Soft liquidity constraints and precautionary saving

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Abstract

This paper considers the implications for consumption and saving behaviour when households are allowed to borrow, but face penalties which increase with the amount borrowed. It shows that the introduction of this type of constraints (soft liquidity constraints) does not lead to consumers behaving very differently from consumers who face constraints which prevent them from borrowing at any time (hard liquidity constraints). However, when hard constraints are relaxed and become soft, the amount of precautionary saving falls.

JEL classification: C6, D91, E21.

Summary

Miles Kimball defines a precautionary motive as 'any aspect of an agent's preferences which causes a risk to affect decisions *other* than the decision of how strenuously to avoid the risk itself and risks correlated with it (which is governed by risk aversion). A precautionary motive leads an agent to respond to a risk by making adjustments that will help to reduce the expected cost of the risk'. Thus, precautionary saving arises when forward-looking consumers accumulate wealth *today* for the purpose of reducing the impact of *future* uncertainty on *future* consumption decisions. Liquidity constraints arise when consumers have difficulties to obtain credit. More specifically, soft liquidity constraints represent the situation where consumers are able to borrow, but incur penalties which increase with the amount borrowed. Hard liquidity constraints refer to the unavailability of credit altogether.

The modern consumption literature has examined the problem of how much to consume and save each period under two polar scenarios. One scenario considers perfect capital markets where no barriers to borrowing exist and where interest rates are the same for savers and borrowers. The other scenario assumes that consumers are not able to borrow at all. Both scenarios, however, do not seem to match what is commonly observed in developed economies: consumers often borrow and face interest rates that are higher for debt than for saving.

The theoretical implications for consumption arising from the two polar cases are summarised by Carroll and Kimball in two papers. Carroll and Kimball (1996) look at the case where consumers are allowed to borrow, whereas Carroll and Kimball (2001) concentrate on the scenario where consumers are not allowed to borrow at all. These two papers provide the conditions under which the introduction of uncertainty and liquidity constraints leads to precautionary saving, and analyse how precautionary saving falls when wealth is accumulated. Technically, these conditions require the interaction of risk (either to labour income or to the rate of return) with liquidity constraints and/or with certain functional forms for the utility function.

Carroll and Kimball (2001) demonstrate three important implications of hard constraints for consumption. First, hard constraints increase precautionary saving around levels of wealth where the constraints bind. Second, if consumers face the possibility of becoming constrained at any point in the future, they will behave as if they were constrained today, even in the absence of a current liquidity constraint. Finally, the introduction of further borrowing constraints does not necessarily lead to an increase in precautionary saving.

This paper considers the implications for consumption behaviour when households are allowed to borrow, but face penalties that increase with the amount borrowed. The introduction of this type of constraint does not lead to consumers behaving very differently from consumers who face hard constraints. A soft constraint increases precautionary saving and affects prior periods, although the introduction of further soft constraints can lead to *lower* precautionary saving. However, a new result is that the amount of precautionary saving is reduced when hard constraints are relaxed and become soft. The intuition behind this result is simple: when consumers cannot borrow, they must have savings to avoid shocks that could leave them with low levels of income. A relaxation of the borrowing constraint means that consumers do not need to have these (high) savings to

avoid adverse shocks to income. More technically, the paper shows the effects that soft liquidity constraints have on the value, marginal value and consumption functions in a dynamic programme. The introduction of a soft constraint makes consumers more averse to risk (since the value function becomes more concave) and also more prudent (since the marginal value function becomes more convex). An implication is that the resulting consumption function becomes *concave* with respect to wealth.

1 Introduction and motivation

Since the late 1970s (Grossman *et al* (1979)) and the early 1980s (Zeldes (1989b)[(1)]), economists have been aware that consumers who cannot borrow at any time (ie who face a hard borrowing constraint) may engage in precautionary saving to protect themselves from those (bad) draws in income which would force them to borrow. Carroll and Kimball (2001) give the theoretical reasons behind that effect: 'when a liquidity constraint is added to the standard optimal consumption problem, the resulting value function exhibits increased prudence'[(2)] (page 1). Carroll and Kimball (2001) derive conditions under which hard constraints increase prudence (in the value function) and therefore lead to increased precautionary saving for the class of utility functions that exhibit hyperbolic absolute risk aversion (HARA).

In a similar vein to Zeldes (1989b) and Deaton (1991) who look at credit rationing, Scott (1996) uses numerical simulations to examine the implications for consumption for agents who face imperfect capital markets. Scott examines the 'consumption behaviour of risk-averse individuals in the presence of an upward-sloping interest rate schedule' (page 1). He finds that 'for even moderate levels of risk aversion [...] the threat of having to pay high borrowing rates means that consumers rarely borrow. Instead, they accumulate precautionary balances and so avoid the threat of having to pay penal interest rates in bad-income states of the world' (page 1). This paper provides the theoretical reasons for Scott's results by examining (a more general) consumption optimisation problem with soft constraints. Soft constraints allow consumers to get into debt, at the expense of having to pay penalties (such as interest payments) which increase with the amount borrowed. This paper also compares the behavioural differences that soft and hard constraints induce to the consumption problem.[(3)] Finally, the paper considers what happens to the consumption rule when soft constraints are introduced and compares results with Carroll and Kimball's (1996) consumption framework, which has no constraints of any type.

The introduction of soft constraints needs perhaps some motivation, which we obtain using the same analysis as in Scott. He bases the increasing interest rate schedule on the observations that in the United Kingdom '(a) consumers hold significant levels of debt and (b) there is wide diversity in the interest rates charged on loans' (page 2). This evidence suggests that not only are many individuals able to borrow, but also that they may use different sources of credit, starting with those that offer the lowest (interest) payment and moving to other sources which incur a higher (interest) payment when cheaper sources are no longer available. This appears to be consonant to a situation where consumers face soft liquidity constraints (with individuals facing increasing interest rate schedules).[(4)]

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(1) Parts of his 1984 MIT PhD thesis were published in 1989 (see references).

(2) A '*precautionary motive* is any aspect of an agent's preferences which causes a risk to affect decisions *other* than the decision of how strenuously to avoid the risk itself and risks correlated with it (which is governed by risk aversion). A precautionary motive leads an agent to respond to a risk by making adjustments that will help to reduce the expected cost of the risk. *Certainty equivalence*, which is the absence of precautionary motives, arises when an agent has no way to affect the expected cost of a risk' (Kimball (1991)). Technically, for a given period utility function, u, prudence is given by $\frac{-u'''}{u''} > 0$.

(3) In fact it is possible to define the soft constraint in our problem in such a manner that the consumer is effectively facing a hard constraint (ie consumers do not borrow at any time).

(4) These suggestions about soft constraints and Scott's work appear to contradict Stiglitz and Weiss's (1981)

Numerical simulations show that precautionary saving occur in consumption problems where a concave utility function has a positive third derivative. With a constant relative risk aversion (CRRA) utility function, Zeldes (1989a) finds that introducing uncertainty in labour income makes the *consumption rule concave* with the marginal propensity to consume everywhere *higher* than in the certainty case, and the *level* of consumption everywhere lower. The lower level of consumption represents precautionary saving. The concave consumption function implies that the marginal propensity to consume (the slope of the consumption function) decreases when wealth is increased.(5) The theoretical literature explaining Zeldes's numerical results dates back to the 1960s: Leland (1968) and Sadmo (1970) prove that a higher level of risk induces more precautionary saving for a given level of wealth. More recently, Kimball (1990a, 1990b) provides a theoretical explanation for the increase in the slope of the consumption function which income uncertainty induces for a *given level of consumption*. Carroll and Kimball (1996) complete the picture when they derive the conditions under which concavity in the consumption rule is induced by the introduction of risk. This paper uses their techniques to examine how the consumption rule is affected by the introduction of soft constraints to the consumer's problem. The resulting consumption function(6) has a higher marginal propensity to consume out of total resources and it is also more concave, compared with a problem where no constraints exist.

Apart from the differences in behaviour that may arise when agents facing hard and soft constraints are compared with agents facing no constraints, the problem examined here is interesting for other reasons. The soft constraint can be defined as a function which represents interest penalties charged by lending institutions to consumers. So interest rates may have a role in the transmission mechanism of the consumption problem, in addition to the impact that they have on the standard intertemporal budget constraint, which assumes that capital markets are perfect (ie that borrowers and lenders face the same interest rate). This new role via the soft constraint can introduce the interest rate differential faced by borrowers and lenders into the problem. Another reason why this problem may be interesting is that the same techniques used here could easily address the effects that the introduction of (asymmetric) adjustment cost functions have upon the solution path in problems that examine factor demands. An important finding from this type of problem is that even if the adjustment cost is not in place at the time an

well-known result that under adverse selection, banks do not raise borrowing rates but simply limit loan supply under some circumstances. Stiglitz and Weiss's result has been used by various authors to motivate the introduction of hard constraints (for example Zeldes (1989b), Deaton (1991), Seater (1997), etc). However, Milde and Riley (1988) provide a theoretical motivation for the introduction of soft constraints. They examine a more general version of the problem considered by Stiglitz and Weiss where agents can choose the size of loan they wish. This extension enables borrowers to use loan size to signal their characteristics to banks. The result is that the more individuals borrow, the higher the interest rate they face. At some point however, credit rationing may occur: at certain levels of debt a consumer may be unable to obtain additional loans at any interest rate. The introduction of soft constraints is therefore not inconsistent with Milde and Riley's arguments.

(5) More specifically, as wealth gets larger, the slope of the consumption function when there is uncertainty, gets closer and closer to the slope of the consumption function when there is no uncertainty. The intuition behind this result is that as wealth approaches infinity, the proportion of future consumption financed out of uncertain (future) labour income approaches zero and labour income uncertainty becomes less relevant to consumption decisions (see Carroll (2001a) for an excellent review of the modern literature on consumption).

(6) In this paper, like in Carroll and Kimball (1996, 2001) and others (for example Zeldes (1989a) and Carroll (2001a, 2001b)), the consumption function is defined as the expression that determines the level of consumption for a given level of wealth. Wealth is in turn defined as the sum of current assets and current labour income, or in Deaton's (1991) terms, 'cash-on-hand'.

agent makes her decision, the possibility that such adjustment cost may exist in the future induces similar behaviour as if the constraint were in place at the point the optimisation decision is made. (This highlights the importance of *horizontal aggregation.*[(7)])

There are three important findings in this paper (the first two are similar to Carroll and Kimball's problem with hard constraints). First, even if consumers are allowed to borrow at all times, the introduction of soft constraints leads to precautionary saving behaviour when the period utility function is quadratic. Second, the introduction of soft constraints does not necessarily lead to increased precautionary saving behaviour if such behaviour is already present before the constraint is introduced. Third, precautionary saving behaviour is never higher in a problem with soft constraints than in a problem with hard constraints.[(8)] What is the intuition behind this result? Consumers want to avoid the possibility of facing a constraint today or in the future and therefore save even if they can borrow. However, because they have the possibility of borrowing at their disposal, they do not need to accumulate as much saving to buffer themselves against a bad shock as consumers who are not able to borrow at all.[(9)] A corollary of all these arguments is that with the introduction of the soft constraint, the consumption rule will always be strictly concave with respect to wealth even if the utility function is quadratic, in a way that looks similar to that one suggested by Zeldes (1989a).

