Long-term interest rates, wealth and consumption

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Abstract

This paper examines the sensitivity of the level of consumption to interest rates in a standard partial equilibrium theoretical framework with no uncertainty. Using a multi-period framework, the consumption function is derived and interest rate effects are decomposed into substitution, income and wealth effects. Drawing on parallels with the finance literature, the paper illustrates two key implications of the theory that are not typically emphasised in the economics literature. First, it shows that wealth effects mean that consumption is much more likely to be negatively related to interest rates than the simple two-period textbook model might suggest. Second, it demonstrates that long-term interest rates are more important than short-term rates – the sensitivity of consumption to interest rate changes depends crucially on how long these are expected to persist. Numerical calibrations provide an indication of the sensitivity of the results to key parameters.

Key words: Consumption, interest rates.

JEL classification: E21.

Summary

Changes in interest rates influence consumption through a number of channels. This paper focuses on the role of wealth, and the importance of expectations of future interest rates. It examines the sensitivity of the level of consumption to interest rates in a standard partial equilibrium theoretical framework with no uncertainty. Using a multi-period framework, the consumption function is derived and interest rate effects are decomposed into substitution, income and wealth effects.

Drawing on parallels with the finance literature, the paper illustrates and quantifies two key implications of the theory that are not typically emphasised in the economics literature. First, it shows that wealth effects - particularly revaluation of human wealth - mean that consumption is much more likely to be negatively related to interest rates than the simple two-period textbook model might suggest. Second, it demonstrates that long-term interest rates are more important than short-term rates – the sensitivity of consumption to interest rate changes depends crucially on how long these are expected to persist.

Numerical calibrations provide an indication of the sensitivity of the results to key parameters. Under plausible parameter assumptions, if future labour income is assumed to be exogenous, the (negative) wealth effect is of a similar order of magnitude to the (positive) income effect. Hence the net effect on consumption of changes in interest rates is similar to the (negative) substitution effect. In a general equilibrium context, income - particularly capital income - will not be fully exogenous to the level of interest rates. However, income may respond slowly to interest rate changes.

The calibrations also show the importance of the persistence of interest rates. Rates that are only expected to be high temporarily will have much less impact than rates that are expected to be high permanently. Even rates that are temporarily high for two years will have much less impact if the time horizon of the representative consumer is long.

These results are best thought of as steady-state ‘comparative dynamic’ comparisons, given the absence of uncertainty. In this framework, the paper shows that the results also apply in the presence of habits in consumption.

The framework and the utility function used are highly stylised. Actual interest rate changes are likely to work in large part through affecting credit constraints and precautionary saving, though their impact also depends on the degree of uncertainty. Nevertheless, the model suggests that more attention should be paid to the role of long-term interest rates in empirical analysis of consumption. It also helps to provide a framework for understanding some recent empirical results in the area of consumption modelling, which stress the interaction with wealth.

1 Introduction

Monetary policy affects demand and inflation through a number of channels.[(1)] The impact on consumption is a key channel and is captured in a variety of empirical macroeconomic models through their consumption function and Euler equation relationships.[(2)] In these models higher interest rates dampen the current level of consumption.

There are a number of reasons why consumption responds in this way. For example, cash-flow effects in the presence of credit constraints mean that debtors cut back on consumption. Such channels may be important in practice. But as a foundation for macroeconomic modelling work, it is the substitution effect of standard optimising theory, arising from the increased incentive to save, that is at the core.[(3)]

However, it is well known that this channel may theoretically be offset by a positive income effect. As Deaton (1992) says 'the direction of the effect of interest rates on consumption is not predicted on theoretical grounds'. And often empirical estimates of the interest elasticity of consumption are not large or well determined: 'In the literature on the time-series consumption function, there are many studies where an interest rate is included in the model, along with income and other variables.... My reading of this literature is that the empirical results are as ambiguous as is the theory' (page 60) Deaton *ibid*. This ambiguity in theory and empirical work is often noted in the literature, and may contribute to a sense that monetary policy may not be very powerful in influencing consumption, particularly where institutional arrangements mean consumers have cash flows that are not so affected by short-term interest rates.[(4)]

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(1) These channels are commonly referred to as the transmission mechanism. There are many papers on the transmission mechanism. As a starting point see the Journal of Economic Perspectives Symposium of the transmission mechanism (1995); the Monetary Policy Committee's view of the transmission mechanism (1999) and the New York Fed's (2002) conference on the transmission mechanism. Most central banks' web sites/working papers report research on the transmission mechanism.

(2) The literature on small macroeconomic models for policy analysis uses the Euler equation to derive the IS curve. See for example McCallum and Nelson (1997), Rotemberg and Woodford (1997), Clarida *et al* (1999) and King (2000).

(3) And which is the main direct channel of the transmission mechanism in standardised macroeconomic models which use the Euler equation to derive the IS curve. There are, of course, other channels of the transmission mechanism such as the exchange rate channel, expectational channels, credit frictions, the impact of interest rates on investment decisions, etc; but the substitution channel in the IS equation tends to be perceived as one of the more important channels.

(4) For example, Britton and Whitley (1997) note that 'The finance of house purchase in different countries is often singled out as a prime reason for differences in how real demand responds to changes in interest rates'. Angeloni, Kashyap, Mojon and Terlizzese (2002) in their analysis of the channels of the transmission mechanism in the euro area assume that there is little impact of interest rates on non-durable consumption. 'Here we appeal to the large

Theory is less ambiguous on its prediction for consumption *growth*: the Euler equation states that higher interest rates between periods increase the growth rate of consumption.[5] Academic economists have used this equation as the basis for most of their research on interest rates and consumption – theoretical and empirical.[6] The path of consumption within general equilibrium models is based on this relationship, and the implications for the level of consumption from different interest rate paths follow from the full dynamic solution to the model.

However, this emphasis on growth rates and numerical simulation may have detracted from a better understanding of what determines the size of the impact of the interest rate on the initial level of consumption. Applied economists may base their views more on results from simple textbook models. Perhaps as a result, two particular features of standard theory do not seem well recognised. Though noted in a number of papers, the importance of wealth in strengthening the impact of interest rates on consumption, and the core role of long-term interest rates, are often underemphasised.

Allowing for wealth effects makes it much more likely that consumption is lower when interest rates are higher. The net present value of a given exogenous income stream will be lower when interest rates are higher, and this can be a powerful effect. Revaluation of future labour income (human capital revaluation) is particularly important. While there is now considerable emphasis on financial wealth effects on consumption in applied macroeconomic work, the link to interest rates is not always explicit. Indeed in some empirical models non-human wealth is determined by the capital stock rather than by market values.[7] Human capital is often not considered at all.

One notable paper that discussed the wealth effect in some detail is Summers (1981). He analysed this in the context of the impact of taxes on saving behaviour, which may in part explain why the

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empirical literature suggesting that non-durable consumption is insensitive to interest rates. The case of weak wealth effects seems to be particularly plausible in the euro area, in light of the short duration of financial wealth and of the limited weight of stock market wealth in private portfolios' (page 15). Note that in this paper we do not analyse durable consumption purchases, which are essentially investment decisions, but include the services from durables as part of total consumption.

(5) In other words, the elasticity of (intertemporal) substitution of consumption is positive. Throughout this paper we refer to the intertemporal elasticity of substitution and not the intratemporal elasticity which (normally) arises from different specifications of the utility function that include components other than consumption, say, for example, money and labour supply decisions.

(6) Most papers which have looked at the Euler equation have mainly done so to test the validity of theoretical variations to the standard life-cycle model of consumption. (Browning and Lusardi (1996) provide an excellent summary of studies that have tested the validity of the Euler equation and the presence of precautionary savings behaviour.)

(7) See for example the ECB consumption function detailed in Fagan *et al* (2001, pages 42-43).

result has not received more attention. The approach is similar to that presented here. He concludes that 'the common two-period formulation of saving decisions yields quite misleading results' and that 'usually, in empirical work, wealth is held constant. Since interest rate changes exert their effect in part through changes in wealth, this procedure is likely to obscure the impact of interest rates'. Elmendorf (1996) has an extensive discussion of the impact of interest rates on different types of wealth.

Similarly, long-term rates and/or expectations of future rates have generally not been emphasised in discussions of consumption demand (8) or been included in applied macroeconomic model equations. (9) Though discussions of the transmission mechanism stress the feed-through of short-term policy rates to market rates, both short and long, the discussion of the impact of long rates often focuses more on investment demand.

This has begun to change more recently. Low inflation rates have stimulated work on how monetary policy might work when short-term nominal rates are close to or at the zero nominal bound (see for example, Eggertsson and Woodford (2003) and Buiter (2003)). This literature illustrates that, in fact, long-term interest rates/interest rate expectations are key to consumer demand, and this has been influential in recent central bank speeches on the continued potency of monetary policy in such circumstances. (10) But again it may not be widely appreciated just how important long-term rates are in such models.

Therefore, to fill this gap in popular perceptions, here we focus on explaining and quantifying the roles of wealth and long-term interest rates using the framework of the consumption function for the level of consumption, derived from standard optimising theory. We do this in a multi-period context, in continuous time, as set out from Section 3 onwards. Since many of the expressions have parallels in the finance literature, we draw on this in the exposition.

Tables A and B illustrate some of these points. (11) They show the size of the interest rate effect for

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(8) For example, Elmendorf (1996) considers only the effects of higher rates across the yield curve.
(9) Small general equilibrium models such as those of McCallum and Nelson (1997) and Rotemberg and Woodford (1997) suggest that the interest rate that matters for consumption is the long rate. But large-scale macro models often include only the short rate in the consumption function; see for instance Bank of England (1999).
(10) See Bernanke and Reinhart (2004).
(11) The assumptions behind these tables are that real consumption and exogenous income growth both are initially 3% per year each; there are no initial or final variable-rate assets; real interest rates are initially 4%. In Table A, the elasticity of substitution is 2 ($1/\theta = 1/0.5$). This implies a subjective rate of time preference of 2.5% per year

Table A: Percentage effect on level of consumption of a 25 basis point rise in real interest rates: High elasticity of substitution

Duration of rate increase	*Horizon/ Lifetime*	Substitution	Income	Wealth	Total
Permanent	10 years	-2.5	+1.3	-1.3	-2.5
	40 years	-9.4	+4.4	-4.4	-9.4
	Infinite	-50	+25	-25	-50
2 years	10 years	-0.9	0.4	-0.4	-0.9
	40 years	-1	0.5	-0.5	-1
	Infinite	-1	0.5	-0.5	-1
NB: Elasticity of substitution >1. Consumption and income growth initially equal at 3%					

Table B: Percentage effect on level of consumption of a 25 basis point rise in real interest rates: Low elasticity of substitution

Duration of rate increase	*Horizon/ Lifetime*	Substitution	Income	Wealth	Total
Permanent	10 years	-1.3	-1.3	-1.3	-1.3
	40 years	-3.8	+4.4	-4.4	-3.8
	Infinite	-21	+25	-25	-21
2 years	10 years	-0.4	0.4	-0.4	-0.4
	40 years	-0.4	0.5	-0.5	-0.4
	Infinite	-0.4	0.5	-0.5	-0.4
NB: Elasticity of substitution <1. Consumption and income growth initially equal at 3%					

the benchmark case with growth of consumption initially equal to that of income, when interest rates are 25 basis points higher.[(12)] Table A shows the effect where the elasticity of substitution is high and Table B shows it for a low elasticity case. For both tables, the (negative) wealth effect is equal to the (positive) income effect, so that the impact on the level of consumption depends crucially on the elasticity of substitution. The overall impact is much greater for long lifetimes/horizons. And the importance of the duration of the rate increase is illustrated by the much smaller effects when rates increase for only two years (particularly when horizons are long).

The framework we use is a restrictive one – optimising behaviour, with simple utility functions where there is perfect foresight. When we discuss the impact of higher interest rates, we are essentially considering one dynamic steady-state environment with another. Our analysis of the

(through the Euler equation with growth rate of consumption of 3%). In Table B, the elasticity of substitution is 0.83 ($1/\theta = 1/1.2$) implying a subjective discount rate of 0.4%.

(12) Using the elasticities in Appendix B, Tables H and I.

impact of higher or lower interest rates might rather be thought of as 'comparative dynamics' – what consumption would be in different scenarios, just as with classic 'comparative static' exercises.[13] To improve the paper's readability, we shall often refer to a comparison between steady states as illustrating the effect of a change or rise in interest rates. However, strictly speaking the framework cannot be directly applied to situations where interest rates change unexpectedly, and where future interest rates are not known with certainty.

Another simplification is that the approach is partial equilibrium. We do not analyse why interest rates differ in our scenarios – we treat them as exogenous. Though in equilibrium interest rates will be equal to rates of return on physical capital[14] we do not consider how this comes about. Since we treat future income as exogenous, we implicitly assume large adjustment costs. More generally different interest rate paths will be associated with different income paths.[15] In any full-scale analysis it is important to consider what underlies different interest rates, and how general equilibrium is brought about. Nevertheless, like other simple models, which often deal with uncertainty by making other simplifying assumptions on utility functions etc, or deal with general equilibrium by assigning short-term rigidities, the results can still be insightful, and provide a baseline framework for thinking about consumption behaviour during periods with different interest rates.

The structure of the paper is as follows. Section 2 summarises the classic two-period model and provides some simple illustrations of the key results using discrete-time models. Section 3 sets out the theoretical framework and derives the consumption function. The responsiveness of the level of consumption to interest rate changes is decomposed into income, substitution and wealth effects, and simple examples are given to illustrate the importance of wealth and long rates. Section 4 derives explicit elasticity expressions for shifts of the future profile of rates, when the yield curve is flat. It also notes the implications of the model, when extended for habits.[16] Section 5 derives expressions for the change in consumption following a temporary rise in interest rates, and illustrates the sensitivity to the length of period over which rates are raised. The income, substitution and wealth effects are all stronger when rate changes are persistent. Section 6

(13) We would like to thank one of the referees for this terminology.
(14) Assuming no risk premium.
(15) As Lantz and Sarte (2001) note 'General equilibrium considerations imply that wealth, the rate of interest, and consumption all contemporaneously react to the various disturbances affecting the economy'. See also Millard and Power (2004).
(16) Appendix E shows that the results generalise when interpreted as comparisons of steady-state paths.

discusses the sensitivity of the results to different parameter values and finds that, for most cases, the interest elasticity is negative. Section 7 concludes, noting some limitations of the framework. Appendix A uses two-period models for consumption to show how different interest rates affect consumption decisions. Appendix B introduces a three-period model to summarise the main results of the paper. Appendix C gives details of the derivation of the elasticity formulae. Appendix D sets out details of the calibrations for different parameter values. Appendix E gives details of the habits model used and its results. Appendix F contains a glossary of financial terms used in the paper.

2 Textbook models of consumption

Since the results in the paper contrast with those from the simple two-period textbook model of consumption, this section first summarises that model (Section 2.1). It also explains the intuition behind the key results of our continuous time analysis, using some simple discrete-time examples (Section 2.2), and highlights some important caveats (Section 2.3).

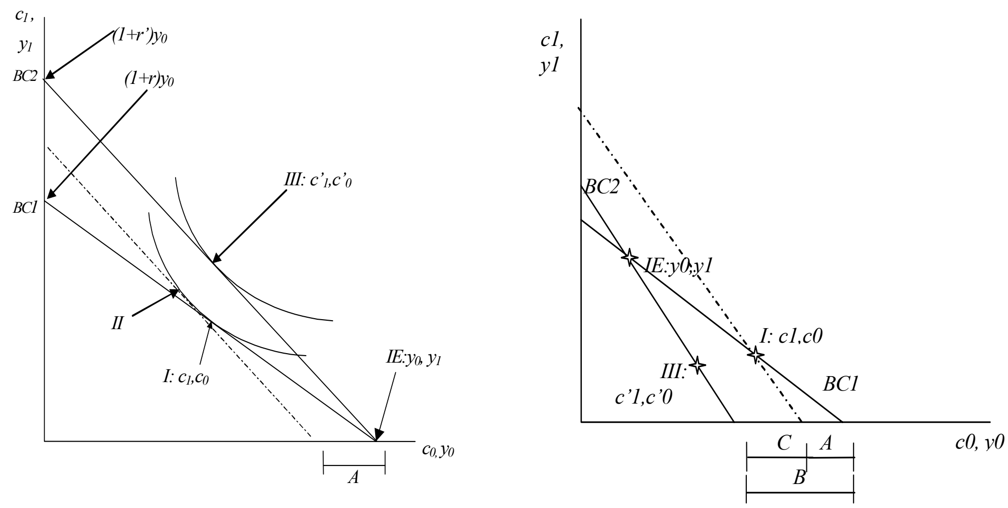
2.1 The two-period model

Few models are as well known in economics as the classic two-period model used to illustrate the theoretical ambiguity of the relationship between interest rates and the level of consumption. Appendix A provides details of the substitution, income and wealth effects using the textbook indifference curve analysis.[17] Here we summarise the key points.

Assume all income is received in the first period. Higher interest rates increase the incentive to save, reducing current consumption (giving up a unit of consumption in the first period allows greater consumption in the second). This is the substitution effect. But first-period consumption may nevertheless rise. The higher interest rate means the consumer is better off, because the increased return on savings means additional consumption is possible in both periods. This is the income effect. If the elasticity of substitution is low, ie the consumer strongly prefers even consumption in each period,[18] consumption in the first period will rise. This is illustrated in Chart 1(a). The negative substitution effect of shifting along a given indifference curve is

(17) See Nicholson (1992, page 711) for a textbook treatment. Elmendorf (1996) provides a nice example of the income, substitution and wealth revaluation effects in a two-period model. See also Romer (1996, page 326).
(18) Adjusted for any effects from the subjective rate of time preference.

Chart 1: Income, substitution and wealth revaluation effects in a two-period case



a) Income only in the first period

b) Income in both periods

outweighed by the positive income effect of the shift out in the budget constraint.[19]

The income effect is conventionally defined as the impact of interest rates on the 'price' of future consumption. Less current consumption needs to be given up to achieve a given level of future consumption. This is captured formally by greater discounting of future consumption (the net present cost/value of future consumption, $c1/(1+r)$, is lower). The income effect, so defined, is always positive.[20] Any given mix of consumption now and consumption in the future is more affordable – the present value cost is lower. Diagrammatically this is captured by segment A in Chart 1(a), obtained by pivoting the budget constraint around the initial consumption choice. With the budget constraint steeper, the initial consumption mix could be bought for a cash sum in period 0 lower by an amount A than it would have been prior to the rate increase.[21]

(19) See Appendix A for more explanation of these graphs. Elmendorf (1996, page 82), also has two-period diagrams.

(20) Note that, even if a consumer is in debt, the income effect so defined is still positive. Assets or liabilities do not affect the price of future consumption. It is possible to define the effect more directly in terms of income/rates of return. This approach captures the negative impact that higher interest rates have on variable-rate debt payments. But, since overall future consumption must be positive, overall current assets must be positive (defined to include all assets, including the future value of exogenous income). So any negative income effect from debts will be more than offset by a positive income effect from assets. Whether a consumer is better or worse off will depend on the wealth effect. If assets are fixed rate (eg human wealth), but debts are variable rate, higher interest rates will make the consumer worse off.

(21) The net present value benefit is $-c1\left(\frac{1}{(1+r')}-\frac{1}{(1+r)}\right)$.

However, higher interest rates do not mean that consumers are necessarily better off. The two-period example shown in Chart 1(a) is misleading because there is no exogenous second-period income. In the more general two-period model, higher interest rates not only reduce the current cost of future consumption, they reduce the current value of future income ($y1/(1+r)$). This is illustrated by the budget constraints shown in Chart 1(b), where some income is in the second period. The positive income effect on the budget constraint, making higher consumption in both periods possible, is again measured by segment A. But now there is a second effect shifting the budget constraint. With higher interest rates, the net present value of the exogenous second-period income is less, other things being equal, making the consumer worse off.[(22)] This negative wealth effect is captured by segment B in Chart 1(b), obtained by pivoting the initial budget constraint around the exogenous income mix. In the case shown in Chart 1(b), the change in wealth more than offsets the positive income effect. The shift in the net budget constraint is captured by segment C, the difference between A and B. Here, the consumer is worse off, and first-period consumption falls from $c0$ to $c'0$.[(23)]

If most income had been in the first period, but most consumption in the second, the opposite would have been the case[(24)] – the income effect would have dominated the wealth effect and consumption would rise (if the negative substitution effect - not shown - was small). But, so long as there is some second-period exogenous income, there will be a negative wealth effect that at least partially offsets the positive income effect.

