5% per annum. According to the official description (Herman (2000)), the 40% rate is derived from Lane (1999). But if depreciation is defined in the correct economic sense as the proportionate difference between the price of a new and used asset of given quality *at a point in time*, then the Lane study supports a lower figure, around 30%.(31)

Use of a lower depreciation rate for computers would strengthen the empirical conclusions of this paper, since the rising share of computer investment would have a smaller effect on aggregate depreciation. But the impact would probably be quite small. For example, depreciation on privately-held computers and peripherals was $60.3 billion in 2000, or 0.61% of GDP (NIPA Fixed asset table 2.4). Using 31.5% per annum as the depreciation rate, one can calculate that this proportion would fall to 0.57% of GDP, a negligible effect.

Why has net investment grown more rapidly than gross investment?

As we have seen, in the US national accounts, gross investment, net investment, and the capital stock of each asset are related by the accumulation equation **(25)**. In a steady state, gross investment, net investment and the capital stock of a particular asset will all grow at the same rate. Now suppose that initially there is a steady state in which all three variables are growing at

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(29) When there are several assets, it is quite possible for a chain index of aggregate gross investment to grow more rapidly than a chain index of the aggregate stock, even in a steady state (Whelan (2000)).

(30) Asset quantities in Table C are Fisher indices which use shares in total asset value as weights. These indices rise more slowly than do quantity indices for investment because equipment and software, whose relative price has been falling rapidly, has a higher weight in investment than it does in asset value.

(31) The BEA seems to have derived its 40% figure from Exhibit 5 (page 26) of Lane (1999). This shows the decline in price of an asset of given quality over time: that is, the current price as a percentage of the historic cost. This gives an exaggerated estimate of economic depreciation since the price of a new asset of comparable quality has declined since the asset was purchased. Combining Exhibit 5 with the BEA price index for computers (reproduced in Lane's Exhibit 10, page 28), we obtain an estimate of the depreciation rate of approximately 30% per annum. I am grateful to Richard Lane for sending me a copy of his study. These comments should not be taken as critical of his study, whose focus was the correct market valuation of second-hand assets for tax purposes, not the estimation of economic depreciation in the national accounts.

3% per period and that the depreciation rate is 13% per annum (approximately the average rate assumed by the BEA for non-ICT equipment). Then assume that the desired growth rate of the capital stock rises to 5%; the growth rate of the *actual* stock is assumed to rise steadily from 3% to 5% over 8 periods.

Chart 3a shows the effects of simulating this change using equation **(25)**; the change in the capital stock growth rate starts in period 4. The growth rate of gross investment rises steadily from period 4 onwards till period 11. It overshoots its new long-run value in period 6, though it falls back to it again in period 12. The growth rate of net investment follows a much more dramatic course. It more than triples in period 4, even though the growth rate of the stock is assumed to rise only from 3.00% to 3.20%. It continues at a high level through period 11 before it too falls sharply to its long-run value in period 12. Its growth rate is above that of gross investment throughout the adjustment period.

The intuition here is that to make the capital stock grow at a faster rate in period 4, an additional amount, say x, has to be invested. But x is a larger percentage of the level of gross investment in period 3 than it is of the capital stock and a still larger percentage of net investment. Hence initially net investment grows faster than gross investment and gross investment grows faster than the stock.

Chart 3a

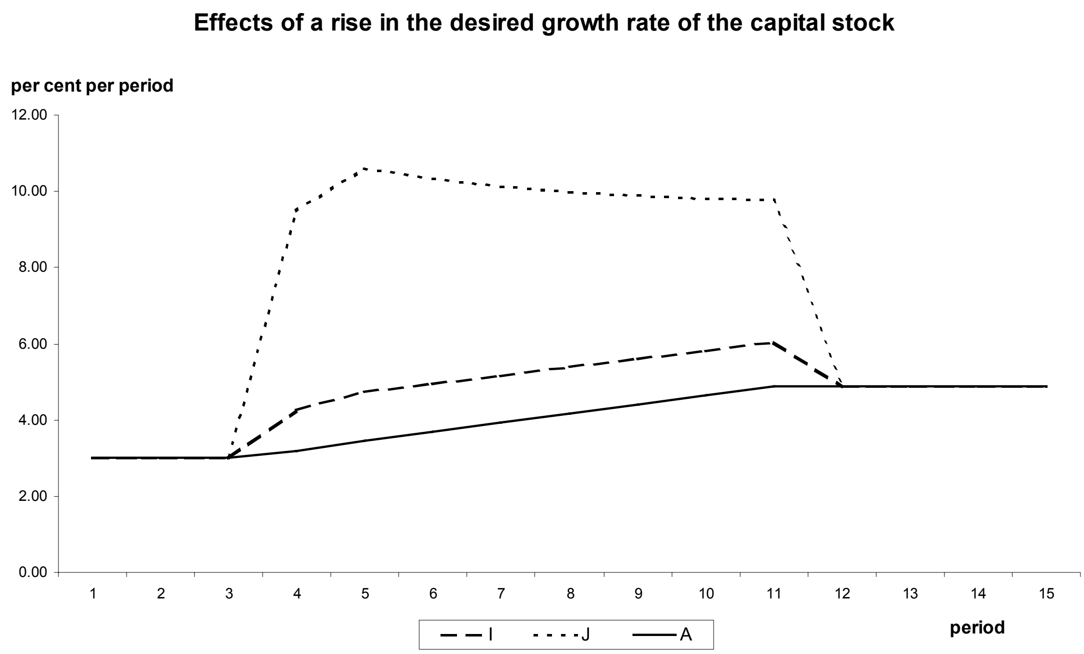
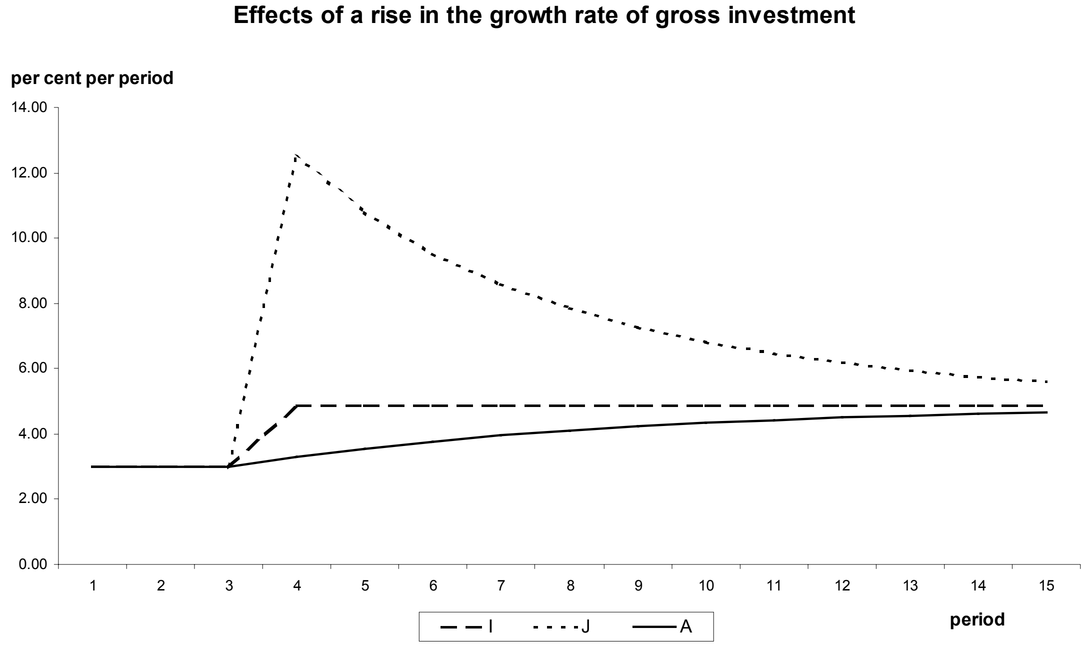


Chart 3b



Note: Simulated results using equation **(25)**.

