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Abstract

This paper estimates yield curve models for the United Kingdom, where the underlying determinants have a macroeconomic interpretation. The first factor is an unobserved inflation target, the second factor is annual inflation, and the third factor is a 'Taylor rule residual', which, among other things, captures the effects of the output gap and monetary policy surprises in the Taylor rule. We find that the long end of the yield curve is primarily driven by changes in the unobserved inflation target. At shorter maturities, yield curve movements reflect short-run inflation and the Taylor rule residual. For holding periods of one month, our preferred model implies that agents require compensation for risks associated with cyclical and inflation shocks but do not require compensation for shocks to the inflation target. For holding periods beyond one month, agents require compensation for all three sources of risks. Time series of risk premia on long forward rates from the preferred yield curve model have declined since the 1970s, which is consistent with perceptions of declining macroeconomic uncertainty or perhaps more efficient macroeconomic stabilisation policies. Model-implied risk premia at short maturities match up reasonably well with survey-based risk premia, which indicates that the model could be useful for the purpose of extracting market-based interest rate expectations.

Key words: Term structure models, factor models, interest rates, risk premium.

JEL classification: C13, C32, E43, E44, E52.

Summary

Understanding which factors drive movements in the term structure of interest rates is of potential interest to policymakers for a number of reasons. For example, the extent to which changes in the short-term policy rate feed through to longer-term yields is important since it represents a key part of the transmission mechanism of monetary policy by affecting the spending, saving and investment behaviour of individuals and firms in the economy. Moreover, the yield curve has been found to be a good predictor of future real activity and inflation. The term structure also contains information about market participants' expectations of the future path of interest rates. But there is strong empirical evidence to suggest that time-varying risk premia drive forward rates away from these expectations. The decomposition of forward rates into expectations of future interest rates and risk premia is one of the key contributions of this paper.

In this paper, we estimate various models of the term structure of interest rates for the United Kingdom, where the underlying factors that drive movements in the term structure have a macroeconomic interpretation. The first factor is an unobserved inflation target, the second factor is annual inflation, and the third factor reflects, among other things, the output gap and monetary policy shocks. We find that the long end of the yield curve is primarily driven by changes in the unobserved inflation target. At shorter maturities, yield curve movements reflect mainly the other two factors.

Our preferred model implies that agents require compensation (ie a risk premium) for risks associated with output gap and inflation shocks but do not require compensation for shocks to the inflation target. This result seems consistent with simple asset pricing models with an assumed representative (homogenous) agent. Our yield curve models can be used to back out a path for an unobserved time-varying inflation target. This path is shown to be closely linked to other measures of long-run inflation expectations, such as those from market-based ten year ahead breakeven inflation rates and long-run Consensus forecasts of inflation.

Time series of risk premia on long forward rates from the preferred yield curve model have declined since the 1970s, which is consistent with perceptions of declining macroeconomic uncertainty or perhaps more efficient macroeconomic stabilisation policies. Model-derived risk premia at short maturities are shown to be highly correlated with survey-based risk premia, which

indicates that the model could be useful for the purpose of extracting market-based interest rate expectations. This is comforting because we have not used survey data for estimation or even model selection. As such, it provides support for the estimated models.

1 Introduction

Understanding which factors drive movements in the term structure of interest rates is of potential interest to policymakers for a number of reasons. For example, the extent to which changes in the short-term policy rate feed through to longer-term yields is important since it represents a key part of the transmission mechanism of monetary policy by affecting the spending, saving and investment behaviour of individuals and firms in the economy. Moreover, the yield curve has been found to be a good predictor of future real activity and inflation (see Harvey (1988); Mishkin (1990); and Estrella and Hardouvelis (1991)). The term structure also contains information about market participants' expectations of the future path of interest rates. But there is strong empirical evidence to suggest that time-varying risk premia drive forward rates away from these expectations (see Cochrane (2005)). Consequently, the decomposition of forward rates into expectations of future interest rates and risk premia is one of the key contributions of this paper.

Early empirical studies typically modelled bond yields within a vector autoregression framework (see Hall, Anderson and Granger (1992); Shea (1992); and Pagan, Hall and Martin (1996)). But it was quickly realised that this methodology was unsatisfactory. To begin with, Litterman and Scheinkman (1991) showed that only three state factors were needed to explain almost all of the variation in bond yields. Moreover, the factors were often found to be distinct in the way they affected the shape of the yield curve. By contrast, vector autoregression models need to be heavily parameterised to explain the cross-section of bond yields leading to poor inference and forecasting power (see Ang and Piazzesi (2003); and Diebold and Li (2006)). Perhaps more importantly, the lack of cross-equation restrictions within a vector autoregression framework means that such models permit arbitrage. But, given the highly liquid nature of most bond markets and the sophisticated tools market participants use to trade away any arbitrage opportunities, this is unlikely to be a sensible feature of a model of the term structure.

Consequently, the empirical literature has now turned towards equilibrium term structure models that explicitly incorporate cross-equation restrictions implied by no-arbitrage. The most common models used assume that bond yields are an affine function of two, or perhaps three, state factors (for the classic papers see Vasicek (1977); and Cox, Ingersoll and Ross (1985)). In early empirical work using affine term structure models the underlying factors were typically unobserved. But these types of models are not very useful for answering questions on how changes in fundamental factors such as expected output and inflation affect the yield curve. A more recent and rapidly growing field of research addresses this issue by fitting the term structure to macroeconomic factors, either, as in this paper, by combining them within unobserved factors (see, Ang and Piazzesi (2003); Ang, Dong and Piazzesi (2005); Bernanke, Reinhart and Sack (2004); and Dai and Philippon (2005)), or by incorporating a no-arbitrage model of the term structure within a fully specified macroeconomic model that exhibits both rational expectations and nominal rigidities (see Hordahl, Tristani and Vestin (2006); Bekaert, Cho and Moreno (2005); Dewachter and Lyrio (2006); and Rudebusch and Wu (2004)).

This paper makes two main contributions to the existing literature. First, from a policy perspective, one of the key advantages of equilibrium term structure models is that they allow us to separate out movements in bond yields into those that reflect changes in expected future interest rates from those that reflect changes in risk premia and convexity. Nevertheless, there has been little discussion in the literature about how these different factors influence the term structure of interest rates. In this paper, however, we not only discuss how each of the underlying factors influence expectations of future interest rates and risk premia across the term structure, but also provide a detailed decomposition of the forward curve into these component parts. Second, to the authors' knowledge, ours is the first paper to fit an affine macro-factor term structure model to bond yields in the United Kingdom. The existing literature, by contrast, has typically examined US data.

The paper is structured as follows. Section 2 sets out our affine macro-factor model of the yield curve. Section 3 goes on to describe the econometric approach we use for estimating the model. Section 4 provides a description of the data set. Section 5 presents and discusses our estimation results. Section 6 concludes.

2 Model

2.1 Model specification

We assume that the monetary authority sets the short-term interest rate y_t^1 by following a simple monetary policy rule (see Taylor (1993))

$$y_t^1 = r^* + \gamma' z_t \qquad \textbf{(1)}$$

where r^* corresponds to the equilibrium real interest rate, and where the vector of state factors z_t includes inflation π_t, a latent time-varying inflation target π_t^*, and a residual u_t ie $z_t = (\pi_t, \pi_t^*, u_t)'$. Moreover, we assume the coefficient vector to be $\gamma = (\gamma, [1 - \gamma], 1)'$. Combining these two assumptions, the monetary policy rule given in equation **(1)** has the following form, (this specification is also used by Rudebusch and Wu (2004))

$$y_t^1 = r^* + \pi_t^* + \gamma \left(\pi_t - \pi_t^*\right) + u_t \quad \textbf{(2)}$$

The monetary authority, therefore, moves the short-term interest rate in response to any deviation of inflation π_t from a time-varying target π_t^*. Note that the residual u_t will capture all features of interest rate setting behaviour that cannot be explained either by the deviation of inflation from target, or the time variation in the inflation target itself. For example, the residual will subsume the impact of deviations of output from trend on the conduct of monetary policy; any persistence in interest rate setting; and any time variation in the equilibrium real interest rate r^*.

Ideally, we would have a direct measure of the output gap in the Taylor rule (equation **(2)**). However, this is problematic because of two reasons. First, the output gap is unobservable. There are different approaches used in the literature to measure the output gap. Most of them are based on the difference of actual output relative to some benchmark level, ie the level of output that occurs when all wages and prices are flexible and adjust to balance supply and demand in all markets. Since this theoretical level of output is not directly observable, the benchmark is commonly interpreted to be the trend level of output. But there are many different ways to estimate trend output, and each can give conflicting signals about the size and the sign of the output gap. Second, the measurement problem and uncertainty about the output gap could explain the low and often insignificant response to output gap in empirically estimated Taylor rules. As shown by Smets (2002), theory suggests that output gap uncertainty reduces the response to the current estimated output gap relative to current inflation and may partly explain why the parameters in estimated Taylor rules are often much lower than suggested by optimal control exercises which assume the state of the economy is known. In addition, most output gap measures are based on quarterly data, while our estimation is based on monthly data. An interpolation from quarterly data could magnify the already inaccurate output gap proxies, biasing our estimation results. We therefore decided to only use the current inflation gap in the specification of our Taylor rule, which makes the policy rule residual more cyclical as it captures movements in the output gap as well as the monetary policy shock.

The state factors z_t are assumed to follow a first-order vector autoregression

$$z_{t+1} = \Phi z_t + \Omega^{1/2} \epsilon_{t+1} \tag{3}$$

Specifically, we assume that inflation π_t follows a first-order autoregressive process with a time-varying mean equal to the inflation target π_t^*. The time-varying inflation target π_t^* itself is assumed to follow a random walk. The motivation for this formulation stems from Kozicki and Tinsley (2001), who stress the importance of non-stationary long-run inflation in terms of understanding and modelling the yield curve. The random walk specification also provides us with a simple and effective way of allowing for (albeit gradual) structural changes in the inflation target and, therefore, the monetary regime. Finally, the residual u_t is assumed to follow a first-order autoregressive process with mean zero. In summary, our model parameterisation is given by

$$\begin{bmatrix} \pi_{t+1} \\ \pi_{t+1}^* \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & 1-\phi_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \phi_3 \end{bmatrix} \begin{bmatrix} \pi_t \\ \pi_t^* \\ u_t \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \\ \epsilon_{3,t+1} \end{bmatrix} \tag{4}$$

where we assume that the innovations ϵ_{t+1} are independently and normally distributed with mean zero and unit variance such that $\epsilon_{t+1} \sim \text{NID}(0, I)$ where I denotes the identity matrix.