The key assumption when considering the impact of the wealth effect is the exogeneity of future income to different interest rates. While the overall income effect may always be positive, the size of the wealth effect will depend on the mix between variable and fixed-rate assets and liabilities. If, for example, assets are all variable rate, there is no wealth effect (because there is no exogenous future income to revalue). This is true whether there are two periods or many periods. But this is extreme. Given the importance of labour income and slow adjustment of the physical capital stock, it is important to allow for the case where much future income may be exogenous and income revaluation effects potentially sizable. In this case, the number of periods included in the

(22) The effect is negative so long as future exogenous income is positive. If there are net fixed-rate liabilities, higher interest rates will reduce the current value of these fixed future payments, making the liability smaller.
(23) The diagram does not show the substitution effect as we have not shown indifference curves.
(24) In that case, with no bequests and more initial income than consumption, but more consumption than income in the second (final) period, the growth rate of consumption is greater than the growth rate of income. As we shall see below, this makes the income effect greater than the wealth effect.

analysis becomes important (the more income in the future, the more sensitive wealth becomes to discounting), and analysis is best carried out in a multi-period framework.

2.2 Multi-period examples

If future exogenous income and consumption are similar, wealth and income effects will be broadly similar. The net present value of future consumption and income will be affected to a similar extent by interest rates. But the sign will be opposite, so the effects will be broadly offsetting, with only negative substitution effects remaining.

To take an illustrative example, consider a consumer whose sole wealth is held in the form of a perpetuity – an infinitely lived bond. With fixed coupon, Y, each year, if the interest rate is r, the value of the bond is $W = (Y/r)$. Assume the coupon income is initially just consumed each period. A higher interest rate would reduce the net present value of the bond. But the coupon is unaffected, so that the rate of return on it would rise proportionately. A consumer could continue to consume the same coupon each period – the income and wealth effects cancel. But there remains a substitution effect, which would mean initial consumption would be lower with higher interest rates. The same analysis would apply to a consumer who received exogenous labour income of Y each year. In this case the value of wealth is still (Y/r), the value of human wealth.

The extent of the substitution depends on the parameters of the utility function. In this example, if the utility function is of the commonly used constant relative risk aversion (CRRA) type (see equation **(7)** below), consumption grows at $g_c = (r - \delta)/\theta$. The consumption function of our consumer with fixed income each period, Y, is then:

$$C = (r - g_c)W = \left[r - \frac{(r-\delta)}{\theta} \right] (Y/r) = Y - \frac{Y}{\theta} - \frac{\delta Y}{r\theta} \qquad \textbf{(1)}$$

This illustrates that the interest rate effect in this simple case is always negative, and determined by the size of the elasticity of substitution. As will be shown later, the impact depends on the lifetime of the consumer – the shorter the lifetime, the smaller the impact.

The multi-period framework also has the advantage that analysis can be carried out of the sensitivity of consumption to expectations of future interest rates as well as to current rates. In this framework of perfect certainty this is equivalent to considering the responsiveness of consumption to long-term as well as short-term interest rates. In particular, since interest rates can vary period by period, we analyse the impact of higher interest rates for a temporary period, compared with the effect of permanently higher rates. The analysis in Section 6 illustrates that the persistence of higher rates is crucial – the longer rates are high, the greater the impact on the initial level of consumption. These results are derived using a continuous-time framework. Similar results could be obtained using a discrete-time model. As an example, Appendix B uses a three-period model to show that, if rates are higher between the first and the second period, the effects on initial consumption are roughly 50% greater if the rate is also higher between the second and the third period. A simple logarithmic approximation to the formula for the consumption level illustrates this:[25]

$$\ln c_0 \approx \ln y_0 + (2/3)(g_y(1) - g_c(1)) + (1/3)(g_y(2) - g_c(2))$$

where $g_c(i) = (r_i - \delta)/\theta$ for $i = 1, 2$. This formula generalises to n periods as below:

$$\ln c_0 \approx \ln y_0 + \left[\sum_{j=1}^{n-1} \left(\frac{n-j}{n}\right) \left(g_y(j) - g_c(j)\right)\right] \qquad \textbf{(2)}$$

This formula becomes less accurate for longer horizons (larger n). But it does suggest that, as n increases, the impact of higher interest rates is roughly linear in their duration. So, for example, a high rate that persists for two periods has twice the impact on the initial level of consumption of a high rate for one period.[26] This is confirmed by the continuous-time analysis (see Chart 7 below, and Table R).

2.3 Caveats

As noted in the introduction, these results are intended to illustrate the potential importance of taking both wealth effects and long-term interest rates explicitly into account in analysis of the interaction between the level of consumption and interest rates. In practice the effects may not be so great, given the simple nature of the model. In particular, in a general equilibrium framework interest rates and future incomes are not exogenous.

For example, it may not be reasonable to assume that a fall in the market value of equity wealth following an interest rate rise will be sustained. Income from the capital stock is not exogenous and, over longer horizons, the rate of return will rise as investment falls. So, in this sense, non-human wealth should behave more like a variable-rate asset than a fixed-rate asset. On the

(25) Assuming low growth rates, and growth rates of consumption and income are initially the same.
(26) The derivation also makes it clear that, where growth rates of consumption and income are initially equal, the income and wealth effects broadly cancel so it is the substitution effect, captured in g_c and determined by the size of the elasticity of substitution, that is crucial.

other hand, there can be significant impacts over a protracted period, as suggested by the reliance of the finance industry on tools such as the dividend discount model – an effect not found in simple general equilibrium models featuring quick adjustment costs. This makes it important to consider the wealth channel for equity. Adjustment costs are likely to be even greater for human capital.

And long-term interest rates may be less important than the model suggests if there is uncertainty about the future short rates that pin them down. This is consistent with the presence of term premia at longer-term maturities (where market rates, adjusted for credit risk are biased predictors of future short-term rates, see eg Bank of England (2002)). But it would seem likely that they are more important than their absence from traditional consumption functions would suggest. So care is required in interpreting macro-level empirical work in the area. It will be difficult to disentangle wealth changes resulting from future income expectations and those reflecting interest rates. But our analysis suggests that it is worth making the attempt.

3 The Euler equation and the consumption function

This section derives the (general) mathematical expressions for the Euler equation and the consumption function used to investigate the impact that changes to interest rates have on consumption.

3.1 The problem

Consider the standard partial equilibrium consumption model without uncertainty where a representative agent aims to maximise overall utility, U, subject to a budget constraint[(27)]

$$U = \int_0^T u[c(t)]e^{-\delta t}dt \quad \textbf{(3)}$$

$$s.t.\ \dot{a}(t) = y(t) + r(t)a(t) - c(t) \qquad a(0) \text{ given} \quad \textbf{(4)}$$

where $u[c(t)]$ denotes the utility function with standard assumptions $u'[c(t)] > 0$ and $u''[c(t)] < 0$, δ the rate of time preference, a is the level of net variable-rate assets, y is exogenous income, r is the instantaneous interest rate and c is consumption.[(28)] Exogenous income may include labour income but also property income which is not affected by interest rate changes, eg

(27) Problems like this one are common in the literature: see Barro and Xala-i-Martin (1995) for a textbook review of this problem, Merton (1969) for an early model.
(28) The following assumptions are made:

bond coupons. Returns on equities may to some extent be independent of interest rates (see discussion of the wealth revaluation effect in Section 3.5.3).

3.2 The Euler equation

The resulting Euler equation for consumption is given by

$$\frac{\dot{c}}{c} = g_c(t) = -\frac{u'}{u''c}\left[r(t) - \delta\right] \qquad \textbf{(6)}$$

Thus, in the absence of uncertainty and under the assumptions made about preferences, **(6)** states that the growth rate of consumption at any time t, $g_c(t)$, must equal the difference between the rate of interest and the rate of time preference, scaled by the intertemporal elasticity of substitution;[(29)] consumers with a high elasticity of substitution are more responsive to the difference between the rate of interest and the rate of time preference, ie are more willing to depart from a flat consumption profile. Loosely speaking, a higher interest rate must warrant higher consumption growth to satisfy the Euler equation under the specific assumptions in the problem considered above.[(30)] If preferences are CRRA

$$u[c(t)] = \frac{c^{1-\theta}}{1-\theta} \qquad \textbf{(7)}$$

1.Felicity functions are time-additive and satisfy the Inada conditions

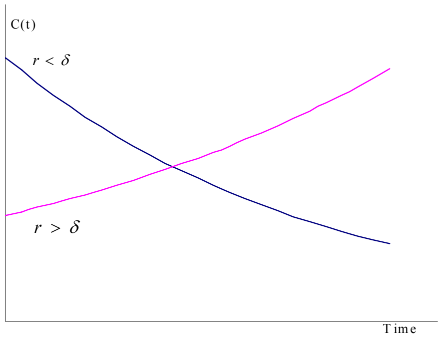
$$\lim_{c\to 0} u'[c(t)] \;\to\; \infty$$
$$\lim_{c\to \infty} u'[c(t)] \;\to\; 0$$

2.The rate of time preference is constant each time period.
3.Capital markets are perfect so any amount of borrowing and lending can be done. To avoid the possibility that in an infinite version of the problem an individual is able to borrow forever without having to repay the principal, the following transversality condition is imposed

$$\lim_{t\to\infty} \left\{ a(t) e^{\left[-\int_0^t r(v)dv\right]} \right\} \geq 0 \qquad \textbf{(5)}$$

so that in the long run an agent's debt cannot grow as fast as r, ie the net present value of assets is asymptotically non-negative.
(29) The reciprocal of the elasticity of marginal utility with respect to consumption, reflecting the degree to which marginal utility diminishes as consumption rises - see Barro and Xala-i-Martin (1995, page 64).
(30) The relationship between the interest rate and consumption growth is different when one considers alternative preferences. For example, if habits are important, consumers will care directly about the growth rate of consumption as well as its level, thereby amending the Euler equation (see equation **(E-4)**).

Chart 2: Consumption profiles from the Euler equation



where θ is the coefficient of risk aversion and $\frac{1}{\theta}$ is the elasticity of substitution,[(31)] the Euler equation for consumption is then

$$\frac{\dot{c}}{c} = g_c(t) = \frac{1}{\theta}\left[r(t) - \delta\right] \qquad \textbf{(8)}$$

From **(8)** we see that a higher level of interest rates warrants a higher *growth* rate of consumption; the extent depends on the elasticity of substitution: if $\frac{1}{\theta} < 1$, an increase in the interest rate has less than a one-to-one effect on the growth rate of consumption, and *vice versa*.[(32)] [(33)] Consumption growth is positive if the interest rate is above the subjective rate of time preference (Chart 2).

3.3 The consumption function

To derive the expression for the level of consumption - the consumption function - we use the budget constraint. Assuming that consumers do not leave any bequests, $a(T) = 0$, the budget

(31) Note that in this framework, a high relative risk aversion parameter, θ, means the gain from higher consumption is much less than the loss from low consumption, thereby implying a low elasticity of substitution over time $(1/\theta)$. Non time-separable utility functions break this close link (see Epstein and Zin (1989) for example). See Deaton (1992, pages 19-21) for an interesting summary of the relative merits of non time-separable utility functions.
(32) In absolute percentage point terms. For example, if $1/\theta < 1$ a 1 percentage point increase in the interest rate will boost consumption growth by less than 1 percentage point.
(33) If $\theta = 0$, utility is linear and the elasticity of substitution is so high that consumers are willing to accept large swings in consumption to take advantage of any changes in the difference between the rate of interest and the rate of time preference; if $\theta = \infty$, then the elasticity of substitution is zero and consumers are unwilling to change their consumption in response to differences between the rate of interest and the rate of time preference. If $\theta = 1$ consumers have logarithmic utility, and the growth rate of consumption, g_c, is $r - \delta$, as often assumed in calibrated general equilibrium models.

constraint, **(4)**, can be rewritten in lifetime terms as

$$\int_0^T c(t)e^{-\overline{r}(t)t}dt = W(0) \quad \textbf{(9)}$$

$$W(0) = \int_0^T y(t)e^{-\overline{r}(t)t}dt + a(0) \quad \textbf{(10)}$$

where $\overline{r}(t)t = \int_0^t r(v)dv$ denotes the *average* interest rate between time 0 and t multiplied by the number of periods, so it represents the rate used to discount a cash flow in period t.(34) The lifetime budget constraint states that the net present value of consumption and the net present value of endowments are equal to initial wealth, $W(0)$. Combining **(8)** with **(9)** gives the consumption function(35)

$$c(0) = \lambda(0)W(0) \quad \textbf{(11)}$$

where $\lambda(0)$ is the marginal propensity to consume at time 0, ie the fraction of wealth consumed. For the CRRA case, this is given by

$$\frac{1}{\lambda(0)} = \int_0^T e^{-\overline{r}(t)t}e^{\left[\frac{\overline{r}(t)-\delta}{\theta}\right]t}dt = \int_0^T e^{-\left[\overline{r}(t)-\overline{g_c}(t)\right]t}dt \quad \textbf{(12)}$$

where $\overline{g_c}(t)$ is the average growth rate of consumption between 0 and t. The marginal propensity to consume guarantees that the net present value of consumption at time 0 equals the net present value of wealth at time 0 along the optimal consumption path. As detailed below it will in general depend on the interest rate. Only for logarithmic utility, $\theta = 1$, will it be independent since **(12)** then reduces to $\lambda(0) = \left[\int_0^T e^{-\delta t}dt\right]^{-1}$ and in that case the size of the (negative) substitution effect will be equal to the size of the (positive) income effect. Note further that for the case $r = \delta$, often assumed in the literature, the marginal propensity to consume is $\lambda(0) = \left[\int_0^T e^{-\overline{r}(t)t}dt\right]^{-1}$ and the only component affecting the marginal propensity to consume comes from the budget constraint.(36) This case has been considered by eg Hall (1978) and Flavin (1981) who then define permanent income as $\left[\int_0^T e^{-\overline{r}(t)t}dt\right]^{-1} W(0)$, which is also known as the Hicksian definition of income. In this case when $T = \infty$ and $r(t) = r$ for all t, the consumption function reduces to $C(0) = rW(0)$.

(34) In financial terms, $\overline{r}(t)$ is the *t-period spot rate* - see Appendix F.
(35) Note that in this substitution we use the relationship

$$c(t) = c(0)e^{\overline{g_c(t)}t}$$

where $\overline{g_c(t)}$ is defined as the average rate of growth of consumption to time t.
(36) Assuming that when r moves so does δ. This interest rate independence also follows if $\theta = \infty$, the case when there is no substitution.

In these cases there is no growth of consumption ($g_c = 0$ in equation **(12)**). In the more general case, where consumption varies over time, the expression for the consumption function can be rewritten in a way that illustrates the key factors determining the initial level. First, expressing the wealth equation **(10)** under the assumption that exogenous income growth, g_y, is constant, yields:

$$W(0) = y(0)\int_0^T y(t)e^{-[\overline{r}(t)-g_y]t}dt + a(0) \qquad \textbf{(13)}$$

Substituting this into equation **(10)**, for the case $a(0) = 0$, gives an expression for the consumption function that illustrates the key role of income and consumption growth rates relative to each other:

$$c(0) = y(0)\frac{\int_0^T e^{-[\overline{r}(t)-g_y]t}dt}{\int_0^T e^{-[\overline{r}(t)-\overline{g_c}(t)]t}dt} \qquad \textbf{(14)}$$

Hence the initial level of consumption will be below the initial level of income if the growth of income is lower than that of consumption, and *vice versa*. And the larger the difference, the greater the effect on the initial level of consumption. If those growth rates are equal, $c(0) = y(0)$.

This expression for the consumption function also illustrates the key role that the differences between the interest rate and the growth rates of consumption and income play. Any percentage point change in interest rates will be potentially more important if the gap is small.[37] We can rewrite this expression in terms of growth-adjusted interest rates ($\omega_c(t) = r(t) - g_c(t)$ and $\omega_y(t) = r(t) - g_y(t)$). This makes it analogous to net present value expressions in finance,[38] and means we can draw on results from that literature to understand the impact of rate changes. In particular, we can think about interest rates as having their impact through annuity-type yields, $\omega_c yield$ and $\omega_y yield$, that capture the average discounting effect of the interest rate (and growth) profiles.

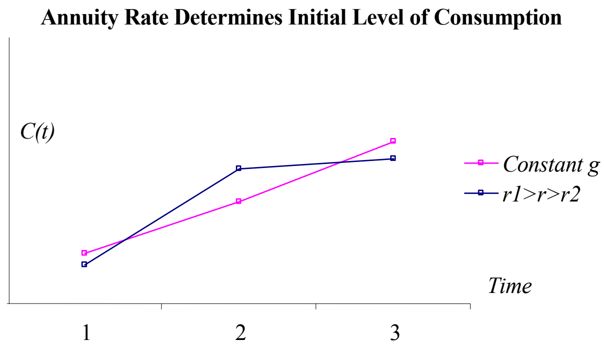
$$c(0) = y(0)\frac{\int_0^T e^{-\omega_y yield\cdot t}dt}{\int_0^T e^{-\omega_c yield\cdot t}dt} = y(0)\frac{\left[1 - e^{-\omega_y yield\cdot T}\right]/\omega_y yield}{\left[1 - e^{-\omega_c yield\cdot T}\right]/\omega_c yield} \qquad \textbf{(15)}$$

3.4 Short versus long rates

This analogy with annuity valuations and yields demonstrates the relative importance of interest rates along the curve. Changes to short-term rates, ie at the short end of the forward curve, will have more impact than changes further out. This is because they affect net present values more,

(37) For example, if the growth rate of consumption were initially zero, a small difference between the interest rate and the growth rate of income would lead to a very high initial level of consumption relative to current income, because future income would be very important in discounted terms, relative to current income.
(38) The integral expressions in this consumption function are the same as net present value formulae for annuities, where the (potentially time-varying discount rates) are w_c and w_y.

Chart 3: Different annuitities and the time profiles for consumption



and this is captured as a different effect on annuity yields. For example, a temporary increase in short-term interest rates will apply to all future cash flows. But a change further out the curve will not.

Chart 3 illustrates how the profile of the interest rate curve can affect the initial level of consumption, even if the average interest rate is the same. It compares two scenarios. Both have the same average growth rate of consumption, reflecting equal average interest rates. But the initial consumption levels are not the same – in the case where the interest rate is high at first, the initial level of consumption is lower, compensated in this case by higher consumption in the second period.[39] This greater importance of rates at closer horizons is fully captured in the annuity yield, which will be higher in the case of the kinked profile.

Having high interest rates for a few periods in the immediate future will therefore have a larger impact on initial consumption than would having high rates for the same duration in the more distant future. In this sense short-term rates are more important than 'long-term' rates. However, while such 'forward' long rates do matter less than short rates, long rates, defined as either spot rates or yields, matter much more (see Appendix F). In this framework, high interest rates will only have a large effect on the level of consumption if they are expected to persist, and these standard long rate measures capture this persistence.

(39) The negative substitution and wealth effects dominate the income effect in this example.

3.5 Decomposing the effects of interest rate changes on consumption

The effects of any change in interest rates on the initial level of consumption can be decomposed as follows:

$$\frac{\partial c(0)}{\partial r(t)} = \underbrace{\frac{\partial \lambda(0)}{\partial r(t)} W(0)}_{inc+sub} + \underbrace{\frac{\partial W(0)}{\partial r(t)} \lambda(0)}_{wealth} \qquad \textbf{(16)}$$

The propensity to consume encompasses the standard income and substitution effects. The income and substitution effects work through changing the propensity to consume out of wealth, $\lambda(0)$. The second term captures the wealth revaluation effect.