Chart 3b shows an alternative simulation where the growth rate of gross investment is assumed to rise from 3% to 5% in period 4 and to continue at this level thereafter. There is an immediate spike in net investment growth which then gradually decays away. Note that even after 12 periods the growth rate of net investment still substantially exceeds that of gross investment.

We might interpret what has happened in the United States since the mid-1990s either as an increase in the desired growth rate of the capital stock or as an increase in the desired growth rate of gross investment. Whichever interpretation we prefer, these simulations suggest that during the adjustment period the growth of net investment will exceed that of gross investment.[32]

GDP, NDP and WNDP compared

Table D compares GDP, official NDP and two measures of WNDP in the 1990s. The first measure, WNDP(1), is obtained by deflating nominal NDP by the price index for private consumption. The second measure, WNDP(2), deflates nominal NDP by a Fisher price index of the prices of private and government (federal, state and local) consumption. For the moment we concentrate on the first measure of WNDP.

Official NDP has been growing on average a little less rapidly than GDP throughout this period (Chart 4). The contrast between NDP and GDP is much the same for the non-farm business sector (Chart 5). But official NDP accelerates almost as much as GDP in the second half of the 1990s (Table E). Chart 6 compares the first measure of WNDP with GDP in the 1990s, while Chart 7 gives a longer perspective, 1973-2000. From these charts and Table D, we can see that GDP generally grows a bit faster than WNDP. But the gap between the two growth rates was

(32) There is a complication that these simulations do not take into account. When the trend growth rate of productivity increases, theory suggests that the desired growth rate of the capital stock increases but the desired initial *level* falls. This paradoxical behaviour has been analysed theoretically by Bakhshi and Larsen (2001) and by Pakko (2001).

actually bigger in the earlier period, 1973-90, than in the later one, 1990-2000. And (surprisingly) WNDP accelerates by the same amount as GDP when we compare the first and second halves of the 1990s (Table E and Chart 6).[(33)]

If what is called government consumption in the national accounts really is a separate category of consumption, then presumably it should be deflated by its own price index, and not by the price of private consumption. This method yields the second measure of WNDP, WNDP(2), in Table D. WNDP(2) has been growing more slowly than WNDP(1), but nevertheless it still shows a substantial acceleration of 1.51% per annum over the second half of the 1990s.

In practice, government services are not usually sold on the open market and their 'price' mainly reflects the assumptions made by the BEA about government output, which is generally measured by labour input, ie productivity growth is assumed to be zero. Not surprisingly therefore the price of government consumption tends to rise more rapidly than that of private consumption. It is likely that a true price index would rise more slowly. In addition, a substantial proportion of government 'consumption' is state and local expenditure on education. Arguably, this should be classified as investment (though if so it measures gross not net investment). In a WNDP measure, any investment component should be deflated by the price index of 'true' consumption. For these reasons, the second measure of WNDP is not necessarily superior to the first. From now on, I use the first measure.

Table D
Gross and net product in the United States: growth rates, % p.a.

	GDP	*WNDP(1)*	*WNDP(2)*	*NDP*
1991	-0.47	-0.93	-0.91	-1.02
1992	3.00	2.45	2.50	2.73
1993	2.62	2.89	2.82	2.75
1994	3.96	3.83	3.71	3.69
1995	2.63	2.61	2.51	2.66
1996	3.51	3.40	3.28	3.34
1997	4.34	4.41	4.30	4.15
1998	4.19	4.33	4.23	3.85
1999	4.00	3.52	3.27	3.55
2000	4.07	3.51	3.11	3.74
Averages annual growth rates, % p.a.				
1973-2000	2.98	2.67	n.a.	2.78
1973-90	2.86	2.48	n.a.	2.68
1990-2000	3.19	3.00	2.88	2.94
1990-95	2.35	2.17	2.13	2.16
1995-2000	4.02	3.83	3.64	3.73

Source: US NIPA and own calculations for WNDP. WNDP(1) deflates nominal NDP by the price index for private consumption; WNDP(2) deflates by a Fisher index of the prices of private and government consumption.

(33) The similarity of the growth rates of WNDP and GDP has also been noted by Wadhwani (2001).