2.2 Bond yields

Bond yields are computed recursively from the fundamental asset pricing equation

$$P_t^n = E_t \left[M_{t+1} P_{t+1}^{n-1} \right] \tag{5}$$

where P_t^n denotes the n-period zero-coupon bond price at time t. We complete our model with the following specification for the stochastic discount factor

$$M_{t+1} = \exp \left[-y_t^1 - \frac{1}{2} (\Lambda_t' \Omega \Omega' \Lambda_t) - (\Lambda_t' \Omega) \epsilon_{t+1} \right] \tag{6}$$

where Λ_t denotes the market prices of risk as these determine the covariance between shocks to the stochastic discount factor and the state factors. Following Duffee (2002) and Dai and Singleton (2000), we assume that the market prices of risk are affine in the state factors such that

$$\Lambda_t = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \begin{bmatrix} \pi_t \\ \pi_t^* \\ u_t \end{bmatrix} \quad \textbf{(7)}$$

In the empirical section of this paper we adopt two versions of our model. In one version we assume that the (non-pure) expectations hypothesis holds and, therefore, that risk premia are constant through time.(1) This requires the restriction that $\beta_{ij} = 0$ for all i and j. In the other version of our model we allow risk premia to be time varying. For this case λ_i and β_{ij} are left unrestricted.

Assuming that bond yields are affine in the state factors, bond prices and the stochastic discount factor are jointly lognormal such that

$$p_t^n = E_t\left[m_{t+1} + p_{t+1}^{n-1}\right] + \frac{1}{2}Var_t\left[m_{t+1} + p_{t+1}^{n-1}\right] \quad \textbf{(8)}$$

where lower-case letters represent logarithms eg $p_t^n = \ln(P_t^n)$. With this assumption in place we show in Appendix A that bond yields are indeed affine in the state factors such that

$$y_t^n = n^{-1}(A_n + B_n^{'} z_t) \quad \textbf{(9)}$$

where y_t^n measures the n-period zero-coupon yield at time t. The recursive intercept and slope coefficients are given by

$$A_n = r^* + A_{n-1} - \frac{1}{2}\left[2\lambda'\Omega B_{n-1} + B_{n-1}'\Omega B_{n-1}\right] \quad \textbf{(10)}$$

$$B_n' = \gamma' + B_{n-1}'\left(\Phi - \Omega\beta\right) \quad \textbf{(11)}$$

with initial conditions $A_0 = 0$ and $B_0' = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.

2.3 Decomposition of the forward curve

From the viewpoint of monetary policy, the forward curve, as opposed to the spot yield curve, is of particular interest because it directly conveys information about market expectations of future interest rates. In practice, however, risk premia (and at longer horizons the convexity effect) are

(1) See Campbell, Lo and MacKinlay (1997) for a discussion of the various forms of the expectations hypothesis.

likely to distort the observed forward curve away from market expectations of future interest rates. Hence, a theoretical understanding and an empirical examination of these effects are relevant from the perspective of a central bank.

Forward rates are defined from spot rates by the relation

$$f_t^n = (n+1)\, y_t^{n+1} - n y_t^n \qquad \textbf{(12)}$$

The affine yield curve model allows us to decompose the forward curve into interest rate expectations, risk premia and a convexity effect, such that

$$f_t^n = E_t\left[y_{t+n}^1\right] + \phi_{t,n} + \omega_{t,n} \qquad \textbf{(13)}$$

where $E_t\left[y_{t+n}^1\right]$ is the expected future short rate n periods ahead, $\phi_{t,n}$ is the risk premium in the forward curve at maturity n, and $\omega_{t,n}$ is the convexity effect at maturity n. In Section 2.3.1 we explain the intuition of this decomposition, and in Section 2.3.2 we show how to compute the individual components of the forward curve in equation **(13)** from an empirical estimate of the yield curve model.

2.3.1 Intuition

In the standard representative agent model, the nominal stochastic discount factor M_{t+1} given by equation **(6)**, measures the marginal rate of nominal substitution between time t and $t+1$. Thus,

$$M_{t+1} = \frac{\delta U'(C_{t+1})}{U'(C_t)} \frac{Q_t}{Q_{t+1}} \qquad \textbf{(14)}$$

where δ is the time discount factor, $U(\cdot)$ is the utility function, C_t measures real consumption at time t, and Q_t measures the price level at time t. To provide some intuition, we first derive an expression for the expected one-period return on an n-period zero-coupon bond from our affine yield curve model. This is given by[(2)]

$$E_t\left(p_{t+1}^{n-1} - p_t^n\right) = \underbrace{y_t^1}_{\text{One-period risk-free rate}} \underbrace{-Cov_t(m_{t+1}, p_{t+1}^{n-1})}_{\text{Risk premium}} - \underbrace{\frac{Var_t\left(p_{t+1}^{n-1}\right)}{2}}_{\text{Convexity effect}} \qquad \textbf{(15)}$$

(2) Equation **(15)** can be derived from equation **(8)**, and makes use of the fact that M_{t+1} and P_{t+1}^{n-1} are jointly lognormally distributed as of time t.

where p_t^n denotes the log price of a zero-coupon bond at time t with n periods to maturity, and m_{t+1} is the log stochastic discount factor at time $t+1$. It is worth stressing that the affine yield curve model as such does not allow for a rich interpretation of this expression, but if we think of our formulation for the stochastic discount factor in equation **(6)** as a reduced-form version of the nominal marginal rate of substitution in equation **(14)**, the expression above is quite intuitive. Buying an n-period bond at time t and selling it off at time $t+1$ is a risky investment because the price of an $(n-1)$-period bond at time $t+1$ is uncertain. Equation **(15)** decomposes the expected return into three components. The first component is the risk-free rate. The second component is a risk premium. If the log stochastic discount factor and the log price of the bond are positively (negatively) correlated, the investor requires a relatively low (high) expected return on the bond because of the property that it has a high (low) price in bad states of the world. In the affine yield curve model this covariance term boils down to

$$\begin{aligned}
Cov_t(m_{t+1}, p_{t+1}^{n-1}) &= Cov_t\left(-\left(r^* + \gamma' z_t\right) - \frac{\Lambda_t' \Omega \Lambda_t}{2} - \Lambda_t' \Omega^{1/2} \epsilon_{t+1}, -A_{n-1} - B_{n-1}' z_{t+1}\right) \\
&= Cov_t\left(\left(r^* + \gamma' z_t\right) + \frac{\Lambda_t' \Omega \Lambda_t}{2} + \Lambda_t' \Omega^{1/2} \epsilon_{t+1}, A_{n-1} + B_{n-1}'\left(\Phi z_t + \Omega^{1/2} \epsilon_{t+1}\right)\right) \\
&= Cov_t\left(\Lambda_t' \Omega^{1/2} \epsilon_{t+1}, B_{n-1}' \Omega^{1/2} \epsilon_{t+1}\right) \\
&= \Lambda_t' \Omega B_{n-1} \\
&= \left(\lambda + \beta z_t\right)' \Omega B_{n-1} \qquad \textbf{(16)}
\end{aligned}$$

where we used equations **(3)**, **(6)**, and **(9)** and the fact that $p_{t+1}^{n-1} = -n y_{t+1}^{n-1}$. It is clear from this expression that Λ_t (ie β and λ) governs risk premia in the sense that λ controls the constant part of risk premia and β controls the time-varying part. Due to the close link between risk premia on holding period returns and risk premia in spot and forward yields, see eg Svensson (1993), expressions for risk premia in spot yields and forward yields could be derived. They are not as simple as the expression above for the holding period return and are therefore not reported here. The important point, though, is that even if risk premia are measured in spot rates or in forward rates, λ and β will continue to govern the constant and time-varying part of risk premia, respectively.

Even if the representative agent were risk-neutral (ie λ and β were zero matrices), expected returns on bonds with different maturities would not equalise due to the presence of the convexity term in equation **(15)**. To illustrate this effect, it is useful to consider expected holding-period returns on two bonds. The first bond is a one-period bond, so there is no uncertainty about its price tomorrow. At time $t+1$, it will mature and pay back 1 unit of account, so

$$P_{t+1}^{0} = 1$$

The other bond is an n-period bond and, therefore, its price at time $t+1$ will be uncertain. The price of the bond at time t is P_t^n and its expected price next period is $E_t\left(P_{t+1}^{n-1}\right)$. Due to risk-neutrality, the relative prices of these two bonds are

$$\frac{P_t^1}{P_t^n} = \frac{1}{E_t\left(P_{t+1}^{n-1}\right)}$$

because risk-neutral investors value assets in terms of *expected pay-offs*. Re-arranging and taking natural log yields

$$\ln E_t\left(P_{t+1}^{n-1}\right) - \ln P_t^n = -\ln P_t^1 \qquad \textbf{(17)}$$

Whenever there is uncertainty about P_{t+1}^{n-1} as of time t, it follows from Jensen's inequality that

$$E_t\left(p_{t+1}^{n-1}\right) = E_t\left(\ln\left(P_{t+1}^{n-1}\right)\right) < \ln\left(E_t\left(P_{t+1}^{n-1}\right)\right) \qquad \textbf{(18)}$$

Hence, equation **(17)** can be rewritten as

$$E_t\left(p_{t+1}^{n-1} - p_t^n\right) < \ln E_t\left(P_{t+1}^{n-1}\right) - \ln P_t^n = -\ln P_t^1 = -p_t^1 \qquad \textbf{(19)}$$

In words, under risk-neutrality the stochastic nature of future bond prices drive expected holding-period returns on multi-period bonds below the one-period rate. One can show that this effect increases with the variability of bond prices. In a deterministic world where future bond prices are known, the expression in equation **(19)** would collapse to

$$E_t\left(p_{t+1}^{n-1} - p_t^n\right) = -p_t^1 \qquad \textbf{(20)}$$

Consequently, it follows directly from equations **(19)** and **(20)**, that under risk-neutrality, the expected holding-period return on the n-period bond is driven below the risk-free rate (the one-period rate) if future bond prices are uncertain at time t and due to the fact that there is a non-linear relationship between future bond prices and holding-period returns. Note that this effect is independent of risk preferences (because the example above assumed a risk-neutral investor). It can be shown that this convexity effect in holding-period returns carries over to the forward curve in the sense that the convexity effect drives a wedge between the forward curve and the term structure of expected future one-period interest rates, even under risk-neutrality.