There are at least three important implications for the consumption/saving problem from the model that we examine in this paper. First, this model is very general and can represent problems where either hard, soft or no constraints exist, provided the function that introduces soft constraints is specified appropriately. Second, (if the function that incorporates soft constraints is modelled appropriately) one can eliminate the discontinuities in the Euler equation which arise from modelling hard constraints. This means that, in principle, it should be possible to solve the consumption problem more easily and one could solve the consumption rule using Taylor series approximations in a way which is similar to Skinner's (1988) approach. Third, with our consumption specification one is able to examine the effects that the soft constraint will have upon vertical and horizontal aggregation and therefore examine the impact of soft liquidity constraints on the consumption rule.

The structure of the paper is as follows. Section 2 describes the problem and makes the distinction between hard and soft constraints. Section 3 discusses the first-order condition for the consumption problem under two different scenarios: the first scenario assumes that the soft liquidity constraint kicks in at the point where the consumer begins to borrow, whereas the second scenario assumes that there exists an asymmetric cost of holding debt (which can be viewed as an asymmetric rate of return on net worth). Section 4 introduces some definitions which are useful to

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(7) The terms 'vertical' and 'horizontal aggregation' were first defined in Carroll and Kimball (1996, page 983). Because the value function is given by the maximised sum of all expected discounted utility functions, it is the result of two operations: first, the sum of discounted utility functions across states of nature at a given point in time and second by maximising the sum of utility in one period and the value from all subsequent periods. The first operation refers to vertical aggregation whilst the second operation is horizontal aggregation.

(8) This is because the marginal value function is more convex when hard constraints exist compared to a situation where soft constraints exist.

(9) To allow borrowing when the Inada condition applies, one needs to modify the nature of the income process; see Ludvigson (1999, footnote 12, page 437) or Carroll (2001a, 2001b).

examine the impact of soft constraints on consumption. Section 5 examines the impact of soft constraints on consumption. Section 6 concludes.

2 Soft constraints and consumption: the framework

The standard consumption problem consists of a representative consumer who is maximising the time-additive discounted value of utility from consumption $u(c)$. If the consumer does not face any future liquidity constraints, then denoting the (possibly stochastic) gross interest rate and time preference factors by $\widetilde{R}_t \in (0,\infty)$ and $\widetilde{\beta}_t \in (0,\infty)$ respectively, and labelling consumption c_t, stochastic labour income $\widetilde{y}_t$ and gross wealth (inclusive of period-t labour income) w_t, the consumer's problem can be written as:

$$
\begin{aligned}
J_t(w_t) &= \max_{c_t} u(c_t) + E_t \sum_{s=t+1}^{T} \prod_{j=t+1}^{s} \widetilde{\beta}_j u(c_s) \qquad \textbf{(1)}\\
s.t.\ w_{t+1} &= \widetilde{R}_{t+1}\left[(w_t - c_t)\right] + \widetilde{y}_{t+1} \text{ and terminal condition}\\
c_T &\leq w_T
\end{aligned}
$$

where $w_t = A_t + \widetilde{y}_t$ and A_t denotes assets. Re-write the problem (using the value function) as:

$$
\begin{aligned}
J_t(w_t) &= \max_{c_t} u(c_t) + E_t \widetilde{\beta}_{t+1} J_{t+1}(w_{t+1})\\
s.t.\ w_{t+1} &= \widetilde{R}_{t+1}\left[(w_t - c_t)\right] + \widetilde{y}_{t+1} \text{ and terminal condition}\\
c_T &\leq w_T
\end{aligned}
$$

with $u(c)$ satisfying Inada conditions:

$$
\begin{aligned}
\lim_{c\to 0} u'(c) &\to \infty\\
\lim_{c\to \infty} u'(c) &\to 0
\end{aligned}
$$

For future purposes denote:

$$\phi_t(s_t) = E_t \widetilde{\beta}_{t+1} J_{t+1}\left(\widetilde{R}_{t+1}(s_t) + \widetilde{y}_{t+1}\right)$$

where $s_t = w_t - c_t \ \forall\ t$. Carroll and Kimball (2001) look at the implications for consumption and savings behaviour associated with hard constraints or the condition that $s_t \geq 0$ at all times. This implies that consumers are not able to borrow at any point in their lifetime. In this paper we examine soft constraints and to this effect we assume that consumers face the following modification to the standard consumption problem:[10]

$$
\begin{aligned}
V_t\left(w_t\right) &= \max_{c_t} u\left(c_t\right) + E_t \sum_{s=t+1}^{T} \left[\prod_{j=t+1}^{s} \widetilde{\beta}_j \right] u\left(c_s\right) \qquad \textbf{(2)} \\
&\quad s.t. \\
w_{t+1} &= \widetilde{R}_{t+1}\left[\left(w_t - c_t\right) - f\left(w_t - c_t\right)\right] + \widetilde{y}_{t+1} \text{ and terminal condition} \\
c_T &\leq w_T
\end{aligned}
$$

We define savings as the difference between 'cash-in-hand' or wealth and consumption $s_t = w_t - c_t \; \forall \; t$.

The difference between our problem and the standard one (examined by Carroll and Kimball (1996)) is found in the term $f_t(s)$, the function that introduces the costs of borrowing/soft constraints. We assume that $f'(.) < 0$, $f''(.) > 0$ and $f'''(.) \leq 0$. Thus, the costs of borrowing increase as borrowing increases. Given that $f(.)$ is (strictly) convex by definition but it is preceded by a negative sign, we have that $w_{t+1} = \widetilde{R}_{t+1}\left[\left(w_t - c_t\right) - f\left(w_t - c_t\right)\right] + \widetilde{y}_{t+1}$ is (strictly) concave in w_t. Note that the function $f(.)$ can be defined in such a way that the problem becomes one where consumers will never borrow. To do this, define a level of wealth w^{hc} such that $\exists \; w^{hc}$ such that $\lim_{w \downarrow w^{hc}} V'_{t+i}(w) = \infty$. Then if:

$$
\begin{aligned}
\lim_{s \downarrow \leq 0} f_{t+i}(s) &= w^{hc}_{t+i} \forall \; i = 0, 1, ... \qquad \textbf{(3)} \\
&\Rightarrow \; V'_{t+i}(w) = \infty \; \forall \; i = 0, 1, ...
\end{aligned}
$$

the consumer would avoid getting into debt.

In the work that follows, we can either assume that the function 'kicks in' when savings are negative (we term this case the discontinuous case), or that we can write a proper specification so that the soft constraint function looks like Chart 1 (ie like an asymmetric adjustment cost function which we term the continuous case). Either way, the three basic conclusions of this research pointed in the introduction arise, but we require different methods to prove them.[11] In this paper we shall first prove the implications for the case where the constraints 'kick in' when savings are not positive (Sections 5.3 to 5.5) and then examine the case when the soft constraint looks like an asymmetric adjustment cost function (Section 5.6).

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(10) For previous literature looking at the consumption problem see Miller (1974, 1976), Sibley (1975), Schechtman (1976), Schechtman and Escudero (1977), Antzoulatos (1994) and Seater (1997).

(11) Essentially, for the discontinuous case the methods required to arrive at the conclusions mentioned above are very similar to Carroll and Kimball's (2001) methods. For the continuous case one can use the methods in Carroll and Kimball (1996) directly.

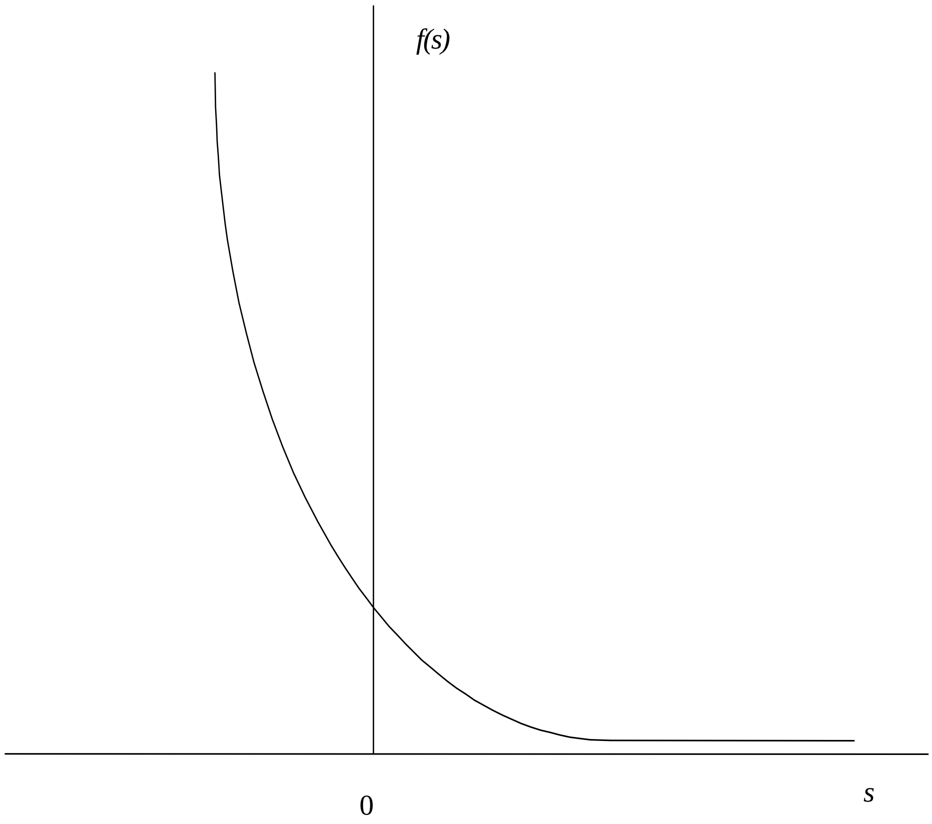

Chart 1: Asymmetric costs of borrowing

3 The effects of soft constraints: first-order conditions

The recursive nature of **(2)** allows us to write:

$$\begin{aligned} V_t(w_t) &= \max_{c_t} u(c_t) + E_t \widetilde{\beta}_{t+1} V_{t+1}(w_{t+1}) \\ &= \max_{c_t} u(c_t) + E_t \widetilde{\beta}_{t+1} V_{t+1}\left(\widetilde{R}_{t+1}\left[(s_t) - f(s_t)\right] + \widetilde{y}_{t+1}\right) \end{aligned} \quad \textbf{(4)}$$

We define $\mu(s_t) = E_t \widetilde{\beta}_{t+1} V_{t+1}\left(\widetilde{R}_{t+1}\left[(s_t) - f_t(s)\right] + \widetilde{y}_{t+1}\right)$ for future purposes and term it the 'expected utility of saving'. The first-order condition for the maximisation of the problem is:

$$c_t : 0 = u'(c_t) + E_t \widetilde{\beta}_{t+1} V'_{t+1}\left(\widetilde{R}_{t+1}\left[(s_t) - f(s_t)\right] + \widetilde{y}_{t+1}\right) \widetilde{R}_{t+1}\left[\frac{\partial s_t}{\partial c_t} - f'(s_t)\frac{\partial s_t}{\partial c_t}\right] \quad \textbf{(5)}$$