Chart 4

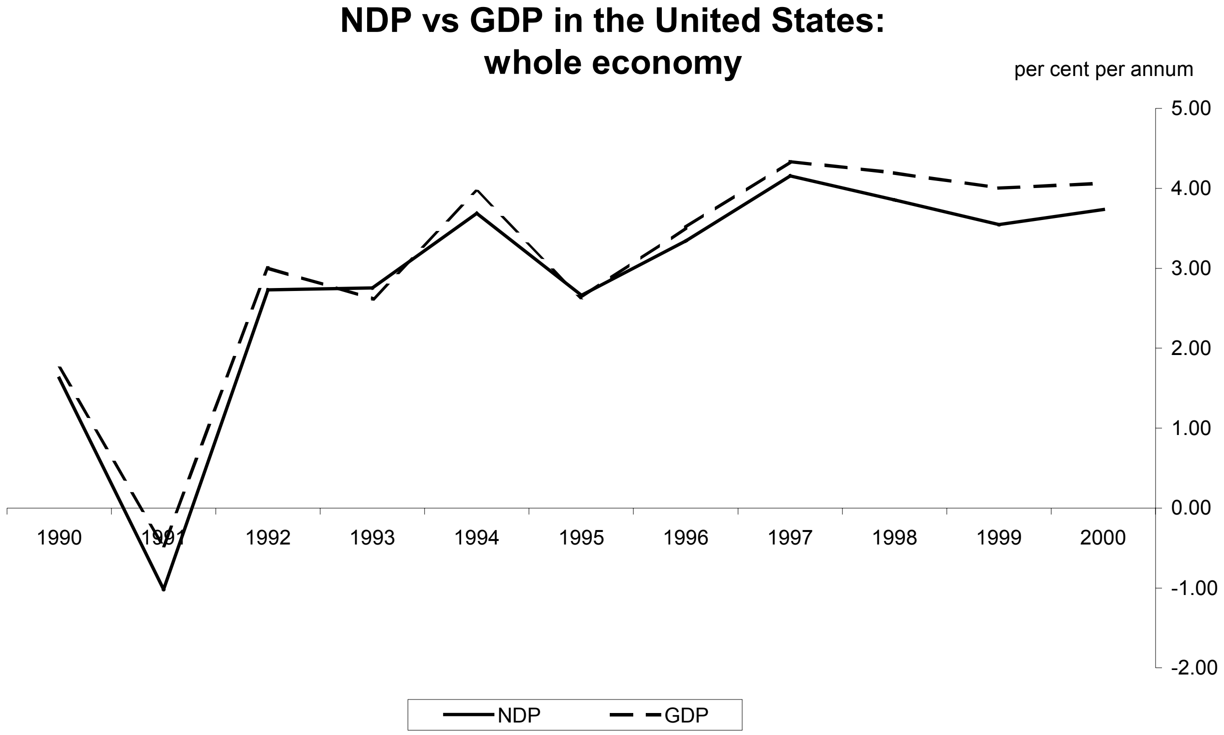


Chart 5

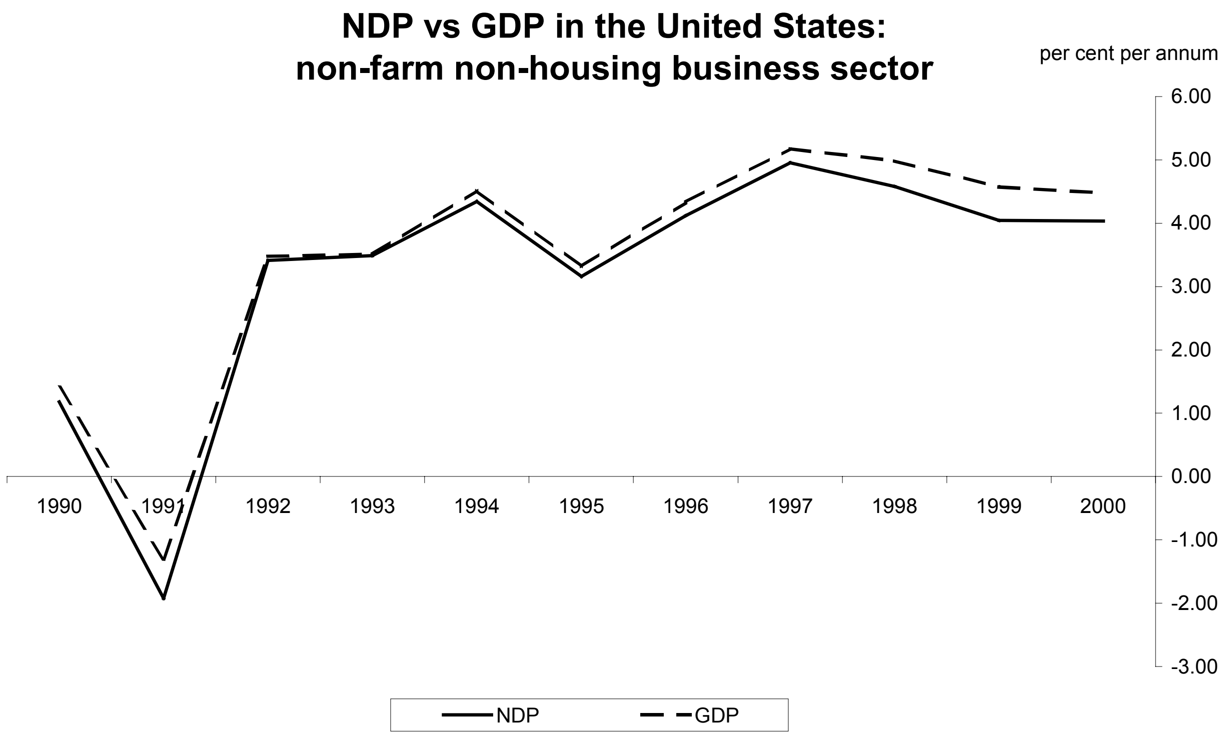


Table E
Acceleration in growth, 1995-2000 over 1990-95
(percentage points per annum)

GDP	*WNDP(1)*	*WNDP(2)*	*NDP*
+1.67	+1.66	+1.51	+1.56

Source: Table D.

Chart 6

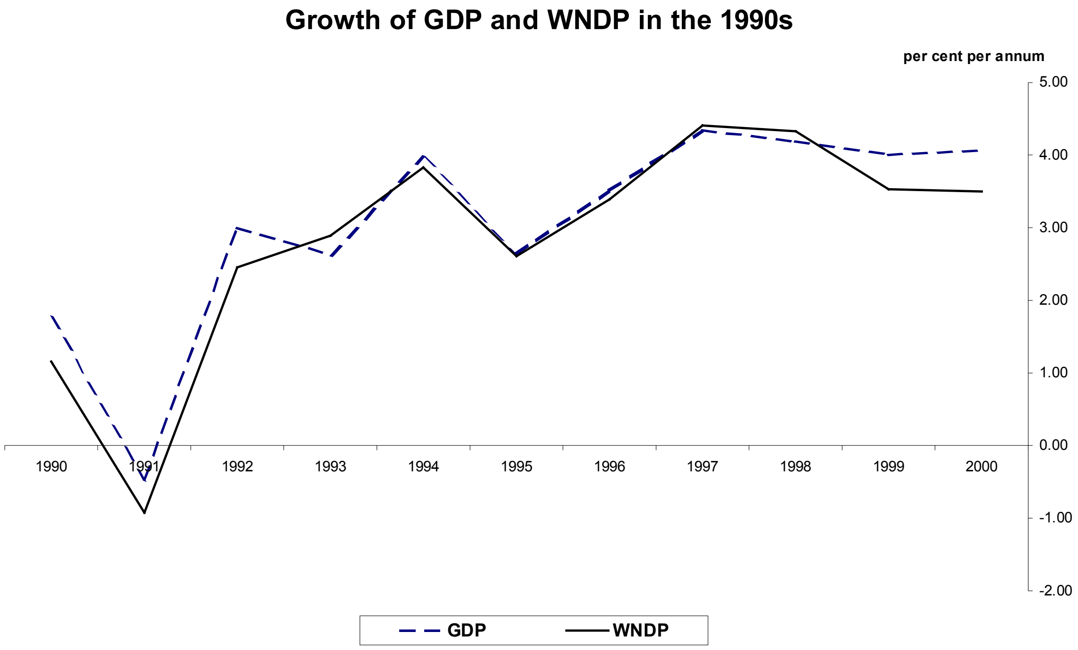
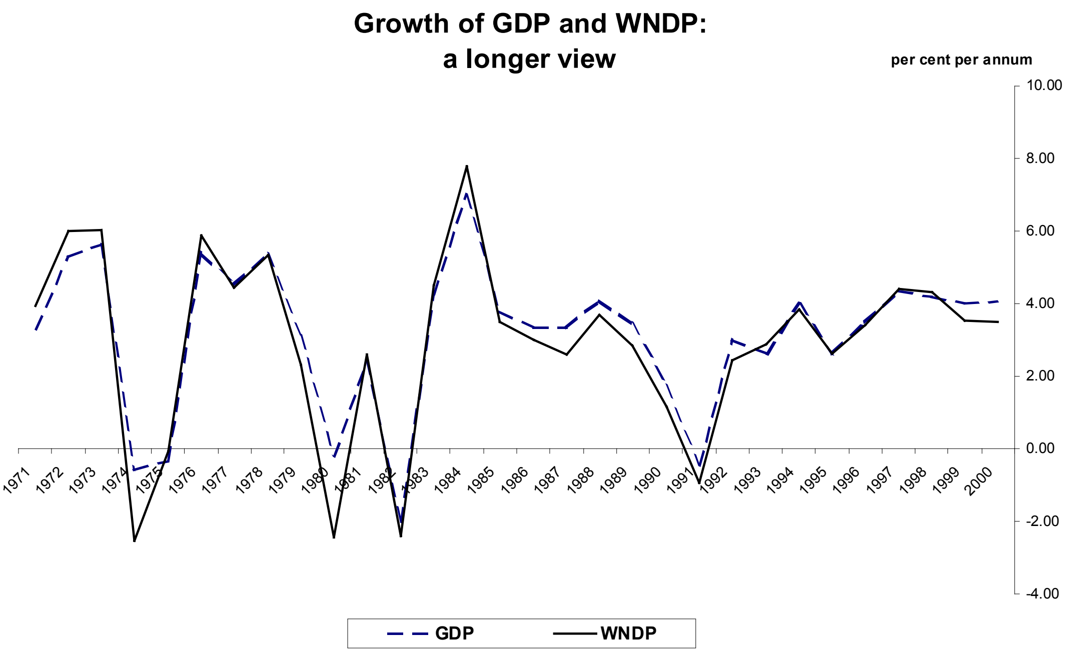


Chart 7



Decomposing the difference between WNDP and GDP

We can employ a discrete version of equation **(7)** to decompose the difference between GDP and WNDP growth in the United States in recent years. In the discrete version, the point-in-time shares are replaced by averages over adjacent years and continuous growth rates are replaced by log differences. The decomposition is a little more complicated in practice since we have to allow for net trade and government consumption. In the WNDP calculation, government consumption, exports and imports, as well as net investment, are deflated by the price index for personal consumption. The results are in Table F.