2.3.2 Computations

In the affine yield curve model, the spot yield curve, equation **(9)**, is given by

$$y_t^n = n^{-1}(A_n + B_n^{'} z_t) \qquad \textbf{(21)}$$

where A_n and B_n are functions of the estimated parameters, and time series for the underlying factors z_t are a by-product of the estimation procedure. The model-implied forward rate curve is computed by transforming the spot curve above into forward space, see equation **(12)**, and inserting equation **(21)**, which gives the forward yield curve

$$\begin{aligned} f_t^n &= A_{n+1} + B_{n+1}' z_t - \left(A_n + B_n' z_t\right) \\ &= (A_{n+1} - A_n) + \left(B_{n+1}' - B_n'\right) z_t \end{aligned} \qquad \textbf{(22)}$$

where A_n and B_n' were defined in equations **(10)** and **(11)**.

We define the risk-neutral forward yield curve as the yield curve that would prevail if investors did not price risk (ie λ and β are equal to zero matrices) and all other parameters remain unchanged. The risk-neutral forward curve is therefore computed as

$$f_t^n\big|_{\lambda=0,\beta=0} = (A_{n+1} - A_n)|_{\lambda=0,\beta=0} + \left(B_{n+1}' - B_n'\right)\big|_{\lambda=0,\beta=0} z_t \qquad \textbf{(23)}$$

where the notation is supposed to indicate that the A_n's and B_n's are computed from the recursions in equations **(10)** and **(11)** with the restriction that $\lambda = 0$ and $\beta = 0$, which correspond to the recursions below

$$\begin{aligned} A_n &= r^* + A_{n-1} - \frac{1}{2} B_{n-1}' \Omega B_{n-1} \\ B_n' &= \gamma' + B_{n-1}' \Phi \end{aligned} \qquad \textbf{(24)}$$

Risk premia in the forward curve are therefore computed as

$$\phi_{t,n} = f_t^n - f_t^n\big|_{\lambda=0,\beta=0} \quad \textbf{(25)}$$

The intuition is that forward rate risk premia can be computed as the difference between the forward curve and an artificial forward curve computed as if investors were risk-neutral.

The convexity effect term from equation **(13)** is computed as the difference between the risk-neutral forward curve from equation **(23)** and a forward curve computed as if investors were risk-neutral and as if future bond prices were deterministic. As explained above, risk-neutrality corresponds to imposing that λ and β are zero matrices. The additional assumption that future bond prices are deterministic is imposed by setting the variance-covariance matrix of the state variables Ω, see equation **(3)**, to a zero matrix. Consequently, the convexity effect in the forward curve is computed as

$$\omega_{t,n} = f_t^n\big|_{\lambda=0,\beta=0} - f_t^n\big|_{\lambda=0,\beta=0,\Omega=0} \quad \textbf{(26)}$$

The intuition is that even under risk-neutrality, the convexity effect drives a wedge between the forward curve and expectations of future interest rates due to the stochastic nature of future bond prices. This effect is independent of risk preferences and is therefore computed under risk-neutrality.

We can now back out the term structure of expected future interest rates by combining equations **(13)**, **(25)**, and **(26)**

$$\begin{aligned} E_t\left[y_{t+n}^1\right] &= f_t^n - \phi_{t,n} - \omega_{t,n} \\ &= f_t^n - \left(f_t^n - f_t^n\big|_{\lambda=0,\beta=0}\right) - \ldots \\ &\ldots - \left(f_t^n\big|_{\lambda=0,\beta=0} - f_t^n\big|_{\lambda=0,\beta=0,\Omega=0}\right) \\ &= f_t^n\big|_{\lambda=0,\beta=0,\Omega=0} \end{aligned} \quad \textbf{(27)}$$

3 Econometric approach

In the literature on yield curve modelling with macro factors, there are two econometric approaches. The first approach, based on Chen and Scott (1993), assumes that a subset of yield data are measured with error and the rest of the observed yields are measured without error. Although the method is easy to implement, the distinction between yields measured with and without error is hard to justify on economic grounds and in practice it becomes arbitrary.

Consequently we adopt the alternative approach which assumes that all yields and macro variables are measured with error. This estimation method is implemented by the Kalman Filter (see Chapter 13 in Hamilton (1994)). The state equation, from equation **(3)**, is given by

$$z_t = \Phi z_{t-1} + \Omega^{1/2}\epsilon_t$$

where

$$z_t = \begin{bmatrix} \pi_t \\ \pi_t^* \\ u_t \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \phi_{11} & 1-\phi_{11} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \phi_{33} \end{bmatrix}, \qquad \Omega = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}$$

The observation equation is

$$\begin{bmatrix} \pi_t \\ y_t^3 \\ y_t^{12} \\ y_t^{24} \\ y_t^{36} \\ y_t^{60} \\ y_t^{120} \\ y_t^{180} \end{bmatrix} = \begin{bmatrix} 0 \\ A_3^* \\ A_{12}^* \\ A_{24}^* \\ A_{36}^* \\ A_{60}^* \\ A_{120}^* \\ A_{180}^* \end{bmatrix} + \begin{bmatrix} 1 \quad 0 \quad 0 \\ B_3^{*\prime} \\ B_{12}^{*\prime} \\ B_{24}^{*\prime} \\ B_{36}^{*\prime} \\ B_{60}^{*\prime} \\ B_{120}^{*\prime} \\ B_{180}^{*\prime} \end{bmatrix} z_t + \begin{bmatrix} \upsilon_{1,t} \\ \upsilon_{2,t} \\ \upsilon_{3,t} \\ \upsilon_{4,t} \\ \upsilon_{5,t} \\ \upsilon_{6,t} \\ \upsilon_{7,t} \\ \upsilon_{8,t} \end{bmatrix} \quad \textbf{(28)}$$

where z_t is a 3×1 matrix and the distribution of measurement errors is

$$\begin{bmatrix} \upsilon_{1,t} \\ \upsilon_{2,t} \\ \upsilon_{3,t} \\ \upsilon_{4,t} \\ \upsilon_{5,t} \\ \upsilon_{6,t} \\ \upsilon_{7,t} \\ \upsilon_{8,t} \end{bmatrix} \sim NID \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \eta^2 I_8 \right) \quad \textbf{(29)}$$

So the vector of measurement errors on the yields and inflation data in equation **(28)** is homoskedastic, ie all measurement errors have the same variance of η^2.

In line with most of the literature on estimating Taylor rules, and due to the large number of estimated parameters, we fix the equilibrium real interest rate r^* from the Taylor rule, equation **(2)**, in our case to 3%.(3) This results in 19 free parameters to estimate for the time-varying risk premium (TRP) model

$$\gamma_1, \phi_{11}, \phi_{33}, \sigma_1^2, \sigma_2^2, \sigma_3^2, \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}, \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix}, \eta^2$$

and 10 parameters for the constant risk premium (CRP) model

$$\gamma_1, \phi_{11}, \phi_{33}, \sigma_1^2, \sigma_2^2, \sigma_3^2, \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}, \eta^2$$

Optimisation of the likelihood function is implemented in GAUSS.(4) Parameter estimates were found to depend on the starting values of the algorithm, which is far from unusual when estimating yield curve models. In practical terms, we generated random starting values by drawing uniform numbers from the constraints specified for individual parameters in the optimisation routine. After approximately 70 estimations with random starting values we picked the estimation with the highest likelihood value. Fortunately, it appeared that several draws of random starting values converged on the optimal parameter estimates which indicates that our results are not sensitive to the choice of starting values. Further details are available on request.

4 Data

Our sample period runs from January 1975 to May 2004, yielding a total of 353 monthly observations. We estimate our models using end-month spot UK interest rates with maturities of 3 months, 1, 2, 3, 5, 10, and 15 years. The 1, 2, 3, 5, 10, and 15-year spot yields are extracted from the Bank of England yield curve data set.(5) These yield curve data are estimated by fitting a spline through general collateral (GC) repo rates and conventional government bonds. Unfortunately, due to lack of a repo market before 1997, the data set does not contain yields with

(3) This was the sample average for the *ex-post* one-period real rate.
(4) GAUSS Version 6 with the Constrained Maximum Likelihood Estimation package Version 2.
(5) See Anderson and Sleath (1999). These data are available at the Bank of England website www.bankofengland.co.uk.

maturities less than one year on a regular basis before 1997. We therefore adopt three-month Treasury bills from Datastream as our short rate.[6] The inflation measure is annual RPI inflation. Table A displays some summary statistics for the data set. The spot yield curve is on average downward sloping from three months out to one year, upward sloping from one to ten years, and slightly downward sloping from ten to fifteen years. Inflation is more volatile than any of the spot yields. To get a feel for the Treasury bill data, we select all calendar days since 1 January 1975, where three-month spot rates are available from the Bank of England yield curve data set. For these days, Chart 1 provides a cross-plot between three-month spot rates measured by Treasury bills and the Bank of England data set, respectively. The chart shows that the two series are highly correlated and it indicates that Treasury bills provide a reasonable measure of a risk-free three-month interest rate. From Table A, note that the shapes of the mean and the median spot yield curves differ and in particular that the short end of the average yield curve is downward sloping on average, and upward sloping according to the median yield curve. This probably reflects the non-normal distribution of spot yields in the sense that measures of skewness and excess kurtosis are different from zero.

Table B provides instantaneous correlations for the variables in the data set. The spot yields are in general positively correlated. Inflation and spot yields are also highly correlated and this effect becomes stronger as we look along the yield curve. It seems as if inflation is related to the yield curve, in particular at the long end.

Our data on spot yields and RPI inflation are in per cent. To transform these annualised data into monthly yields, we divide the data by 1200, so for example a ten-year spot yield of 7.14% is transformed to 0.00595.

5 Results

The estimation results for the constant (CRP) and time-varying risk premia (TRP) models are presented in Tables D and E respectively. The feedback from deviations in inflation from target to the short-term interest rate is given by the γ_1 parameter. In both models this coefficient is significantly below unity, which is consistent with previous work on estimating Taylor rules for the United Kingdom. For instance, Nelson (2000) finds that over the period 1972-97 the feedback

(6) Datastream code: LDNTB3M. These data are from FT. According to FT, UK Treasury bill rates are from RBS/Reuters.

coefficient on inflation was close to 0.2. In contrast, Taylor (1999) showed theoretically under reasonably general assumptions that it needs to be larger than one in order to ensure that inflation on average is kept on target. If this result extends to our assumption of a time-varying inflation target, then the estimates of γ_1 in Tables D and E look quite low.[7] One possible reason is that, up until adoption of inflation targeting in October 1992, interest rates were set with reference to other economic objectives, such as targeting a variety of monetary aggregates or the exchange rate, which may have diluted the response of short-term interest rates to the level of inflation.

Moving on to the factor equations we find that the inflation and policy rule residual factors are strongly persistent. This is unsurprising given the results in Table C, which show that both inflation and the yield data were highly autocorrelated. The conditional volatility of the factors are given by the sigma parameters. Intuitively, we find that in both models the volatility of the unobserved time-varying inflation target is much less pronounced than that for inflation and the policy rule residual.