Note the following envelope condition between $V'_t(w_t)$ and $u'(c_t)$ along the optimal path: consider the effect of a small change in w_t on both sides of **(4)**:

$$w_t : V'_t(w_t) = E_t \widetilde{\beta}_{t+1} V'_{t+1}\left(\widetilde{R}_{t+1}\left[(s_t) - f(s_t)\right] + \widetilde{y}_{t+1}\right) \widetilde{R}_{t+1}\left[\frac{\partial s_t}{\partial w_t} - f'(s_t)\frac{\partial s_t}{\partial w_t}\right] \quad \textbf{(6)}$$

Note that $\frac{\partial s_t}{\partial w_t} = -\frac{\partial s_t}{\partial c_t} = 1$. Thus we have:

$$u'(c_t) = V'_t(w_t) \quad \textbf{(7)}$$

Using this in **(5)** yields our modified Euler equation for consumption in the *discontinuous* case[12]:

$$u'(c_t) = \max[E_t\widetilde{\beta}_{t+1}u'(c_{t+1})\widetilde{R}_{t+1}\left[1 - f'(s_t)\right], E_t\widetilde{\beta}_{t+1}u'(c_{t+1})\widetilde{R}_{t+1}] \qquad \textbf{(8)}$$

For the *continuous* case the Euler equation is simply:

$$u'(c_t) = E_t\widetilde{\beta}_{t+1}u'(c_{t+1})\widetilde{R}_{t+1}\left[1 - f'(s_t)\right] \qquad \textbf{(9)}$$

What are the implications of **(8)** and **(9)**? In the standard consumption framework, the marginal utility of consumption at time t is equated to the discounted marginal utility of consumption at time $t+1$. Because in our case $\left[1 - f'(s_t)\right] > 1$ (when the constraint bites), to equate the discounted value of marginal utilities cannot be a solution to the consumption problem, for then the right-hand side of **(8)** or **(9)** will be greater than the left-hand side. Because $u' > 0$ and $u'' < 0$, to make both sides equal the left-hand side must increase; this is achieved by reducing consumption at time t and increasing it at time $t+1$. Doing this means increasing the amount of savings at time t, given our assumptions about f, this means that the term $\left[1 - f'(s_t)\right]$ becomes closer to 1.[13][14]

4 Some preliminary definitions

In this paper we will be assuming that the consumption rule is drawn from HARA utility functions, ie those functions that satisfy the condition:

$$\frac{u'''u'}{u''^2} = k$$

Fama (1970) proves that the value function inherits monotonicity and concavity from the utility function if the budget constraint is *linear.*[15] Since our budget constraint is strictly concave (rather than linear) when the constraint bites, then the value function will inherit monotonicity and strict concavity from the utility function and the budget constraint. Thus if we assume that the utility function is of the HARA class, then value function will have the following characteristics: $V' > 0$, $V'' < 0$. Given the assumptions about f' being non-concave, the marginal value function will be strictly convex, ie $V''' > 0$. Moreover, the more convex $-f'$ becomes, the more convex V' will be. These claims are proved in Appendix A.

We define $E_t\widetilde{\beta}_{t+1}V'(w_{t+1})\widetilde{R}_{t+1}\left[1 - f'(s_t)\right] = \mu'_t(s_t)$ as the expected marginal utility of saving. Note that from the first-order condition and the envelope relations we have $\mu'_t(s_t) = u'(c_t) = V'_t(w_t)$. Denoting $\lambda_t = \mu'_t(s_t)$ we define the following inverses:

$$z_t(\lambda_t) = u'^{-1}(\lambda_t) = c_t \qquad \textbf{(10)}$$

(12) Compare to Zeldes (1989b)'s Euler equation $u'(c_t) = E_t\widetilde{\beta}_{t+1}u'\left(c_{t+1}\right)\widetilde{R}_{t+1} + \lambda_t$, where λ_t is the Lagrange multiplier associated with the hard constraint, or Deaton's $u'(c_t) = \max[E_t\widetilde{\beta}_{t+1}u'\left(c_{t+1}\right)\widetilde{R}_{t+1}, u'(w_t)]$.

(13) We require certain assumptions about the derivatives of the utility function and the cost of borrowing function to avoid a corner solution where consumption will be zero at time t. (See Grossman *et al* (1979) for the conditions in a case where there are hard constraints.)

(14) This is the result that will give us precautionary savings or prudence. Obviously, the more negative $f'(s)$ is the higher precautionary savings will be.

(15) This implies that as wealth increases total utility increases but at a decreasing rate. Moreover, the value function exhibits risk aversion.

$$g_t(\lambda_t) = \mu_t'^{-1}(\lambda_t) = s_t \quad \textbf{(11)}$$

$$h_t(\lambda_t) = V_t'^{-1}(\lambda_t) = w_t \quad \textbf{(12)}$$

We shall use these definitions to demonstrate that the introduction of the cost of borrowing term will lead to precautionary saving. By definition we have $s = w - c$ which implies that $h = z + g$ $\forall\, t$.

5 The effect of soft constraints: the slope of the value function and the consumption rule

This section examines the implications of the introduction of soft constraints on the value, marginal value and consumption functions. The introduction of soft constraints convexifies the marginal value function, leading to precautionary saving only when the convex marginal value function interacts with risk. Sections 5.3 to 5.5 demonstrate for the discontinuous case, that soft constraints induce precautionary saving when utility is quadratic, that the convex marginal value function propagates back to previous periods and that prudence is not necessarily enhanced when soft constraints are introduced to a problem where prudence already exists. Section 5.6 shows similar implications for the continuous case.

5.1 Why introducing soft constraints changes the slope of the marginal value function

Carroll and Kimball (2001) prove that introducing the restriction that $s > 0$ at all times makes the marginal value function V_t' convex at the point where the constraint begins to bind, even if utility functions are quadratic. With quadratic utility, this leads to precautionary saving behaviour in the face of liquidity constraints because the expected marginal utility of savings increases. To prove that the marginal value function becomes more convex, Carroll and Kimball show that introducing the liquidity constraint leads to an increase in the slope of the marginal value function V' at the points where the constraint is binding. This creates a kink in the marginal value function, making it convex. If this kink interacts with risks associated to either the labour income process, the rate of return or both, then the expected marginal utility of savings will increase. We shall prove that our problem also introduces more convexity to the marginal value function and can therefore lead to increased precautionary saving behaviour in the face of liquidity constraints as the expected marginal utility of savings increases.

In lemma 1 below we prove how the slope of the marginal value function changes (it becomes more convex) as the soft constraint binds. The proof is valid for both the continuous and discontinuous cases.[16] In lemma 2 we explain how the change in the marginal value function has to interact with risks to change the expected marginal value of savings and therefore change prudence. Proving these two lemmas enables us to say that the introduction of the soft constraint leads to precautionary saving when utility functions are quadratic (theorem 1).

Lemma 1: *The introduction of the soft constraint for a given level of savings makes the marginal value function more convex.*

(16) One could use the same techniques as Carroll and Kimball (2001) for the discontinuous case to demonstrate how the value function becomes strictly convex.

Proof: See Appendix A for a proof.

An important implication of this lemma is that the more convex $-f'$ is, the more convex the value function will become and therefore the behaviour of the consumer who faces the soft constraint will be closer to the behaviour of a consumer who faces a hard constraint.

5.2 The effect of the change in the slope of the marginal value function on consumption

To consider the effects that the introduction of the soft constraint has on the consumption/savings decision consider Chart 2.(17) With a linear marginal value function (and no constraints) there are no precautionary saving. As we know from Appendix A, the introduction of the soft constraint makes the value function strictly convex at the point where the constraint bites. Now consider a situation without any uncertainty between periods t and $t+1$, such that the optimal savings decision would lead the consumer to be at point A. Now consider the effect of introducing a two point mean-zero risk $\widetilde{\epsilon}$, which, if period t saving is held constant, leads to two possible outcomes, $A-\epsilon$ and $A+\epsilon$, where $A-\epsilon$ is to the right of the point at which the soft constraint begins to bite. In this case, the addition of the risk has no effect on the expected marginal value function. Consider on the other hand, the introduction of a larger two point mean-zero risk $\widetilde{\eta}$ where $\widetilde{\eta} > \widetilde{\epsilon}$ and where $A-\widetilde{\eta}$ is to the left of the point at which the soft constraints begin to bite. In this case it is easy to see that the introduction of the risk increases the expected marginal value function, which in turn will increase the marginal utility of savings and therefore increase the level of savings. Thus, in this case, the risk $\widetilde{\eta}$ will induce precautionary saving. We see that it is important in this framework for *the risk to interact with the point at which the soft constraint begins to bite*. Thus, the requirements for precautionary saving when soft constraints exist and utility is quadratic are the same as those required when there are hard constraints and utility is *quadratic* (theorem 1).(18)

(17) Chart 2 depicts a situation where the utility function is quadratic and thus the marginal value function is linear, in the absence of any constraints, since the value function inherits the characteristics of the utility functions.
(18) In the proof we are assuming that the probability that future constraints will not bind is 1.

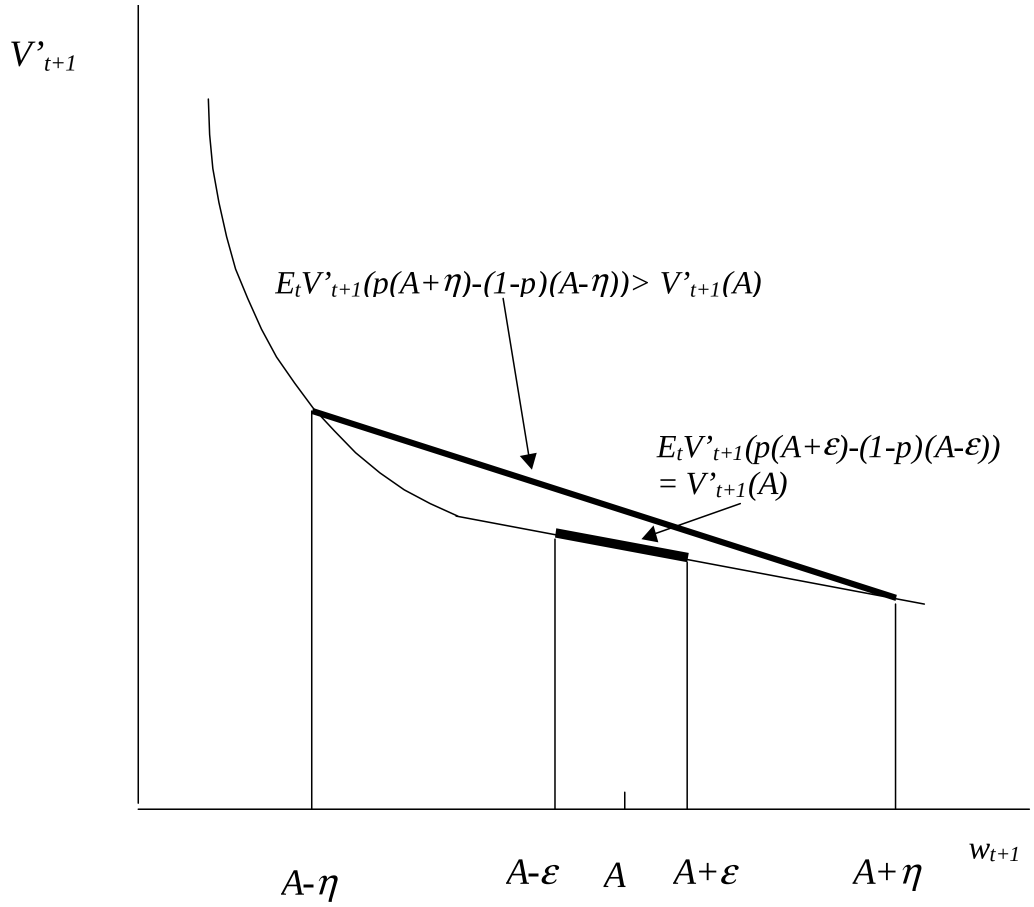

Chart 2: Convexity in the marginal value function when utility is quadratic

We can express these arguments mathematically, using the same definitions and lemma 6 used by Carroll and Kimball (2001). We use the concepts of a support to a mean-preserving spread.[19]