Table F
Decomposition of difference between GDP growth and WNDP growth (percentage points per annum)

		1990-95	*1995-2000*
1.	$(w_C - v_C)\cdot(\hat{C} - \hat{I})$	+0.12	+0.43
2.	$v_J(\hat{I} - \hat{J})$	-0.01	-0.37
3.	$-v_J\hat{p}$	+0.10	+0.16
4.	Net trade	-0.06	-0.13
5.	Government consumption	-0.05	-0.16
6.	Other	+0.08	+0.16
7.	Total: GDP growth minus WNDP growth	+0.18	+0.10

Note: w_C, v_C: shares of consumption GDP, NDP; v_J: share of net investment in NDP. Government consumption calculated as Fisher index of federal, state and local consumption. Gross investment calculated as Fisher index of private and government gross investment; similarly for net investment. The components need not sum exactly to the total but in practice they do, to two decimal places.
Source: US NIPA and own calculations.

If the components identified in equation **(7)**, the first three rows of Table F, had been the only factors, then the gap between GDP and WNDP would have widened modestly from 0.21 to 0.22 percentage points per annum over the 1990s. The difference between the consumption and gross investment growth rates widened (row 1), and the relative price effect increased (row 3), but these effects were offset by a widening of the gap between the gross and net investment growth rates (row 2). In addition, the net trade and government effects worked to reduce the GDP-WNDP growth gap, and to an increasing extent. This is mainly because the price of personal consumption was rising less rapidly than that of government consumption but more rapidly than that of imports, and both these differences increased in the latter half of the 1990s.

In summary, there are three reasons why one might expect WNDP to have grown more slowly than GDP. First, the weight on consumption is higher in GDP than in WNDP (or NDP) and consumption has been growing more slowly than investment. Second, the relative price of investment goods has been falling which reduces WNDP growth. Third, one might have expected capital consumption to have risen as a ratio to GDP, thus raising the share of consumption in WNDP still further. In practice as we have just seen, WNDP and GDP have been growing at very similar rates, for several reasons. The ratio of capital consumption to GDP has in fact been stable, despite the growing importance of short-lived assets. And there have been two countervailing factors which have tended to raise WNDP relative to GDP growth. The first factor is that net investment has grown more rapidly than gross investment. A second, subsidiary factor is that the price indices for government consumption and for net trade have grown more rapidly than the price index for consumption: this raises WNDP growth relative to GDP growth.

The growth rate of the WNDP frontier (λ)

The growth rate of the WNDP frontier is given by equation **(16)**, while that of the GDP frontier is given by equation **(17)**. Here we adopt a simplified approach, aggregating all types of investment together. Then the equations become:

$$\lambda = (1+d)\cdot\hat{A} + d\cdot\hat{B} \qquad \textbf{(27)}$$

$$\mu = \hat{A} + s\cdot\hat{B} \qquad \textbf{(28)}$$

Oliner and Sichel (2000) have provided estimates of μ for the US non-farm non-housing business sector for the period 1973-99. We can estimate the TFP growth differential in the investment good industry by the rate of growth of the relative price of the investment good to that of the consumption good, ie we assume that $\hat{B} = -\hat{p}$.[34] Then, using nominal data on gross investment and GDP in the business sector, we can solve for $\hat{A}$ from **(28)**. Finally, we can use nominal data on GDP, NDP and depreciation to find λ from **(27)**: see Table G.

Table G
Average rates of growth of the GDP and WNDP frontiers, % p.a.:
US non-farm business sector (excluding housing)

	μ	$\hat{B}$	$s\cdot\hat{B}$	$\hat{A}$	$d\cdot\hat{B}$	λ
1973-90	0.33	1.05	0.17	0.16	0.14	0.32
1990-95	0.48	2.14	0.30	0.18	0.28	0.48
1995-99	1.16	3.01	0.48	0.68	0.39	1.15
Acceleration, 1995-99 over 1990-95	+0.68	+0.87	+0.18	+0.50	+0.11	+0.67

Note: λ and $\hat{A}$ are calculated from equations **(27)** and **(28)**, from estimates of μ, $\hat{B}$, s and d.

Sources: μ : Oliner and Sichel (2000, Table1).[35] $\hat{B}$: growth rate of price index of personal consumption minus growth rate of price index of private non-residential gross fixed investment (Tables 7.1 and 7.6 of US NIPA). s: ratio of private non-residential gross fixed investment to GDP in non-farm business sector (excluding housing), current prices. d: [GDP – NDP]/NDP in the non-farm business sector (excluding housing), current prices. Private non-residential gross fixed investment in current prices from US NIPA, Table 5.2. GDP and NDP in current prices in the non-farm business sector (excluding housing) from US NIPA, Tables 1.7 and 1.12.

There is a large difference between TFP growth in the consumption sector and TFP growth in the investment goods industries, as measured by $\hat{B}$. This difference averaged 3.01% per annum in 1995-99 and 2.14% per annum in 1990-95. Despite this, it turns out that λ and μ are virtually identical. Moreover they show an almost identical acceleration after 1995.

[34] This is only exactly true when the two industries have identical production functions up to a scalar multiple; see the proof of Theorem 3 in the Annex. But it is likely to be a reasonable approximation.
[35] I use a different convention for naming periods. What I call 1990-95 corresponds to what Oliner and Sichel call 1991-95; similarly, my 1995-99 corresponds to their 1996-99.

Oliner and Sichel have calculated the impact of information technology (IT), defined as computers, software and communications equipment, on the growth of GDP per unit of labour, as in equation **(23)**. They split the effect into two parts. First, there is the contribution of TFP growth in the IT sector to aggregate TFP growth. Second, there is the contribution of capital accumulation of IT goods. We consider the second contribution in Table H. This compares the effect of IT capital deepening on GDP growth with its effect on WNDP growth (the latter measured by equation **(22)**). To recall, capital deepening in any type of capital input (eg computers) is measured as the growth rate of this input per unit of labour weighted by its income share. For the GDP calculation, the income share is gross profit attributable to the input as a proportion of GDP. For the WNDP calculation the income share is net profit as a proportion of NDP.

The discussion in Section 2 above suggested that the importance of IT capital deepening is likely to be greater for GDP than it is for WNDP and this turns out to be the case. In absolute terms, the contribution of computers and software is roughly halved, with a somewhat smaller effect on communications. In relative terms, IT capital deepening accounted for 37.9% of the growth of labour productivity in 1996-99 from a GDP perspective, but 31.3% from a WNDP perspective. However the conclusions on what accounts for the *acceleration* in labour productivity growth are very similar. For GDP, the acceleration in IT capital deepening accounts for 44% of the acceleration in labour productivity growth (ie 0.46 as a percentage of 1.04), while for WNDP the corresponding figure is 38% (ie 0.38 as a percentage of 1.01). Incidentally, in the business sector the acceleration in the growth of WNDP was almost exactly the same as that of GDP, thus confirming the results of Table D for the whole economy.

Table H
Capital deepening in IT in the non-farm non-housing business sector: contributions to GDP and WNDP compared, 1990-1999

	GDP			*WNDP*		
	1990-95	*1996-99*	*Change*	*1990-95*	*1996-99*	*Change*
Growth rates (% per annum)						
Output	2.75	4.82	2.07	2.32	4.36	2.04
Labour productivity	1.53	2.57	1.04	1.10	2.11	1.01
Income shares (%)						
Computers	1.4	1.8	0.4	0.8	1.3	0.5
Software	2.0	2.5	0.5	1.0	1.1	0.1
Communications	1.9	2.0	0.1	1.3	1.5	0.2
Contribution to capital deepening (percentage points per annum)						
Computers	0.23	0.61	0.38	0.12	0.43	0.30
Software	0.24	0.27	0.03	0.11	0.15	0.03
Communications	0.05	0.10	0.05	0.05	0.09	0.04
Total	0.51	0.97	0.46	0.28	0.66	0.38
Contribution to capital deepening (% of growth in labour productivity)						
Computers	14.9	23.6	8.67	11.3	20.2	8.98
Software	15.5	10.5	-5.07	10.1	6.9	-3.19
Communications	3.0	3.9	0.90	4.1	4.1	0.04
Total	33.4	37.9	4.50	25.5	31.3	5.83

Source: All GDP-related figures, the growth of labour input, and the growth of capital inputs from Oliner and Sichel (2000, Tables 1 and 2). All WNDP-related figures: own calculations. Income shares: for GDP, gross profit as a proportion of GDP; for WNDP, net profit (gross profit *less* depreciation, latter from US NIPA) as a proportion of NDP. Contributions to labour productivity growth: growth rate of capital input per unit of labour *times* income share.