In the CRP model the lambda estimates govern the market prices of risk. Note that a positive (negative) value for λ_i corresponds to a positive (negative) covariance between the stochastic discount factor and bond prices, and thus a negative (positive) risk premium. In this model we find that all the lambdas except that related to the unobserved inflation target, λ_2, are significant, and moreover that they are everywhere positive. Note that the insignificant value for λ_2 suggests that agents did not require compensation for the risk attached to shocks to the target. In Section 5.2, we are going to show that the inflation target mainly affects the level of the yield curve. The interpretation, therefore, is that agents did not require compensation for shocks to the level of the yield curve. In the TRP model the market prices of risk are also a function of the state variables via the beta parameters. Once we allow for time-varying risk premia we find that all the lambdas are rendered insignificant. By contrast, a number of the beta parameters, specifically $(\beta_{31}, \beta_{32}, \beta_{13})$, are found to have t-statistics larger than one in absolute value. We therefore impose a set of restrictions on the matrix of beta parameters such that only β_{13}, β_{31} and β_{32} are estimated in order to reduce the dimensionality of the model. The results of the restricted TRP model are presented in Table F. In general, the model parameters are broadly unchanged with the exception of the lambda related to inflation which becomes significantly different from zero. The remaining beta

(7) Taylor (1999) assumed that the inflation target was constant over time.

parameters turn out to be highly significant with t-values exceeding 9 in absolute values.(8)

The parameter estimates of measurement error variance, η^2, in Tables E and F correspond to standard deviations of 27-30 basis points for annualised yields/inflation.(9) Charts 2 and 3 show plots of the actual and fitted yields. Both models provide a close fit of the data, particularly inflation, which is consistent with the relatively low estimates of measurement variance, η^2.

The assumption that the measurement variances on bond yields and inflation are identical, see equations **(28)** and **(29)**, was made to keep the model simple and parsimonious. To explore robustness to this simplifying assumption, we re-estimated the preferred restricted TRP model with two measurement variances, one for bond yields and one for inflation. The results turned out to be almost identical to Table F.(10) The measurement variance on bond yields remains at the level of η^2 in Table F, the measurement variance on inflation falls below that level and all other parameters remain roughly unchanged. In other words, our empirical results seem to be robust to alternative specifications of measurement errors.

5.1 Factor evolutions

Chart 4 shows the evolution of the three factors generated in each of our models. The factors correspond to inflation, and the two unobservable factors z_{2t} and z_{3t}, the first of which we interpret as a time-varying inflation target, or a measure of long-run inflation expectations. The considerable fall in inflation over the period is clearly evident from the chart. More encouragingly, the unobservable factor z_{2t} in each model appears to correlate well with notions of a time-varying inflation target. In particular, since the early 1990s z_{2t} has declined gradually from a little over 6% before stabilising at around 2% since 1998. In fact, across the three models there is little discernable difference between each of the factor evolutions. Further evidence on the link between z_{2t} and measures of long-run inflation is provided by Chart 5. The chart shows that, particularly for both the TRP models, the estimated time-varying inflation factor has moved closely with measures of long-run inflation expectations derived both from index-linked debt and from surveys.

---

(8) Note that $\beta_{13} \approx -\beta_{32}$. The restriction $\beta_{13} = -\beta_{32}$ would imply that risk premia were stationary due to the cointegrating vector of $(1, -1)$ between $z_{1,t}$ and $z_{2,t}$.We have estimated the model with the restriction $\beta_{13} = -\beta_{32}$ and the estimates are almost identical to when β_{13} and β_{32} are unrestricted.

(9) For Table D, $1200 \cdot \sqrt{0.61 \cdot 10^{-7}} \approx 0.30$, and for Table E, $1200 \cdot \sqrt{0.51 \cdot 10^{-7}} \approx 0.27$.

(10) Estimation results are available upon request.

We have also experimented with a four-factor extension of the TRP model, where the fourth factor only drives risk premia. Overall, inclusion of a fourth factor did not materially improve the model so the results are not reported here.[(11)]

5.2 Impulse responses

Chart 6 shows how shocks to the factors affect the forward yield curve. More specifically, each curve plots the reaction of the forward curve to a one unit standard deviation shock to each of the factors. In the CRP model we find that inflation and the unobservable factor z_{3t} primarily affect the short end of the yield curve. But whereas the inflation loading dies away relatively quickly towards zero, the more persistent nature of z_{3t} means that it also affects yields out to ten years. Finally, and consistent with our *a priori* expectations, the unobservable time-varying inflation target z_{2t} predominately affects the longer end of the forward curve. The inferences from the restricted TRP model are somewhat similar for the z_{2t} and z_{3t} factors. But the model suggests that the influence of inflation is modest, and perhaps counterintuitively is suggests that positive shocks to inflation result in a fall in forward yields. The unrestricted TRP model stands out from the other two models. This model has positive shocks to inflation driving yields down at the short end of the curve, but pushing yields higher at the long end. This curvature effect is also seen for the Taylor rule residual. But here the opposite is found, with shocks to the z_{3t} factor having a positive effect on short yields and a growing negative influence on the longer end of the curve.

The forward curve can be decomposed into interest rate expectations, forward rate risk premia and a convexity effect, see equation **(13)**. Charts 7 and 8 decompose the forward rate responses for the TRP and restricted TRP model, see Chart 6, into changes in expectations and risk premia.[(12)] In Charts 7 and 8, the top figures show the reaction in interest rate expectations to a shock in each of the three factors, while the bottom figures show the equivalent response in forward rate risk premia. In the restricted TRP model we noted above that an inflation shock drives down the forward curve at maturities greater than a couple of months ahead. Chart 8 sheds further light on this possibly counterintuitive result. It appears this fall in the forward curve covers a slight rise in

(11) Not surprisingly, this four-factor model fits data very well in the sense that estimates of the measurement error variance, η^2, corresponds to standard deviations of 18 basis points for annualised yields/inflation. However, some parameter estimates and the implied factor evolutions were not appealing. Further results are available upon request.
(12) The profile of the convexity effect is increasing with maturity, but does not change over time, due to our assumption of constant volatilities. Consequently, changes in the convexity effect do not contribute to forward rate changes.

interest rate expectations and a larger fall in risk premia. Hence the negative impact of an inflation shock on the forward curve in the restricted TRP model makes more sense because, due to the persistent nature of inflation, interest rate expectations do rise in the face of an inflation shock. It is an open question whether the negative relation between inflation shocks and forward rate risk premia is sensible. Economic theory has little to say about this.

More generally, Charts 7 and 8 also shed light on the relative importance of changes in expectations and changes in risk premia on yield curve movements. For example, these charts indicate that the time-varying inflation target factor, z_{2t}, impacts the long end of the yield curve primarily via the expectations channel.[13] In other words, a majority of movements at the long end of the forward curve are driven by shocks to the time-varying inflation target which shifts interest rate expectations.

Charts 7 and 8 reveal a marked difference between the TRP and the restricted TRP model in the sense that positive shocks to the Taylor rule residual drive long risk premia down and up, respectively, in the two models. Our priors are that if long forward risk premia do vary over the cycle, they should be procyclical. In the absence of other discriminatory evidence, this leads us to select the restricted TRP as the preferred model.

5.3 Risk premia

Chart 9 plots the estimates of the risk premium for each of our models. In the CRP model the estimate of the constant risk premium at the one-year horizon is close to zero, but the estimate rises to around 1.2 percentage points at the ten-year horizon. In the TRP models the risk premium is allowed to vary across time. Chart 9 shows that the time profile for the estimated risk premium in the two TRP models are broadly similar for the one-year horizon. Both models suggest that the risk premium was relatively more volatile in the earlier part of the sample. Thereafter, the risk premium appears to have fluctuated gently around a positive mean. To provide a cross-check on our one-year risk premium estimates, Chart 10 plots the time-varying risk premium from the TRP and the restricted TRP model together with a measure of the risk premium from a survey that was conducted by Merrill Lynch up until January 2001. This measure equals the difference between the two-week government forward rate one year ahead and the survey expectation of the policy

(13) Specifically, because the impulse responses for the long-run inflation factor in the upper panels of Charts 7 and 8 exceed those in the lower panels.

rate at the same horizon. It is clear that there is a close link between the survey-based and model-based risk premia from the two models and the model-based risk premia appear to provide a smoothed version of the survey-based measure. Over the period the correlations between survey and model-based measures are quite high, at around 0.7. Perhaps more importantly, the model-based measures capture each of the observed turning points in the survey-based risk premium.

Chart 9 also shows the time-varying risk premium generated by the TRP models at the ten-year horizon. The general picture is that they have come down over time, possibly due to increased credibility of the Bank of England and/or reduced macroeconomic uncertainty.

5.4 Forward curve decomposition

Chart 11 shows a decomposition of the fitted forward curve as at May 2004 into its three main determinants: expectations of the future short-term interest rate; a risk premium; and the convexity effect. Note that in each model the convexity effect is constant across time, but that the effect is somewhat larger in the CRP model.(14) In the CRP model the profile for risk premia is also constant across time. However, the risk premia profiles in the TRP models vary through time. Our preferred restricted TRP model, Chart 11b, suggests that risk premia were relatively low in May, bounded between ± 0.5 percentage points across the forward curve. Despite the differences in convexity and risk premium estimates across the models, the implications for expectations of future short-term interest rates are broadly similar. In each model, the profile suggests that expectations in May were for interest rates to rise modestly from a little under 5% to around that level by the fifteen-year horizon.

6 Conclusion

Using a no-arbitrage affine yield curve framework, where the underlying factors have a macroeconomic interpretation, we have presented three empirical yield curve models for the United Kingdom. A general result is that the long end of the yield curve is primarily driven by changes in the unobserved inflation target. The short end of the yield curve reflects movements in short-run inflation and other factors subsumed in the Taylor rule residual.

(14) It can be shown that the convexity effect is time-invariant and depends only on Ω and Φ in our models.

The preferred yield curve model implies that both instantaneous inflation risk and instantaneous Taylor rule residual risk are priced, whereas instantaneous risk associated with changes in the time-varying inflation target is not priced. This seems intuitive, because for example, in a simple representative agent model with log utility, the stochastic discount factor is given by nominal consumption growth. It seems likely that nominal monthly consumption growth is linked to shocks to annual inflation and the Taylor rule residual. But it is far less obvious why monthly shocks to the inflation target would be correlated with monthly nominal consumption growth. However, at horizons beyond one period, investors take into account all three sources of risk because future inflation risk is partly made up of inflation-target risk.