Decomposing between the income and substitution effects is straightforward. In the equation for the propensity to consume, $\lambda(0)$, (equation **(12)**), the second (consumption growth) term comes from the Euler equation, whereas the first term comes from the budget constraint. Since the substitution effect represents the change in consumption while holding the present value of all future consumption constant, the second term is associated with the substitution effect, whereas the first term represents the income effect. Intuitively, this is because the value of the elasticity of substitution parameter, θ, only affects the second term. Thus, we can calculate the substitution effect by holding constant the first $\bar{r}(t)$ in the formula for $\lambda(0)$, and the income effect by holding constant the second $\bar{r}(t)$ in the same formula.(40)

3.5.1 The substitution effect

As just noted, the substitution effect works through the propensity to consume and derives from the Euler equation. A higher interest rate - over any period - means greater average consumption growth. For a given budget constraint this means that initial consumption must fall, since future consumption is higher. If interest rates are higher along the curve the profile pivots (Chart 4). Even if the current short rate is unchanged, if forward interest rates are higher some time in the future, initial consumption will fall. Intuitively, it is worth saving more today to build up savings to take advantage of a high rate of return in the future. Given a preference for consumption

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(40) The decomposition between income, substitution and wealth effects can also be carried out in terms of elasticities:

$$\frac{\partial c(0)}{\partial r(t)} \frac{r(t)}{c(0)} = \underbrace{\frac{\partial \lambda(0)}{\partial r(t)} \frac{r(t)}{\lambda(0)}}_{inc+sub} + \underbrace{\frac{\partial W(0)}{\partial r(t)} \frac{r(t)}{W(0)}}_{wealth} \qquad \textbf{(17)}$$

Equation **(17)** shows that the elasticity of consumption to interest rate changes equals the elasticity of the marginal propensity plus the elasticity of wealth.

smoothing this is preferred to reducing consumption just before the interest rate actually rises.[41] As noted above, the net effect is captured by an annuity-type yield.

The substitution effect is determined by the following parameters: the elasticity of substitution, the rate of time preference, the interest rate and the length over which interest rates are increased. The elasticity of substitution determines the consumer's willingness to smooth/substitute consumption between periods; the less concerned consumers are about consumption smoothing, the more responsive they will be to interest rate changes and the greater the substitution effect. The rate of time preference determines how patient consumers are; the greater the rate of time preference, the more impatient consumers will be as they will discount the future utility of consumption more. Consumers will be less willing to substitute current consumption for future consumption the more impatient they are. Both the interest rate and the time horizon exert their impact on the substitution effect via discounting. The higher the interest rate, the smaller the present discounted values of future consumption and therefore the greater the effect of an interest rate change. The longer the time horizon, the greater the amount of discounting and therefore the greater the impact of a change to interest rates.

3.5.2 The income effect

A change in interest rates affects the consumption component of the budget constraint - the affordability of a given consumption profile - leading to shifts in the level of the consumption profile. These may or may not offset the negative substitution effect.

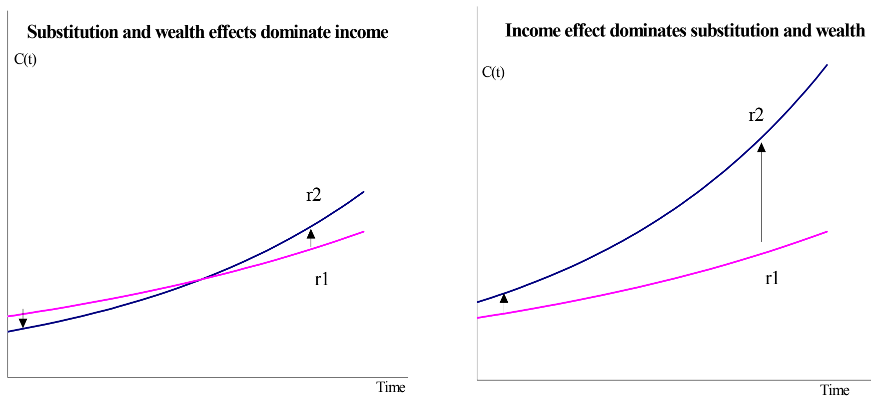
For given wealth, $W(0)$, a rise in interest rates reduces the present value cost of future consumption. Future consumption flows are discounted by more when interest rates rise, so they are more affordable out of current wealth; see **(9)**.[42]

As noted above, in terms of the consumption function, the income effect, like the substitution effect, operates through the marginal propensity to consume. The first term in **(12)** captures the effect. The substitution effect pivots the consumption path, giving it a new slope. Initial

(41) This contrasts with the relationship between interest rates and investment in stylised optimising models. With no adjustment costs only current short rates affect current investment (there is no need to change the optimal capital stock now if the required rate of return is changing in the future).

(42) Each period's consumption is translated into present value terms by scaling using the period t discount rate ie the average interest rate to t - the t-period spot long rate.

Chart 4: Impact on consumption path of a higher interest rate profile (r2>r1)



consumption falls, but future consumption rises. In contrast, the income effect shifts the whole consumption path up (as in the second panel of Chart 4). Like the substitution effect, the income effect is greater when future consumption has a higher weight. Hence the elasticity of substitution, the rate of time preference, the rate of interest and the time horizon all affect the size of the jump.

As the simple textbook model shows, whether or not this positive income effect offsets the (negative) substitution effect depends on the strength of the elasticity of substitution. These substitution and income effects will be equal and opposite if the elasticity is unity (logarithmic utility), in which case the marginal propensity to consume is determined only by the subjective rate of time discount (see equation **(12)**). This is true irrespective of whether long or short interest rates are changing.

3.5.3 The wealth effect

What simple textbook models often ignore is that, in general, there will also be substantial wealth revaluation effects. Key parameters affecting the wealth effect are: the growth rate of exogenous income, the interest rate and the time horizon. Why? The level of wealth depends on future exogenous income, but also on the extent to which this is discounted, see **(10)**. Just as the income effect reflects changes to the discounting applied to future consumption flows, the wealth effect reflects changes to the discounting applied to future exogenous income flows. Higher interest rates lead to greater discounting. If consumers have future income flows that are unaffected by interest rate changes, the current present value of these will fall when interest rates rise.(43)

The revaluation of bond coupons is a good example. Bond prices fall when interest rates rise. Equity dividend returns may also be partly independent of interest rate changes,(44) as may labour income flows - the return from human wealth. Other things equal, such effects will shift the consumption profile down when interest rates rise since $W(0)$ falls.

The presence of future exogenous income therefore makes it more likely that higher interest rates reduce the initial level of consumption, though the overall impact remains ambiguous. The net effect will depend on the importance and time profile of exogenous income. If exogenous income is growing strongly, interest rates changes will be more powerful, as discounting is applied to larger cash flows. And longer-term interest rates will be relatively more important (by contrast, for example, if exogenous income is all in the near future, the path of interest rates further out will have no effect on wealth).

3.5.4 What determines the net impact?

The income and wealth effects are both determined by the impact of discounting on present values, as illustrated by the budget constraint **(9)** and the consumption function itself (equations **(11)-(14)**). If the starting profiles of consumption and exogenous income are the same, then the income and wealth effects will be of the same magnitude but will have opposite signs (mathematically they 'cancel' each other out). Both the 'asset' (income) and 'liability' (consumption) sides of the lifetime budget constraint will be affected in the same way, so the

(43) The wealth effect consists of two effects: a human wealth revaluation effect and a financial wealth revaluation effect. An increase in the interest rate lowers the present discounted value of future labour income - the human wealth revaluation effect - and the present discounted value of some future property income - the financial wealth revaluation effect (eg for existing fixed-rate assets). For variable-rate assets or liabilities there is no revaluation effect. The future stream of income from variable-rate assets will be discounted more heavily if interest rates rise. But the actual stream of income will also be higher, and the two will exactly offset each other. This difference between variable-rate and fixed-rate assets does not arise because of different 'income' effects. For all types of asset, the rate of return on a given level of wealth will vary with the interest rate. For variable-rate assets this translates into a higher future stream of interest payments. But for fixed-rate assets, the income stream is unaffected. So to achieve the higher rate of return the market value of those assets needs to fall.

(44) Future cash flows from equity are valued more highly in present value terms when interest rates fall. But these cash flows will, in general, not be exogenous but will depend on the level of interest rates. Lower interest rates will stimulate investment and hence, through diminishing returns to capital, reduce the returns on all equity wealth (eventually, the value of the capital stock falls to replacement cost levels when Tobin's Q equals unity). However, if investment is slowed by large capital adjustment costs, this process may take many years. Empirically, it seems that the response of investment is sufficiently slow that revaluations of the existing capital stock can be large when interest rates change, with Q deviating from unity for many years.

consumer is no better or worse off in this respect. The consumer is fully hedged. Wealth is revalued down, but consumption becomes cheaper, and there is no net effect.

Table C[(45)] illustrates the equality between the wealth and income effects when growth rates of consumption and income are initially the same. (It also shows the more general property that the substitution effect is of the same magnitude as the income effect but scaled by the elasticity of substitution - they are of the same magnitude when $\theta = 1$, ie logarithmic utility.)

Table C: Relative sizes of substitution, income and wealth effects

	General	$g_c = g_y$	$\theta = 1$	$\theta = 1$, $g_c = g_y$	$\theta = \infty$, $g_c = g_y$
Substitution	$-\frac{X}{\theta}$	$-\frac{X}{\theta}$	$-X$	$-X$	0
Income	X	X	X	X	X
Wealth	$-Y$	$-X$	$-Y$	$-X$	$-X$
Total	$-Y + \left(1 - \frac{1}{\theta}\right) X$	$-\frac{X}{\theta}$	$-Y$	$-X$	0

Intuitively, the higher rate of return on wealth exactly offsets the lower initial level of wealth, and the total income from wealth is unchanged, as is consumption. In the simple case of consumption totally financed by exogenous income, and no substitution effects, the level of interest rates does not matter - consumption is equal to exogenous income in each period. For the case of a perpetuity we could write:

$$C = rW = r(Coupon/r) = Coupon$$

The coupon is permanent income and is unaffected by interest rates.

Therefore, under the assumption that exogenous income comprises all income, and that its growth rate is the same as that of consumption, the income and wealth effects of a change in interest rates 'cancel' each other. Higher interest rates then unambiguously reduce the initial level of consumption, to an extent that depends only on the substitution effect.[(46)]

We next examine the quantitative impact that interest rate changes have on consumption for different parameter values. We examine two scenarios, in both of which interest rates are initially constant and equal throughout the yield curve. In the first scenario, the whole yield curve shifts up

(45) Table C assumes $a(0) = a(T) = 0$. $\frac{1}{\theta}$ is the elasticity of substitution. X and Y are notional positive amounts, designed to illustrate relative effects only. Future exogenous income is assumed positive.
(46) This would help to justify modelling consumption as a function of income and interest rates, with no wealth term. But wealth may change for other reasons, for example changes in future expected income, so including wealth may still be necessary.

by the same amount (Section 4). In the second scenario, interest rates increase temporarily before reverting back to their initial level (Section 5).

4 Constant interest rates: a flat yield curve

4.1 The level of consumption

If we assume that interest rates are constant and equal to r every period then **(11)** is

$$c(0) = \beta(0)W(0) \tag{18}$$

where β (0) is the marginal propensity to consume. Equation **(12)** simplifies to:

$$\beta(0) = \frac{r - g_c}{1 - e^{-\{r-g_c\}T}} = \frac{\omega_c}{1 - e^{-\omega_c T}} \tag{19}$$

where g_c is the constant growth rate of consumption ($= \frac{(r-\delta)}{\theta}$) and ω_c is a normalised interest rate adjusted by the constant consumption growth rate ($\omega_c = r - g_c$). As noted earlier, this is the same formula as that for the net present value of an annuity (inverted), with constant payment of 1, where the interest rate is ω_c.(47) As $T \to \infty$ the marginal propensity becomes equal to the yield(48)

$$\lim_{T\to\infty} \beta(0) = \omega_c \tag{20}$$

In the infinite case, if the interest rate equals the subjective rate of time preference ($r = \delta$), $g_c = 0$, $\omega_c = r$ and the propensity to consume, $\beta(0)$, is simply the constant rate of interest. Each period consumption equals the return from wealth, leaving the level of wealth (and consumption) unchanged, and the profile for consumption flat ($c(0) = rW(0)$). Consumption in this case is equal to permanent income, which is the return from wealth.

Using **(18)** we can analyse the impact on the level of consumption of an interest rate change, across a flat yield curve, as done by Elmendorf (1996). Chart 4 illustrates the shifts we might see in the profile of consumption. To quantify the substitution, income and wealth effects we assume that exogenous income growth is constant at rate g_y, $y(t) = y(0)e^{g_y t}$ (where $y(0)$ denotes the amount of income received by the consumer at time 0). Then wealth (with $\overline{r}(t) = r$) becomes

$$W(0) = a(0) + \frac{y(0)}{r - g_y}\left[1 - e^{-(r-g_y)T}\right] = a(0) + \frac{y(0)}{\omega_y}\left[1 - e^{-\omega_y T}\right] \tag{21}$$

where $\omega_y = r - g_y$ is the income growth adjusted interest rate. The last term is equivalent to the present value formula for an annuity with payments ending at time T, with discount rate ω_y (see

(47) The marginal propensity is then like the annuity rate (fixed payment over initial value), which will be more than the annuity yield when the lifetime is finite.
(48) In this case the annuity is a perpetuity (see glossary).

Appendix F). For convergence, if $T \to \infty$, we require the assumption that $r > g_y$.[49] In that case we have

$$\lim_{T\to\infty} W(0) = a(0) + \frac{y(0)}{r - g_y} = a(0) + \frac{y(0)}{\omega_y} \qquad \textbf{(22)}$$

With constant growth of consumption or income, and $a(0) = 0$, equation **(18)** simplifies to

$$\begin{aligned} c(0) &= \frac{\omega_c / \left[1 - e^{-\omega_c T}\right]}{\omega_y / \left[1 - e^{-\omega_y T}\right]} y(0) \\ \lim_{T\to\infty} c(0) &= \frac{\omega_c}{\omega_y} y(0) \end{aligned} \qquad \textbf{(23)}$$

ie the initial level of consumption depends on the initial level of exogenous income, the rate of interest, and the relative growth rates of income and consumption. If those growth rates are equal, $c(0) = y(0)$. This does not mean that the level of consumption is then independent of interest rates. If we consider a higher interest rate, the consumption path will be different (the growth rate of consumption will be higher) and in such a case ω_c will not equal ω_y after the rate change. The assumption therefore provides a plausible but simple initial starting point, which we highlight as an interesting benchmark case. Such an assumption is consistent with general equilibrium in a closed economy.[50]

4.2 The impact of an increase in the yield curve on consumption

When the yield curve is flat, the effect of any exogenous change in *all interest rates along a flat yield curve on the level of consumption* can be decomposed into:

$$\frac{\partial c(0)}{\partial r} = \underbrace{\frac{\partial \beta(0)}{\partial r} W(0)}_{inc+sub} + \underbrace{\frac{\partial W(0)}{\partial r} \beta(0)}_{wealth} \qquad \textbf{(24)}$$

General expressions for the substitution, income and wealth elasticities when the yield curve is flat are given in Table D.[51] The derivation is set out in Appendix C. Table E depicts a number of

(49) If $g_y > r$, then:

$$\lim_{T\to\infty} e^{-(r-g_y)T} \to \infty \text{ and } \lim_{T\to\infty} W(0) \to \infty$$

The model does not converge to a finite level of wealth.

(50) It is not required for general equilibrium since endogenous income, produced as a return from the capital stock, could grow to ensure that total income grows in line with total consumption.

(51) The decomposition between income, substitution and wealth effects can also be carried out in terms of elasticities:

$$\frac{\partial c(0)}{\partial r}\frac{r}{c(0)} = \underbrace{\frac{\partial \beta(0)}{\partial r}\frac{r}{\beta(0)}}_{inc+sub} + \underbrace{\frac{\partial W(0)}{\partial r}\frac{r}{W(0)}}_{wealth}$$

special cases under certain parameter restrictions.[52]

Table D: Substitution, income and wealth elasticities (a(0)=0)

	$T < \infty$	$T \to \infty$
Sub	$-\frac{1}{\theta}\frac{r}{\omega_c}\phi_c$	$-\frac{1}{\theta}\frac{r}{\omega_c}$
Inc	$\frac{r}{\omega_c}\phi_c$	$\frac{r}{\omega_c}$
Wealth	$-\frac{r}{\omega_y}\phi_y$	$\frac{-r}{\omega_y}$
Total	$r\left[\left(\frac{\theta-1}{\theta}\right)\frac{1}{\omega_c}\phi_c - \frac{1}{\omega_y}\phi_y\right]$	$r\left[\left(\frac{\theta-1}{\theta}\right)\frac{1}{\omega_c} - \frac{1}{\omega_y}\right]$
$\omega_c = r - g_c$, $\omega_y = r - g_y$, $g_c = \frac{(r-\delta)}{\theta}$, $\phi_c = \left[\frac{1-(1+\omega_c T)e^{-\omega_c T}}{(1-e^{-\omega_c T})}\right]$, $\phi_y = \frac{\left[1-(1+\omega_y T)e^{-\omega_y T}\right]}{(1-e^{-\omega_y T})}$		

Table E: Substitution, income and wealth elasticities: some special cases

	$T < \infty$	$T < \infty$	$T \to \infty$	$T \to \infty$	$T \to \infty$	$T \to \infty$	$T \to \infty, \theta = \infty$
	$g_y = g_c$	$g_y = g_c = 0$	$r = \delta$	$r = \delta, \theta = 1$	$g_y = g_c$	$g_y = g_c = 0$	$g_y = g_c$
Sub	$-\frac{1}{\theta}\frac{r}{\omega}\phi$	$-\frac{1}{\theta}\phi_{g0}$	$-\frac{1}{\theta}$	-1	$-\frac{1}{\theta}\frac{r}{\omega}$	$-\frac{1}{\theta}$	0
Inc	$\frac{r}{\omega}\phi$	ϕ_{g0}	1	1	$\frac{r}{\omega}$	1	$\frac{r}{\omega}$
Wealth	$-\frac{r}{\omega}\phi$	$-\phi_{g0}$	$\frac{-r}{(r-g_y)}$	$\frac{-r}{(r-g_y)}$	$\frac{-r}{\omega}$	-1	$\frac{-r}{\omega}$
Total	$-\frac{1}{\theta}\frac{r}{\omega}\phi$	$-\frac{1}{\theta}\phi_{g0}$	$-\left(\frac{r-g_y(1-\theta)}{\theta(r-g_y)}\right)$	$\frac{-r}{(r-g_y)}$	$-\frac{1}{\theta}\frac{r}{\omega}$	$-\frac{1}{\theta}$	0
where $\phi = \phi_c = \phi_y$, $\omega = \omega_c = \omega_y$, $\phi_{g0} = \left[\frac{1-(1+rT)e^{-rT}}{(1-e^{-rT})}\right]$							

As we can see from Table D the income and substitution effects have opposite signs, as in standard two-period theory. The substitution effect is equal and opposite to the income effect, multiplied by the elasticity of substitution. So which one dominates depends on the elasticity of substitution: the income effect will be greater than the substitution effect if $\theta > 1$ and *vice versa*. The wealth effect is always negative.[53] Here, if $\theta \leq 1$, the overall impact of a rise in interest rates is to reduce the initial level of consumption. But if $\theta > 1$, the overall effect of an interest rate rise depends on all the parameters in the model. However, if the wealth effect is reasonably large, the net impact of an interest rate rise is likely to be negative for consumption.

The parameter ϕ acts as a scaling variable whose damping effect depends crucially on the time horizon T : as T gets larger, changes in r have greater effect because the future becomes more important. As $T \to \infty$, $\phi \to 1$.[54]

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(52) Throughout we assume that $a(0) = a(T) = 0$.
(53) Since we assume future exogenous income is positive.
(54) The level of ω_c (the interest rate adjusted by consumption growth) is also important: the greater it is initially, the

As discussed in the general case in Section 3, the results demonstrate that, in many circumstances, the income, substitution and wealth effects will be broadly similar in magnitude. For example, as shown in Table E, the income and substitution effects will be equal in magnitude but different in sign if the elasticity of substitution is equal to one and the wealth and income effects will be equal in magnitude but different in sign if the growth rates of consumption and income are initially the same.