4. Conclusions

We have argued that Weitzman's NDP is superior to GDP as a welfare measure. This does *not* mean that GDP is in some way conceptually flawed, only that it is measuring something different, namely output. So even if GDP had grown significantly faster than WNDP, this would not be a cause for disquiet. By contrast, official NDP does not have a clear interpretation either as a measure of welfare or as a measure of output.

The rate at which the WNDP frontier shifts out over time, which I have called the growth rate of total factor welfare, is a weighted average of the TFP growth rates in the individual industries. In this sense it is analogous to the growth rate of aggregate TFP, which is also a weighted average of industry TFP growth rates, though the weights differ in the two measures. Hence we can analyse the growth rate of TFW using the same theoretical tool as we use to analyse aggregate TFP growth: the industry-level gross output production function.

Turning from theory to empirics, we find that in the United States in the 1990s, the composition of investment and the aggregate capital stock has been shifting towards assets with shorter lives. Consequently, the aggregate depreciation rate has been rising. Both gross and net investment have been rising more rapidly than GDP, which might suggest that the aggregate capital stock has been growing more rapidly too. If so, this would have led to a rise in the ratio of depreciation (capital consumption) to GDP and a tendency for GDP to grow ever more rapidly than WNDP (and NDP).

In fact this has *not* happened. Although the aggregate depreciation rate has indeed risen, depreciation as a proportion of GDP has been flat. And the aggregate capital stock has grown *less* rapidly than GDP. The explanation for this apparent paradox is that both gross investment and the capital stock are chain indices, which use different weights. Equipment and software have been growing very rapidly, but structures rather slowly. The weight for structures in the capital stock is higher than their weight in investment. So it is quite possible for gross investment to grow more rapidly than GDP while the capital stock grows less rapidly.

In the 1990s, Weitzman's NDP and GDP grew at very similar rates in the United States. In the second half of the 1990s, WNDP growth rose by 1.66 percentage points, almost the same as the increase in the growth rate of GDP. We also find that in the 1990s the growth rate of TFW in the US non-farm business sector was almost identical to that of TFP and that they showed the same increase post 1995, about 0.67 percentage points. Hence it seems that in practice GDP has provided a 'true and fair' view of US economic performance over this period, even though WNDP is superior as a welfare measure.

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Annex
Proofs of propositions about WNDP in the text

A.1 Equivalence of wealth maximisation and utility maximisation

Consider an economy which behaves as if it is solving the problem of maximising the present value of the utility (U) from aggregate consumption, subject to technology and initial capital stocks:

$$\text{Problem 1:} \quad \max_{c(t)} U = \int_0^\infty u[c(t)] \cdot L(t) \cdot e^{-\rho t} dt, \quad \rho > 0 \qquad \textbf{(29)}$$

where c is consumption per household, $u[c(t)]$ is the one-period utility function of the representative household, L is the number of households and also the labour force (taken to be exogenous), and ρ is the discount rate. In Problem 1, utility is to be maximised subject to the production functions **(9)**, the capital accumulation equations $\dot{K}_i = I_i - \delta_i K_i$, $i = 1,...,m$, and given initial capital stocks. We assume that the utility function and the production functions are strictly concave.

Now consider the associated problem of maximising the present value of aggregate consumption, ie maximising wealth (W), subject to the same constraints:

$$\text{Problem 2:} \quad \max_{c(t)} W = \int_0^\infty c(t) \cdot L(t) \cdot e^{-R(t)} \cdot dt,$$

$$e^{-R(t)} = \exp\left[-\int_0^t r(\tau) \cdot d\tau \right] \qquad \textbf{(30)}$$

Here $\exp[-R(t)]$ is the discount factor which discounts a receipt at time t to time 0 and $\{r(t)\}$ is a sequence of real interest rates.(36) The real interest rate is the nominal rate minus the growth rate of the price of consumption. In Problem 2, these real interest rates are taken as given.

As will be shown, the two problems are connected in the following sense. The optimal sequence of consumption $\{c^*(t)\}$ which maximises the present value of utility, Problem 1, also maximises the present value of consumption, Problem 2, *provided that discounting is done at the real interest rates which maximise utility in Problem 1*. In other words, maximising the present value of utility is equivalent to maximising wealth, a result that goes back to Irving Fisher (1930), as Hulten (1992) has pointed out. Along the optimal path of Problem 1, the real interest rate equals the marginal product of (any type of) capital net of depreciation, valued in consumption units. More formally:

Theorem 1

The optimal sequence $\{c^*(t)\}$ which is the solution to Problem 1 is also the solution to Problem 2, provided that the sequence of real interest rates $\{r^*(t)\}$ used for discounting in Problem 2 is the same as the sequence implied by the solution to Problem 1.

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(36) If the real interest rate were constant at rate r, then $\exp[-R(t)] = \exp[-rt]$. However, we do not assume that the real interest rate is necessarily constant.

Proof of Theorem 1
By concavity, the optimal solution for Problem 1 is unique. This solution gives the capital-labour ratios in each industry, which by constant returns to scale fix the marginal products of each type of capital. The real interest rate is then determined by these marginal products via the Hall-Jorgenson cost of capital formula. For example, in the case of capital of type *i* in the consumption good industry, the optimal real interest rate $r^*(t)$ must satisfy

$$[r^*(t)+\delta_i-\hat{p}_i^*(t)]\cdot p_i^*(t)=A(t)\cdot\left[\frac{\partial f^c(K_{1c}^*/L_c^*,...,K_{mc}^*/L_c^*)}{\partial K_{ic}}\right]$$

Now suppose that, contrary to the theorem, the optimal solution to Problem 2 is a different sequence of consumption, $\{\tilde{c}(t)\}$, which must entail different sequences of capital stocks $\{\tilde{K}_{1c}(t),...,\tilde{K}_m(t);\tilde{K}_{11}(t),...,\tilde{K}_{m1}(t);\ ...\ ;\tilde{K}_{1m}(t),...,\tilde{K}_{mm}(t)\}$ and of labour allocations $\{\tilde{L}_c(t),\tilde{L}_1(t),...,\tilde{L}_m(t)\}$. Then the real interest rate $\{\tilde{r}(t)\}$ sequence implied by the sequence $\{\tilde{c}(t)\}$ cannot be the same as the sequence of discount rates $\{r^*(t)\}$. Hence $\{\tilde{c}(t)\}$ is not an efficient program. But then it cannot be the solution to Problem 2. This is a contradiction, which proves the theorem.

Remark 1
It is reasonable to assume that in Problem 1 discounting is done at the constant rate ρ. This is in order to avoid time-inconsistency problems of the type analysed by Strotz (1956). But this does not imply that the corresponding discount rate *r* in Problem 2 is also constant. For example, if we assume that the household utility function takes the familiar isoelastic form

$$u(c)=\frac{c^{1-\theta}-1}{1-\theta},\qquad \theta>0,\ \theta\neq 1$$

$$u(c)=\ln(c),\qquad \theta=1$$

then, as is well known (see Barro and Sala-i-Martin (1995, chapter 2)), the first-order conditions for a maximum in Problem 1 imply that

$$r(t)=\theta\cdot\hat{c}(t)+\rho \qquad \textbf{(31)}$$

Since the growth rate of consumption $\hat{c}(t)$ is only constant in steady state, the same is true of the real interest rate. In other words, on the utility-maximising path the real interest rate is only constant when the economy is in steady state.