Definition 1: *Consider the interval* $[\underline{w}, \bar{w}]$ *such that* $F_1\left(\underline{w}\right) = F_2\left(\underline{w}\right) = 0$ *and* $F_1\left(\bar{w}\right) = F_2\left(\bar{w}\right) = 1$, *and let the distribution* F_2 *be a mean-preserving spread of* F_1; *that is,* $G_2\left(w\right) > G_1\left(w\right)$ *and* $G_2\left(\bar{w}\right) = G_1\left(\bar{w}\right)$ *where* $G_1\left(w\right) = \int_{\underline{w}}^{w} F_1\left(w\right) dw$ *and* $G_2\left(w\right) = \int_{\underline{w}}^{w} F_2\left(w\right) dw$.

Definition 2: *The open support of the mean-preserving spread is the set* $\{w \mid G_2(w) > G_1(w)\}$. *The support is the closure of the open support.*

To illustrate these definitions, in Chart 2, the support of the mean-preserving spread $\widetilde{\epsilon}$ is the region from $A - \epsilon$ to $A+ \epsilon$, and the support for the mean-preserving spread $\widetilde{\eta}$ is the region from $A - \eta$ and $A+ \eta$. The support of the mean-preserving spread caused by going from $\widetilde{\epsilon}$ to $\widetilde{\eta}$ is the union of the region from $A - \epsilon$ to $A - \eta$ and the region from $A + \epsilon$ to $A + \eta$. With these two concepts in mind we are in a position to make the following lemma which is equivalent to Carroll and Kimball's (2001) Lemma 6.

(19) See Rothchild and Stiglitz (1970) for an early paper looking at the mean-preserving spread in consumption.

Lemma 2: *For a given level of saving s_t, let Ψ be the open support of a mean-preserving spread in w_{t+1}, and let Γ be the set of points at which $V'_{t+1}(w_{t+1})$ is strictly convex. Then the expected marginal value of saving at s_t is increased by the mean-preserving spread iff $\Psi \cap \Gamma \neq \emptyset$.*

Proof: Drop the $t+1$ subscripts for clarity. The change in the expectation of next period's value function as a result of the mean-preserving spread is:

$$\begin{aligned} & \int_{\underline{w}}^{\bar{w}} V'(w)dF_2(w) - \int_{\underline{w}}^{\bar{w}} V'(w)dF_1(w) \\ = & \int_{\underline{w}}^{\bar{w}} [G_2(w) - G_1(w)]dV''(w) \end{aligned}$$

The last integral demonstrates the proposition of the lemma because the integral will only be positive if there is some set of points at which $G_2(w) - G_1(w) > 0$ and $dV''(w) > 0$. These are in fact the points at which the integral interacts with the convexity of the marginal value function (see definitions 1 and 2) and imply $\Psi \cap \Gamma \neq \emptyset$.

The principal difference between a soft and hard constraint framework is that with soft constraints (in the discontinuous case), the marginal value function is strictly convex *at all points where savings are negative* rather than just at the kink.[20] For the continuous case it is straightforward to demonstrate that $\Psi \cap \Gamma \neq \emptyset$ by noting that $dV''(w) > 0 \; \forall \; w$, a straightforward consequence of the proofs in Appendix A.

Theorem 1: *If $f' < 0$, $f'' > 0$ and $f''' \leq 0$, then the introduction of the cost of borrowing constraint can lead to precautionary saving when the utility function is quadratic, only under the case where the introduction of a risk leads to a probability strictly between 0 and 1 that a soft constraint will bind in period $t+1$.*

Proof: Given the last two lemmas, it is trivial to prove the theorem. The proof is simply to note that lemma 1 makes the marginal value function more convex and that lemma 2 implies that the introduction of the risk increases the expected marginal utility of saving, and that an increase in the expected marginal utility of saving will induce the consumer to save more and therefore consume less.

So far we have proved that the introduction of the liquidity constraint in the next time period leads to precautionary saving this period. In the next four subsections we characterise the nature of the consumption rule; the first three subsections examine the discontinuous case. We look at the properties of vertical and horizontal aggregation, which allow us to determine whether the convexity of the marginal value function is preserved when expectations are taken and also whether the precautionary saving motive propagates back to prior periods, which will enable us to see whether a consumer who faces the possibility of being constrained in the future (not just next time period), engages in precautionary saving behaviour today.

(20) In Chart 2 these represent all points to the left of the point at which the constraints begin to bite.

5.3 Implications for quadratic utility: the discontinuous case

In this section we look at the quadratic utility function case in detail, since, in the absence of borrowing constraints, it yields certainty equivalence. We show that with borrowing restrictions, precautionary saving behaviour arises.

The imposition of the constraint in period $t+1$ makes the marginal value function more convex at time $t+1$. Given this observation, what are the implications for consumption behaviour in period t? To address this question we look at the concepts of horizontal and vertical aggregation.

5.3.1 Vertical aggregation

Lemma 3a: *$\mu_t'(s_t)$ is strictly convex at all values s_t such that there is a positive probability that wealth in period $t+1$ will be at a level such that $V_{t+1}'(w_{t+1})$ is strictly convex.*

Proof: Since taking expectations is merely a weighted sum across convex (concave) states if $V_{t+1}'(w_{t+1})$ is convex (concave), $\mu_t'(s_t)$ will also be convex since weighting does not affect convexity.

5.3.2 Horizontal aggregation

Lemma 4a: *If $\mu_t'(s_t)$ is convex after the introduction of the constraint, then $V_t'(w)$ will be convex.*

Proof: Note that $V_t'(w) = u'(c_t) + \max[\mu_t'(s_t), \phi'(s_t)]$. Since we know that summation preserves convexity, we know that $V_t'(w)$ will be convex iff $u'(c_t)$ and $\max[\mu_t'(s_t), \phi'(s_t)]$ are both convex. Now, since we know that for quadratic utility functions $u'(c_t)$ is convex (not strictly convex) then $V_t'(w)$ will be strictly convex iff $\max[\mu_t'(s_t), \phi'(s_t)]$ is strictly convex. Because convexity is preserved by the max operation if $\mu_t'(s_t)$ is strictly convex then $V_t'(w)$ will be strictly convex. As we showed in lemma 1, $\mu_t'(s_t)$ is strictly convex when the constraint bites and therefore $V_t'(w)$ is strictly convex.

With lemmas 3a and 4a we can say something about the impact that soft constraints have on a problem where utility is quadratic. This is summarised in theorem 2.

Theorem 2: *Adding a liquidity constraint to an optimisation problem with quadratic utility induces precautionary saving at any level of wealth such that when the constraint is introduced there is a probability $0 < p < 1$ that the constraint will bind in some future period.*

Proof: Note that the introduction of the constraint in any future period $t+n$ makes the marginal value function at that time period strictly convex. Using lemma 2 we know that the expected marginal value function at the time the constraints are introduced $(t+n)$ will be strictly convex. Using lemma 4a implies that the marginal value function the previous time period $(t+n-1)$ will be strictly convex (even if the constraints do not bind at time $t+n-1$). Continued iteration using lemmas 3a and 4a means that the marginal value function at time t will be strictly convex.

5.4 Soft constraints, hard constraints and prudence for CRRA utility: the discontinuous case

Carroll and Kimball (2001) show that introducing a hard constraint into a previously unconstrained problem with risky income does not necessarily increase the prudence of the value function when the initial value function already exhibits positive prudence.(21) The reason for the result is that the introduction of the constraint 'can 'hide' certain points on the marginal value function that are exposed if the constraint is not present' (page 34).

In this section we show that Carroll and Kimball's proof has to be modified to show the same conclusion when soft constraints are present and in doing so we demonstrate some of the differences that exist between a framework where consumers are not able to borrow compared to a scenario where consumers can but face increasing costs of borrowing.

5.4.1 *A simple experiment*

Consider a consumer with CRRA utility functions(22) who faces income uncertainty but no constraints of any type. Such a consumer is represented by problem **(1)**, where the value function satisfies the Inada condition at some point $\underline{w}$ such that

$$\exists\, \underline{w} \text{ such that } \lim_{w \downarrow \underline{w}} V'_{t+1}(w) = \infty. \qquad \textbf{(13)}$$

For this proof we assume that $\widetilde{R} = \widetilde{\beta} = 1$ for simplicity.

We consider three consumers. Consumer A faces a hard constraint of the type examined by Carroll and Kimball (2001). Consumer B faces a soft constraint of the type examined in this problem. For both A and B, income in period $t+1$ is non-stochastically equal to $\overline{y}$ and both have an initial amount of wealth w_t in period t. The maximisation problem for both types of consumers is:

$$\begin{aligned} A \quad &: \quad \max_{c_t^A} u(c_t^A) + V_{t+1}(s_t^A + \overline{y}) && \textbf{(14)} \\ s.t.\ s_t^A &= w_t - c_t^A > 0 \\ B \quad &: \quad \max_{c_t^B} u(c_t^B) + V_{t+1}(s_t^B - f\left(s_t^B\right) + \overline{y}) && \textbf{(15)} \end{aligned}$$

Consumer C is a non liquidity-constrained consumer with the same $u(c_t)$, V_{t+1} and initial wealth as both A and B, but whose income has a small probability of going to $\underline{w}$ next period. If this probability does not materialise, the consumer will have $\overline{y}$ next period. This consumer solves:

$$C: \max_{c_t^C} u(c_t^C) + pV_{t+1}(s_t^C + \underline{w}) + (1-p)\, V_{t+1}(s_t^C + \overline{y}) \qquad \textbf{(16)}$$

We want to show that as the probability p of the bad shock approaches zero, the behaviour of the unconstrained consumer facing the risk becomes arbitrarily closer to the behaviour of the

(21) For the continuous case equivalent see Section 5.6.

(22) A CRRA utility function is given by the following functional form:

$$u(c) = \frac{c^{1-\theta}}{1-\theta}$$

whereas a CARA utility function is commonly given by:

$$u(c) = \frac{1}{\sigma} e^{-\sigma c}$$

consumer facing the hard constraint, but differs slightly from the consumer with the soft constraint. It is only trivial to show that if one changes the nature of condition **(13)**, one can show that the behaviour of the unconstrained consumer becomes arbitrarily closer to the behaviour of the consumer facing the soft constraint. This will require that the probability of wealth falling to a lower bound leads to a situation where the Inada condition is avoided.