All three elasticities are dependent on the initial levels of interest rates and growth - for the income and substitution effects the initial rate of consumption growth is crucial, for the wealth elasticity it is the growth of exogenous income. The closer the rate of interest and the growth rates (ie the smaller ω_c and ω_y), the more powerful the effects of a change in interest rates.[55] It is easiest to see how all the effects relate to each other in the limiting case when $T \to \infty$. If $a(0) = 0$ then spelling out the ω terms, equation **(23)** implies:

$$\begin{aligned} \frac{\partial c(0)}{\partial r} &= \underbrace{\frac{y(0)}{r-g_y}}_{inc} - \underbrace{\frac{y(0)}{\theta\left(r-g_y\right)}}_{sub} - \underbrace{\left(r - \frac{r-\delta}{\theta}\right)\frac{y(0)}{\left(r-g_y\right)^2}}_{wealth} \\ &= \frac{y(0)}{r-g_y}\left(1 - \frac{1}{\theta} - \frac{\left(r-\frac{r-\delta}{\theta}\right)}{\left(r-g_y\right)}\right) = -\frac{1}{\theta}\frac{y(0)}{r-g_y} \text{ if } g_c = g_y \end{aligned} \quad \textbf{(25)}$$

The income, substitution and wealth effects will cancel if $(1-\theta)\,g_y = \delta$. Since $\delta > 0$ and assuming $g_y > 0$, then for this condition to be true $\theta < 1$. The income and substitution effect will exceed the wealth effect if $(1-\theta)\,g_y > \delta$, this condition holding regardless of $y(0)$.

The income and the wealth effects cancel each other under certain conditions. When $a(0) = 0$, the condition is simply

$$\frac{y(0)}{r-g_y}\left(1 - \frac{(r-g_c)}{\left(r-g_y\right)}\right) = 0 \quad \textbf{(26)}$$

less the dampening effect of short horizons. Thus, for an elasticity of substitution less than 1, the higher r, the higher ω_c will be and the less the dampening effect. The opposite is true if the elasticity of substitution is less than 1.

(55) This is due to the impact of ω_c and ω_y on discounting future paths of consumption and income. The lower these two factors, the greater both lifetime consumption and lifetime wealth will be and the greater the impact of the interest rate.

Thus both effects cancel when the growth rates of consumption and income are equal, $g_c = g_y$.[56]

In terms of elasticities when $T \to \infty$, we see from Table D, combined with equation **(23)** for $c(0)$ that the total effect (the sum of the income, substitution and wealth effects) is given by

$$\frac{\partial c(0)}{\partial r}\frac{r}{c(0)} = \underbrace{\frac{r}{\omega_c}}_{inc} - \underbrace{\frac{r}{\theta\omega_c}}_{sub} - \underbrace{\frac{r}{\omega_y}}_{wealth} \; iff \; a(0) = 0 \qquad \textbf{(28)}$$

Again, if $(1-\theta)g_y = \delta$ the total effect will be zero. In Section 6 we calibrate the sensitivity of the elasticity to different parameters.

These results generalise to a model that incorporates habits in consumption through the utility function, so that the growth of consumption matters. Comparing steady-state growth paths, a multiplicative habits model gives consumption elasticities that are simple multiples of those for the standard case. Appendix E derives this result and discusses the interpretation.

5 A non-flat yield curve

To illustrate the implications of a non-flat yield curve in our framework we assume that the forward interest rate is r_1 between 0 and t_1 and r_2 between t_1 and T. The budget constraint (equations **(9)** and **(10)**) can now be expressed as

$$\int_0^{t_1} c(t)e^{-r_1 t}dt + e^{-r_1 t_1}\int_{t_1}^{T} c(t)e^{-r_2(t-t_1)}dt = W(0) \qquad \textbf{(29)}$$

where

$$W(0) = a(0) + \int_0^{t_1} y(t)e^{-r_1 t}dt + e^{-r_1 t_1}\int_{t_1}^{T} y(t)e^{-r_2(t-t_1)}dt \qquad \textbf{(30)}$$

In the analysis that follows, we assume that r_1 and r_2 are the same prior to a change in the interest rate at the short end, r_1. We do this to compare results with the cases that we examined in Section 4 where all interest rates were the same at all maturities.

The growth of consumption is still determined by the Euler equation, but varies by period, according to the interest rate applying in each ($g_c(t) = (r(t) - \delta))/\theta$). At the start of the second period the growth of consumption changes suddenly, but there is no discontinuity in the level - consumers continue to smooth consumption along the curve (Chart 5 below).

(56) In the case $T \to \infty$ and $a(0) \neq 0$, the condition for all effects to cancel each other is

$$\frac{\partial c(0)}{\partial r} = 0 = \left(1 - \frac{1}{\theta}\right)a(0) + \frac{y(0)}{r - g_y}(1 - \frac{1}{\theta} - \frac{(r - g_c)}{(r - g_y)}) \qquad \textbf{(27)}$$

5.1 The level of consumption

The expression for the level of consumption at time 0 is given by substituting the Euler equation into the budget constraint **(30)**:[57]

$$c(0) = \kappa(0) W(0) \tag{31}$$

where

$$\frac{1}{\kappa(0)} = \frac{1 - e^{-\left\{r_1 - \frac{1}{\theta}[r_1 - \delta]\right\}t_1}}{r_1 - \frac{1}{\theta}\left[r_1 - \delta\right]} + \frac{e^{-\left\{r_1 - \frac{1}{\theta}[r_1 - \delta]\right\}t_1}}{r_2 - \frac{1}{\theta}\left[r_2 - \delta\right]}\left(1 - e^{-\left\{r_2 - \frac{1}{\theta}[r_2 - \delta]\right\}[T - t_1]}\right) \tag{32}$$

$$\frac{1}{\kappa(0)} = \frac{\left(1 - e^{-\omega_c(1)t_1}\right)}{\omega_c(1)} + e^{-\omega_c(1)t_1}\frac{\left(1 - e^{-\omega_c(2)(T - t_1)}\right)}{\omega_c(2)} \tag{33}$$

$$\lim_{T \to \infty} \frac{1}{\kappa(0)} = \frac{\left(1 - e^{-\omega_c(1)t_1}\right)}{\omega_c(1)} + \frac{e^{-\omega_c(1)t_1}}{\omega_c(2)} \tag{34}$$

where $\omega_c(i) = r_i - g_c(i)$ is the normalised interest rate, adjusted for consumption growth, in period $i = 1, 2$. If either $t_1 \to T$, $t_1 \to 0$ or $r_1 = r_2$, then κ (0) reduces to $\beta(0)$, the expression for the marginal propensity to consume when the yield curve is flat. Again, each term within the equation is similar to an annuity valuation formula. The first for the period to t_1, the second for the period t_1 to T (discounted back to the initial period). So rather than the formula containing one individual annuity-like term, as in the constant yield curve equation, it contains the (discounted) sum of annuity terms. As noted in Section 2, more generally, with rates varying by period, the equation could be rewritten in terms of an annuity yield.

Assuming a constant growth rate of exogenous income, g_y, **(30)** simplifies to

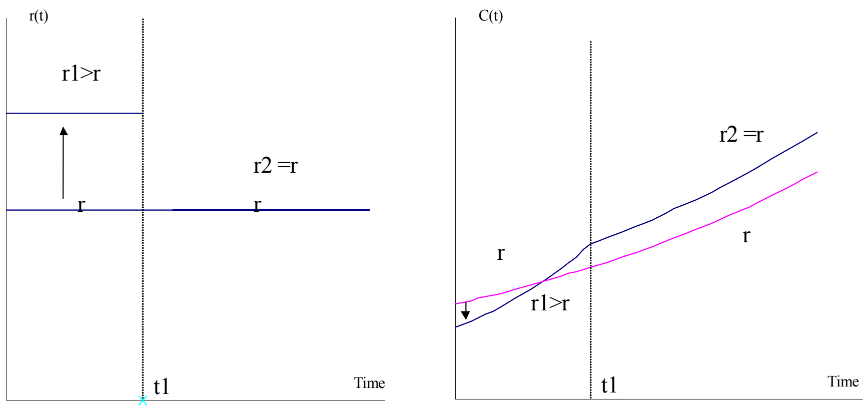
$$W(0) = a(0) + \frac{y(0)\left(1 - e^{-\omega_y(1)t_1}\right)}{\omega_y(1)} + y(0)e^{-\omega_c(1)t_1}\frac{\left(1 - e^{-\omega_y(2)(T - t_1)}\right)}{\omega_y(2)} \tag{35}$$

$$\lim_{T \to \infty} W(0) = a(0) + y(0)\left[\frac{\left(1 - e^{-\omega_y(1)t_1}\right)}{\omega_y(1)} + \frac{e^{-\omega_y(1)t_1}}{\omega_y(2)}\right] \tag{36}$$

where $\omega_y(i) = r_i - g_y(i)$ is the income growth adjusted interest rate in period i.

(57) $C(t) = C(0)e^{(1/\theta)(r_1 - \delta)t}$ up to t_1 and $C(t) = C(0)e^{(1/\theta)(r_1 - \delta)t1 + (1/\theta)(r_2 - \delta)(t - t1)}$ after t_1.

Chart 5: Change in short rates and the consumption profiles



5.2 The impact of higher interest rates between period 0 *and* t_1 *on the level of consumption*

To illustrate the impact on consumption of a rate change at the short end of the yield curve, we compare an increase in r_1 (between 0 and t_1) with no change in r_2 (Chart 5).

We can decompose the impact on consumption between the income, substitution and wealth effects as:

$$\frac{\partial c(0)}{\partial r_1} = \underbrace{\frac{\partial \kappa\,(0)}{\partial r_1} W(0)}_{inc+sub} + \underbrace{\frac{\partial W\,(0)}{\partial r_1} \kappa\,(0)}_{wealth}$$

In terms of elasticities:

$$\frac{\partial c(0)}{\partial r_1}\frac{r_1}{c(0)} = \frac{\partial \kappa\,(0)}{\partial r_1}\frac{r_1}{\kappa(0)} + \frac{\partial W\,(0)}{\partial r_1}\frac{r_1}{W(0)}$$

A key determinant of the response of the level of consumption is the length of the first period, t_1. The first-period interest rate, r_1, enters the expressions for the marginal propensity and wealth (equations **(32)** and **(36)** above), in two places – through the annuity-like valuation expression for the period up to t_1, and via the discounting of the second-period annuity expression. Both of these elements will be relatively more important the greater is the length of the first period, t_1 (for a given lifetime, T). Hence we would expect that the impact of a temporary interest rate rise will be greater the longer it persists, ie the greater is t_1.

The importance of the persistence of the interest rate change is confirmed by the elasticity formulae. Table F summarises the elasticities along the same lines as Table D did for the flat yield curve, assuming throughout that $r_1 = r_2$. Appendix C gives details of the derivations.

Table F: Substitution, income and wealth elasticities

	$T<\infty$	$T\to\infty$
Sub	$-\frac{1}{\theta}\frac{r_1}{\omega_c(1)}\phi_c(1)$	$-\frac{1}{\theta}\frac{r_1}{\omega_c(1)}\psi_c(1)$
Inc	$\frac{r_1}{\omega_c(1)}\phi_c(1)$	$\frac{r_1}{\omega_c(1)}\psi_c(1)$
Wealth	$-\frac{r_1}{\omega_y}\phi_y(1)$	$\frac{-r_1}{\omega_y}\psi_y(1)$
Total	$r_1\left[\left(\frac{\theta-1}{\theta}\right)\frac{1}{\omega_c(1)}\phi_c(1)-\frac{1}{\omega_y(1)}\phi_y(1)\right]$	$r_1\left[\left(\frac{\theta-1}{\theta}\right)\frac{1}{\omega_c(1)}-\frac{1}{\omega_y(1)}\right]$
$\phi_c(1)=\left[\frac{1-\left(e^{-\omega c(1)t_1}+\omega_c(1)t_1e^{-\omega_c(1)[T-t_1]}\right)}{\left(1-e^{-\omega_c(1)[T-t_1]}\right)}\right]$, $\phi_y(1)=\frac{\left[1-\left(e^{-\omega_y(1)t_1}+\omega_y(1)t_1e^{-\omega_y(1)[T-t_1]}\right)\right]}{\left(1-e^{-\omega_y(1)[T-t_1]}\right)}$, $\psi_c(1)=\left(1-e^{-\omega_c(1)t_1}\right)$, $\psi_y(1)=\left(1-e^{-\omega_y(1)t_1}\right)$		

Table G: Substitution, income and wealth elasticities: some special cases

	$T<\infty$	$T<\infty$	$T\to\infty$	$T\to\infty$	$T\to\infty$	$T\to\infty$	$T\to\infty$, $\theta=\infty$
	$g_y=g_c$	$g_y=g_c=0$	$r_1=\delta$	$r_1=\delta,\theta=1$	$g_y=g_c$	$g_y=g_c=0$	$g_y=g_c=0$
Sub	$-\frac{r_1\phi(1)}{\theta\omega(1)}$	$-\frac{1}{\theta}\phi(1)_{g0}$	$-\frac{\pi(1)}{\theta}$	$-\pi(1)$	$-\frac{r_1}{\theta\omega(1)}\psi(1)$	$-\frac{1}{\theta}\psi(1)$	0
Inc	$\frac{r_1}{\omega(1)}\phi(1)$	$\phi(1)_{g0}$	$\pi(1)$	$\pi(1)$	$\frac{r_1}{\omega(1)}\psi(1)$	$\psi(1)$	$\psi(1)$
Wealth	$-\frac{r_1}{\omega(1)}\phi(1)$	$-\phi(1)_{g0}$	$\frac{-r_1}{\omega_y}\psi_y(1)$	$\frac{-r_1}{\omega_y}\psi_y(1)$	$\frac{-r_1}{\omega(1)}\psi(1)$	$-\psi(1)$	$-\psi(1)$
Total	$-\frac{r_1}{\theta\omega(1)}\phi(1)$	$-\frac{1}{\theta}\phi(1)_{g0}$	$\pi(1)\left(1-\frac{1}{\theta}\right)-\frac{r_1}{\omega_y}\psi_y(1)$	$\frac{-r_1}{\omega_y}\psi_y(1)$	$-\frac{r_1}{\theta\omega(1)}\psi(1)$	$-\frac{1}{\theta}\psi(1)$	0
where $\phi(1)=\phi_c(1)=\phi_y(1)$, $\omega(1)=\omega_c(1)=\omega_y(1)$, $\psi(1)=\psi_c(1)=\psi_y(1)$, $\phi(1)_{g0}=\left[\frac{1-\left(e^{-r_1t_1}+\omega_c(1)t_1e^{-r_1[T-t_1]}\right)}{\left(1-e^{-r_1[T-t_1]}\right)}\right]$, $\pi(1)=\left(1-e^{-r_1t_1}\right)$							

As Table F shows, the key role of the duration of the interest rate change is clearest in the limiting case when T approaches infinity, when the elasticities contain dampening factors, similar to those for lifetime, T, in Table D, but this time with terms in t_1 (see $\psi_j(1)$ and $\phi_j(1)$, $j=c,y$). Table G demonstrates the size of the elasticities under special scenarios. We can see that the impact of increasing rates at the short end of the curve is going to have less impact on consumption. The clearest case is where the consumer's lifespan is infinite and where the growth rates of consumption and income are equal; the elasticity of consumption with respect to a change in the rate of interest in this case is scaled down by $\psi(1)=\left(1-e^{-\omega(1)t_1}\right)$.

The expressions when T is less than infinity are more complex, but given that we are assuming that initially r_1=r_2=r, it is also the case that the effect of a temporary change in interest rates is less. We shall show the extent of this sensitivity in the calibrations in the next section.

As discussed earlier, persistence is important because short-lived changes to rates have only a limited effect on the net present value of both future income and future expenditure. If rates rise at the short end all future values will be discounted by more – the average discount factor for all periods rises. But the impact is much less than if forward rates in the future also change in the same direction.

The individual substitution, income and wealth effects are all small for a temporary rate change. For the substitution effect, though the growth rate of consumption over the initial period may change markedly, any shift in the level will be small. As noted earlier, intuitively, the rise in the incentive to save will be affected by the rate of return on the cumulated saving in the future – if this is low consumers will be discouraged from saving now. In the case of two interest rate periods, as here, consumers will want to enter the second period with low savings/high consumption. And so, with a desire for consumption smoothing, they will want to have low savings/high consumption in the lead-up to the second period (as shown in Chart 5, only a minimal degree of pivoting can take place if rates change only at the short end).[58]

6 Calibrations

6.1 What the theory tells us

Higher interest rates mean a lower level of consumption if the sum of the wealth and substitution effects exceeds the income effect. Whether this is the case will depend on the parameters in the model. Thus, to get a feel for the sensitivity of these results to the model's parameters we now report figures for the size of the substitution, income and wealth effect elasticities for given parameter values using numerical methods.

6.2 Choice of parameters

To explore the sensitivity of the consumption elasticities to different parameter values, such as the period over which interest rates differ, t_1, we have chosen a limited set of benchmark coefficients for core variables such as the subjective rate of time preference, δ. These have been chosen on the

(58) Clearly this would not be true of alternative, plausible utility functions. For example, if people are happy to switch consumption between periods in the very short term, eg within a year or two, but not over several years, then they will be more responsive to temporary rate changes than this model suggests.

basis of the range of values that have been estimated or assumed in the literature. Elmendorf (1996) provides a discussion for the parameters chosen to calibrate the impact that interest rates have on savings behaviour and many of the arguments that follow come from that paper.

Estimates from the literature for the elasticity of substitution, $\frac{1}{\theta}$, tend to range from negative values to around $+2$, although estimates vary depending on whether aggregate data or microdata are used.[59] In the calibrations that follow, we assume that the elasticity of substitution is around 0.66 ($\theta = 1.5$) but examine the consequences of changes in the elasticity of substitution between 0.5 ($\theta = 2$) and 2 ($\theta = 0.5$).

Empirical evidence on the rate of time preference, δ, is less prolific. Elmendorf documents that estimates for the rate of time preference vary between -4% to 2%.[60] We assume that the benchmark rate of time preference is 2.5%, but we allow it to vary between 0% and 6%.

The growth rate of real exogenous income, g_y, is assumed to be 3% per annum. This is slightly higher than the average growth rate of real labour income observed in the United Kingdom of around 2%-2.5% since the early 1960s and of real post-tax disposable income of around 2.8% over the same period. The real interest rate assumed in our main calibrations is 4% per annum, although we allow the interest rate to vary between just over 3% and 10% to test the sensitivity of the elasticities to the level of interest rates. A high level of real interest rates needs to be imposed to satisfy the condition that the real interest rate is greater than the rate of growth of income (see footnote 46). For simplicity we assume that the initial level of variable-rate assets is equal to zero, ie $a(0) = 0$.

(59) For example, Hall (1988) finds little evidence of intertemporal substitution in aggregate US data, while Campbell and Mankiw (1989) find estimates of the elasticity of substitution between 0 and 0.2. However, using microdata, Zeldes (1989), in the context of liquidity constraints and using the Panel Study of Income Dynamics (PSID), calculates the elasticity of substitution to be between -1.5 to 1.9. Runkle (1991), also examining the PSID and liquidity constraints, finds elasticities of substitution that vary between 0.3 and 0.6. (Runkle does limit his sample and uses somewhat different econometric techniques from Zeldes; see Deaton (1992, pages 149-55) for more details.) Lawrance (1991) finds that the elasticity of substitution varies between 0.8 and 1.8. Using UK panel data, Attanasio and Weber (1993) found an elasticity of substitution of around 0.8 in the Family Expenditure Survey (FES). A similar figure has been obtained in aggregate UK studies. Calibrations in the business cycle literature tend to use values between 0.25 to 1. Deaton (1992) assumes a figure between 0.33 to 0.5, Elmendorf (1996) assumes 0.3, Carroll (1997) uses 0.5, and more recently, Gourinchas and Parker (2002) find evidence to calibrate a consumption model with the elasticity of substitution varying between 0.8 and 2.

(60) Calibrations on consumption models tend to assume a rate of time preference between 1% and 5%: King and Rebelo (1999) use 1.6%, Millard and Wells (2003) use 3%, and Deaton (1992) uses 5%.