Our empirical yield curve models can be used to back out a path for an unobserved time-varying inflation target within a simple backward-looking Taylor rule. This path is shown to be closely linked to other measures of long-run inflation expectations, such as a ten year ahead breakeven inflation rate and long-run Consensus forecasts of inflation.

The paths for forward rate risk premia derived from our models with time-varying risk premia are shown to be highly correlated with survey-based risk premia over the period June 1990 to January 2001. This is comforting, and surprising, because we have not used survey data for estimation or even model selection. As such, it provides support for the estimated models.

Our model also allows us to generate an estimate of the convexity effect across the yield curve, although this effect is restricted to be constant across our sample period. Our results suggest that the convexity effect rises monotonically to around 1.0 to 1.5 percentage points by the fifteen-year horizon. However, it is unlikely that this estimate is representative of the current convexity effect, as it is probably biased upwards due to the relatively high volatility of interest rates in the 1970s and early 1980s.

The results of the paper are subject to at least one caveat. The parameter γ_1 in our Taylor rule is assumed to be constant. Due to the fact that our sample period covers several monetary policy regimes, it would be ideal to let γ_1 vary over time. As mentioned at the beginning of Section 5, this property of the model might bias the estimate of γ_1 downwards.

Potentially, there are several useful extensions to the framework of this paper. In particular, it would be useful to introduce a real/nominal split in the model, which would allow us to decompose nominal forward rates into real forward rates, inflation expectations and inflation risk premia. For example, this would allow us to gauge the importance of inflation risk premia at the long end of the yield curve. We have only had limited success with estimating the model using output gap data. But given the importance of the output gap in standard Taylor rule specifications, it might be useful to consider alternative output gap series. Finally, our assumption of constant volatilities for inflation, the inflation target, and the Taylor rule residual appears at odds with the perceived fall in macroeconomic uncertainty over the sample period. Relaxing this assumption would make our risk premium specification more flexible and allow for a time-varying convexity effect.

Appendix A: Derivation of yield curve model

The state variables are governed by equations **(3)** and **(4)**. The stochastic discount factor from equation **(6)**, is rewritten in logs below

$$\ln(M_{t+1}) = m_{t+1} = -y_t^1 - \frac{1}{2}(\Lambda_t'\Omega\Lambda_t) - (\Lambda_t'\Omega^{1/2})\epsilon_{t+1} \quad \textbf{(A-1)}$$

The coefficients in equation **(9)** are defined by the recursions in equations **(10)** and **(11)**:

$$A_n = r^* + A_{n-1} - \frac{1}{2}\left[2\lambda'\Omega B_{n-1} + B_{n-1}'\Omega B_{n-1}\right] \quad \textbf{(A-2)}$$

$$B_n' = \gamma' + B_{n-1}'(\Phi - \Omega\beta) \quad \textbf{(A-3)}$$

$$A_0 = 0 \text{ and } B_0' = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad \textbf{(A-4)}$$

We assume that log bond prices, p_t^n, are linear in the state variables

$$\begin{aligned} -\ln P_t^n &= -p_t^n = \underset{1\times 1}{A_n} + \underset{1\times k k\times 1}{B_n' \, z_t} \\ B_n' &= \begin{bmatrix} B_{n,1} & B_{n,2} & B_{n,3} \end{bmatrix} \end{aligned} \quad \textbf{(A-5)}$$

From **(A-5)**, yields are linear in state variables

$$y_t^n = n^{-1}(A_n + B_n z_t)$$

To verify the recursions **(A-2)** and **(A-3)**, rewrite equation **(5)** in log form, using joint log-normality of M_{t+1} and P_{t+1} conditional on the information set at time t

$$p_t^n = E_t\left[m_{t+1} + p_{t+1}^{n-1}\right] + \frac{1}{2}Var_t\left[m_{t+1} + p_{t+1}^{n-1}\right]$$

and substitute in equations **(A-1)**, **(A-5)**, **(A-2)**, and **(A-3)**, to get (the following computations assumes that the reader is familiar with solving affine yield curve models, see eg Backus *et al* (1999))

$$
\begin{aligned}
-p_t^n &= -E_t\left[m_{t+1}+p_{t+1}^{n-1}\right]-\frac{1}{2}Var_t\left[m_{t+1}+p_{t+1}^{n-1}\right] \\
&= E_t\left[r^*+\gamma' z_t+\frac{(\lambda+\beta z_t)'\,\Omega\,(\lambda+\beta z_t)}{2}+(\lambda+\beta z_t)'\,\Omega^{1/2}\epsilon_{t+1}+A_{n-1}+B_{n-1}^{'}z_{t+1}\right] \\
&\quad -\frac{1}{2}Var_t\left[r^*+\gamma' z_t+\frac{(\lambda+\beta z_t)'\,\Omega\,(\lambda+\beta z_t)}{2}+(\lambda+\beta z_t)'\,\Omega^{1/2}\epsilon_{t+1}+A_{n-1}+B_{n-1}^{'}z_{t+1}\right] \\
&= r^*+\gamma' z_t+\frac{(\lambda+\beta z_t)'\,\Omega\,(\lambda+\beta z_t)}{2}+A_{n-1}+B_{n-1}^{'}E_t\left[z_{t+1}\right] \\
&\quad -\frac{1}{2}Var_t\left[(\lambda+\beta z_t)'\,\Omega^{1/2}\epsilon_{t+1}+B_{n-1}^{'}z_{t+1}\right] \\
&= r^*+\gamma' z_t+\frac{(\lambda+\beta z_t)'\,\Omega\,(\lambda+\beta z_t)}{2}+A_{n-1}+B_{n-1}^{'}\Phi z_t \\
&\quad -\frac{1}{2}Var_t\left[(\lambda+\beta z_t)'\,\Omega^{1/2}\epsilon_{t+1}+B_{n-1}^{'}\Omega^{1/2}\epsilon_{t+1}\right] \\
&= r^*+\gamma' z_t+\frac{(\lambda+\beta z_t)'\,\Omega\,(\lambda+\beta z_t)}{2}+A_{n-1}+B_{n-1}^{'}\Phi z_t \\
&\quad -\frac{1}{2}Var_t\left[\left((\lambda+\beta z_t)'\,\Omega^{1/2}+B_{n-1}^{'}\Omega^{1/2}\right)\epsilon_{t+1}\right] \\
&= r^*+\gamma' z_t+\frac{(\lambda+\beta z_t)'\,\Omega\,(\lambda+\beta z_t)}{2}+A_{n-1}+B_{n-1}^{'}\Phi z_t \\
&\quad -\frac{1}{2}\left((\lambda+\beta z_t)'\,\Omega^{1/2}+B_{n-1}^{'}\Omega^{1/2}\right)\left((\lambda+\beta z_t)'\,\Omega^{1/2}+B_{n-1}^{'}\Omega^{1/2}\right)' \\
&= r^*+\gamma' z_t+\frac{(\lambda+\beta z_t)'\,\Omega\,(\lambda+\beta z_t)}{2}+A_{n-1}+B_{n-1}^{'}\Phi z_t \\
&\quad -\frac{1}{2}\left((\lambda+\beta z_t)'\,\Omega^{1/2}+B_{n-1}^{'}\Omega^{1/2}\right)\left(\Omega^{1/2}(\lambda+\beta z_t)+\Omega^{1/2}B_{n-1}\right) \\
&= r^*+\gamma' z_t+\frac{(\lambda+\beta z_t)'\,\Omega\,(\lambda+\beta z_t)}{2}+A_{n-1}+B_{n-1}^{'}\Phi z_t \\
&\quad -\frac{1}{2}(\lambda+\beta z_t)'\,\Omega\,(\lambda+\beta z_t)-(\lambda+\beta z_t)'\,\Omega B_{n-1}-\frac{1}{2}B_{n-1}^{'}\Omega B_{n-1} \\
&= r^*+\gamma' z_t+A_{n-1}+B_{n-1}^{'}\Phi z_t-B_{n-1}^{'}\Omega\,(\lambda+\beta z_t)-\frac{1}{2}B_{n-1}^{'}\Omega B_{n-1}
\end{aligned}
$$

the left-hand side can be re-written as

$$
A_n+B_n^{'}z_t=r^*+\gamma' z_t+A_{n-1}+B_{n-1}^{'}\Phi z_t-B_{n-1}^{'}\Omega\,(\lambda+\beta z_t)-\frac{1}{2}B_{n-1}^{'}\Omega B_{n-1}
$$

Rewriting the equation

$$
A_n+B_n^{'}z_t=r^*+A_{n-1}-B_{n-1}^{'}\Omega\lambda-\frac{1}{2}B_{n-1}^{'}\Omega B_{n-1}+\left(\gamma'+B_{n-1}^{'}\Phi-B_{n-1}^{'}\Omega\beta\right)z_t \quad \textbf{(A-6)}
$$

yields that

$$
\begin{aligned}
B_n^{'} &= \gamma' + B_{n-1}^{'}\Phi - B_{n-1}'\Omega\beta \\
A_n &= r^* + A_{n-1} - B_{n-1}'\Omega\lambda - \frac{1}{2}B_{n-1}^{'}\Omega B_{n-1}
\end{aligned}
$$

for equation **(A-6)** to hold for all values of z_t, which verifies equation **(A-2)** and equation **(A-3)** and the assumption that log bond prices are linear in state variables from equation **(A-5)**.