We consider two scenarios, one where the constrained consumers decide to save and one where both the soft and hard constraints hold. If the initial level of wealth is high enough, none of the constrained consumers will be forced to borrow and therefore they will save. Their first-order conditions are:

$$u'(w_t - s_t^A) = V'_{t+1}(s_t^A + \overline{y})$$

and

$$u'(w_t - s_t^B) = V'_{t+1}(s_t^B + \overline{y})$$

The FOC for the unconstrained consumer is:

$$u(c_t^C) = pV'_{t+1}(s_t^C + \underline{w}) + (1-p)\, V'_{t+1}(s_t^C + \overline{y})$$

Thus given the nature of the assumed risk,

$$\lim_{p\downarrow 0}[pV'_{t+1}(s_t^C + \underline{w}) + (1-p)\, V'_{t+1}(s_t^C + \overline{y})] = V'_{t+1}(s_t^C + \overline{y})$$

Since saving is determined by the FOCs, this implies that:

$$\lim_{p\downarrow 0} s_t^C = s_t^A = s_t^B$$

thus, the behaviour of the unconstrained consumer becomes arbitrarily close to both constrained consumers.

Now, consider a second scenario where both hard and soft constraints hold. The first-order conditions for the two constrained consumers are:

$$u'(c_t^A) = V'_{t+1}(\overline{y}) + \lambda_t \qquad \textbf{(17)}$$

and

$$u'(w_t - s_t^B) = V'_{t+1}(s_t^B - f(s_t^B) + \overline{y})\left[1 - f'(s_t^B)\right] \qquad \textbf{(18)}$$

where λ_t represents the Lagrange multiplier associated with the hard constraint. The difference between A and B, is that A cannot borrow and is forced to consume all its initial wealth, $c_t^A = w_t$. For A, the marginal utility of spending in period t exceeds the marginal utility of having more income in the next period. On the other hand, because consumer B is allowed to borrow, it can therefore increase its consumption above the initial level of wealth but in doing so it incurs a cost when borrowing. This means that:

$$u'(w_t) > u'(w_t - s_t^B) > V'_{t+1}(s_t^B - f(s_t^B) + \overline{y}) > V'_{t+1}(\overline{y})$$

(where the second inequality comes from $\left[1 - f'(s_t^B)\right] > 1$). Thus the introduction of both constraints results in behaviour which differs for the two types of constrained agents. The consumer with soft borrowing restrictions consumes more and therefore has a lower level of precautionary saving compared to a consumer with hard constraints. To complete the argument we set out to prove, consider the unconstrained consumer C. Clearly if C were to save 0 and then experience the bad shock, the Inada condition would hold and then the marginal value function would be ∞. Thus, dissaving is ruled out for this consumer given the nature of the bad shock.[23] [24]

Given the nature of the bad shock and the fact that consumer C will not save, we now need to show that if C were to choose any amount greater than 0, say $\delta > 0$, then as p approaches 0, there will always be some point where C could improve its utility by saving less. If consumer C saves $\delta > 0$ then its marginal utility in period t will be $u'(w_t - \delta)$. The first-order condition for the problem will be:

$$u'(w_t - \delta) > \lim_{p \downarrow 0} \left[p V'_{t+1}(\delta + \underline{w}) + (1-p) V'_{t+1}(\delta + \overline{y}) \right] = V'_{t+1}(\delta + \overline{y})$$

this is because

$$u'(w_t - \delta) > u'(w_t) > V'_{t+1}(\overline{y}) > V'_{t+1}(\overline{y} + \delta)$$

by the concavities of the utility and value functions. Hence, as $p \downarrow 0$, C will be able to improve total utility by saving less and therefore consuming more at t but less at $t+1$. Since this is true for any $\delta > 0$, this argument shows that as p goes to zero, there is no positive level of saving at which C will be better off. Hence saving goes to zero. Thus given these two examples, the behaviour of A and C becomes arbitrarily so close that the type of risk examined above is indistinguishable from a hard liquidity constraint. We can modify these arguments for a scenario where consumer B will behave like C, and in that case, the behaviour of a consumer with soft constraints will be very similar to a consumer that faces no constraints. To do this take equation **(3)** and apply the same proofs as above.

5.4.2 Implications

The implications of these arguments are twofold:

1) For different (income) risk specifications (given by different experiments of the type considered by **(13)**), the behaviour of constrained individuals becomes indistinguishable from that one of a consumer without constraints. Carroll and Kimball prove this for the case of hard constraints.

2) The difference between the two constrained individuals has implications for the level of precautionary saving associated with the two types of constraints. It shows that individuals that face soft constraints do not have a higher level of precautionary saving than individuals with hard constraints.

The implications of 1) above for any type of constraints, result in conclusions that are equal to those obtained by Carroll and Kimball. We know that the introduction of a constraint (hard or soft) induces a change of curvature in the marginal value function at the point where the constraint begins to bind. This means that prudence in the value function, defined as $-\frac{V'''(w)}{V''(w)}$ will be affected by the introduction of the constraint at the level of wealth which activates the constraint (ie at the kink). This jump at the kink leads to infinite prudence at that point (since we are differentiating

(23) This is the result that drives Carroll's buffer-stock framework of consumption.

(24) One can revisit the nature of the bad shock and make it such that the individual dissaves an amount that is equivalent to that consumed by consumer B in our work.

the value function at that point of jump). What does this mean if we consider the arguments in Subsection 5.4.1?

In the limit as $p \downarrow 0$, a future risk (which has to be different for C to mimic the behaviour of consumers A and B) like **(13)** becomes indistinguishable from a (hard) liquidity constraint. Thus, introducing liquidity constraints (soft or hard) in period t where there is a pre-existing risk of this kind is in principle not distinguishable from introducing a second liquidity constraint. Another way to see this is to note that a point which was a kink point before the new liquidity constraint is imposed will not necessarily remain a kink point after the new constraint is imposed. Since the prudence of the value function at the kink point was infinite before the constraint was imposed and may be finite after the constraint is introduced, the introduction of the constraint could reduce the level of precautionary saving at the level of wealth corresponding to the kink point. Thus, this period's constraints can hide the effects of future risk by making the consumer save so much that those future risks are less consequential than before the liquidity constraint was introduced.

5.5 When does adding a constraint to a nonquadratic problem increase precautionary saving?: the discontinuous case

As discussed above, adding a hard or soft constraint to a consumption problem where precautionary saving behaviour is already observed does not necessarily lead to increased precautionary saving behaviour. The reason for this is that introducing any constraint can hide certain points on the marginal value function that are exposed if the constraint is not present. To examine the conditions which would induce precautionary saving when a constraint is introduced, one has to look at the definition of prudence in the value function, $\frac{-V'''}{V''}$. If we start with the envelope condition and differentiate it three times one can obtain an expression of prudence in terms of the consumption rule and the period utility function:

$$
\begin{aligned}
u'(c[w]) &= V'(w) \qquad\qquad \textbf{(19)}\\
u''(c[w])c'[w] &= V''(w)\\
u'''(c[w])(c'[w])^2 + u''(c[w]) &= V'''(w)\\
-\frac{V'''(w)}{V''(w)} &= (-\frac{u'''(c[w])}{u''(c[w])})c'[w] - \frac{c''[w]}{c'[w]}
\end{aligned}
$$

Denote the value function and consumption function after the introduction of the constraint as $\overline{V}$ and $\overline{c}[w]$. What we want to show is that:

$$
-\frac{\overline{V}'''(w)}{\overline{V}''(w)} > -\frac{J'''(w)}{J''(w)} = (-\frac{u'''(c[w])}{u''(c[w])})c'[w] - \frac{c''[w]}{c'[w]}
$$

which will hold if:

$$
(-\frac{u'''(\overline{c}[w])}{u''(\overline{c}[w])})\overline{c}'[w] - \frac{\overline{c}''[w]}{\overline{c}'[w]} > (-\frac{u'''(c[w])}{u''(c[w])})c'[w] - \frac{c''[w]}{c'[w]}
$$

In Appendix B we show that $\overline{c}''[w] < c''[w]$, or $-\overline{c}''[w] > -c''[w]$ and that for a given level of wealth that $\overline{c}'[w] > c'[w]$. Thus introducing the constraints in the problem makes it difficult to say something about $-\frac{\overline{c}''[w]}{\overline{c}'[w]}$ compared to $-\frac{c''[w]}{c'[w]}$ unless we look at certain scenarios or make assumptions about this ratio. Carroll and Kimball consider the cases where in the unconstrained problem $c''[w] = 0$. This occurs in cases where utility is quadratic, where utility is of the CARA form and the only future risk is additive (labour income risk) and when utility is CRRA and the only future risk is multiplicative (rate of return risk). If these aforementioned scenarios apply we know that $-\frac{\overline{c}''[w]}{\overline{c}'[w]} > -\frac{c''[w]}{c'[w]} = 0$. This leads us to investigate the conditions by which $(-\frac{u'''(\overline{c}[w])}{u''(\overline{c}[w])})\overline{c}'[w] > (-\frac{u'''(c[w])}{u''(c[w])})c'[w]$. For this condition to hold we have to observe that the marginal propensity to consume out of wealth does not fall[25] and that either 1) the utility function exhibits decreasing absolute prudence (which implies that $(-\frac{u'''(\overline{c}[w])}{u''(\overline{c}[w])}) > (-\frac{u'''(c[w])}{u''(c[w])})$), or 2) that the utility function exhibits constant absolute prudence (which implies that $(-\frac{u'''(\overline{c}[w])}{u''(\overline{c}[w])}) = (-\frac{u'''(c[w])}{u''(c[w])})$).

Thus, the conditions needed to prove that more prudence is induced when the soft constraint is introduced are the same as the conditions examined in Carroll and Kimball (2001). In that paper it was proved that when the consumption function is concavified (or that the marginal propensity to consume out of wealth decreases with wealth at an increasing rate), either by the introduction of hard constraints or uncertainty, prudence then increases compared to a situation where there was no concavity in the consumption function (either because there was no uncertainty or liquidity constraints). But in our case, as the preceding arguments suggest, the introduction of the soft constraint concavifies the consumption function compared to a situation where the consumption function was linear (ie $c''[w] = 0$). This increases prudence. So the answer to the question we set out to prove is that a soft constraint will increase prudence if the consumption function was not concave prior to the introduction of the soft constraint.

5.6 *The effect of the change in the slope of the marginal value function on consumption: the continuous case*

Having proved in Sections 5.1 and 5.2 that for the continuous case the introduction of soft constraints leads to precautionary saving, we now characterise the nature of the consumption rule.

5.6.1 *Vertical aggregation*

Our next lemma, the continuous version of lemma 3a, states that convexity of the marginal value function is preserved when expectations are taken (ie when aggregating across states of nature) provided some conditions are met. This lemma gives an intuition for the way in which Chart 2 was drawn where there is precautionary saving behaviour.

Lemma 3b: *Vertical aggregation: if* $\frac{V'''_{t+1}V'_{t+1}}{V''^2_{t+1}} > k$, $k \leq 3/2$ *and* $\frac{(-f''')(1-f')}{f''^2} > k$ *then when the soft constraint binds in* $t+1$, $\frac{\mu'''_t \mu'_t}{\mu''^2_t} \geq x > k$.