6.3 Results

Appendix D gives detailed figures for substitution, income and wealth elasticities for a range of specific parameter values. For each of these scenarios, results are shown for both an increase in interest rates along the whole (flat) yield curve, and for a temporary increase in rates lasting two years. Tables H and I show the elasticities when the growth of consumption initially equals the growth of exogenous income. Tables J and K show the impact of different elasticities of substitution. Tables L and M show the impact of different elasticities of substitution when there is initially zero income and consumption growth. Tables N and O show the sensitivity of results to the rate of time preference. Tables P and Q show the effect of different initial interest rates. Table R illustrates the sensitivity of the results for a change to the period of the temporary rate change, t_1.

Note that these represent the percentage increases in the level of real consumption at time 0, for a 1% increase in the level of interest rates. So, for example, this represents a rise in interest rates from 4.00% to 4.04%. Hence for a 25 basis point rise in interest rates from 4.00%, elasticity figures would need to be multiplied by $6.25 (= 0.25/0.04)$.[61]

Charts 6 to 11 show how varying some of the key parameters in this framework affect the results. Note that in all cases the substitution effect is a simple multiple of the income effect, as shown in Table C. Please note that for Charts 6 to 10 the dashed and dotted lines represent the case of a non-flat yield curve, while the solid lines represent a flat yield curve. Results are not shown for the case of habits – these would be simple multiples of the standard elasticities, as discussed in Appendix E.

Some benchmark cases

Tables H and I in Appendix D focus on a benchmark calibration that coincides with the typical values we might expect to see for aggregate consumption, wealth and income initially growing along a balanced growth path, ie consumption growth, $\frac{(r-\delta)}{\theta} = g_c$, in line with income growth, g_y. We use $g_y = 3\%$ per annum and a real interest rate of $r = 4\%$ per annum implying that $(0.04 - \delta) = \theta \times 0.03$. If $\theta = 1$, this implies $\delta = 1\%$. If $\theta = 1.2$ (lower substitution), it implies $\delta = 0.4\%$. If $\theta = 0.5$ (higher substitution) it implies, $\delta = 2.5\%$. In this benchmark case, we see

(61) As is done to obtain the figures in Tables A and B in the introduction using Tables H and I in the appendix.

that the income and wealth elasticity effects exactly offset each other for all T, as shown analytically in Table E.

The time horizon, T and period of rate change t_1

When interest rates change along the yield curve, all the elasticities increase in absolute value with the time horizon, T, as shown in all the tables in Appendix D. The intuition is that, as the interest rate affects how we value the future, the effect is higher the longer the time horizon. When interest rates change temporarily, however, the impact of increasing T is relatively small (at least when T is already well beyond the period over which rates are raised). It is the length of time over which rates increase that is more crucial (see Table R – and Chart 7).

Chart 6 shows the impact that different lifetimes have on the substitution, income and wealth elasticities under baseline parameters.(62) The dashed and dotted lines plot the impact of changes to interest rates for two years at the short end, the solid lines depict the effects of changing interest rates throughout the yield curve. The charts confirm that, for the case where there is a parallel shift in the flat yield curve, the longer the consumer's lifetime, the stronger the effect of a change in the yield curve on all elasticities. When interest rates are changed for two years again the longer the consumer's lifetime, T, the more impact a change in interest rates at the short end will have; but the curves are very flat. Chart 7 shows the impact that increasing the duration of a short rate increase has on the different elasticities. The longer the period over which the short rate is increased, the bigger the effect on the different elasticities, with a fairly linear relationship evident, though the overall impact is fairly small.

The elasticity of substitution, $\frac{1}{\theta}$

The value of the elasticity of substitution has a large impact on the overall interest rate elasticity of consumption; halving the elasticity of substitution parameter roughly halves the interest rate elasticity of consumption.(63) The substitution effect elasticity decreases as the elasticity of substitution decreases. If consumption growth is not zero, then the income effect elasticity decreases as the elasticity of substitution decreases, for the cases where all rates change along the

(62) Baseline parameters are: $g_y = 3\%$, $r = 4\%$, $\delta = 2.5\%$, $\theta = 1.5$, $T = 40$, $t_1 = 2$ for short rate changes.
(63) Tables D to G above might suggest an exact halving. However, though the direct effect of the substitutability parameter would be to halve the total elasticitity, θ also enters via the normalised interest rate, since the consumption growth rate changes - so long as this is not held constant by fixing another parameter, as we did in Tables H and I.

Chart 6: Substitution, income and wealth elasticities and the length of time horizon for the household, T

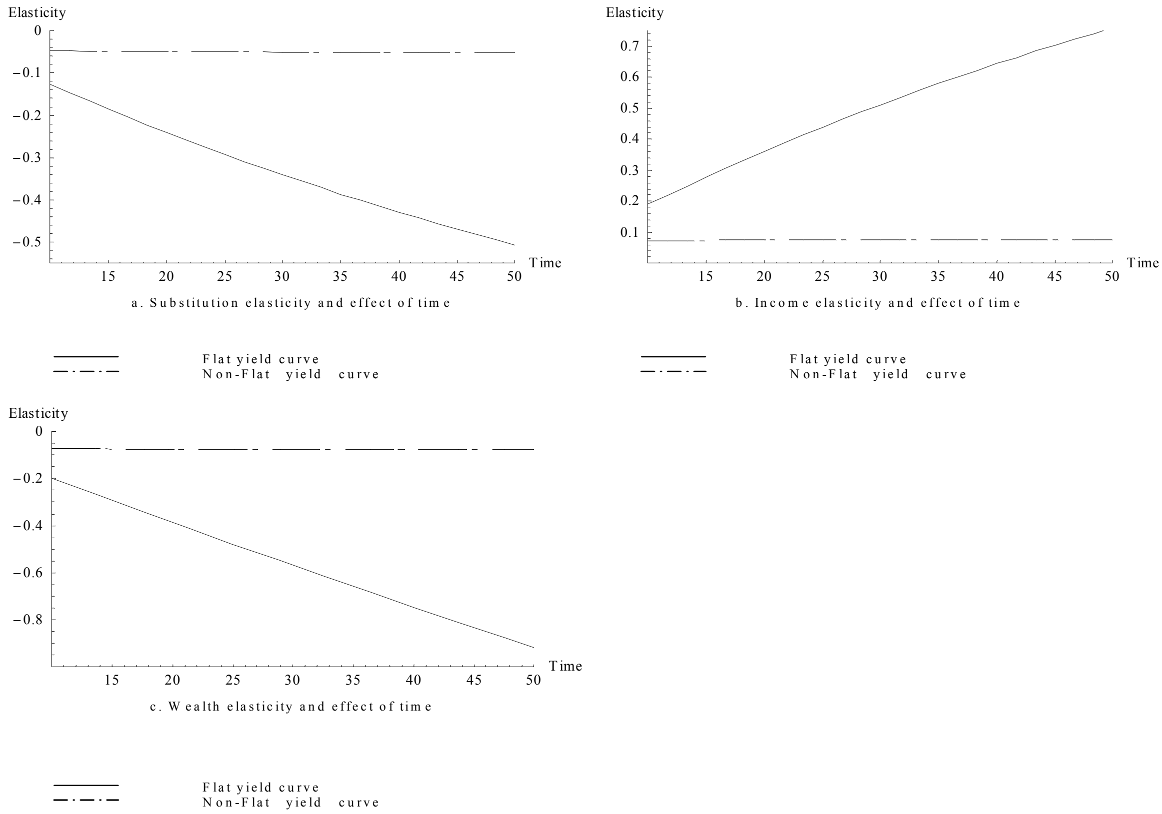
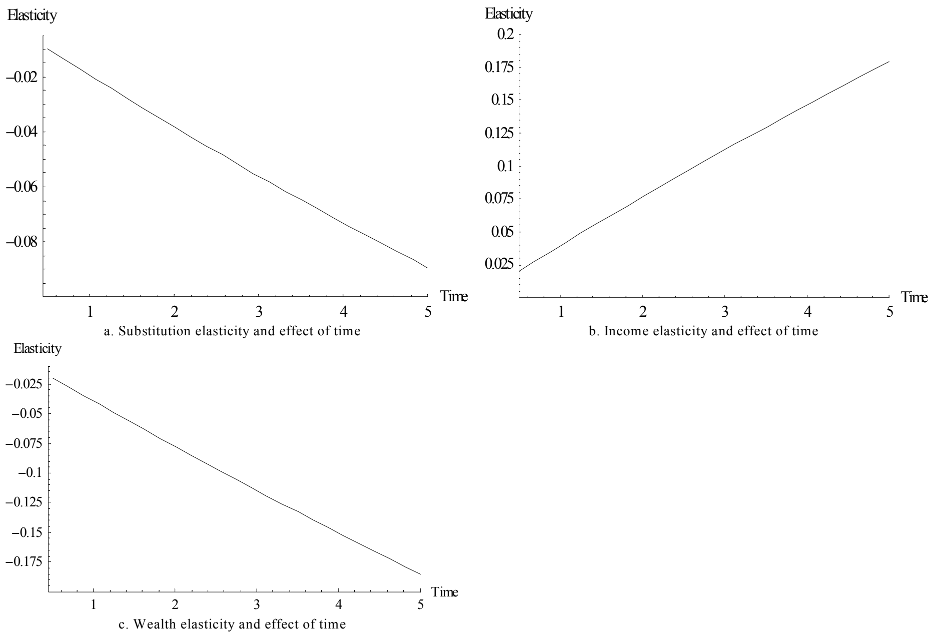


Chart 7: Substitution, income and wealth elasticities for different durations of short rate increases, t_1

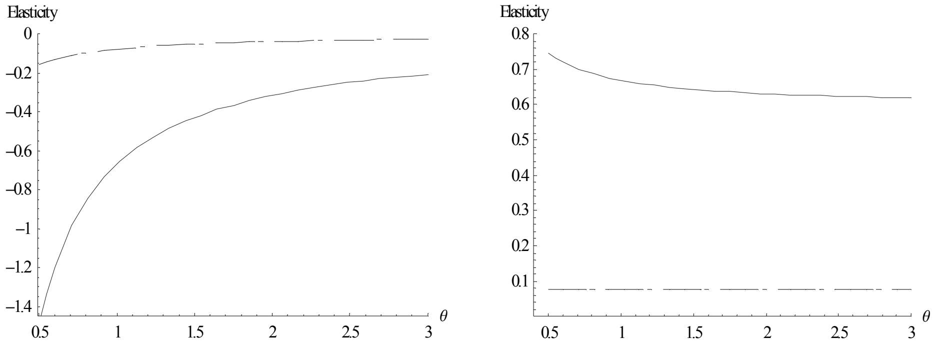


curve (Table J), and remains broadly unchanged when interest rates increase only at the short end (Table K). As one would expect, the wealth effect elasticity is not affected by the elasticity of substitution, see Tables D and F. All these results make sense: if the elasticity of substitution decreases, then consumers are less willing to substitute between periods and the elasticity of substitution effect drops. The income effect will also be reduced to some extent because a lower growth rate of consumption means interest rates have less impact on the present value of future consumption.[64] When interest rates are different for only two years this is less important.

Chart 8 shows the impact of the inverse of the elasticity of substitution upon the substitution and income elasticities (under the assumption of baseline parameters but varying θ). These charts validate the results reported in Table J. Note that short rates have less impact on all of the substitution and income elasticities (Table K).

(64) In terms of Tables D to G, note that the income elasticity is a function of ω_c and $\omega_c(1)$, which in turn are functions of θ.

Chart 8: Inverse of the elasticity of substitution parameter and impact on the substitution and income elasticities



Substitution elasticity and inverse of the elasticity of substitution

Income elasticity and inverse of the elasticity of substitution

Flat yield curve
Non-Flat yield curve

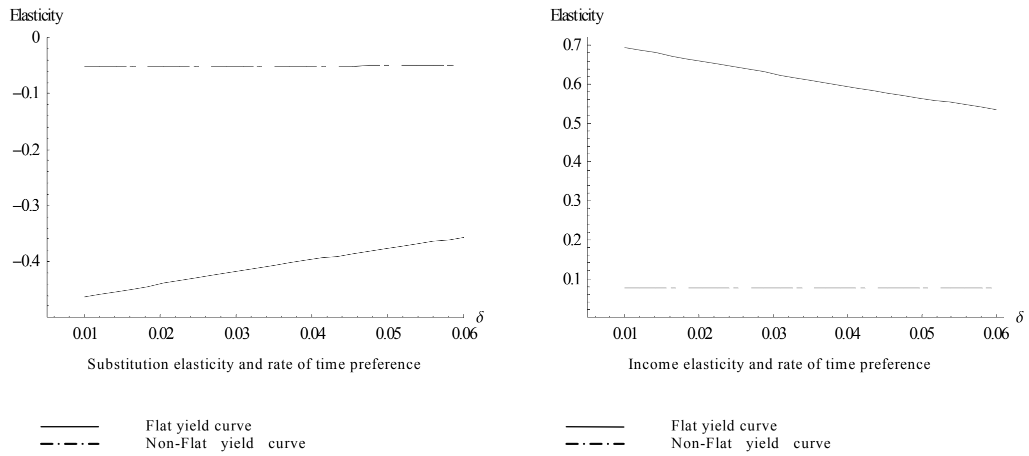
Flat yield curve
Non-Flat yield curve

The rate of time preference, δ

Increases in the rate of time preference lead to decreases in both the substitution effect elasticity and income effect elasticity for changes along the yield curve (Table N). When interest rates change only at the short end, increases in the rate of time preference leave all elasticities broadly unchanged. There is also little variation when the time horizon, T, is short. The wealth effect elasticity is not affected by changes in the rate of time preference. The intuition behind these results is as follows. Like changes in θ, increases in the rate of time preference lead to decreases in the growth rate of consumption ($g_c = \frac{r-\delta}{\theta}$), leading to a decrease in both the income and the substitution elasticities (see Table D). Hence the crucial gap between the interest rate and the rate of consumption growth, ω_c, becomes larger, and the impact of a change in interest rates on consumption is smaller. With higher rates of time preference, consumers become more impatient and are more adverse to postponing consumption. The wealth elasticity, on the other hand, does not depend on this behavioural parameter since changes in this parameter do not affect the revaluation of wealth.

The impact of the rate of time preference on the substitution and income elasticities is given in Chart 9 under our chosen baseline parameters. The dotted lines represent the effect of different short rates while the solid lines represent the effect of different long rates. For the case for short rate changes, we observe the same effects as in the case where interest rates increase across the curve, but the overall impact is less.

Chart 9: Rate of time preference parameter and impact on the substitution and income elasticities



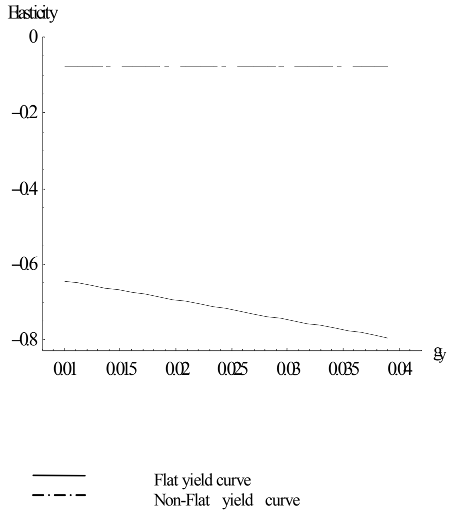
Growth rates of income and consumption

The wealth effect is generally greater than the income effect if the growth rate of income is higher than the growth rate of consumption and *vice versa* (eg see the case where $r = 10\%$ in Table P and equation **(28)** in the text). This is clearest in the infinite case (Table D). The impact of different growth rates of income on the wealth elasticity is given in Chart 10 under the assumption of baseline parameters. We can see that the higher the growth rate of income the greater the wealth elasticity with the impact on the wealth elasticity greater when interest rates change all along the yield curve.

The level of the interest rate

The level at which interest rates were prior to them being increased has a large impact on the elasticity of consumption. As the base rate of interest increases (Table P) all the elasticities increase and the total elasticity of consumption (the sum of all the effects) increases in absolute value with a higher base rate. As the level of the interest rate increases, so does the growth rate of consumption, increasing income and substitution elasticities. In addition, any given percentage change in interest rates has a larger impact on the gap between the interest rate and growth rates. Chart 11 shows the impact of the level of the interest rate on all the elasticities. In this case, the dotted lines on panels a and c denote the wealth elasticity.

Chart 10: Wealth elasticity and the growth rate of income



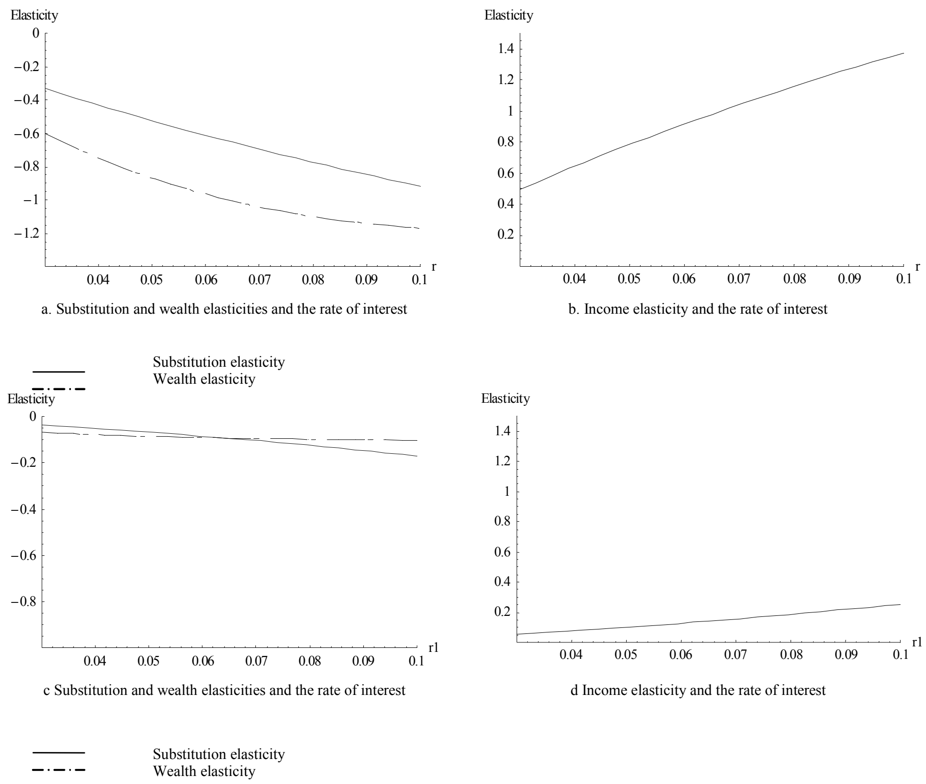
7 Conclusions and suggestions for further research

In this paper we have examined the sensitivity of the level of consumption to interest rates in a simple model of consumption with no uncertainty. We find that, for most parameter values examined, higher interest rates lead to a lower level of consumption. The role of wealth effects is crucial here. The extent to which income flows are indeed exogenous is therefore crucial to the applicability of the results. Introducing habits in a simple way does not change the results fundamentally, as is shown in Appendix E.[65] We also find that the effect of rate changes is much more pronounced when rates change along the whole yield curve rather than only at the short end, and when time horizons are long. For most parameter values, the longer the period over which interest rates are increased along the curve, the greater the impact on the level of consumption.

The results in this paper are helpful in interpreting wealth and interest rate effects in standard macroeconomic model consumption functions. The effects of sustained interest rate changes in our calibrations are large relative to the direct interest rate effects embodied in the consumption equations of many macroeconomic models. One possible explanation is that the income and substitution effects are broadly similar in magnitude (on average over long sample periods), and so roughly offsetting, meaning there is no great direct effect. The impact of interest rate changes may still be captured, indirectly via wealth. But this may not be an explicit channel, for example if

(65) Here it is particularly important to recognise that the underlying comparative dynamics approach may not be applicable to unexpected changes in interest rates, since past consumption affects the current choice of consumption.