Appendix B: Tables

Table A: Summary statistics for full data sample

	y^3	y^{12}	y^{24}	y^{36}	y^{60}	y^{120}	y^{180}	π
Mean	8.82	8.68	8.83	8.95	9.11	9.32	9.29	6.81
Median	8.94	9.19	9.46	9.50	9.68	9.65	9.24	4.20
Standard deviation	3.38	3.00	2.86	2.82	2.83	2.97	3.12	5.86
Skewness	0.26	0.04	−0.05	−0.08	−0.13	−0.17	0.04	1.53
Excess kurtosis	−0.98	−1.06	−0.98	−0.92	−0.92	−1.05	−0.92	1.72
Min	3.33	3.24	3.33	3.50	3.77	4.15	4.26	0.70
Max	16.27	14.96	15.12	15.26	15.54	15.44	15.69	26.90

Table B: Correlations for full data sample

	y^3	y^{12}	y^{24}	y^{36}	y^{60}	y^{120}	y^{180}	π
y^3	1.00							
y^{12}	0.98	1.00						
y^{24}	0.95	0.99	1.00					
y^{36}	0.92	0.97	0.99	1.00				
y^{60}	0.88	0.94	0.98	0.99	1.00			
y^{120}	0.81	0.88	0.93	0.96	0.98	1.00		
y^{180}	0.74	0.80	0.87	0.90	0.93	0.98	1.00	
π	0.60	0.61	0.64	0.66	0.68	0.73	0.80	1.00

Table C: Auto correlations for full data sample

	y^3	y^{12}	y^{24}	y^{36}	y^{60}	y^{120}	y^{180}	π
Lag 1	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.99
Lag 7	0.82	0.84	0.85	0.85	0.87	0.91	0.91	0.80
Lag 13	0.66	0.72	0.75	0.76	0.79	0.84	0.84	0.61
Lag 19	0.55	0.63	0.69	0.71	0.75	0.79	0.78	0.53
Lag 25	0.42	0.53	0.60	0.63	0.68	0.72	0.70	0.46
Lag 31	0.34	0.46	0.53	0.57	0.62	0.67	0.64	0.36

Table D: QML estimates of constant risk premia model (CRP)

$\ln \mathbb{L}$:-18363.71	Estimates	t-ratio
γ_1	0.49	14.1
$\begin{bmatrix} \phi_{11} & \phi_{33} \end{bmatrix}$	$\begin{bmatrix} 0.920 & 0.970 \end{bmatrix}$	$\begin{bmatrix} 199.6 & 1576.0 \end{bmatrix}$
$10^5 \cdot \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{bmatrix}$	$\begin{bmatrix} 57.3 & 29.3 & 49.4 \end{bmatrix}$	$\begin{bmatrix} 33.0 & 28.3 & 26.6 \end{bmatrix}$
$\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}$	$\begin{bmatrix} 448.3 & 17.7 & 208.9 \end{bmatrix}$	$\begin{bmatrix} 2.6 & -1.2 & 3.9 \end{bmatrix}$
$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix}$	$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$	$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$
$10^7 \cdot \eta^2$	0.61	45.4

Table E: QML estimates of time-varying risk premia model (TRP)

$\ln \mathbb{L}$:-18571.07	Estimates	t-ratio
γ_1	0.13	2
$\begin{bmatrix} \phi_{11} & \phi_{33} \end{bmatrix}$	$\begin{bmatrix} 0.97 & 0.95 \end{bmatrix}$	$\begin{bmatrix} 139.6 & 68.1 \end{bmatrix}$
$10^5 \cdot \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{bmatrix}$	$\begin{bmatrix} 59.0 & 27.0 & 47.9 \end{bmatrix}$	$\begin{bmatrix} 39.9 & 3.5 & 11.3 \end{bmatrix}$
$\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}$	$\begin{bmatrix} 2.71 & -89.5 & 70.3 \end{bmatrix}$	$\begin{bmatrix} 0.0 & -1.2 & 0.1 \end{bmatrix}$
$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix}$	$\begin{bmatrix} -4948 & 46098 & 291874 \\ -35668 & 37880 & 26439 \\ 87623 & -114501 & -31171 \end{bmatrix}$	$\begin{bmatrix} -0.2 & 0.4 & 6.5 \\ -0.9 & 0.8 & 0.3 \\ 6.6 & -1.5 & -0.4 \end{bmatrix}$
$10^7 \cdot \eta^2$	0.51	46.8

Table F: QML estimates of restricted version of time-varying risk premia model (TRP)

$\ln \mathbb{L}$:-18491.59	Estimates	t-ratio
γ_1	0.10	2.2
$\begin{bmatrix} \phi_{11} & \phi_{33} \end{bmatrix}$	$\begin{bmatrix} 0.93 & 0.96 \end{bmatrix}$	$\begin{bmatrix} 181.1 & 343.8 \end{bmatrix}$
$10^5 \cdot \begin{bmatrix} \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \end{bmatrix}$	$\begin{bmatrix} 56.6 & 24.5 & 51.0 \end{bmatrix}$	$\begin{bmatrix} 41.0 & 26.6 & 23.8 \end{bmatrix}$
$\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}$	$\begin{bmatrix} 366.6 & 44.2 & 74.6 \end{bmatrix}$	$\begin{bmatrix} 1.8 & 1.2 & 0.6 \end{bmatrix}$
$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix}$	$\begin{bmatrix} - & - & 180603 \\ - & - & - \\ 95533 & -113026 & - \end{bmatrix}$	$\begin{bmatrix} - & - & 9.3 \\ - & - & - \\ 9.9 & -10.1 & - \end{bmatrix}$
$10^7 \cdot \eta^2$	0.55	46.4