Proof: The expression $\frac{\mu'''_t \mu'_t}{\mu''^2_t}$ will be $\geq k$ if $\mu'''_t \mu'_t - k\mu''^2_t$ is non-negative. This expression is the determinant of the following matrix:

$$\Phi_t = \begin{bmatrix} \mu'''_t & \sqrt{k}\mu''_t \\ \sqrt{k}\mu''_t & \mu'_t \end{bmatrix}$$

(25) But we know from Appendix B that this will not happen.

Therefore $\frac{\mu_t'''\mu_t'}{\mu_t''^2} \geq k$ if Φ_t is positive semidefinite. Note that we have:

$$\begin{aligned}\mu_t''' &= E_t\widetilde{\beta}_{t+1}\left\{V_{t+1}'''\left(\widetilde{R}_{t+1}\left[1-f'(s_t)\right]\right)^3\right.\\ &\left.+3V_{t+1}''\widetilde{R}_{t+1}^2\left[1-f'(s_t))\right]\left[-f_t''(s)\right]+V_{t+1}'\widetilde{R}_{t+1}\left[-f'''(s_t)\right]\right\}\end{aligned}$$

$$\mu_t'' = E_t\widetilde{\beta}_{t+1}\left\{V_{t+1}''\left(\widetilde{R}_{t+1}\left[1-f'(s_t)\right]\right)^2+V_{t+1}'\widetilde{R}_{t+1}\left[-f''(s_t)\right]\right\}$$

$$\mu_t' = E_t\widetilde{\beta}_{t+1}\left\{V_{t+1}'\widetilde{R}_{t+1}\left[1-f'(s_t)\right]\right\}$$

Since $V_{t+1}''' \geq 0$, $V_{t+1}'' \leq 0$ and $V_{t+1}' \geq 0$, and $(1-f'(s_t)) > 1$, $(-f''(s_t)) < 0$ and $(-f'''(s_t)) > 0$ then:

$$\mu_t''' > E_t\widetilde{\beta}_{t+1}\left\{V_{t+1}'''\left(\widetilde{R}_{t+1}\left[1-f'(s_t)\right]\right)^3\right\} > 0$$

$$\mu_t'' < E_t\widetilde{\beta}_{t+1}\left\{V_{t+1}''\left(\widetilde{R}_{t+1}\left[1-f'(s_t)\right]\right)^2\right\} < 0$$

and

$$\mu_t' > E_t\widetilde{\beta}_{t+1}\left\{V_{t+1}'\widetilde{R}_{t+1}\left[1-f'(s_t)\right]\right\} > 0$$

This implies that for the following conditions to be true:

$$\frac{\mu_t'''\mu_t'}{\mu_t''^2} > \frac{V_{t+1}'''V_{t+1}'}{V_{t+1}''^2} \geq k$$

we require as a sufficient condition that:

$$\begin{aligned}&V_{t+1}''V_{t+1}'\widetilde{R}_{t+1}^3\left[1-f'(s_t))\right]^2\left[-f_t''(s)\right]\{3-2k\}\\ &+V_{t+1}'^2\widetilde{R}_{t+1}^2\left\{\left[-f'''(s_t)\right]\left[1-f'(s_t)\right]-k\left[-f''(s_t)\right]\right\} > 0\end{aligned}$$

since we have assumed that $\frac{V_{t+1}'''V_{t+1}'}{V_{t+1}''^2} > k$. To guarantee that this last expression is non-negative, we require:

$$\begin{aligned}\frac{(-f'''(s_t))(1-f'(s_t))}{[f''(s_t)]^2} &> k\\ k &\leqslant 3/2\end{aligned}$$

This guarantees that the determinant of Φ_t is positive semidefinite for all positive realisations of $\widetilde{R}$, $\widetilde{\beta}$ and $\widetilde{y}$ since $\mu_{t+1}'''\mu_{t+1}' - k\mu_{t+1}''^2 \in [0,\infty)$ and thus:

$$\frac{\mu_t'''\mu_t'}{\mu_t''^2} > \frac{V_{t+1}'''V_{t+1}'}{V_{t+1}''^2} \geq k$$

The implication of this lemma is that the expected marginal value function will be more convex than the marginal value function itself under certain circumstances. Note that if $\frac{V_{t+1}'''V_{t+1}'}{V_{t+1}''^2} = k = 0$

and soft constraints exist such that $\frac{(-f'''(s_t))(1-f'(s_t))}{[f''(s_t)]^2} > 0$, then $\frac{\mu_t'''\mu_t'}{\mu_t''^2} > 0$ and therefore precautionary saving will occur. Obviously, the greater the ratio $\frac{(-f'''(s_t))(1-f'(s_t))}{[f''(s_t)]^2}$ is the greater the difference between $\frac{\mu_t'''\mu_t'}{\mu_t''^2}$ and $\frac{V_{t+1}'''V_{t+1}'}{V_{t+1}''^2}$ and k.

5.6.2 Horizontal aggregation

Our next lemma uses the results from the last lemma to prove that introducing the cost of borrowing term at any time propagates back to prior periods and thus increases the convexity of the marginal value function across all states. Thus, if consumers face a constraint at $t+i$, $i=1,2,...$, their behaviour at time t may be equivalent to one where the constraint was faced at t. Again, our framework has the advantage that it is able to make statements about the impact that horizontal aggregation will have on the problem.

Lemma 4b: *Horizontal aggregation: If* $\frac{\mu_t'''\mu_t'}{\mu_t''^2} \geq x > k$ *and* $\frac{u'''u'}{u''^2} = k$ *then* $\frac{V_t'''V_t'}{V_t''^2} > k$.

Proof: We drop the time subscripts from z, g and h for convenience. We have:

$$z' = -\frac{1}{u''},$$

$$z'' = \frac{u'''}{u''^2}z' = -\frac{u'''}{u''^3}$$

thus

$$-\frac{\lambda z''}{z'} = \frac{u'''u'}{u''^2} = k.$$

Similarly,

$$-\frac{\lambda g''}{g'} = \frac{\mu_t'''\mu_t'}{\mu_t''^2} \geq x$$

and

$$-\frac{\lambda h''}{h'} = \frac{V_t'''V_t'}{V_t''^2}$$

but $h = z + g$, $h' = z' + g'$ and $h" = z" + g"$, so

$$-\frac{\lambda h''}{h'} = \frac{z'}{z'+g'}\left(-\frac{\lambda z''}{z'}\right) + \frac{g'}{z'+g'}\left(-\frac{\lambda g''}{g'}\right)$$

since we have, $\frac{z'}{z'+g'}, \frac{g'}{z'+g'} > 0$, $-\frac{\lambda z''}{z'} > k$ and $-\frac{\lambda g''}{g'} \geq x$ then we have:

$$\frac{V_t'''V_t'}{V_t''^2} = d > k$$

What this lemma implies is that the introduction of the soft constraint which makes the marginal value function more convex at time $t+1$, will make the marginal value function more convex at time t. Using the same principles, if the marginal value function is more convex at time t, it will then be more convex at time $t-1$ and so on. Moreover, note that the greater the ratio $\frac{(-f'''(s_t))(1-f'(s_t))}{[f''(s_t)]^2}$ is, the greater the difference between x and k and therefore the greater the difference between d and k.

5.6.3 *Consumption rule*

Our next lemma states that the optimal consumption rule is (strictly) concave.

Lemma 5: *If* $\frac{V_t'''V_t'}{V_t''^2} = d > k$ *and* $\frac{u'''u'}{u''^2} = k$, *then the optimal consumption policy rule is (strictly) concave,* $c_j^{*\prime\prime}(w_t) < 0$.

Proof: Define a function that yields the amount of saving corresponding to any optimally chosen level of consumption: $\theta_t^*(c_t) = w_t^*(c_t) - c_t$ where $w_t^*(c_t)$ is the inverse of the optimal consumption rule. If we can prove that $\theta_t^*(c_t)$ is convex and thus $w_t^*(c_t)$, then we will have shown that the consumption rule must be concave since it is the inverse of $w_t^*(c_t)$. Note that we can write $\theta_t^*(c_t) = g(z^{-1}(c_t))$. Differentiate this expression with respect to c_t :

$$\theta_t^{*\prime}(c_t) = \frac{g'(z^{-1}(c_t))}{z'(z^{-1}(c_t))}$$

$$\theta_t^{*\prime\prime}(c_t) = \frac{\left[g''(z^{-1}(c_t))z'(z^{-1}(c_t)) - g'(z^{-1}(c_t))z''\left(z^{-1}(c_t)\right)\right]/z'(z^{-1}(c_t))}{\left[z'(z^{-1}(c_t))\right]^2}$$

$$= \frac{g'\left(z^{-1}(c_t)\right)}{\left[z'(z^{-1}(c_t))\right]^2}\left[\frac{g''\left(z^{-1}(c_t)\right)}{g'\left(z^{-1}(c_t)\right)} - \frac{z''\left(z^{-1}(c_t)\right)}{z'\left(z^{-1}(c_t)\right)}\right]$$

which from the last lemma,

$$= \frac{g'(\lambda_t)}{[z'(\lambda_t)]^2}\frac{1}{\lambda_t}\left[\frac{-\lambda_t z''(\lambda_t)}{z'(\lambda_t)} - \frac{-\lambda_t g''(\lambda_t)}{g'(\lambda_t)}\right]$$

thus we have $\frac{-\lambda_t z''(\lambda_t)}{z'(\lambda_t)} = k$, $\frac{-\lambda_t g''(\lambda_t)}{g'(\lambda_t)} \geq x > k$, $\frac{1}{\lambda_t} > 0$. This means that

$$sign\left(\theta_t^{*\prime\prime}(c_t)\right) = -sign\left(g'(\lambda_t)\right) = -sign\left(\frac{1}{\mu''(s_t)}\right) > 0$$

since $\mu_t'' = E_t\widetilde{\beta}_{t+1}\left\{V_{t+1}''\left(\widetilde{R}_{t+1}\left[1 - f_t'(s)\right]\right)^2 + V_{t+1}'\widetilde{R}_{t+1}\left[-f''(s_t)\right]\right\} < 0$. Thus $\theta_t^{*\prime\prime}(c_t)$ is strictly convex implying that the optimal consumption rule is (strictly) concave.

This lemma shows the effect that the concavity/convexity of the utility, marginal utility, expected utility of saving and marginal expected utility of saving functions have on the consumption rule. Such concavity/convexity determines the level of precautionary saving and the marginal propensity to consume out of wealth.