Chart 11: Effect of the level of the interest rate on all elasticities



wealth is exogenous or does not capture the full effect of short and long rate changes. Other possibilities are that, on average, rate changes are not expected to persist for very long, or that there is uncertainty about this, so that estimations do not pick up the true long-term effects. The importance of long rates in this standard framework suggests it may be worth exploring the use of long rate terms in estimated consumption functions in addition to short rates.[(66)]

The calibrations also help in understanding what might underlie some of the recent empirical results of Lettau and Ludvigson (2001) and Fernandez-Corugedo *et al* (2003), who find that consumption to wealth ratios predict stock market movements and investment. Interest rate changes - or more generally changes to the cost of capital including risk premia - can lead to large changes in wealth. Our illustrative calibrations show that, if the elasticity of substitution is low, changes in wealth arising from this shock may be largely offset by the income effect. Hence they need not have large impacts on consumption, relative to effects from other shocks to wealth, so that consumption may vary less than wealth.

(66) A role for both can be justified since short rates may pick up important short-run liquidity, confidence and distributional effects.

There are many simplifications in this analysis that mean the results are illustrative of potentially important factors, rather than providing estimates of likely empirical relationships. We have not studied the impact that interest rates have on consumption when there is uncertainty: CRRA preferences do not have analytical solutions when there is (labour income) uncertainty. Carroll and Kimball (1996) have shown that interest rate uncertainty leads to a decrease in the level of consumption and therefore implies precautionary savings. The framework we have examined is simple and it therefore does not consider extensions to the model, such as retirement behaviour, durables, target/buffer stock savers, redistribution effects or liquidity constraints.

In addition the analysis is partial equilibrium. Interest rate changes will not generally be fully exogenous and will often be associated with shocks to income growth or preferences. And, even if exogenous, interest rate changes will mean incomes change. In particular, a higher interest rate will tend to reduce investment and push up rates of return on capital. To the extent that aggregate wealth has the characteristics of variable-rate assets, then there will be more limited wealth effects. In a one-good model, for example, capital depreciates fully each time period, and so, in essence, saving is always via variable-rate assets.

But with adjustment costs that make investment slow to respond, there may be substantial interest rate valuation effects. The model's focus on the wealth channel therefore highlights a potentially key channel through which interest rates affect consumption, which may be missed both in empirical work and some general equilibrium models.[67]

Similarly, the simplicity of the model, in particular the lack of uncertainty, means that it may exaggerate the importance of long rates. It nevertheless highlights that long rates have an important role in standard theory, and suggests that, in applied work, it is important to consider long-term as well as short-term interest rates, when thinking about the impact of rates on consumption.

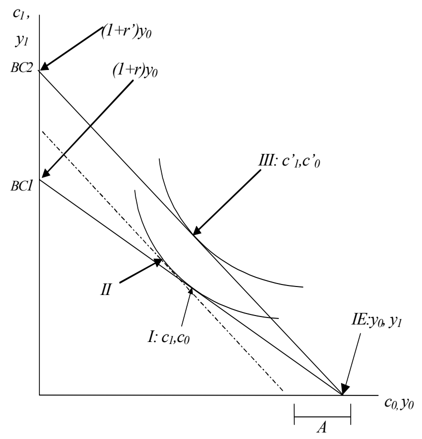
(67) By contrast, imposing general equilibrium constraints from the outset (eg $r = \delta$) may obscure important channels, if such constraints do not fully hold in practice.

Appendix A: Two-period model for consumption

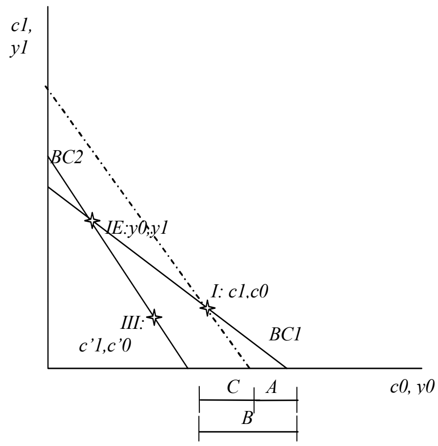
Consider the standard two-period textbook treatment of the impact of a change in interest rates on consumption when all income is earned in the first-period, period 0, $y0 \neq 0$ (see for example Nicholson (1992, page 711)). This problem was depicted in Chart 1(a) which is replicated below. The consumer's initial endowment is given by point *IE* (where all income is earned in the first period; $y1 = 0$ and $y0 \neq 0$). The consumer faces an initial budget constraint given by line $BC1$. The functional form of $BC1$ is $(1 + r)c0 + c1 = (1 + r)y0$. The slope of the budget constraint (in terms of $c1$) is therefore a function of the interest rate, $-(1 + r)$, denoting a downward-sloping budget constraint (the intercept is $(1 + r)y0$). For a given set of preferences, we can draw an indifference curve whose tangent with the budget constraint determines the optimal level of consumption in periods 0 and 1. This point is denoted I in the diagram, with resulting consumption levels $c1$ and $c0$.

The impact of an increase in the interest rate. An increase in the interest rate (from r to r') increases the slope of the budget constraint. If all income is earned in period 0 such an increase makes the budget constraint pivot around IE (the intercept changes in terms of $c1$ since $(1 + r')y0 > (1 + r)y0$, but not in terms of $c0$). The new budget constraint is given by $BC2$ (no income is earned in period 1, thus such income cannot be revalued in present discounted terms and IE does not change). To find the effect of the increase in the interest rate on consumption decisions, we must project the (shape of the previous) indifference curve onto the new budget constraint until these two are tangential. This is point III, yielding consumption levels $c'1$, $c'0$. In this case, the increase in the interest rate leads to slightly higher consumption at time 0 ($c0 < c'0$).

The substitution and income effects. The substitution effect in consumption problems means that a reduction in the price of one good relative to the price of another good entices agents to consume more of the good whose relative price has decreased. Their appetite for such substitution will be given by their preferences (represented by their indifference curve). But there is a second effect which arises because the reduction in the price of one of the goods, *ceteris paribus*, leaves the agent with more money to spend on all goods. This is the income effect.



a) Income only in the first period



b) Income in both periods

To decompose the overall impact of interest rate changes into the income and substitution effects, we note that the substitution effect is always found on the indifference curve (since this curve gives you the willingness consumers have to substitute one good for another) and the income effect is found in the differences between the budget constraints (how the reduction in the price of one good allows for more of all goods to be consumed). Thus, we project the slope of $BC2$ (the slope of the budget constraint after the rate increase) onto the initial indifference curve until these two lines are tangential.(68) The dotted line in the diagram is parallel to the budget constraint $BC2$ and produces a tangent with the old indifference curve given by point II. The income effect is therefore given by A (the difference between $BC2$ and the dotted line) and the substitution effect is given by the difference between point I and point II (both found on the old indifference curve).

The overall impact on consumption decisions following a change in interest rates is given by the difference between points I and III. The income effect is given by the difference between points II and III and the substitution effect is given by the difference between I and II. Note that the substitution effect is always negative (provided the indifference curve is convex and downward sloping) and that the income effect is positive. The overall effect depends on which effect is larger.

Consider now the situation where income is earned in both periods (Chart 1(b)). The initial endowment is given by IE, but in this case $y1 \neq 0$. The budget constraint is now given by $(1 + r)c0 + c1 = (1 + r)y0 + y1$ and shown by $BC1$.

(68) Or the slope of the old budget constraint, $BC1$, onto the new indifference curve.

The impact of an increase in the interest rate. A rise in the rate of interest pivots the budget constraint around IE, moving it from $BC1$ to $BC2$. Higher consumption in both periods is possible if first-period consumption is initially low. As in Chart 1(a), this positive influence on consumption is captured by pivoting the budget constraint around the initial consumption mix $(c0, c1)$, and the shift in the budget constraint is measured by segment A on the horizontal axis.

But now there is an offsetting negative wealth effect, reflecting the downward valuation of second-period exogenous income in discounted terms. Segment B in Chart 1(b), which measures the difference between the budget constraints $BC1$ and $BC2$ on the horizontal axis, captures this wealth reduction.[69] The net budget constraint shift is captured by the difference between segments A and B (segment C).

This wealth effect may be smaller or larger than the positive income effect. In the case shown in Chart 1(b) the income effect on the budget constraint is less than the wealth effect, with the net impact (segment C) negative. The net effect would be positive if the initial consumption mix had first-period consumption lower than first-period income (as in the simple textbook case).

More generally, the negative wealth effect will dominate if exogenous income is concentrated in the future to a greater extent than consumption, so that the discounting has a greater impact on wealth than on the affordability of future consumption.

The net impact on consumption will also need to take into account the negative substitution effect. For this indifference curves are required as in Chart 1(a).

(69) If all consumption were to take place in the first period (ie we are on the horizontal axis, where there is borrowing on the basis of second-period income) then the first-period consumption would have to fall.

Appendix B: A three-period model

Consider a discrete-time three-period problem with CRRA preferences:

$$\begin{aligned} &\max\left[\frac{1}{1-\theta}\right]\left[c_0^{1-\theta}+\frac{1}{1+\delta}c_1^{1-\theta}+\left(\frac{1}{1+\delta}\right)^2 c_2^{1-\theta}\right] \\ st\ W &= c_0+\frac{c_1}{1+r_1}+\frac{c_2}{(1+r_1)(1+r_2)} \end{aligned}$$

Where W denotes wealth:

$$W = A_0 + y_0 + \frac{y_1}{1+r_1} + \frac{y_2}{(1+r_1)(1+r_2)}$$

and where r_i, $i = 1, 2$ is the interest rate that applies in period $i-1$ until i. Setting the Lagrangian:

$$L = \left[\frac{1}{1-\theta}\right]\left[c_0^{1-\theta}+\frac{1}{1+\delta}c_1^{1-\theta}+\left(\frac{1}{1+\delta}\right)^2 c_2^{1-\theta}\right] - \pi\left[c_0+\frac{c_1}{1+r_1}+\frac{c_2}{(1+r_1)(1+r_2)}-W\right]$$

The first-order conditions are:

$$\begin{aligned} c_i &: \frac{c_i^{-\theta}}{(1+\delta)^{i-1}} - \frac{\pi}{\Pi_{j=0}^{i-1}\left(1+r_{j+1}\right)} = 0,\ i = 0,1,2, r_0 = 0 \\ \pi &: c_0+\frac{c_1}{1+r_1}+\frac{c_2}{(1+r_1)(1+r_2)} = W \end{aligned}$$

Combining these equations yields the Euler equation:

$$c_i^{-\theta} = \left(\frac{1+r_i}{1+\delta}\right)c_{i+1}^{-\theta}$$

Substitution of the consumption Euler equation expression into the budget constraint yields the consumption function:

$$c_0 = \lambda \cdot W,\ \lambda = \left[1+\frac{\left(\frac{1+r_1}{1+\delta}\right)^{\frac{1}{\theta}}}{1+r_1}+\frac{\left[\left(\frac{1+r_2}{1+\delta}\right)\left(\frac{1+r_1}{1+\delta}\right)\right]^{\frac{1}{\theta}}}{(1+r_1)(1+r_2)}\right]^{-1} \qquad \textbf{(B-1)}$$

If interest rates are constant and equal, $r = r_1 = r_2$, the last expression reduces to

$$c_0 = \overline{\lambda}\cdot\overline{W},\ \overline{\lambda} = \left[1+\frac{\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\theta}}}{1+r}+\left(\frac{\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\theta}}}{(1+r)}\right)^2\right]^{-1},\ \overline{W} = A_0+y_0+\frac{y_1}{1+r}+\frac{y_2}{(1+r)^2}$$

If we assume labour income growth of g_y, ie $y_{i+1} = \left(1+g_y\right) y_i$, $\overline{W}$ reduces to

$$\overline{W} = A_0 + y_0 + \frac{\left(1+g_y\right) y_0}{1+r} + \frac{\left(1+g_y\right)^2 y_0}{(1+r)^2}$$

so that we can write the consumption expression as

$$c_0 = \left[1 + \frac{\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\theta}}}{1+r} + \left(\frac{\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\theta}}}{(1+r)}\right)^2\right]^{-1} \left\{A_1 + y_1 + \frac{\left(1+g_y\right) y_1}{1+r} + \frac{\left(1+g_y\right)^2 y_1}{(1+r)^2}\right\} \quad \textbf{(B-2)}$$

The impact of a change in interest rates on consumption. We now consider the impact of increasing r and r_1 on equations **(B-1)** and **(B-2)**. For simplicity assume $A_0 = 0$. Taking differentials of **(B-1)**, with respect to r_1 yields

$$Sub = -W\lambda^2 * \frac{\left(\frac{1+r_1}{1+\delta}\right)^{\frac{1}{\theta}}}{\theta\left(1+r_1\right)^2}\left(1 + \frac{\left(\frac{1+r_2}{1+\delta}\right)^{\frac{1}{\theta}}}{\left(1+r_2\right)}\right) \quad \textbf{(B-3)}$$

$$inc = W\lambda^2 * \frac{\left(\frac{1+r_1}{1+\delta}\right)^{\frac{1}{\theta}}}{\left(1+r_1\right)^2}\left(1 + \frac{\left(\frac{1+r_2}{1+\delta}\right)^{\frac{1}{\theta}}}{\left(1+r_2\right)}\right) \quad \textbf{(B-4)}$$

$$wealth = -\lambda * y_1 \frac{\left(1+g_y\right)}{\left(1+r_1\right)^2}\left(1 + \frac{\left(1+g_y\right)}{\left(1+r_2\right)}\right) \quad \textbf{(B-5)}$$

Doing the same for **(B-2)** with respect to r yields

$$Sub = -\overline{W\lambda}^2 * \frac{\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\theta}}}{\theta(1+r)^2}\left(1 + \frac{2\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\theta}}}{(1+r)}\right) \quad \textbf{(B-6)}$$

$$inc = \overline{W\lambda}^2 * \frac{\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\theta}}}{(1+r)^2}\left(1 + \frac{2\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\theta}}}{(1+r)}\right) \quad \textbf{(B-7)}$$

$$wealth = -\overline{\lambda} * y_0 \frac{\left(1+g_y\right)}{(1+r)^2}\left(1 + \frac{2\left(1+g_y\right)}{(1+r)}\right) \quad \textbf{(B-8)}$$

The equivalent elasticities for **(B-3)**-**(B-8)** are:

$$
\begin{aligned}
Sub &= -r_1\lambda * \frac{\left(\frac{1+r_1}{1+\delta}\right)^{\frac{1}{\theta}}}{\theta\,(1+r_1)^2}\left(1+\frac{\left(\frac{1+r_2}{1+\delta}\right)^{\frac{1}{\theta}}}{(1+r_2)}\right)\\
inc &= -r_1\lambda * \frac{\left(\frac{1+r_1}{1+\delta}\right)^{\frac{1}{\theta}}}{(1+r_1)^2}\left(1+\frac{\left(\frac{1+r_2}{1+\delta}\right)^{\frac{1}{\theta}}}{(1+r_2)}\right)\\
wealth &= -\frac{r_1 * y_1}{W}\frac{\left(1+g_y\right)}{(1+r_1)^2}\left(1+\frac{\left(1+g_y\right)}{(1+r_2)}\right)
\end{aligned}
$$

$$
\begin{aligned}
Sub &= -r\overline{\lambda} * \frac{\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\theta}}}{\theta\,(1+r)^2}\left(1+\frac{2\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\theta}}}{(1+r)}\right)\\
inc &= r\overline{\lambda} * \frac{\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\theta}}}{(1+r)^2}\left(1+\frac{2\left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\theta}}}{(1+r)}\right)\\
wealth &= -\frac{ry_0}{\overline{W}}\frac{\left(1+g_y\right)}{(1+r)^2}\left(1+\frac{2\left(1+g_y\right)}{(1+r)}\right)
\end{aligned}
$$

The equations above illustrate the greater impact that a higher interest rate has if sustained. The expressions are essentially the same, except for the additional '2' when the interest rate is higher for both periods. So, for example, if interest and growth rates are small, the impact in the case of rates higher in both periods is around one and a half times the size of the impact in the case where rates are higher for just one period, ie 50% more.

To aid intuition we can also see this relative importance if we rewrite equation **(B-1)** and take a logarithmic approximation. Assuming $A_0 = 0$, writing the growth rate of the consumption term as $g_c(t)$, and allowing income growth, $g_y(t)$, also to vary by period, gives[(70)]

$$
c_0 = y_0 * \left[\frac{1+(1+g_y(1))/(1+r_1)+(1+g_y(1))*(1+g_y(2))/(1+r_1)*(1+r_2)}{1+(1+g_c(1))/(1+r_1)+(1+g_c(1))*(1+g_c(2))/(1+r_1)*(1+r_2)}\right]
$$

(70) Clearly if the growth rates of consumption and income are equal then $c_0 = y_0$. But since the growth rate of consumption depends on the interest rate this cannot be assumed for the elasticity calculations or to illustrate the relative importance of short and long rate changes.

For low growth rates, and particularly when consumption growth and income growth rates are initially similar[71]

$$\ln c_0 \approx \ln y_0 + (2/3) * (g_y(1) - g_c(1)) + (1/3) * (g_y(2) - g_c(2))$$

So again, if the rate change (which is within the growth of the consumption term) persists for both periods, the effect is about 50% greater than if the change is for one period only. This formula generalises to an n-period context as below

$$\ln c_0 \approx \ln y_0 + \left[\sum_{j=1}^{n-1} \left(\frac{n-j}{n} \right) \left(g_y(j) - g_c(j) \right) \right]$$

Even if growth rates of consumption and income are not similar, persistence will generally be important. For example, in the simple textbook case where there is no exogenous income after the first period, we have

$$c_0 = y_0/[1 + (1 + g_c(1))/(1 + r1) + (1 + g_c(1)) * (1 + g_c(1))/(1 + r1) * (1 + r1)]$$

In this case the logarithmic approximation is

$$\ln c_0 \approx \ln(y_0/3) + (2/3) * (r_1 - g_c(1)) + (1/3) * (r_2 - g_c(2))$$

(71) Dividing top and bottom through by 3, using $\ln(1 + g) \approx g$, ignoring second-order terms.