A.2 Weitzman's result when the real interest rate varies over time
In this sub-section I extend the results of Weitzman (1976) and (1997) and Löfgren (1992) to the case where the real interest rate can vary over time. I consider an economy with one consumption and one investment good, where TFP growth rates may be non-zero and can differ between the two sectors. It would be simple to extend the result to the case of many investment goods as in the Weitzman papers. There is however one restriction which I impose for tractability reasons. I assume that the production functions are the same up to a time-varying scalar multiple (TFP). This enables me to provide a self-contained proof.

Both the Weitzman papers assume a constant real interest rate. In both papers, Weitzman's proof relies on features of the cost function for net output, an unfamiliar concept. Löfgren (1992) does consider the case where the utility of consumption rather than consumption itself appears in the social planner's maximand, but he does not make the link between these two cases as I have done in the preceding section. He too assumes a constant real interest or discount rate. His proof is more self-contained than either one of Weitzman's. But he assumes a one-good economy.[37]

The social planning problem can now be written as follows:

$$\max_{c(s)} W(t) = \int_t^{\infty} C(s) \cdot e^{R(t)-R(s)} \cdot ds \qquad \textbf{(32)}$$

subject to

$$\dot{K} = I - \delta K$$

$$C = A \cdot L_c \cdot f(K/L)$$

$$I = A \cdot B \cdot L_i \cdot f(K/L)$$

$$L = L_c + L_i$$

where $C(s) = c(s) \cdot L(s)$ and, as before, the discount factor $\exp[-R(s)]$, which discounts payments received at time *s* back to time 0, is defined as

$$e^{-R(s)} = e^{-\int_0^s r(\tau)d\tau} \qquad \textbf{(33)}$$

Hence the factor which discounts payments received at time *s* back to time *t* is $\exp[R(t)-R(s)]$.

Because of the assumption about technology, efficiency requires that the capital-labour ratio is the same in both industries and thus equal to the economy-wide capital-labour ratio. This justifies the way the production functions have been written above.

We can note the following properties of the discount factor:

$$(d/dt)e^{-R(t)} = -r(t) \cdot e^{R(t)} \qquad \textbf{(34)}$$

$$e^{R(t)-R(s)} = e^{-r(s-t)}, \text{ if } r(\tau) = r \text{ for } \tau \geq t \qquad \textbf{(35)}$$

By substituting the other constraints into the first, we can reduce the four constraints to one:

$$\dot{K} = L \cdot A \cdot B \cdot f(K/L) - B \cdot C - \delta K \qquad \textbf{(36)}$$

The current value Hamiltonian for this problem is

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[37] Sefton and Weale (2000) have also criticised the assumption of a constant real interest rate. They point out that, in the absence of TFP growth, constancy of the real interest rate implies constancy of the capital stock. Then WNDP reduces just to consumption. They derive under more general conditions a result which is analogous to equation **(42)** below (their Proposition 1). However, in another respect their model is less general since they assume that TFP growth is zero.

$$Y = C + p\dot{K} \qquad \textbf{(37)}$$

where p is the shadow price of capital. In our set-up this will turn out to equal the actual market price of the investment good in terms of the consumption good. So WNDP is the current value Hamiltonian, which justifies the use of the same symbol Y. From **(37)** and **(36)**, the first-order conditions are:

$$\frac{\partial Y}{\partial C} = 1 - pB = 0 \Rightarrow p = 1/B \qquad \textbf{(38)}$$

$$\frac{\partial Y}{\partial K} = -(\dot{p} - rp) \qquad \textbf{(39)}$$

plus the transversality condition:

$$\lim_{s \to \infty} e^{-R(s)} p(s) K(s) = 0 \qquad \textbf{(40)}$$

Totally differentiating the Hamiltonian with respect to time,

$$\frac{dY}{dt} = \frac{\partial Y}{\partial C}\frac{dC}{dt} + \frac{\partial Y}{\partial K}\frac{dK}{dt} + \frac{\partial Y}{\partial p}\frac{dp}{dt} + \frac{\partial Y}{\partial t}$$

Using the first-order conditions **(38)** and **(39)**, we find that along the optimal path

$$\begin{aligned} \frac{dY}{dt} &= -(\dot{p} - r(t)p)\dot{K} + \dot{K}\dot{p} + \frac{\partial Y}{\partial t} = r(t)p\dot{K} + \frac{\partial Y}{\partial t} \\ &= r(t)(Y - C) + \frac{\partial Y}{\partial t} \end{aligned} \qquad \textbf{(41)}$$

Theorem 2
The solution to this differential equation **(41)** is:

$$Y(t) = \int_t^\infty r(s) \cdot C(s) \cdot e^{R(t)-R(s)} \cdot ds - \int_t^\infty \frac{\partial Y}{\partial s} \cdot e^{R(t)-R(s)} \cdot ds \qquad \textbf{(42)}$$

If the real interest rate is constant and equal to r on the interval $[t, \infty]$, the solution becomes:

$$\begin{aligned} Y(t) &= r\int_t^\infty C(s) \cdot e^{-r(s-t)} \cdot ds - \int_t^\infty \frac{\partial Y}{\partial s} \cdot e^{-r(s-t)} \cdot ds \\ &= rW(t) - \int_t^\infty \frac{\partial Y}{\partial s} \cdot e^{-r(s-t)} \cdot ds \end{aligned} \qquad \textbf{(43)}$$

Proof of Theorem 2
We can of course easily verify by differentiation that **(42)** is *a* solution of **(41)**. But we want to show that it is the general solution.

The integrating factor for solving equation **(41)** is $e^{-R(t)}$. Multiplying through by this in **(41),**

$$e^{-R(t)}[\dot{Y}-r(t)Y]=-e^{-R(t)}r(t)C(t)+e^{-R(t)}(\partial Y/\partial t)$$

Applying **(34)** to the left-hand side,

$$(d/dt)[e^{-R(t)}Y(t)]=-e^{-R(t)}r(t)C(t)+e^{-R(t)}(\partial Y/\partial t) \qquad \textbf{(44)}$$

The fundamental theorem of integral calculus implies that, given any two functions $F(s)$ and $f(s)$, related by $F'(s)=f(s)$, we have

$$\int_t^\infty f(s)ds = F(\infty)-F(t)$$

assuming that $F(\infty)$ exists. Rearranging,

$$F(t)=F(\infty)-\int_t^\infty f(s)ds \qquad \textbf{(45)}$$

Now we apply this to the problem at hand and set

$$F(t)=e^{-R(t)}Y(t)$$

$$F(\infty)=\lim_{t\to\infty}e^{-R(t)}Y(t)=0 \quad \left(\text{from the transversality condition}\right) \qquad \textbf{(46)}$$

$$f(s)=-e^{-R(s)}r(s)C(s)+e^{-R(s)}(\partial Y/\partial s)$$

So in this notation, equation **(44)** states that $\dot{F}(t)=f(t)$. Substituting from **(46)** into **(45)**, we get

$$e^{-R(t)}Y(t)=\int_t^\infty r(s)\cdot C(s)\cdot e^{-R(s)}\cdot ds-\int_t^\infty \frac{\partial Y}{\partial s}\cdot e^{-R(s)}\cdot ds$$

Multiplying through by $\exp[R(t)]$, we obtain finally

$$Y(t)=\int_t^\infty r(s)\cdot C(s)\cdot e^{R(t)-R(s)}\cdot ds-\int_t^\infty \frac{\partial Y}{\partial s}\cdot e^{R(t)-R(s)}\cdot ds$$

which is equation **(42)**. Equation **(43)**, the special case of a constant real interest rate, then follows by using **(35)**. □

Remark 2
Equation **(43)** is the basic result of Weitzman (1976), as amended by Weitzman (1997): in the absence of TFP growth, WNDP is the yield on wealth. Equation **(42)** is a further generalisation to the case where the interest rate can vary. In this latter case, we see that it is no longer quite true that WNDP is the yield on wealth, even in the absence of TFP growth, since the interest rate appears inside instead of outside the first integral in equation **(42)**. But we can restore the flavour of Weitzmans's original result by defining a weighted average discount rate $\overline{r}(t)$:

$$\bar{r}(t) = \frac{\int_t^\infty r(s)\cdot C(s)\cdot e^{R(t)-R(s)}\cdot ds}{\int_t^\infty C(s)\cdot e^{R(t)-R(s)}\cdot ds}$$

Here the weights are the present value of consumption in each future period. (Note that this average discount rate is a function of *t*.) Now assuming for the moment that TFW growth is zero, we have from **(44)**

$$Y(t) = \bar{r}(t)\int_t^\infty C(s)\cdot e^{R(t)-R(s)}\cdot ds = \bar{r}(t)W(t)$$

So WNDP can still be interpreted as the yield on wealth, ie permanent income, even when the real interest rate is variable.