Appendix C: Charts

Chart 1: Cross-plot between three-month spot rates and Treasury bills

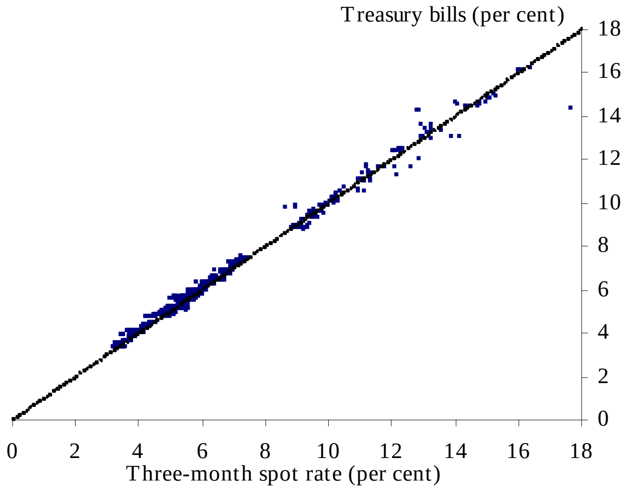


Chart 2: Actual and fitted values for the CRP model

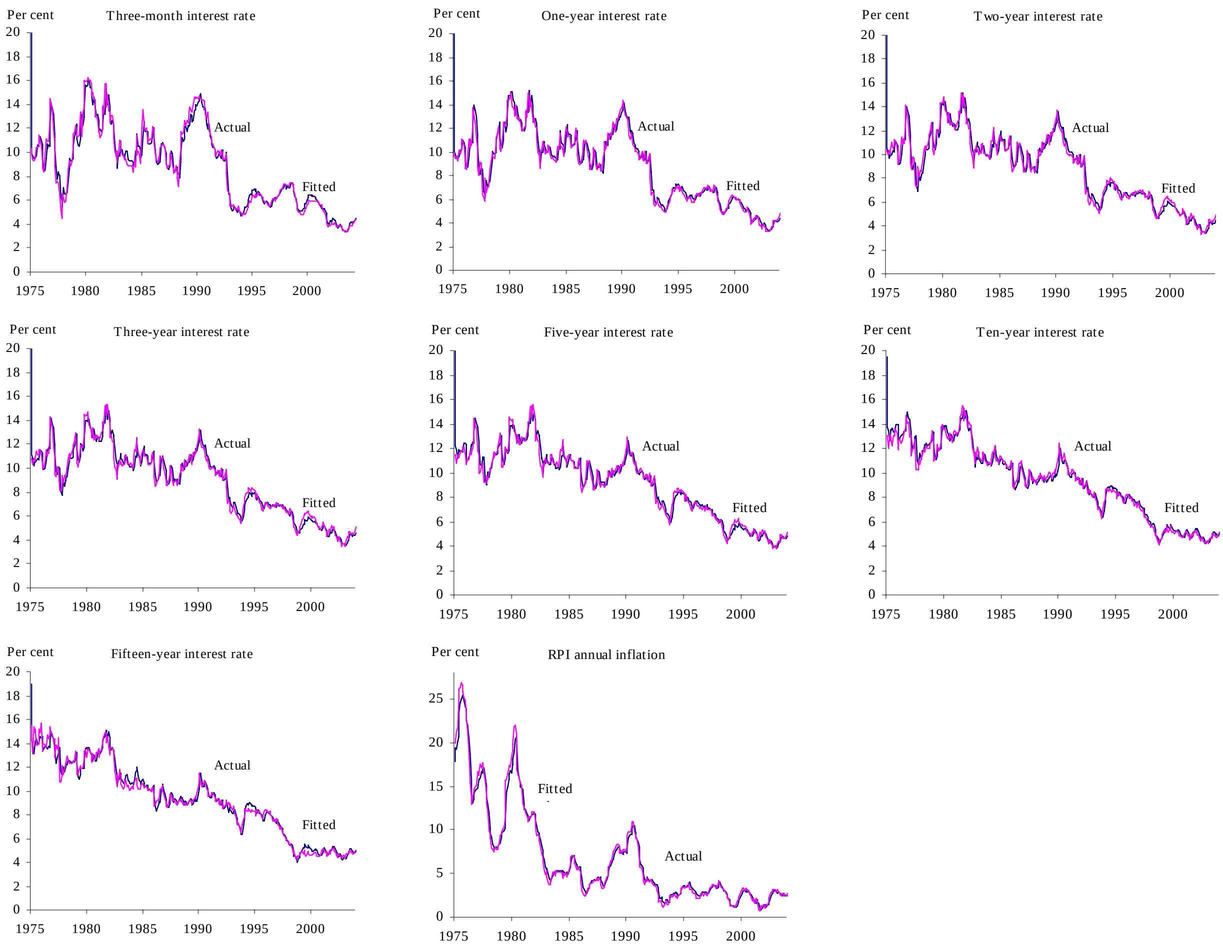


Chart 3: Actual and fitted values for the TRP model

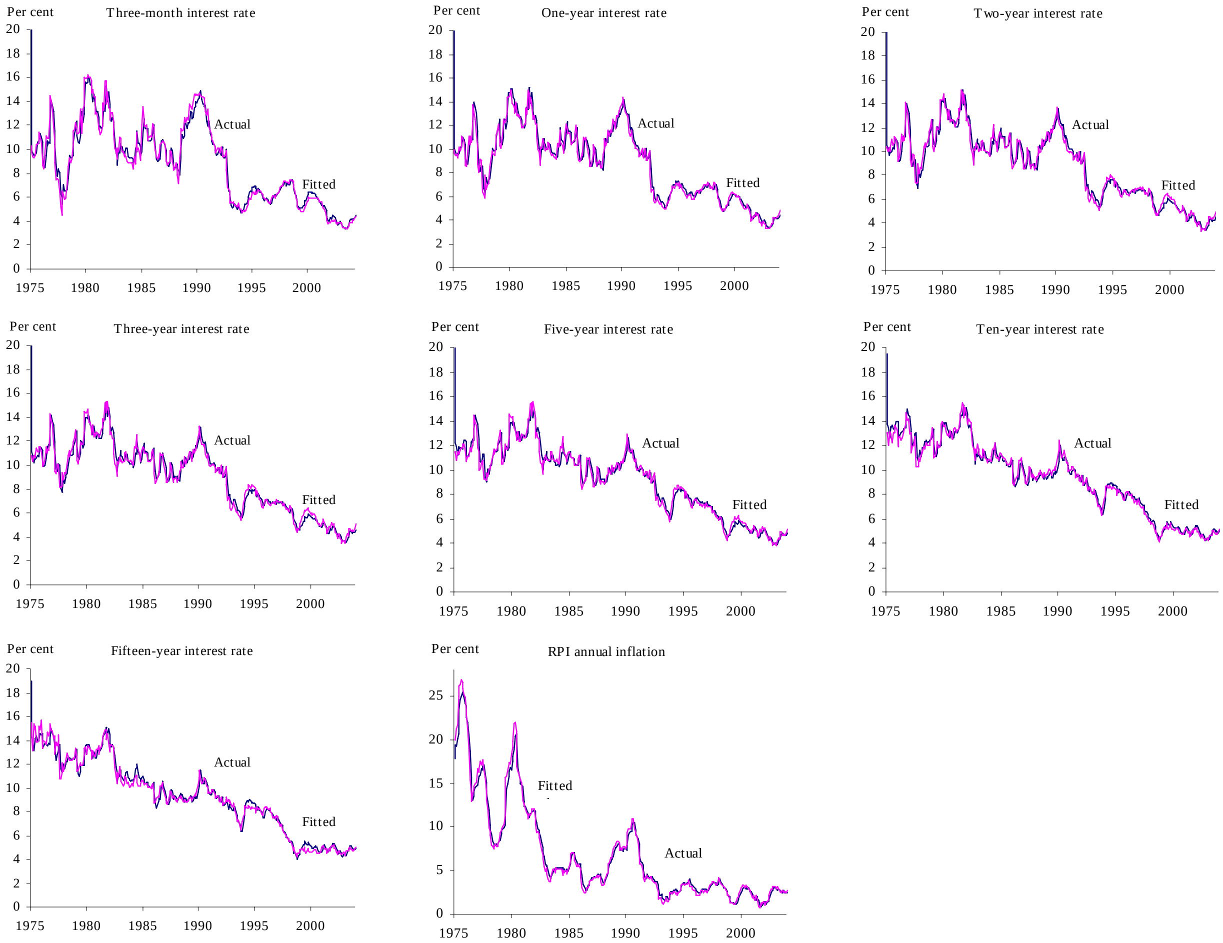
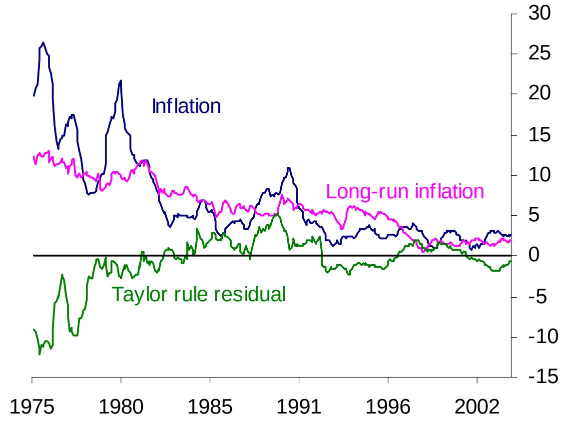
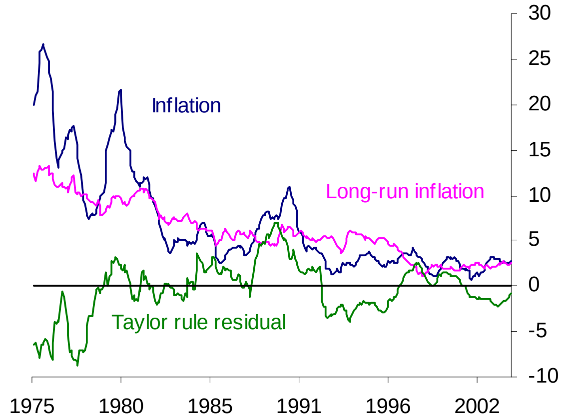


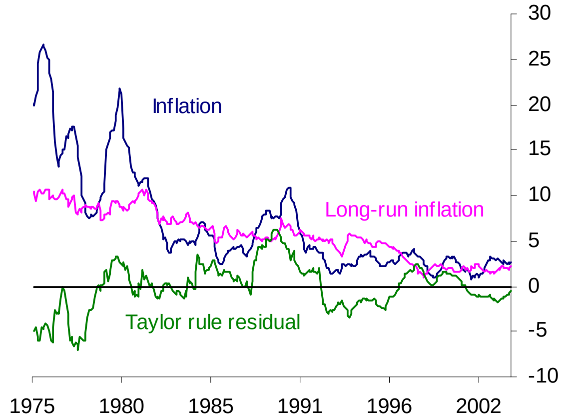
Chart 4: Factor evolution for the estimated models



CRP model

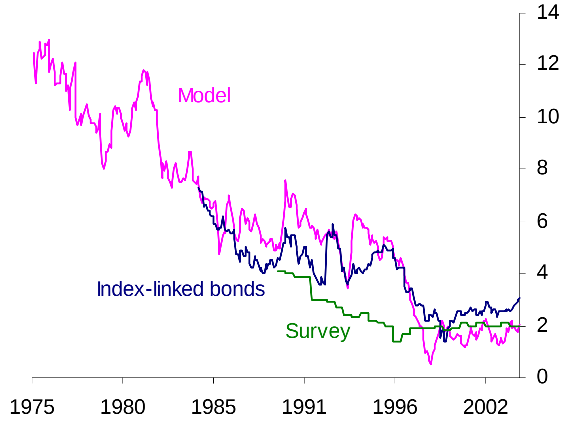


Restricted TRP model

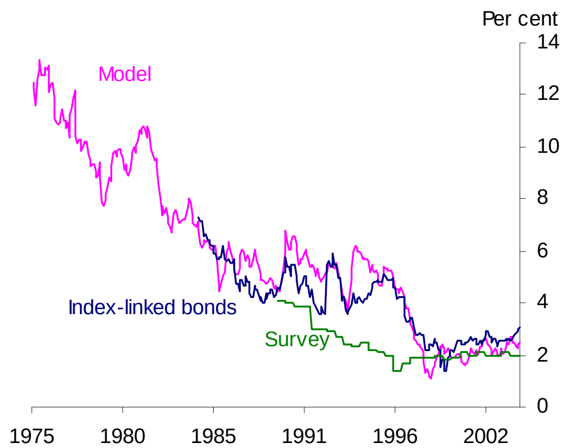


TRP model

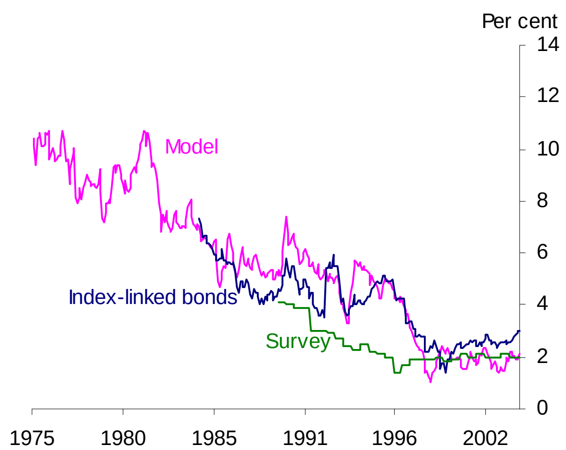
Chart 5: Long-run inflation from estimated models



CRP model



Restricted TRP model



TRP model

Chart 6: Impulse responses for estimated models

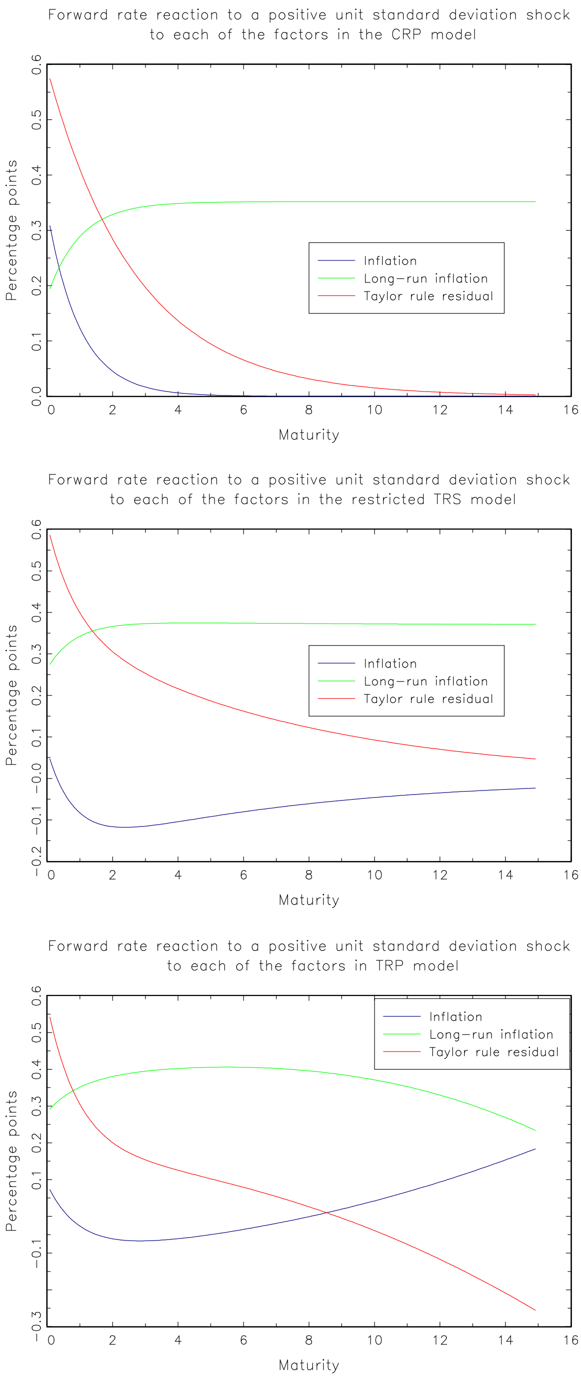
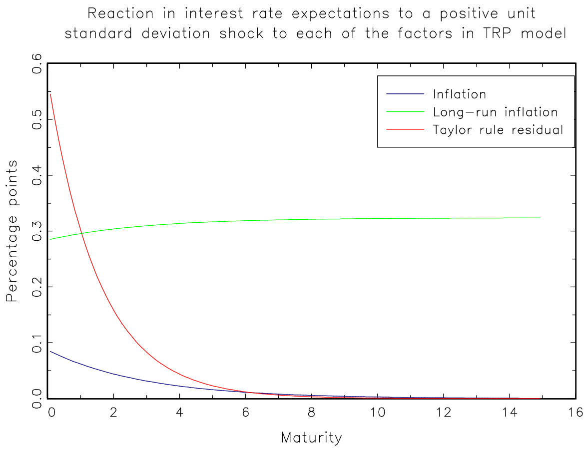