Appendix C: Derivation of expressions in Tables D-G

Substitution, income and wealth revaluation expressions when the yield curve is flat

In this section we spell out the expressions that were found in the main text. We start with the consumption function **(18)**

$$c(0) = \beta(0) W(0) \qquad \textbf{(C-1)}$$

where β (0) is the marginal propensity to consume given by

$$\beta(0) = \frac{r - g_c}{1 - e^{-\{r-g_c\}T}} = \frac{\omega_c}{1 - e^{-\omega_c T}} \qquad \textbf{(C-2)}$$

with g_c denoting the growth rate of consumption ($= \frac{(r-\delta)}{\theta}$) and ω_c the constant consumption growth adjusted interest rate. If we assume that exogenous income growth is constant at rate g_y, wealth is given by

$$W(0) = a(0) + \frac{y(0)}{r - g_y}\left[1 - e^{-(r-g_y)T}\right] = a(0) + \frac{y(0)}{\omega_y}\left[1 - e^{-\omega_y T}\right] \qquad \textbf{(C-3)}$$

where $\omega_y = r - g_y$ is the income growth adjusted interest rate. Assuming further that $a(0) = 0$, **(18)** simplifies to

$$c(0) = \frac{\omega_c / \left[1 - e^{-\omega_c T}\right]}{\omega_y / \left[1 - e^{-\omega_y T}\right]} y(0) \qquad \textbf{(C-4)}$$

The income elasticity works through the marginal propensity to consume and is derived by taking the growth rate of consumption as constant and taking the differential of **(C-4)** with respect to r :

$$\frac{\partial \beta(0)}{\partial r}|_{g_c \text{ given}} \times \frac{r}{\beta(0)} = r \times \frac{\left(1 - e^{-\{r-g_c\}T}\right) - (r - g_c)\, T e^{-\{r-g_c\}T}}{\left(1 - e^{-\{r-g_c\}T}\right)^2 \left(\frac{r-g_c}{1-e^{-\{r-g_c\}T}}\right)}$$

$$= \frac{r}{(r - g_c)} \times \frac{1 - (1 + (r - g_c)\, T)\, e^{-\{r-g_c\}T}}{\left(1 - e^{-\{r-g_c\}T}\right)} = \frac{r}{\omega_c} \times \frac{1 - (1 + (\omega_c)\, T)\, e^{-\omega_c T}}{\left(1 - e^{-\omega_c T}\right)}$$

$$= \frac{r\phi_c}{\omega_c} \text{ where } \phi_c = \left[\frac{1 - (1 + \omega_c T)\, e^{-\omega_c T}}{\left(1 - e^{-\omega_c T}\right)}\right]$$

This is the general expression reported in Table D. To derive the substitution elasticity we assume that the interest rate associated with the budget constraint in the marginal propensity to consume is constant:

$$\frac{\partial \beta(0)}{\partial r}|_{\text{budget constraint r given}} \times \frac{r}{\beta(0)} = -\frac{r}{\theta} \times \frac{\left(1-e^{-\{r-g_c\}T}\right) - (r-g_c)\, T e^{-\{r-g_c\}T}}{\left(1-e^{-\{r-g_c\}T}\right)^2 \left(\frac{r-g_c}{1-e^{-\{r-g_c\}T}}\right)}$$

$$= -\frac{r}{\theta\,(r-g_c)} \times \frac{1-(1+(r-g_c)\,T)\,e^{-\{r-g_c\}T}}{\left(1-e^{-\{r-g_c\}T}\right)} = -\frac{r}{\theta\omega_c} \times \frac{1-(1+(\omega_c)\,T)\,e^{-\omega_c T}}{\left(1-e^{-\omega_c T}\right)} = -\frac{r\phi_c}{\theta\omega_c}$$

To derive the expression for the wealth revaluation effect we inspect the impact that interest rates have on wealth. The wealth elasticity is given by

$$\frac{\partial W(0)}{\partial r} \times \frac{r}{W(0)} = r \times \frac{\left(r-g_y\right) T e^{-\left(r-g_y\right)T} - \left(\left[1-e^{-\left(r-g_y\right)T}\right]\right)}{\left(r-g_y\right)^2 \frac{\left(1-e^{-\left(r-g_y\right)T}\right)}{\left(r-g_y\right)}}$$

$$= -\frac{r \times \left(1-(1+\left(r-g_y\right)T)e^{-\left(r-g_y\right)T}\right)}{\left(r-g_y\right)\left(1-e^{-\left(r-g_y\right)T}\right)} = -\frac{r \times \left(1-(1+\omega_y T)e^{-\omega_y T}\right)}{\omega_y\left(1-e^{-\omega_y T}\right)}$$

$$= -\frac{r\phi_y}{\omega_y} \text{ where } \phi_y = \left[\frac{1-\left(1+\omega_y T\right)e^{-\omega_y T}}{\left(1-e^{-\omega_y T}\right)}\right]$$

Substitution, income and wealth revaluation expressions when interest rates change at the short end

For the derivation of the substitution, income and wealth elasticities for the case where interest rates change at the short end we start with the consumption function given by

$$c(0) = \kappa(0)W(0) \qquad \textbf{(C-5)}$$

where

$$\frac{1}{\kappa(0)} = \frac{1-e^{-\left\{r_1-\frac{1}{\theta}[r_1-\delta]\right\}t_1}}{r_1-\frac{1}{\theta}[r_1-\delta]} + \frac{e^{-\left\{r_1-\frac{1}{\theta}[r_1-\delta]\right\}t_1}}{r_2-\frac{1}{\theta}[r_2-\delta]}\left(1-e^{-\left\{r_2-\frac{1}{\theta}[r_2-\delta]\right\}[T-t_1]}\right)$$

and

$$W(0) = y(0)\left[\frac{\left(1-e^{-\left(r_1-g_y\right)t_1}\right)}{r_1-g_y} + e^{-\omega_c(1)t_1}\frac{\left(1-e^{-\left(r_2-g_y\right)(T-t_1)}\right)}{r_2-g_y}\right]$$

Letting $\varsigma(0) = \frac{1-e^{-\left\{r_1-\frac{1}{\theta}[r_1-\delta]\right\}t_1}}{r_1-\frac{1}{\theta}[r_1-\delta]} + \frac{e^{-\left\{r_1-\frac{1}{\theta}[r_1-\delta]\right\}t_1}}{r_2-\frac{1}{\theta}[r_2-\delta]}\left(1 - e^{-\left\{r_2-\frac{1}{\theta}[r_2-\delta]\right\}[T-t_1]}\right)$ to save notation, the income effect is found in the same way as in the previous section.

$$
\begin{aligned}
& \frac{\partial \kappa(0)}{\partial r_1}|_{g_c(1)\text{ given}} \times \frac{r_1}{\kappa(0)} \\
= \; & \frac{r_1 \times \varsigma(0)}{\varsigma(0)^2} \times \left(\frac{\partial \varsigma(0)}{\partial r_1}|_{g_c(1)\text{ given}}\right) \\
= \; & r_1\kappa(0) \times \{\left[1 - e^{-(r_1-g_c(1))t_1} - (r_1 - g_c(1))\, t_1 e^{-(r_1-g_c(1))t_1}\right] \frac{-t_1 e^{-(r_1-g_c(1))t_1}}{r_2 - g_c(2)} \left(1 - e^{-(r_2-g_c(2))[T-t_1]}\right)\} \\
= \; & r_1\kappa(0) \times \left[1 - (1 + [(r_1 - g_c(1)) + \frac{\left(1 - e^{-(r_2-g_c(2))[T-t_1]}\right)}{r_2 - g_c(2)}]t_1)e^{-(r_1-g_c(1))t_1}\right] \\
= \; & \frac{r_1\left[\frac{1-\left(e^{-(r_1-g_c(1))t_1}+(r_1-g_c(1))t_1 e^{-(r_1-g_c(1))[T-t_1]}\right)}{\left(1-e^{-(r_1-g_c(1))[T-t_1]}\right)}\right]}{(r_1 - g_c(1))} \; (\text{if } r_1 = r_2) = \frac{r_1\phi_c(1)}{\omega_c(1)}
\end{aligned}
$$

For the substitution effect we take the same steps as in the income effect but taking the interest rate associated with the budget constraint as given. The only difference between the income and the substitution effect is the term $-\frac{1}{\theta}$. To derive the wealth revaluation effect we evaluate the following function

$$
\begin{aligned}
\frac{\partial W(0)}{\partial r_1} \times \frac{r_1}{W(0)} \; = \; & \frac{r_1 y(0)\left[t_1 e^{-\left(r_1-g_y\right)t_1}\left(r_1 - g_y\right) - \left(1 - e^{-\left(r_1-g_y\right)t_1}\right)\right]}{W(0)\left(r_1 - g_y\right)^2} \\
& - \frac{r_1 y(0)\left[t_1 e^{-\left(r_1-g_y\right)t_1}\left(1 - e^{-\left(r_2-g_y\right)t_1}\right)\right]}{W(0)\left(r_2 - g_y\right)} \\
= \; & -\frac{r_1\left[\frac{\left[1-\left(e^{-\left(r_1-g_y\right)t_1}+\left(r_1-g_y\right)t_1 e^{-\left(r_1-g_y\right)[T-t_1]}\right)\right]}{\left(1-e^{-\left(r_1-g_y\right)[T-t_1]}\right)}\right]}{\left(r_1 - g_y\right)} \; (\text{if } r_1 = r_2) = -\frac{r_1\phi_y(1)}{\omega_y}
\end{aligned}
$$

Appendix D: Elasticity calibrations

Table H: Flat yield curve: consumption growth equals income growth

Initial parameters, $g_y = 3\%, r = 4\%$, $a(0) = a(T) = 0$		Substitution	Income	Sub+Inc	Wealth	Total
$\delta = 0.4\%$	$T = 10$	-0.2	0.2	0	-0.2	-0.2
$\theta = 1.2$	$T = 40$	-0.6	0.7	0.1	-0.7	-0.6
$g_c = 3\%$	$T = \infty$	-3.3	4	0.7	-4	-3.3
$\delta = 1\%$	$T = 10$	-0.2	0.2	0	-0.2	-0.2
$\theta = 1$	$T = 40$	-0.7	0.7	0	-0.7	-0.7
$g_c = 3\%$	$T = \infty$	-4	4	0	-4	-4
$\delta = 2.5\%$	$T = 10$	-0.4	0.2	-0.2	-0.2	-0.4
$\theta = 0.5$	$T = 40$	-1.5	0.7	-0.8	-0.7	-1.5
$g_c = 3\%$	$T = \infty$	-8	4	-4	-4	-8

Table I: Non-flat yield curve: consumption growth equals income growth

Initial parameters, $g_y = 3\%, r = 4\%,\ t_1 = 2$ $a(0) = a(T) = 0$		Substitution	Income	Sub+Inc	Wealth	Total
$\delta = 0.4\%$	$T = 10$	-0.06	0.07	0.01	-0.07	-0.06
$\theta = 1.2$	$T = 40$	-0.06	0.08	-0.02	-0.08	-0.06
$g_c = 3\%$	$T = \infty$	-0.07	0.08	0.01	-0.08	-0.07
$\delta = 1\%$	$T = 10$	-0.07	0.07	0	-0.07	-0.07
$\theta = 1$	$T = 40$	0.08	0.08	0	-0.08	-0.08
$g_c = 3\%$	$T = \infty$	0.08	0.08	0	-0.08	-0.08
$\delta = 2.5\%$	$T = 10$	-0.14	0.07	-0.07	-0.07	-0.14
$\theta = 0.5$	$T = 40$	-0.16	0.08	-0.07	-0.07	-0.16
$g_c = 3\%$	$T = \infty$	-0.16	0.08	-0.07	-0.08	-0.16

Table J: Flat yield curve: different elasticities of substitution

Initial parameters, $g_y = 3\%, r = 4\%,$ $a(0) = a(T) = 0, \delta = 2.5\%$		Substitution	Income	Sub+Inc	Wealth	Total
$\theta = 0.5$ $g_c = 3\%$	$T = 10$	-0.4	0.2	-0.2	-0.2	-0.4
	$T = 40$	-1.4	0.7	-0.7	-0.7	-1.4
	$T = \infty$	-8	4	-4	-4	-8
$\theta = 1$ $g_c = 1.5\%$	$T = 10$	-0.2	0.2	0	-0.2	-0.2
	$T = 40$	-0.7	0.7	0	-0.7	-0.7
	$T = \infty$	-1.6	1.6	0	-4	-4
$\theta = 2$ $g_c = 0.75\%$	$T = 10$	-0.1	0.2	0.1	-0.2	-0.1
	$T = 40$	-0.3	0.6	0.3	-0.7	-0.1
	$T = \infty$	-0.6	1.2	0.6	-4	-3.4

Table K: Non-flat yield curve: different elasticities of substitution

Initial parameters, $g_y = 3\%, r = 4\%,\ t_1 = 2$ $a(0) = a(T) = 0, \delta = 2.5\%$		Substitution	Income	Sub+Inc	Wealth	Total
$\theta = 0.5$ $g_c = 3\%$	$T = 10$	-0.14	0.07	-0.07	-0.07	-0.14
	$T = 40$	-0.15	0.07	-0.07	-0.07	-0.15
	$T = \infty$	-0.16	0.08	-0.08	-0.08	-0.16
$\theta = 1$ $g_c = 1.5\%$	$T = 10$	-0.07	0.07	0	-0.07	-0.07
	$T = 40$	-0.08	0.08	0	-0.08	-0.08
	$T = \infty$	-0.08	0.08	0	-0.08	-0.08
$\theta = 2$ $g_c = 0.75\%$	$T = 10$	-0.04	0.07	0.03	-0.07	-0.04
	$T = 40$	-0.04	0.07	0.03	-0.08	-0.04
	$T = \infty$	-0.04	0.08	0.04	-0.08	-0.04

Table L: Flat yield curve: different elasticities of substitution and zero growth in consumption and income

Initial parameters, $\delta = 4\%,\ r = 4\%,$ $a(0) = a(T) = 0,$		Substitution	Income	Sub+Inc	Wealth	Total
$\theta = 0.5$	$T = 10$	-0.4	0.2	-0.2	-0.2	-0.4
$g_c = 0\%$	$T = 40$	-1.1	0.6	-0.5	-0.6	-1.1
$g_y = 0\%$	$T = \infty$	-2	1	-1	-1	-2
$\theta = 1$	$T = 10$	-0.2	0.2	0	-0.2	-0.2
$g_c = 0\%$	$T = 40$	-0.6	0.7	0	-0.6	-0.6
$g_y = 0\%$	$T = \infty$	-1	1	0	-1	-1
$\theta = 2$	$T = 10$	-0.1	0.2	-0.1	-0.2	-0.1
$g_c = 0\%$	$T = 40$	-0.3	0.6	-0.3	-0.6	-0.3
$g_y = 0\%$	$T = \infty$	-0.5	1	0.5	-1	-0.5

Table M: Non-flat yield curve: different elasticities of substitution and zero growth in consumption and income

Initial parameters, $g_y = 0\%, r = 4\%, t_1 = 2$, $a(0) = a(T) = 0, \delta = 4\%$,		Substitution	Income	Sub+Inc	Wealth	Total
$\theta = 0.5$, $g_c = 0\%$	$T = 10$	-0.14	0.07	-0.07	-0.07	-0.14
	$T = 40$	-0.15	0.08	-0.07	-0.08	-0.15
	$T = \infty$	-0.15	0.08	-0.07	-0.08	-0.16
$\theta = 1$, $g_c = 0\%$	$T = 10$	-0.07	0.07	0	-0.07	-0.07
	$T = 40$	-0.08	0.08	0	-0.08	-0.08
	$T = \infty$	-0.08	0.08	0	-0.08	-0.08
$\theta = 2$, $g_c = 0\%$	$T = 10$	-0.04	0.07	0.03	-0.07	-0.04
	$T = 40$	-0.04	0.08	0.04	-0.08	0.04
	$T = \infty$	-0.04	0.08	0.04	-0.08	-0.04

Table N: Flat yield curve: different rates of time preference

Initial parameters, $g_y = 3\%, r = 4\%$, $a(0) = a(T) = 0,\ \theta = 1.5$		Substitution	Income	Sub+Inc	Wealth	Total
$\delta = 0\%$, $g_c = 2.6\%$	$T = 10$	-0.1	0.2	0.1	-0.2	-0.1
	$T = 40$	-0.5	0.7	0.2	-0.7	-0.5
	$T = \infty$	-2	3	1	-4	-3
$\delta = 4\%$, $g_c = 0\%$	$T = 10$	-0.1	0.2	0.1	-0.2	-0.1
	$T = 40$	-0.4	0.6	0.2	-0.7	-0.5
	$T = \infty$	-0.7	1	0.3	-4	-3.7
$\delta = 6\%$, $g_c = -1.3\%$	$T = 10$	-0.1	0.2	0.1	-0.2	-0.1
	$T = 40$	-0.4	0.5	0.1	-0.7	-0.6
	$T = \infty$	-0.5	0.7	0.2	-4	-3.7

Table O: Non-flat yield curve: different rates of time preference

Initial parameters, $g_y = 3\%, r = 4\%, t_1 = 2$, $a(0) = a(T) = 0,\ \theta = 1.5$		Substitution	Income	Sub+Inc	Wealth	Total
$\delta = 0\%$, $g_c = 2.6\%$	$T = 10$	-0.05	0.07	0.02	-0.07	-0.05
	$T = 40$	-0.05	0.08	0.03	-0.08	-0.05
	$T = \infty$	-0.05	0.08	0.03	-0.08	-0.05
$\delta = 4\%$, $g_c = 0\%$	$T = 10$	-0.05	0.07	0.02	-0.07	-0.05
	$T = 40$	-0.05	0.08	0.03	-0.08	-0.05
	$T = \infty$	-0.05	0.08	0.03	-0.08	-0.05
$\delta = 6\%$, $g_c = -1.3\%$	$T = 10$	-0.05	0.07	0.02	-0.07	-0.05
	$T = 40$	-0.05	0.08	0.03	-0.08	-0.05
	$T = \infty$	-0.05	0.08	0.03	-0.08	-0.05

Table P: Flat yield curve: different initial interest rates

Initial parameters, $g_y = 3\%, \delta = 2.5\%$ $a(0) = a(T) = 0, \theta = 1.5$		Substitution	Income	Sub+Inc	Wealth	Total
$r = 3.01\%$ $g_c = 0.3\%$	$T = 10$	-0.1	0.1	0	-0.1	-0.1
	$T = 40$	-0.3	0.5	0.2	-0.6	-0.4
	$T = \infty$	-0.7	1.1	0.4	-300	-300.6
$r = 5\%$ $g_c = 1.7\%$	$T = 10$	-0.2	0.2	0	-0.2	-0.2
	$T = 40$	-0.5	0.8	0.3	-0.9	-0.6
	$T = \infty$	-1	1.5	0.5	-2.5	-2
$r = 10\%$ $g_c = 5\%$	$T = 10$	-0.4	0.5	0.1	-0.4	-0.3
	$T = 40$	-0.9	1.4	0.5	-1.2	-0.7
	$T = \infty$	-1.3	2	0.7	-1.4	-0.8

Table Q: Non-flat yield curve: different initial interest rates

Initial parameters, $g_y = 3\%, \delta = 2.5\%, t_1 = 2$ $a(0) = a(T) = 0, \theta = 1.5$		Substitution	Income	Sub+Inc	Wealth	Total
$r = 3.01\%$ $g_c = 0.3\%$	$T = 10$	-0.04	0.05	0.01	-0.05	-0.04
	$T = 40$	-0.04	0.06	0.02	-0.06	-0.04
	$T = \infty$	-0.04	0.06	0.02	-0.06	-0.04
$r = 5\%$ $g_c = 1.7\%$	$T = 10$	-0.06	0.09	0.03	-0.09	-0.06
	$T = 40$	-0.06	0.1	0.04	-0.1	-0.06
	$T = \infty$	-0.06	0.1	0.04	-0.1	-0.06
$r = 10\%$ $g_c = 5\%$	$T = 10$	-0.12	0.18	0.06	-0.18	-0.12
	$T = 40$	-0.12	0.19	0.07	-0.19	-0.12
	$T = \infty$	-0.13	0.19	0.06	-0.19	-0.12

Table R: Flat yield curve: different length of periods for short rate changes

Initial parameters, $g_y = 3\%, \delta = 0.5\%, r_1 = 4\%$ $a(0) = a(T) = 0, \theta = 1.5$		Substitution	Income	Sub+Inc	Wealth	Total
$t_1 = 1$ $g_c = 3\%$	$T = 10$	-0.03	0.04	0.01	-0.04	-0.03
	$T = 40$	-0.03	0.04	0.01	-0.04	-0.03
	$T = \infty$	-0.03	0.04	0.01	-0.04	-0.03
$t_1 = 2$ $g_c = 3\%$	$T = 10$	-0.05	0.07	0.02	-0.07	-0.05
	$T = 40$	-0.05	0.08	0.03	-0.08	-0.05
	$T = \infty$	-0.05	0.08	0.03	-0.08	-0.05
$t_1 = 4$ $g_c = 3\%$	$T = 10$	-0.08	0.13	0.05	-0.13	-0.08
	$T = 40$	-0.1	0.15	0.05	-0.15	-0.1
	$T = \infty$	-0.1	0.16	0.05	-0.16	-0.1

Appendix E: Derivation of a consumption expression in the case of habits

Habits have recently become popular in the consumption literature. Introducing habits means utility in time t no longer depends only on consumption in that period. A common specification is the 'relative' formulation as presented by Abel (1990), Gali (1994) and Carroll *et al* (1997, 2000)[72]

$$u(c,z) = \frac{\left(\frac{c}{z^{\gamma}}\right)^{1-\theta}}{1-\theta} \qquad \textbf{(E-1)}$$

where z is the reference stock. The parameter γ determines the importance of z: if $\gamma = 0$ the utility function reverts to the standard CRRA utility function implying that consumers only care about the *level* of consumption; if $\gamma = 1$ then only consumption *relative* to the reference stock is all that matters. It is assumed that $0 \leq \gamma < 1$ and $\theta > 1$ is imposed to guarantee that the utility function is strictly concave in both c and z. The reference stock is a weighted average of past consumption evolving according to the following equation:

$$\dot{z} = \rho(c - z) \qquad \textbf{(E-2)}$$

Parameter ρ determines the relative weight of consumption at different times. The lower ρ is the less important recent consumption becomes, so that if $\rho = 0$ then past values of z do not matter in the utility function which reduces to the standard CRRA case. Therefore, not only the level but also the growth of consumption matters to utility. The Euler equation in the case of habits is

$$\begin{aligned} & \rho\left(\frac{c}{z}-1\right)(\rho\gamma\ (\gamma\ (1-\theta))+1)\left(\theta\left(1-\frac{c}{z}\right)-1\right)+\gamma\ (1-\theta)\,(2\delta+2\rho-r) \qquad \textbf{(E-3)} \\ & +\frac{\dot{c}}{c}\left[\theta\,(\theta+1)\,\frac{\dot{c}}{c}+2\theta\gamma\ (1-\theta)\,\rho\left(\frac{c}{z}-1\right)+\theta\,(2\delta+\rho-r)\right] \\ = \ & \frac{c}{z}\rho\gamma\ (\delta-r)-\theta\frac{\ddot{c}}{c}+(\rho+\delta)\,(\delta-r) \end{aligned}$$

This modified version of the Ramsey rule gives a relationship between the level of consumption, the level of the reference stock and the first and second time derivatives of consumption along an

(72) Carroll *et al* (1997, 2000) argue for using this expression to model habits instead of the additive functions used by Muellbauer (1988) and Deaton (1992). Multiplicative habits prevent the undesirable property that the argument inside the utility function is negative if past values of consumption are very high.

optimal path. The difference between **(8)** and **(E-3)** is in the importance of the preference stock, $\frac{c}{z}$ and the second time derivative of the level of consumption, $\frac{\ddot{c}}{c}$. Because consumer's utility is now a function of both the growth rate and the level of consumption (ie the ratio $\frac{c}{z^{\gamma}}$), then the temporal evolution of the growth rate must satisfy an optimality condition. Intuitively, concavity in the utility function implies that consumers will desire to smooth the level of consumption and its growth rate. Note that if either $\gamma = 0$ or $\rho = 0$ then the standard Euler equation **(8)** applies.