Remark 3
The second term on the right-hand side of equation **(42)** or **(43)** is the present value of future increments to net output resulting from exogenous technical progress. Using equation **(16)** specialised to one capital good, the rate of change of WNDP due to technical progress at time *s* is

$$\frac{\partial Y}{\partial s} = \lambda(s)Y(s) = Y(s)\cdot\left[(1+d)\cdot\hat{A} + d\cdot\hat{B}\right] \qquad \textbf{(47)}$$

Here $d = \delta pK/Y = \delta K/BY$, using the first-order condition for *C*. So if TFP growth is positive in both industries, WNDP (*Y*) underestimates the yield on wealth by an amount equal to the present value of future growth in the WNDP frontier. Intuitively, this is because, in the presence of TFP growth, the value of net investment does not capture the full value of future growth in consumption.

Corollary to Theorem 2
When the real interest rate is constant, the growth rate of WNDP equals the growth rate of wealth, even though the *level* of WNDP understates the yield on wealth: see Weitzman (1997).

Proof of Corollary
In a steady state, λ must be constant and *Y* will also grow at a constant rate, say *g* ($g < r$). So from **(43)** and **(47)**,

$$Y(t) = rW(t) - \lambda\int_t^\infty Y(s)\cdot e^{-r(s-t)}\cdot ds$$

$$= rW(t) - \lambda Y(t)\int_t^\infty e^{-(r-g)(s-t)}\cdot ds$$

$$= rW(t) - \lambda Y(t)/(r-g)$$

Hence

$$Y(t) = \left[\frac{r-g}{r-g+\lambda}\right]rW(t)$$

and *Y* grows at the same rate as *W*. □

A.3 Does WNDP give the same ranking of alternatives as the present value of utility?
It is tempting to assert a stronger result than Theorem 1, namely that WNDP and the present value of utility yield the same ranking of alternative situations. Consider two alternative steady states, A and B. These may differ in technology and initial capital stocks, so the growth rates of consumption and the initial level of consumption per household may differ. But both situations are to be evaluated using the same preferences (ie the same parameters ρ and θ in the utility function). Associated with these two states are WNDP levels $Y_A(0), Y_B(0)$ and optimal utility levels $U_A^*(0)$, $U_B^*(0)$ at time zero (where a star denotes optimal). Then we have the following

Conjecture If $Y_A(0) > Y_B(0)$, then $U_A^*(0) > U_B^*(0)$. That is, Weitzman's NDP, the yield on wealth, gives the same ranking as does (unobservable) utility.

In fact, we can show by a simple example that this conjecture is *false*. Let utility be logarithmic, $u(c(t)) = \ln(c(t))$, and assume a steady state in which consumption per household grows at rate *g* and the number of households at rate *n*. Then the present value of utility in the logarithmic case ($\theta = 1$) is given by:

$$U^*(0) = \int_0^\infty \ln[c^*(0)e^{gt}]L(0)e^{nt}e^{-\rho t}$$

$$= \ln(c^*(0))\int_0^\infty e^{(n-\rho)t}dt + g\int_0^\infty te^{(n-\rho)t}dt$$

$$= \frac{\ln(c^*(0))}{\rho - n} + \frac{g}{(\rho - n)^2}$$

where we have set $L(0) = 1$. Here $c^*(0)$ is the initial, optimally chosen, consumption level. The optimised level of wealth is given by

$$W^*(0) = \int_0^\infty c^*(0)e^{gt}L(0)e^{nt}e^{-rt}dt = c^*(0)L(0)\int_0^\infty e^{(g+n-r)t}dt$$

$$= \frac{c^*(0)}{r - n - g}$$

$$= \frac{c^*(0)}{\rho - n}$$

Here we have used the fact that the same consumption stream maximises both wealth and utility and that with logarithmic utility $r = g + \rho$ (see equation **(32)**).[38] WNDP is therefore given by

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[38] Incidentally, this shows that wealth is a poor measure of welfare when utility is logarithmic. In this case wealth is independent of the growth rate of consumption per household (*g*). So situations with the same initial consumption level but different growth rates would receive equal ranking by wealth. WNDP is a better measure, but still not ideal as will be shown in a moment.

$$Y(0) = rW^*(0) = \frac{(g+\rho)c^*(0)}{\rho - n}$$

To see that this is compatible with an optimising model of economic growth, consider the simple 'AK' model:

$$Z = AK,\ \ A > 0$$

Assuming utility maximisation, the solution to this model is (see Barro and Sala-i-Martin (1995, pages 141-43)):

$$g = (A - \delta - \rho)/\theta$$

$$r = A - \delta$$

$$c^*(0) = \phi K(0)/L(0),\ \ \phi = (A-\delta)(\theta - 1)/\theta + (\rho/\theta) - n$$

$$= (\rho - n)K(0)/L(0) \text{ if } \theta = 1$$

In this model TFP growth is zero. There are no transition dynamics so the economy is always in steady state. The initial consumption level is proportional to the given initial capital stock; the factor of proportionality depends only on exogenous parameters. In the logarithmic utility case the factor depends only on ρ and n. The equilibrium growth rate depends positively on the productivity of capital. So one country could have a lower initial consumption level but a higher growth rate than another, hence possibly (though not necessarily) a higher present value of utility.

Inspection of the formulas for $Y(0)$ and $U^*(0)$ suggests that they will not necessarily yield identical rankings and this turns out to be the case. Consider the following numerical example where in situation A the initial consumption level is assumed to be below that of situation B. We first set some parameters:

$$g_B = n_B = n_A = 0;\ \rho = 0.05;\ \theta = 1;\ L_A(0) = L_B(0) = 1;\ c_A^*(0) = 0.5;\ c_B^*(0) = 1$$

leaving g_A free to vary. For a range of values of g_A, we obtain the following results.

Table A.1
Comparison of WNDP and utility rankings

g_A	$Y_A(0)$	$Y_B(0)$	$U_A^*(0)$	$U_B^*(0)$	*Do rankings agree?*
0.030	0.80	1	-1.86	0	*YES*
0.045	0.95	1	4.14	0	*NO*
0.055	1.05	1	8.14	0	*YES*

Thus the WNDP and utility rankings agree for low and high growth rates but disagree for an intermediate growth rate. The reason is that when using WNDP as the yardstick we are assuming in effect that the marginal utility of income is the same in both situations. But when using the present value of utility levels as the yardstick we are allowing marginal utility to differ. In the logarithmic case, marginal utility at time zero is $1/c^*(0)$. So in the example it is twice as high in situation A as in situation B. The conclusion is that WNDP is an indicator of welfare but it may mislead us when the initial consumption levels in the situations being compared are very different. On the other hand, when the initial consumption levels are the same and only growth rates differ, then WNDP gives the correct ranking.