Chart 7: Decomposition of impulse responses for TRP model



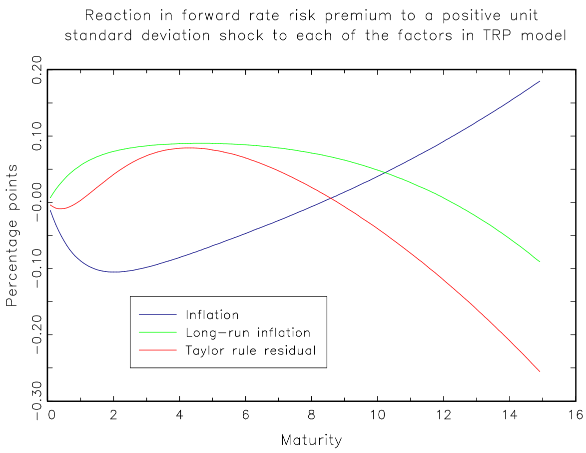
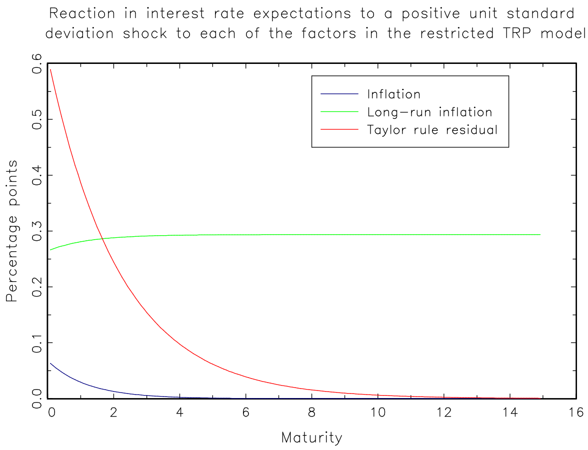


Chart 8: Decomposition of impulse responses for restricted TRP model



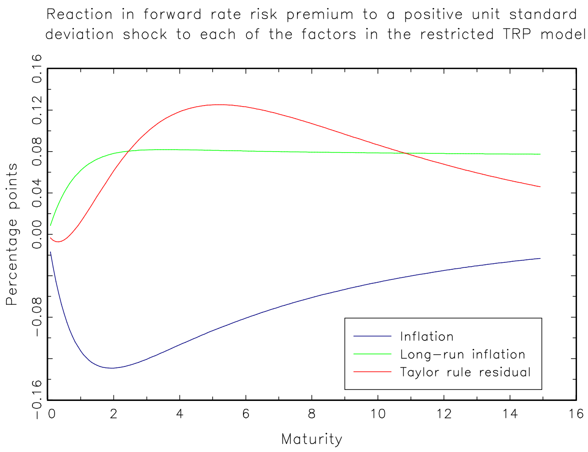
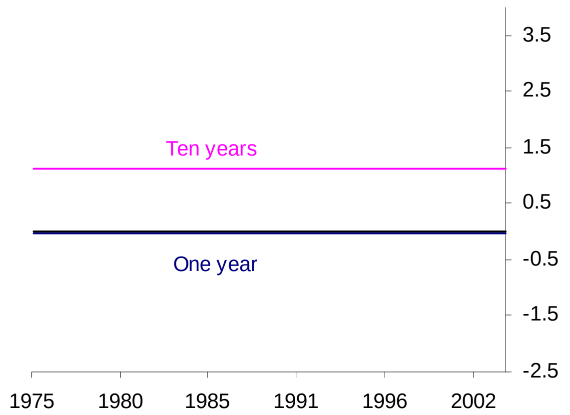
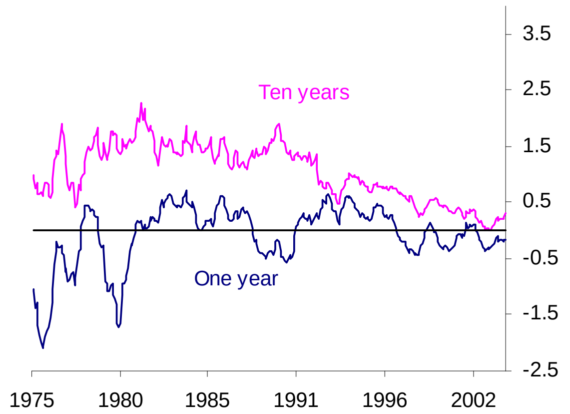


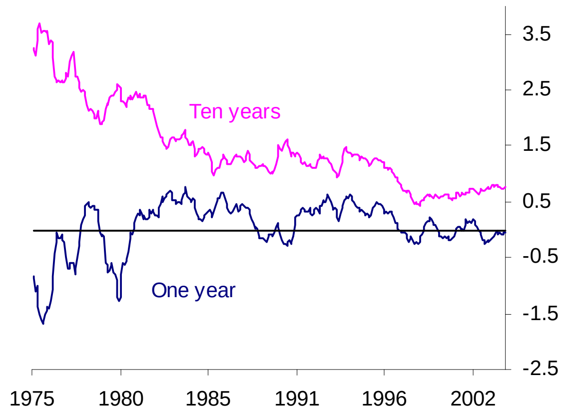
Chart 9: Forward rate risk premia in estimated models



CRP model

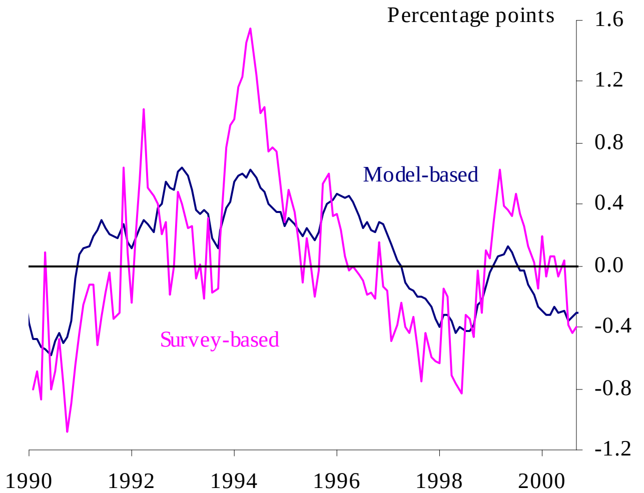


Restricted TRP model

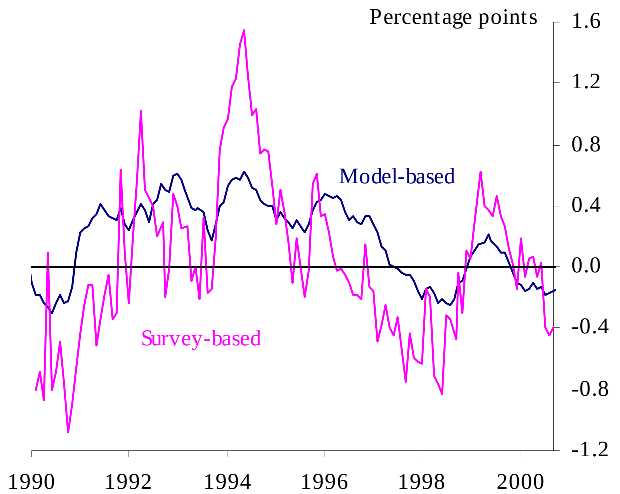


TRP model

Chart 10: Time-series plot of forward rate risk premia at one-year horizon

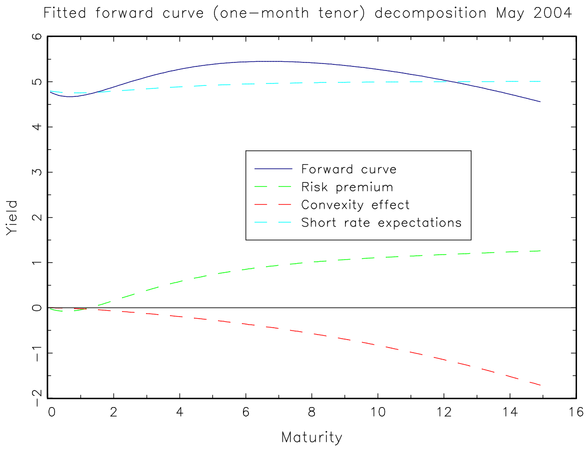


Restricted TRP model (correlation between model and survey-based risk premium: 0.70)

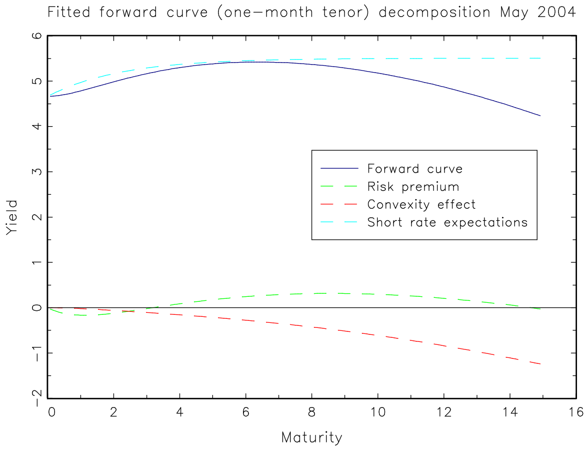


TRP model (correlation between model and survey-based risk premium: 0.70)

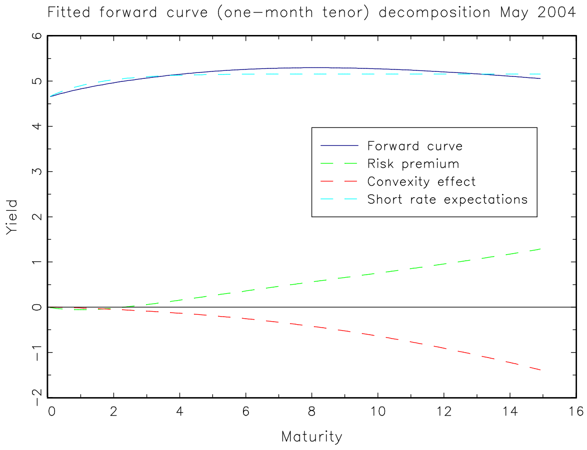
Chart 11: Fitted forward curve decomposition for estimated models at May 2004



CRP model



Restricted TRP model



TRP model

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