In steady state, when the growth rate of consumption is constant at g_{ch}, **(E-3)** reduces to[(73)]

$$\alpha_1 \left(\frac{\dot{c}}{c}\right)^2 + \alpha_2 \left(\frac{\dot{c}}{c}\right) + \alpha_3 = 0 \qquad \textbf{(E-4)}$$

$$\begin{aligned} \text{where } \alpha_1 &= \theta^2,\ \alpha_2 = 2\theta\gamma\,(1-\theta)\,\rho\,(\chi-1) + \theta\,(\rho - r + 2\delta)\,, \\ \alpha_3 &= \rho\,(\chi-1)\,(\rho\gamma\,(\gamma\,(1-\theta)+1)\,(\theta\,(1-\chi)-1) + \gamma\,(1-\theta)\,(2\delta+2\rho-r)) \\ &\quad -\chi\rho\,(\delta - r) + (\rho+\delta)\,(\delta - r) \end{aligned}$$

and $\frac{c}{z} = \chi$ is constant if the growth rate of the preference stock is constant. Equation **(E-4)** is a quadratic equation and has two solutions. Substituting

$$\frac{c}{z} = 1 + \frac{\dot{c}}{\rho c} \qquad \textbf{(E-5)}$$

into **(E-4)** yields the following pair of solutions for the growth rate of consumption and the ratio of consumption to the preference stock:

$$\left(\frac{\dot{c}}{c}\right) = \frac{r-\delta}{\gamma\,(1-\theta)+\theta} = g_{ch1},\ \frac{c}{z} = 1 + \frac{1}{\rho}\left(\frac{r-\delta}{\gamma\,(1-\theta)+\theta}\right) \qquad \textbf{(E-6)}$$

$$\left(\frac{\dot{c}}{c}\right) = \frac{\rho(1-\gamma)+\delta}{\theta(\gamma-1)} = g_{ch2},\ \frac{c}{z} = 1 + \frac{1}{\rho}\left(\frac{\rho(1-\gamma)+\delta}{\theta(\gamma-1)}\right) \qquad \textbf{(E-7)}$$

where $g_{chi},\ i = 1, 2$ denote the growth rates of consumption for the two roots when there are habits. Carroll *et al* (1997) demonstrate that the solution associated with g_{ch2} is unstable and violates transversality conditions so we ignore it and use only g_{ch1}.[(74)]

(73) Note that $\frac{\ddot{c}}{c} = (\frac{\dot{c}}{c})^2$ if the growth rate of consumption is assumed constant.
(74) We therefore denote g_{ch1} as g_{ch}.

Given the assumptions that $0 \leq \gamma < 1$ and $\theta > 1$, $\gamma(1-\theta)+\theta = \theta_h < \theta$, so the steady-state *growth rate of consumption when there are habits exceeds the growth rate of consumption when there are no habits, given the same instantaneous degree of risk aversion,* θ_h is, loosely speaking, the finite-case equivalent of the inverse of the elasticity of substitution when there are habits.(75) In the case of habits the elasticity of substitution, $\frac{1}{\theta_h}$, is greater than in the case of no habits, $\frac{1}{\theta}$.

To obtain an expression for the level of consumption substitute $c(t) = c(0)e^{-g_{ch}t}$ into **(9)**:

$$c(0) = \lambda_h(0)\, W(0) \qquad \textbf{(E-8)}$$

$$\text{where } \frac{1}{\lambda_h(0)} = \int_0^T e^{-\bar{r}(t)t} e^{-\frac{[\bar{r}(t)-\delta]}{\theta_h}} dt \qquad \textbf{(E-9)}$$

and $W(0)$ is still given by **(10)** the unchanged budget constraint. Given $\theta_h < \theta$, with a higher growth rate of consumption, the level at time 0 must be lower.(76)

The impact of interest rates on consumption when there are habits: comparison with the case of no habits

The principal difference between a steady-state consumption expression with habits and one without habits is found in the elasticity of substitution term. As a result, the mathematical expressions for the impact of a change in interest rates on the level of consumption are equivalent to those derived in the main text. The difference is that with habits the elasticity of substitution term is higher. Thus, one only needs to look at Chart 8 and note that with habits, $\theta_h < \theta$. Therefore, in the case of habits both the income and substitution elasticities must be higher. Since θ does not affect the wealth elasticity (Tables D and E), introducing habits does not affect this elasticity.

(75) The elasticity of substitution is defined as the response of consumption growth to interest rates. See Carroll *et al* (2000, page 347) for more about the correct definition of the elasticity of substitution when the time horizon is infinite and when it is only small.
(76) Another way to see this is to think that habit-forming consumers have a higher elasticity of substitution than no habit-forming consumers and are therefore more willing to substitute between periods, leading them to face higher growth rates of consumption. Habit-forming consumers are more willing to substitute consumption intertemporally - ie have a higher elasticity of substitution - because any increase or decrease in consumption will lead to a lower gain or loss in utility as a result of adjustments to the reference stock, z.

Appendix F: Glossary of financial terms

For extensive glossaries of financial terms and details of interest rate and bond mathematics see textbooks such as *Principles of corporate finance* by Brealey and Myers (1996) and *Financial markets and corporate strategy* by Grinblatt and Titman (1998). This appendix summarises some of the main concepts and formulae used in this paper.

Annuity. *A standard annuity* is an investment whose returns are regular payments, of a constant amount, for a limited number of periods. It is like a constant coupon bond, but with no repayment of the capital at the maturity date (so that the regular payments are therefore higher than on a bond of the same value and maturity).

A standard perpetuity is an infinite annuity. An example is a government bond known as a consol.

The net present value of a standard annuity is the present discounted value of the future stream of constant payments. The algebraic representations are shown in Table S below, along with the summary formula.

Note that the present value formula for an annuity of maturity T can be derived from the difference between the value of an infinite annuity (perpetuity), with payments starting immediately, and the value of an infinite annuity, starting in period T (discounted to today's value). This derivation is analogarithmicous to that used to estimate the effect on the level of consumption of interest rates that are higher for several periods only, before returning to their benchmark level.

The annuity rate (often quoted in the financial press) is the annuity payment divided by the initial value of the annuity. For maturities less than infinity, it is more than the yield on the annuity, as it involves repayment of the capital element. In this paper the marginal propensity to consume has the same form as an annuity rate.

A growing annuity has payments that grow over the life of the annuity. The formulae are essentially the same as those for a standard constant-payment annuity, but with a lower discount rate $(r - g)$. This is very much like our adjusted interest rate, w, used in the consumption function

Table S: Alternative specifications of annuity net present value*

Discount factor/interest rate	Discrete-time formula	Continuous-time formula
Constant-rate case	$\sum_{t=1}^{T} \frac{1}{(1+r)^t} = \frac{1}{r}\left(1 - \frac{1}{(1+r)^T}\right)$	$\int_0^T e^{-rt}dt = \frac{1}{r}\left(1 - e^{-rT}\right)$
Using forward (short) rates	$\sum_{t=1}^{T} \frac{1}{\prod_{\upsilon=1}^{t}(1+r(\upsilon))}$	$\int_0^T e^{-\int_0^t r(\upsilon)d\upsilon}dt$
Using spot (zero-coupon) long rates	$\sum_{t=1}^{T} \frac{1}{(1+\overline{r}(t))t}$	$\int_0^T e^{-\overline{r}(t)t}dt$
Using yield	$\sum_{t=1}^{T} \frac{1}{(1+r_{yield})t} = \frac{1}{r_{yield}}\left(1 - \frac{1}{(1+r_{yield})^T}\right)$	$\int_0^T e^{-r_{yield}t}dt = \frac{1}{r_{yield}}\left(1 - e^{-r_{yield}T}\right)$

*With payments of 1 per period from period 1 until T.

expressions.

Average interest rate $\overline{\mathbf{r}}(\mathbf{t})$. The average interest rate is the t-period spot long interest rate (zero-coupon).

Coupon. A coupon is the regular payment on a bond (as opposed to the final repayment of the notional capital value).

Forward interest rate. An interest rate fixed today, for one or more periods in the future, eg the rate on a loan to be made at time s, repaid at time T. Though the forward's maturity date is T, informally it may sometimes be said to have maturity $T - s$ eg a three-month rate, two years (24 months) forward, would mature in 27 months, but be a three-month (short) contract.

Instantaneous short rate, $\mathbf{r}(\mathbf{v})$. The instantaneous short rate is a short rate for an infinitely short contract length, at period υ, and is used in continuous-time formulae. The instantaneous forward rate for period υ is the rate set at time 0.

Internal rate of return. The internal rate of return is the interest rate that just discounts a set of future cash flows so that the net present value is zero. Or excluding the initial outlay/purchase price, the interest rate that discounts future cash flows (the coupons and capital repayment in the case of a bond) so that their discounted sum just equals the initial outlay/price.

Long rate. A loose term referring to interest rates covering long periods of time eg five or ten years. Usually refers to spot rates (from today), but can be used to refer to forward rates (forward long rates as opposed to forward short rates).

Maturity. The maturity of a loan or investment is the end date of the contract (when it matures). It is sometimes loosely used to mean the length of the contract (eg a bond that pays out for ten years is said to have a ten-year maturity).

Short rate. A loose term referring to interest rates covering short periods of time eg overnight or three months. Usually refers to spot rates (from today), but can be used to refer to forward rates (forward short rates as opposed to forward long rates).

Spot interest rate. An interest rate fixed today for one or more periods from today, ie the rate on a loan applying between time 0 and time T. Time T is known as the maturity T. The spot rate is not informative about the profile of interest rates during the period of the investment. If there is arbitrage and no risk, the spot rate to time T will be the compounded sum of period-by-period rates eg for two periods $(1 + r) = (1 + r_1) * (1 + r_2)$ where the rates are not annualised.

Yield (to maturity). The yield to maturity (T) is the internal rate of return on a bond. If the bond has no coupons, its yield is the same as the spot rate for maturity T. Otherwise the yield will reflect the profile of coupons, and the spot rates for each coupon date. So, for example, if short-term interest rates are low, but longer-term rates high, the yield on a coupon bond will be lower than on a zero-coupon bond, because the rate on the coupons is lower and reduces the (weighted) average.

Yield curve/term structure of interest rates. The yield curve is a loose term for the relationship between interest rates of different maturities. It can be expressed in terms of forward rates, spot rates or yields.

Zero-coupon bond. A zero-coupon bond has no coupon payments, repaying the notional capital at maturity.

Zero-coupon rate. The zero-coupon interest rate is the rate that discounts a cash flow at a certain date, t, repayment of capital. It is the rate that is used to discount cash flows at particular dates. It is the same as the yield on a zero-coupon bond, where all the return is in the form of a capital gain (representing the total interest cumulated over the period of the loan as the market price rises to the notional issue capital value).

References

Abel, A B (1990), 'Asset prices under habit formation and catching up with the Joneses', *American Economic Review*, Vol. 40, pages 38-42.

Angeloni, I, Kashyap, A, Mojon, B and Terlizzese, D (2002), 'Monetary transmission in the euro area: where do we stand?', *European Central Bank Working Paper no. 114.*

Attanasio, O P and Weber, G (1993), 'Consumption growth, the interest rate, and aggregation', *Review of Economic Studies*, Vol. 60, pages 631-49.

Bank of England (1999), *The transmission mechanism of monetary policy*.

Bank of England (2002), Box on interest rate term premia, *Bank of England Quarterly Bulletin*, Spring, page 9.

Barro, R and Xala-i-Martin, X (1995), *Economic growth*, McGraw-Hill Series in Economics.

Bernanke, B S and Reinhart, V R (2004), 'Conducting monetary policy at very low short-term interest rates', speech given at the International Center for Monetary and Banking Studies Lecture, Geneva, Switzerland.

Brealey, R A and Myers, S C (1996), *Principles of corporate finance*, 6th edition, McGraw-Hill.

Britton, E and Whitley, J (1997), 'Comparing the monetary transmission mechanism in France, Germany and the United Kingdom: some issues and results', *Bank of England Quarterly Bulletin*, May, pages 152-62.

Browning, M and Lusardi, A (1996), 'Household saving: micro theories and macro facts', *Journal of Economic Literature*, XXXIV, pages 1,797-855.

Buiter, W (2003), 'Deflation: prevention and cure', *NBER Working Paper no. 9623*.

Campbell, J Y and Mankiw, N G (1989), 'Consumption, income and interest rates: reinterpreting the time series evidence', in Blanchard, O J and Fischer, S (eds), *NBER Macroeconomics Annual: 1989*, MIT Press, pages 185-216.

Carroll, C (1997), 'Buffer stock saving and the life cycle/permanent income hypothesis', *The Quarterly Journal of Economics*, CXII, pages 1-57.

Carroll, C (2000), 'Solving consumption models with multiplicative habits', *Economics Letters*, Vol. 68(1), pages 67-77.

Carroll, C and Kimball, M S (1996), 'On the concavity of the consumption function', *Econometrica*, Vol. 64, pages 981-92.

Carroll, C, Overland, J R and Weil, D N (1997), 'Comparison utility in a growth model', *Journal of Economic Growth*, Vol. 2(4), pages 339-67.

Carroll, C, Overland, J R and Weil, D N (2000), 'Saving and growth with habit formation', *American Economic Review*, Vol. 90(3), pages 341-55.

Clarida, R, Gali, J and Gertler, M (1999), 'The science of monetary policy: a new Keynesian perspective', *Journal of Economic Literature*, Vol. 37(4), pages 1,661-707.

Deaton, A (1992), *Understanding consumption*, Clarendon Press.

Eggertsson, G and Woodford, M (2003), 'The zero bound on interest rates and optimal monetary policy', *Brookings Papers on Economic Activity*, Vol. 1, pages 139-233.

Elmendorf, D W (1996), 'The effect of interest-rate changes on household saving and consumption: a survey', *Federal Reserve Board Finance and Economics Discussion Series Paper no. 27*.

Epstein, L G and Zin, S E (1989), 'Substitution, risk aversion, and the temporal behaviour of consumption and asset returns: a theoretical framework', *Econometrica*, Vol. 57, pages 937-69.

Fagan, G, Henry, J and Mestre, R (2001), 'An area-wide model (AWM) for the euro area', *European Central Bank Working Paper no. 42*.

Federal Reserve Bank of New York (2002), 'Financial innovation and monetary transmission', proceedings of a conference sponsored by the Federal Reserve Bank of New York, 5-6 April 2001. Published in the Federal Reserve Bank of New York's *Economic Policy Review*, Vol. 8(1).

Fernandez-Corugedo, E W, Price, S and Blake, A (2003), 'The dynamics of consumers' expenditure: the UK consumption ECM redux', *Bank of England Working Paper no. 204*.

Flavin, M (1981), 'The adjustment of consumption to changing expectations about future income', *Journal of Political Economy*, Vol. 89, pages 974-1,009.

Gali, J (1994), 'Keeping up with the Joneses: consumption externalities, portfolio choice and asset prices', *Journal of Money, Credit, and Banking*, Vol. 26, pages 1-8.

Gourinchas, P O and Parker, J (2002), 'Consumption over the life-cycle', *Econometrica,* Vol. 70(1), pages 47-91.

Grinblatt, M and Titman, S (1998), *Financial markets and corporate strategy*, McGraw-Hill.

Hall, R E (1978), 'Stochastic implications of the life cycle-permanent income hypothesis: theory and evidence', *Journal of Political Economy*, Vol. 86, pages 971-87.

Hall, R E (1988), 'Intertemporal substitution in consumption', *Journal of Political Economy*, Vol. 96, pages 339-57.

Journal of Economic Perspectives (1995), Symposium on 'The transmission mechanism'.

King, R (2000), 'The new IS-LM model: language, logic, and limits', *Federal Reserve Bank of Richmond Economic Quarterly*, Vol. 86(3), pages 45-103.

King, R G and Rebelo, S T (1999), 'Resuscitating real business cycles', in Taylor, J B and Woodford, M (eds), *Handbook of macroeconomics,* Vol. 1B, Elsevier Science, North Holland.

Lantz, C D and Sarte, P-D G (2001), 'Consumption, savings, and the meaning of the wealth effect in general equilibrium', *Federal Reserve Bank of Richmond Economic Quarterly*, Vol. 87 (3), pages 53-71.

Lawrance, E C (1991), 'Poverty and the rate of time preference: evidence from panel data', *Journal of Political Economy*, Vol. 99, pages 54-77.

Lettau, M and Ludvigson, S (2001), 'Consumption, aggregate wealth, and expected stock returns', *Journal of Finance*, Vol. 56, pages 815-49.

McCallum, B and Nelson, E (1997), 'An optimizing IS-LM specification for monetary policy and business cycles analysis', *Journal of Money, Credit, and Banking*, Vol. 31(2), pages 296-316.

Merton, R C (1969), 'Lifetime portfolio selection under uncertainty: the continuous-time case', *The Review of Economics and Statistics*, Vol. 51, pages 247-57.

Millard, S and Power, J (2004), 'The effects of stock market movements on consumption and investment: does the shock matter?', *Bank of England Working Paper no. 236*.

Millard, S and Wells, S (2003), 'The role of asset prices in transmitting monetary and other shocks', *Bank of England Working Paper no. 188*.

Muellbauer, J N (1988), 'Habits, rationality and myopia in the life cycle consumption function', *Annales d' ѐconomie et de Statistique*, Vol. 9, pages 47-70.

Nicholson, W (1992), *Microeconomic theory: basic principles and extensions*, 5th edition, The Dryden Press.

Romer, D (1996), *Advanced macroeconomic*s, McGraw-Hill Series in Economics.

Rotemberg, J and Woodford, M (1997), 'An optimisation-based econometric framework for the evaluation of monetary policy: expanded version', *NBER technical paper no. 233*.

Runkle, D E (1991), 'Liquidity constraints and the permanent-income hypothesis: evidence from panel data', *Journal of Monetary Economics*, Vol. 27, pages 73-98.

Summers, L (1981), 'Capital taxation and accumulation in a life cycle growth model', *American Economic Review*, Vol. 71(4), pages 533-44.

Zeldes, S P (1989), 'Consumption and liquidity constraints: an empirical investigation', *Journal of Political Economy*, Vol. 97, pages 305-46.