5.6.4 *Consumption rule across periods*

We are now ready to prove the following theorem:

Theorem 3: *For utility functions of the HARA class, for any permissible income process,*(26) *if* $u'(c) > 0$, $u''(c) < 0$, *and* $u'''(c) \geq 0$, *the introduction of a borrowing constraint induces a strictly concave consumption rule.*

Proof: In the last period of life, $c_T = w_T$ and $V_T(w_t) = u(w_t)$, so $\frac{V_T'''V_T'}{[V_T'']^2} = \frac{u'''u'}{u''^2} = k$. Therefore, by lemma 3b, $\frac{\mu_{T-1}'''\mu_{T-1}'}{\mu_{T-1}''^2} \geq x > k$ and by lemma 4b $x \geq \frac{V_{T-1}'''V_{T-1}'}{V_{T-1}''^2} \geq d > k$. Continuous iteration

(26) A permissible income process is defined as any income process 'which permits the agent to ensure that consumption remains within the domain over which $u(c)$ is defined' (Carroll and Kimball (1996, page 984)).

using lemmas 3b and 4b demonstrates that for any $t < T$, $\frac{V_t''' V_t'}{V_t''^2} \geq d > k$. Then lemma 5 shows that the consumption rule is strictly concave in all time periods t.

Corollary 1: *When utility is quadratic and soft constraints bite at some point in the future, the consumption rule is strictly concave.*

Proof: Substitute $k = 0$ in the theorem and use lemmas 3b, 4b and 5 to prove that the consumption rule is strictly concave.

To reconcile this corollary to theorem 3 note that when the consumption rule across periods is strictly concave, precautionary saving exist. This theorem with corollary 1 represents the first result of our paper which implies that the introduction of soft constraints induces precautionary saving in a problem where the utility function is quadratic.

Corollary 2: *The greater the ratio* $\frac{(-f'''(s_t))(1-f'(s_t))}{[f''(s_t)]^2}$ *is, the more concave the consumption function will be.*

Proof: The proof is obvious, the greater the ratio $\frac{(-f'''(s_t))(1-f'(s_t))}{[f''(s_t)]^2}$ is, x will be greater.

Corollary 3: *If prudence already exists, we cannot say whether the consumption rule will be more concave than what it already is.*

Proof: With CARA and CRRA utility functions, we do not know whether $\frac{V_{t+1}''' V_{t+1}'}{V_{t+1}''^2} = k \leq 3/2$ and thus we cannot make exact statements the convexity of the expected marginal value function, ie the condition $\frac{\mu_t''' \mu_t'}{\mu_t''^2} > k$. Thus, when prudence already exists in the value function, either through CARA, CRRA or because a soft constraint bites in the future when utility is quadratic, then we cannot say definite statements about whether prudence increases after the introduction of the constraint. This is consistent with the results we found in Section 5.4.

6 Conclusions and future work

In this paper we have shown that the introduction of soft constraints does not lead to behaviour that is fundamentally different to that under hard constraints. But consumers who face soft constraints *have lower levels of precautionary saving than those who face hard constraints*. An important aspect of this framework is that we have been able to make exact statements about vertical and horizontal aggregation and therefore about the nature of the consumption rule. We have shown that consumption rules that originate from this type of problem are strictly concave and will therefore look like Zeldes's (1989a) and Scott's (1996) solutions obtained using numerical methods. This implies that precautionary saving behaviour exists in this framework. Finally, we have shown that the more constrained the individual becomes (ie as the consumer moves from a situation with soft constraints to a situation with hard constraints), the more concave the consumption rule will be.

This framework has important implications for problems other than the traditional consumption/saving decision. A natural extension for the problem examined in this paper would be to apply the same principles to problems where decision-makers (for instance, firms and governments) face a constraint (soft or hard) in their borrowing. The results of our work suggest that even if they do not face constraints today, but face the possibility of constraints in the future, agents may engage in precautionary behaviour today (this is the implication of horizontal aggregation).

These results also have important implications for models other than those which consider borrowing decisions. First, our model suggests that the introduction of adjustment costs, either today or in the future (for instance, adjustment costs that are asymmetric, like firing and hiring costs in labour demand), should have the same effect on optimal intertemporal decisions as the effect the function f has on our problem. This means that it is possible to gain information about the effects of adjustment costs in intertemporal optimisation decisions by looking at the first-order conditions rather than requiring a solution to the problem, which is often impossible to solve analytically.

Second, if the f function is defined appropriately, it could be possible to examine the effects of the credit channel on consumption/saving decisions. This may allow us to examine ways in which the policy-maker can affect the shape of f today or in the future and thus see how consumption decisions change. For instance one could consider the implications of increased competition in the credit market by reducing the slope of the soft constraint, making the soft constraint less convex, etc.

Finally, Skinner (1988) solves the consumption problem **(2)** with CRRA utility and without constraints using Taylor approximations and obtained a solution for the consumption rule. Skinner identifies the effect of risks (to income) on consumption and showed how precautionary saving enters the problem. In an extension to this paper, we could check the accuracy of the Taylor series approximation when soft constraints are introduced to the consumption problem and examine their implications for precautionary saving.

Appendix A: Characteristics of the value function

In this section we prove the assertion that the value function inherits the characteristics of the one-period utility function.(27)

Lemma 6: *If the one-period utility function is monotone increasing and strictly concave (m.i.s.c.) then the value function* $V(w)$ *will be m.i.s.c..*

Proof: (By induction.) In the last period, $w_T = c_T$ thus $V_T(w) = u_T(c) = u_T(w)$ is m.i.s.c.. We now prove that the value function is m.i.. Noting that $c_t(w+\delta)$ need not be the same as $c_t(w)+\delta$,we have that

$$\begin{aligned}
V_t(w+\delta) &= \max_{c_t}[u\left(c_t\left(w+\delta\right)\right) \\
&\quad +E_t\widetilde{\beta}_{t+1}V_{t+1}\left(\widetilde{R}_{t+1}\left[\left(w_t-c_t\left(w+\delta\right)\right)-f\left(w_t-c_t\left(w+\delta\right)\right)\right]+\widetilde{y}_{t+1}\right)] \\
&\geq [u\left(c_t\left(w\right)+\delta\right) \\
&\quad +E_t\widetilde{\beta}_{t+1}V_{t+1}(\widetilde{R}_{t+1}\left(w_t+\delta-c_t\left(w\right)-\delta\right) \\
&\quad -\widetilde{R}_{t+1}f\left(w_t+\delta-c_t\left(w\right)+\delta\right)+\widetilde{y}_{t+1})] \\
&> [u\left(c_t\left(w\right)\right) \\
&\quad +E_t\widetilde{\beta}_{t+1}V_{t+1}\left(\widetilde{R}_{t+1}\left[\left(w_t-c_t\left(w\right)\right)-f\left(w_t-c_t\left(w\right)\right)\right]+\widetilde{y}_{t+1}\right) \\
&= V_t(w)
\end{aligned}$$

so $V_t(w)$ is m.i.. This proof holds even if $f(s)=0$. Concavity is also proved by induction. We have already proved that $V_T(w) = u_T(c) = u_T(w)$ is m.i.s.c.. Now we need to prove that $V_t(w)$ is concave. Assume that $V_{t+1}(w)$ is concave although not necessarily strictly concave. Then, the function $\vartheta_{t+1}(w) = V_{t+1}\left[\widetilde{R}\left[(w-c)-f\left(w-c\right)\right]+\widetilde{y}\right]$ is strictly concave if $f \neq 0$ and quasi-concave if $f=0$. As the sum of concave functions is itself concave, $E_t\widetilde{\beta}_{t+1}V_{t+1}\left(\widetilde{R}\left[w-c-f\left(w-c\right)\right]+\widetilde{y}\right)$ is concave in w and c if $f_t=0$ and strictly concave if $f_t \neq 0$. The same applies to $\bar{\vartheta}_t(w) = u(c)+E_t\widetilde{\beta}_{t+1}V_{t+1}\left(\widetilde{R}\left[w-c-f\left(w-c\right)\right]+\widetilde{y}\right)$ which is strictly concave by the nature of $u(c)$. By nature of the strict concavity of $\vartheta_t(w)$, we have $\vartheta_t(\lambda w_1+(1-\lambda)w_2) = \vartheta_t(w) > \lambda\vartheta_t(w_1)+(1-\lambda)\vartheta_t(w_2)$ where $\lambda w_1+(1-\lambda)w_2 = w$ and w, w_1 and w_2 are different levels of wealth. Thus we have:

$$\begin{aligned}
V_t\left(w\right) &= \max_c \vartheta_t(w) > \lambda \max_c \vartheta_t(w_1)+(1-\lambda)\max_c \vartheta_t(w_2) \\
&= \lambda V_t(w_1)+(1-\lambda)V_t(w_2)
\end{aligned}$$

which completes the proof.

(27) See Benveniste and Scheinkman (1979) and Lippman (1987) for more on the value function when there are no borrowing constraints.

Corollary 4: *The introduction of the soft constraint makes the value function more concave for a level of savings that warrants the introduction of the constraint.*

Proof: Note that when the soft constraint binds, $\vartheta_{t+1}(w)$ is strictly concave if $f_t \neq 0$ and quasi-concave if $f_t = 0$.

A direct implication of corollary 4 is that the introduction of the soft constraint increases risk aversion.

Lemma 7: *If the one-period marginal utility function is convex, then the marginal value function will be convex.*

Proof: (By induction.) Consider the FOCs:

$$V_T'(w) = u'[c(w)] \tag{A-1}$$

and

$$u'[c(w)] = \widetilde{\beta}_T \widetilde{R}_T u'(c_T)\left[1 - f'(s_{T-1})\right] \tag{A-2}$$

Let $w = \lambda w_1 + (1-\lambda) w_2$ and $c(w_1) \neq c(w_2)$; then if:

$$c[\lambda w_1 + (1-\lambda) w_2] \geq \lambda c[w_1] + (1-\lambda) c[w_2] \tag{A-3}$$

it follows that:

$$\begin{aligned} u'\{c[\lambda w_1 + (1-\lambda) w_2]\} &\leq u'[\lambda c[w_1] + (1-\lambda) c[w_2]] \\ &< \lambda u'[c(w_1)] + (1-\lambda) u'[c(w_2)] \end{aligned}$$

where the last inequality arises if u' is strictly convex. Hence, if **(A-3)** holds, **(A-1)** implies that:

$$V_T'[\lambda w_1 + (1-\lambda) w_2] \leq \lambda V_T'[w_1] + (1-\lambda) V_T'[w_2] \tag{A-4}$$

$$< \lambda V_T'[w_1] + (1-\lambda) V_T'[w_2] \tag{A-5}$$

On the other hand, suppose that:

$$c[\lambda w_1 + (1-\lambda) w_2] < \lambda c[w_1] + (1-\lambda) c[w_2] \tag{A-6}$$

Then

$$\lambda w_1 + (1-\lambda) w_2 - c[\lambda w_1 + (1-\lambda) w_2] \tag{A-7}$$

$$> \lambda w_1 + (1-\lambda) w_2 - \lambda c[w_1] - (1-\lambda) c[w_2] \tag{A-8}$$

Now to prove convexity in **(A-2)** we have to prove it in two parts. We look at $\widetilde{\beta}_T \widetilde{R}_T u'(c_T)$ first. Note that by the convexity of f and **(A-6)** we have:

$$f(\lambda w_1 + (1-\lambda) w_2 - c[\lambda w_1 + (1-\lambda) w_2]) \tag{A-9}$$

$$< \lambda f(w_1 - c[w_1]) + (1-\lambda) f(w_2 - c[w_2]) \tag{A-10}$$

thus:

$$
\begin{aligned}
& -f\left(\lambda w_1+(1-\lambda)\, w_2-c\left[\lambda w_1+(1-\lambda)\, w_2\right]\right) && \textbf{(A-11)}\\
> \;& -\lambda f\left(w_1-c\left[w_1\right]\right)-(1-\lambda)\; f\left(w_2-c\left[w_2\right]\right) && \textbf{(A-12)}
\end{aligned}
$$