Money metric utility
An alternative justification for WNDP comes from setting up the problem explicitly as utility maximisation:[39]

$$\max_{c(s)} U = \int_t^\infty u[c(s)] \cdot L(s) \cdot e^{-\rho(s-t)} ds, \quad \rho > 0 \qquad \textbf{(48)}$$

where $c(s)$ is consumption of the representative household at time s and $L(s)$ is the number of households. An important feature of this formulation is that it assumes that the instantaneous utility function is cardinal. In other words, the ranking yielded by U is unique only up to a positive affine transformation of the instantaneous utility function u, to say $v(\cdot) = au(\cdot) + b,\ a > 0$. Applying this transformation leads to a similar transformation of U to say V:

$$V = aU + (b/\rho)$$

and clearly this preserves rankings. But if instead we applied a more general positive monotonic transformation to u, say from $u(c(s)) = \ln(c(s))$ to $v(c(s)) = c(s)$, then this would *not* in general preserve the ranking yielded by U.

The Hamiltonian in the problem of equation **(33)** is

$$H(s) = u(c(s)) \cdot L(s) + q(s) \cdot \dot{K}(s)$$

where $q(s)$ is the shadow price in utility units of net investment. Under perfect competition,

$$q(s) = \omega(s) p(s)$$

where $\omega(s)$ is the marginal utility of income. Weitzman (1999) and (2002) shows that the maximised Hamiltonian is related to the maximised utility function by

$$H^*(t) = \rho U^*(t)$$

where stars indicate optimal values. Utility is unobservable. But Weitzman shows that one can use WNDP itself as the utility function ('money metric utility'). The reason is as follows. We can apply a positive affine transformation to u, $v(\cdot) = au(\cdot) + b,\ a > 0$, and choose values for the parameters a and b such that in the base period, here time t, $v(c(t)) = c(t)$. The following choices satisfy this requirement

$$a = 1/\omega(t), \quad b = c^*(t) - \left[u(c^*(t))/\omega(t)\right]$$

[39] See Weitzman (1999) and Weitzman (2002, chapter 6), who also proves a more general result since he allows for multiple consumption goods (see his Theorem 1). In the latter case he shows that welfare comparisons involve an additional term in consumer surplus.

This is money metric utility. Under this transformation, marginal utility at time t is $\tilde{\omega}(t) = v'(c(t)) = 1$ and the new maximised Hamiltonian $\tilde{H}^*(t)$ now equals WNDP:

$$\tilde{H}^*(t) = \rho V^*(t) = v(c^*(t)) \cdot L(t) + \tilde{\omega}(t) p(t) \dot{K}^*(t)$$

$$= c^*(t) \cdot L(t) + p(t) \dot{K}^*(t)$$

Notice however that this transformation uses the value of marginal utility at the base time t. So all rankings based on WNDP must use this as a standard. Therefore this approach does not escape the difficulty revealed by the numerical example above. Of course, this difficulty is not unique to WNDP but affects all indicators (including GDP) that use money rather than utility as the measuring rod.[40]

A.4 The top down measure of TFW growth

Theorem 3

If the economy is competitive, if any given input receives the same price in all industries, and if the production functions (or equivalently the unit cost functions) are all the same up to a scalar multiple (TFP), then the growth rate of TFW can be expressed as in equation **(21)**:

$$\lambda = \hat{Y} - \beta \hat{K} - (1-\beta)\hat{L}$$

where K is an index of capital and β is the share of *net* profits in *net* domestic product.

Proof of Theorem 3

We prove the theorem for the case of one consumer good and one investment good. The extension to many goods is straightforward. The equations of the model are now

$$Y = C + p(I - \delta K) = C + p\dot{K} \qquad \textbf{(49)}$$

$$C = A \cdot L_c \cdot f(K/L) \qquad \textbf{(50)}$$

$$I = A \cdot B \cdot L_i \cdot f(K/L) \qquad \textbf{(51)}$$

$$L = L_c + L_i \qquad \textbf{(52)}$$

As before, p is the relative price of the investment good in terms of the consumption good: $p = P_i / P_c$ where P_i, P_c are the nominal prices. The production functions can be written in terms of the aggregate capital/labour ratio because of (a) constant returns to scale and (b) the

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[40] An alternative welfare measure has been suggested by Basu and Fernald (2002) and by Basu *et al* (2001). They argue that TFP growth (the Solow residual), adjusted for economic profit (which however they argue is small), is a measure of welfare change. They show that, following a 'small' shock to productivity, the change in welfare is proportional to TFP growth. But this argument relies on the shock being small. Consider a one-good model and assume that utility is linear in consumption. Then the steady-state growth of GDP equals $\mu/(1-\alpha)$ where (as before) μ is TFP growth (the Solow residual) and α is the share of capital. This is also the steady-state growth rate of consumption. Then by the assumption of linear utility welfare grows at the same rate too, ie $\mu/(1-\alpha)$, not μ. The reason is that productivity changes are in general not small but induce additional capital accumulation. So welfare accounting must allow for this, as indeed does WNDP. However, their approach does yield additional insight since they show that TFP growth still has a welfare interpretation even in the presence of imperfect competition. That is, though TFP is a biased measure of what they call technology, it still accurately measures welfare change, though I would add, only for small changes in technology.

assumption that they are identical up to a scalar multiple. The corresponding unit cost functions are

$$P_c = A^{-1} \cdot P_L \cdot g(P_K / P_L)$$

$$P_i = A^{-1} \cdot B^{-1} \cdot P_L \cdot g(P_K / P_L)$$

where P_K, P_L are the prices of capital and labour services. It follows immediately that

$$p = 1/B$$

$$\hat{p} = -\hat{B}$$

(Note that in this special case the total derivative of the log of the relative price with respect to time equals the partial derivative: compare equation **(12)**.) From the production functions, equations **(50)** and **(51)**, and the last result, we find that the allocation of labour between the two sectors is

$$\begin{aligned} L_c / L &= C/(C + pI) \\ L_i / L &= pI/(C + pI) \end{aligned} \qquad \textbf{(53)}$$

Let Π be aggregate profits deflated by the price of consumption. Then we can define the current price share of gross profit in GDP as:

$$\alpha = \Pi/(C + pI)$$

The current price share of *net* profit in *net* domestic product can be defined as

$$\beta = (\Pi - \delta pK)/Y$$

As before, we also define the current price ratio of depreciation to NDP as

$$d = \delta pK / Y$$

and we can note that the ratio of GDP to NDP is $(C + pI)/Y = (1+d)$. Finally, recall that we have already proved in the main text that the bottom up measure of TFW growth is given by

$$\lambda = (1+d)\hat{A} + d\hat{B}$$

(see equation **(16)**). Now differentiate equation **(49)** totally with respect to time:

$$\hat{Y} = (C/Y)\hat{C} + (pI/Y)\hat{I} - (\delta pK/Y)\hat{K} + [(pI - \delta pK)/Y]\hat{p} \qquad \textbf{(54)}$$

Under the assumption of competition and equality of input prices across industries, $\hat{C} = \alpha(\hat{K} - \hat{L}) + \hat{L}_c + \hat{A}$, $\hat{I} = \alpha(\hat{K} - \hat{L}) + \hat{L}_i + \hat{A} + \hat{B}$ (note that α is the same in both sectors), and, as just shown, $\hat{p} = -\hat{B}$. Substituting these last results into equation **(54)** and collecting terms,

$$\hat{Y} = \left[\frac{\Pi - \delta pK}{Y} \right] (\hat{K} - \hat{L}) + \left(\frac{C}{Y} \right) \hat{L}_c + \left(\frac{pI}{Y} \right) \hat{L}_i + \left(\frac{C + pI}{Y} \right) \hat{A} + \left(\frac{\delta pK}{Y} \right) \hat{B} \qquad \textbf{(55)}$$

Then substituting into this equation the expressions for the proportions of labour in the two sectors (equations **(53)**), the definition of the net profit share (β), and the expression for λ , and rearranging, we finally obtain

$$\lambda = \hat{Y} - \beta \hat{K} - (1 - \beta) \hat{L}$$