Note that because we assume that in the last period everything is consumed, then:

$$c_T=w_T=\widetilde{R}_T\left[\left(w_{T-1}-c_{T-1}\right)-\, f_{T-1}(w-c)\right]+\widetilde{y}_T$$

Thus

$$\widetilde{\beta}_T \widetilde{R}_T u'\left(c_T\right)=\widetilde{\beta}_T \widetilde{R}_T u'\left(\widetilde{R}_T\left[\left(w_{T-1}-c_{T-1}\right)-\left[1-\, f'(s_{T-1})\right](w-c)\right]+\widetilde{y}_T\right)$$

from **(A-7)** and **(A-11)**, we have:

$$
\begin{aligned}
& \widetilde{\beta}_T \widetilde{R}_T u'(\widetilde{R}_T\left[\left(\lambda w_1+(1-\lambda)\, w_2-c\left[\lambda w_1+(1-\lambda)\, w_2\right]\right)\right]\\
& -\widetilde{R}_T\, f\left(\lambda w_1+(1-\lambda)\, w_2-c\left[\lambda w_1+(1-\lambda)\, w_2\right]\right)+\widetilde{y}_T)\\
< \;& \widetilde{\beta}_T \widetilde{R}_T u'(\widetilde{R}_T\left[\left(\lambda w_1+(1-\lambda)\, w_2-c\left[\lambda w_1+(1-\lambda)\, w_2\right]\right)\right]\\
& \widetilde{R}_T\left[-\lambda\, f\left(w_1-c\left[w_1\right]\right)-(1-\lambda)\; f\left(w_2-c\left[w_2\right]\right)\right]+\widetilde{y}_T)\\
< \;& \widetilde{\beta}_T \widetilde{R}_T u'(\widetilde{R}_T\left[\lambda\left(w_1-c\left[w_1\right]-\, f\left(w_1-c\left[w_1\right]\right)\right)\right]\\
& +(1-\lambda)\, \widetilde{R}_T\left(w_2-c\left[w_2\right]-\, f\left(w_2-c\left[w_2\right]\right)\right)+\widetilde{y}_T)\\
= \;& \widetilde{\beta}_T \widetilde{R}_T u'(\left[\lambda\left(\widetilde{R}_T\left[w_1-c\left[w_1\right]-\, f\left(w_1-c\left[w_1\right]\right)\right]+\widetilde{y}_T\right)\right]\\
& +(1-\lambda)\left[\widetilde{R}_T\left(w_2-c\left[w_2\right]-\, f\left(w_2-c\left[w_2\right]\right)\right)+\widetilde{y}_T\right])\\
< \;& \widetilde{\beta}_T \widetilde{R}_T \lambda u'\left(\widetilde{R}_T\left(w_1-c\left[w_1\right]-\, f\left(w_1-c\left[w_1\right]\right)\right)+\widetilde{y}_T\right)\\
& +\widetilde{\beta}_T \widetilde{R}_T \lambda u'\left(\widetilde{R}_T\left(w_2-c\left[w_2\right]-\, f\left(w_2-c\left[w_2\right]\right)\right)+\widetilde{y}_T\right)
\end{aligned}
$$

where the first inequality comes from **(A-11)**, the second one from **(A-6)** and the final one from the convexity of the marginal utility function. It is the first inequality which arises from the impact that the binding constraints have, that makes the value function more convex and also what makes the value function convex even when marginal utility is linear. To complete the proof we need to prove that since $\left[1-\, f_t'(s)\right]\geq 1$, then

$$
\begin{aligned}
& -f'\left(\lambda w_1+(1-\lambda)\, w_2-c\left[\lambda w_1+(1-\lambda)\, w_2\right]\right)\\
\leq \;& -\lambda f'\left(w_1-c\left[w_1\right]\right)-(1-\lambda)\; f'\left(w_2-c\left[w_2\right]\right)
\end{aligned}
$$

and so

$$
\begin{aligned}
1 \;<\;& \left[1-\, f'\left(\lambda w_1+(1-\lambda)\, w_2-c\left[\lambda w_1+(1-\lambda)\, w_2\right]\right)\right]\\
\leq\;& \left[1-\lambda f'\left(w_1-c\left[w_1\right]\right)-(1-\lambda)\; f\left(w_2-c\left[w_2\right]\right)\right]
\end{aligned}
$$

which is true given the quasi-concavity of the f' function. Hence

$$\begin{aligned} V_T'[\lambda w_1 + (1-\lambda)\, w_2] &= u'[c_T(\lambda w_1 + (1-\lambda)\, w_2)] \\ &< \lambda u'[c_T(w_1)] + (1-\lambda)\, u'[c_T(w_2)] \\ &= \lambda V_T'\,[w_1] + (1-\lambda)\, V_T'\,[w_2] \end{aligned}$$

Thus, in either case $V_T'[\lambda w_1 + (1-\lambda)\, w_2] < \lambda V_T'\,[w_1] + (1-\lambda)\, V_T'\,[w_2]$. Also, the general case is similar. Now suppose that $V_{t+1}'(w)$ is convex. Then the following equations must be satisfied:

$$V_t'(w) = u'\,[c_t(w)]$$

and

$$u'\,[c_t(w)] = E_t \widetilde{\beta}_{t+1} V_{t+1}' \left(\widetilde{R}_{t+1}\,[(s_t) - f_t(s)] + \widetilde{y}_{t+1}\right) \widetilde{R}_{t+1} \left[1 - f_t'(s)\right]$$

hence since u' and $V_{t+1}'(w)$ are convex, it follows that $V_t'\,(w)$ is convex by exactly the same argument as above.

Corollary 5: *The introduction of the f function makes the marginal value function more 'convex' even if the one-period utility function is quadratic.*

Proof: It is obvious that the introduction of the f function has made the value function more convex.

This proof is equivalent to Carroll and Kimball's (2001) Lemma 5, thus showing that the introduction of the constraint makes the marginal value function more convex.

Appendix B: Soft constraints, the marginal propensity to consume and concavity in the consumption function

Lemma 8: *The level of consumption is lower in the case where there are constraints compared to the case where there are no constraints.*

Proof: Note that from the first-order condition

$$
\begin{aligned}
u'\left(\overline{c}_t(w_t)\right) &= \overline{V}'(w_t) \\
&= \max[E_t\widetilde{\beta}_{t+1}\overline{V}'\left(w_{t+1}\right)\widetilde{R}_{t+1}\left[1-f'(s_t)\right], E_t\widetilde{\beta}_{t+1}\overline{V}'\left(w_{t+1}\right)\widetilde{R}_{t+1}] \\
&\leq V'\left(w_t\right) = u'(c_t(w_t))
\end{aligned}
$$

where the inequality comes from the proofs in Appendix A which demonstrate that the introduction of the soft constraint convexifies the marginal value function. Thus, from concavity of u then $c_t(w_t) \geq \overline{c}_t(w_t)$.

For the continuous case note

$$
\begin{aligned}
u'\left(\overline{c}_t(w_t)\right) &= \overline{V}'(w_t) = E_t\widetilde{\beta}_{t+1}\overline{V}'\left(w_{t+1}\right)\widetilde{R}_{t+1}\left[1-f'(s_t)\right] \\
&\leq V'\left(w_t\right) = u'(c_t(w_t))
\end{aligned}
$$

so that again concavity in u leads to $c_t(w_t) \geq \overline{c}_t(w_t)$.

Lemma 9: *Introducing a soft constraint means that the marginal propensity to consume out of wealth is higher than a case where there are no constraints for a given level of consumption.*

Proof: Begin with the function $\theta_t^*\left(c_t\right) = w_t^*\left(c_t\right) - c_t$ where $w_t^*\left(c_t\right)$ is the inverse of the optimal consumption rule. Write the last expression as $\theta_t^*\left(c_t\right) + c_t = w_t^*\left(c_t\right)$. All the variables that represent the introduction of the constraint are termed $\overline{\theta}_t^*\left(c_t\right), \overline{c}_t$ and $\overline{w}_t^*\left(c_t\right)$. Thus if we can prove that $w_t^{\prime *}\left(c_t\right) = \theta_t^{\prime *}\left(c_t\right) + 1 > \overline{\theta}_t^{\prime *}\left(c_t\right) + 1 = \overline{w}_t^{\prime *}\left(c_t\right)$, then $\overline{c}_t'[w] > c_t'[w]$. Note that we can write $\theta_t^*\left(c_t\right) = g(z^{-1}(c_t))$. Differentiate this expression with respect to c_t :

$$
\theta_t^{*\prime}\left(c_t\right) = \frac{g'(z^{-1}(c_t))}{z'(z^{-1}(c_t))} = -\frac{u''(c_t)}{\phi_t''(s_t)}
$$

now,

$$
\overline{\theta}_t^{*\prime}\left(c_t\right) = \frac{\overline{g}'(\overline{z}^{-1}(\overline{c}_t))}{\overline{z}'(\overline{z}^{-1}(c_t))} = -\frac{u''(\overline{c}_t)}{\mu_t''(\overline{s}_t)}
$$

We know that $\mu_t''(\overline{s}_t) < \phi_t''(s_t) \leq 0$, thus $-\mu_t''(\overline{s}_t) > -\phi_t''(s_t) > 0$. Thus for a given level of consumption, $\overline{\theta}_t^{*\prime}\left(c_t\right) < \theta_t^{*\prime}\left(c_t\right)$. This means that $\overline{w}_t^{\prime *}\left(c_t\right) < w_t^{\prime *}\left(c_t\right)$. This implies that $\overline{c}_t'[w] > c_t'[w]$. Thus the marginal propensity to consume out of wealth increases when the constraint is introduced.

Another way to prove this lemma is to note that the marginal propensity to consume can be defined as

$$\frac{\partial c_t}{\partial w_t} = \frac{V_t''(w)}{V_t'(w)} \frac{u'(c_t(w_t))}{u''(c_t(w_t))}$$

thus for the same utility function in the unconstrained and constrained cases, if the risk aversion of the value function is greater when soft constraints are introduced compared to the case where there are not constraints, then the marginal propensity to consume will be higher in the constrained case. But as we saw in Appendix A, this is true since introducing the soft constraint concavifies the value function. (28)

Lemma 10: *At a given level of consumption, the consumption function is more concave when soft constraints exist than when they are absent.*

Proof: The introduction of the soft constraint convexifies the marginal value function. Using the first-order condition

$$u'\left(\overline{c}_t(w_t)\right) = \overline{V}'(w_t) \leq V'(w_t) = u'(c_t(w_t))$$

this implies that in the constrained case, the marginal utility function is more convex than in the case where there are no constraints. This can only happen if the consumption function has been concavified. (29)

(28) See Carroll and Kimball (2001, footnote 17) for an analogous statement about prudence and concavity of the consumption function.
(29) Again, this proof is close in spirit to Pratt (1964), Carroll and Kimball (1996) and Carroll and Kimball (2001), see their footnote 17, or page 24 in that paper.

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