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Abstract

Long-horizon interest rates in the major international bond markets fell sharply during 2004 and 2005, at the same time as US policy rates were rising; a phenomenon famously described as a 'conundrum' by Alan Greenspan the Federal Reserve Chairman. But it was arguably the decline in international long real rates over this period which was more unusual and, by the end of 2007, long real rates in the United Kingdom remained at recent historical lows. In this paper, we try to shed light on the recent behaviour of long real rates, by estimating several empirical models of the term structure of real interest rates, derived from UK index-linked bonds. We adopt a standard 'finance' approach to modelling the real term structure, using an essentially affine framework. While being empirically tractable, these models impose the important theoretical restriction of no arbitrage, which enables us to decompose forward real rates into expectations of future short (ie risk-free) real rates and forward real term premia. One general finding that emerges across all the models estimated is that time-varying term premia appear to be extremely important in explaining movements in long real forward rates. Although there is some evidence that long-horizon expected short real rates declined over the conundrum period, our results suggest lower term premia played the dominant role in accounting for the fall in long real rates. This evidence could be consistent with the so-called 'search for yield' and excess liquidity explanations for the conundrum, but it might also partly reflect strong demand for index-linked bonds by institutional investors and foreign central banks.

Key words: Yield curve, term premia, conundrum.

JEL classification: C32, E43, E44, G12.

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Summary

Long-term interest rates in the major international bond markets fell sharply during 2004 and 2005, at the same time as US policy rates were rising. This phenomenon was famously described as a 'conundrum' by Alan Greenspan the Federal Reserve Chairman at the time. But it was arguably the decline in international real rates (interest rates adjusted for inflation) which was more unusual. And by the end of 2007, although real (and nominal) rates had recovered slightly in the United States and euro area, long real rates in the United Kingdom remained at recent historical lows.

Understanding the causes of low long real rates matters for monetary policy makers, not least because different explanations have correspondingly different implications for monetary conditions. If, for example, low real rates are due to lower investor risk aversion, the response of monetary policy may differ from the scenario where they reflect expectations of weaker long-term growth. There are also implications regarding the risks of long rates reverting to more normal, higher levels. For example, if low long real rates reflect a temporary rather than a permanent shock, there is a greater risk of a sharp upward adjustment in borrowing rates, which would be disruptive for the real economy.

A large number of potential explanations for the conundrum have been put forward. Some have emphasised the role of saving and investment: either high global saving (the so-called Asian 'saving glut') or low investment (particularly in the industrialised countries). Others have focused on looser monetary policy or 'excess liquidity'. Other explanations have related the conundrum to lower risk premia (the amount by which the market rewards the holders of more risky assets). This may have reflected perceptions of greater macroeconomic stability, or the so-called 'search for yield', which could have driven up the demand for riskier but higher yielding assets. And search for yield itself has been seen by some as a possible consequence of excess liquidity, which has depressed nominal risk-free rates and increased investors' demand for risky assets to meet their nominal return aspirations. Finally, other explanations have focused on the role of imbalances between market demand and supply, arising from either large portfolio inflows into bonds from Asian central banks or strong demand from pension funds. Each of these explanations has some plausibility and it is probably fair to say that no firm consensus has yet emerged on which was the most important. But the fact that the fall in long nominal interest

rates during the conundrum period mainly reflected a decline in long real rates, as opposed to lower inflation compensation, suggests that understanding the behaviour of real rates may be particularly fruitful.

In this paper, we try to shed light on what accounts for the phenomenon of low long real rates, by estimating several empirical models of the term structure of real interest rates, derived from UK government index-linked bonds. We adopt a standard 'finance' approach to modelling the real term structure, based on the assumption that there are no risk-free profits to be made by trading between different government bonds (in other words, there are no arbitrage opportunities). Importantly, the assumption of no arbitrage enables us to decompose forward real rates into expectations of future short (ie risk-free) real rates and forward real term premia in a theoretically consistent way.

Although we find some evidence that long-horizon risk-free real rates of interest have declined, the results from the models we examine suggest that reductions in term premia played the more important role in explaining the decline in UK long real rates over the 2004-05 period. This could be consistent with both the search for yield/excess liquidity explanation of the conundrum or heavy demand for index-linked bonds by institutional investors and central banks, although the global nature of the conundrum inclines us to put more weight on the former explanation. More recently, however, it seems likely that real rates have been depressed by a 'flight to quality' from risky assets triggered by the sub-prime crisis in the United States. Taking our results at face value would suggest that there are risks that real rates may rise in the future, as they currently remain below the long-run equilibrium levels implied by our models. But it should be borne in mind that there are a number of caveats with our analysis. In particular, the model set-up does not directly allow for structural changes in the level of the long-run equilibrium real interest rate, and the estimates themselves may be less reliable as a result of the relatively short data sample available.

1 Introduction

Long-horizon interest rates in the major international bond markets fell sharply during 2004 and 2005; a phenomenon famously described as a 'conundrum' by Alan Greenspan the Federal Reserve Chairman at the time because it accompanied rising policy rates in the United States (see Greenspan (2005)).

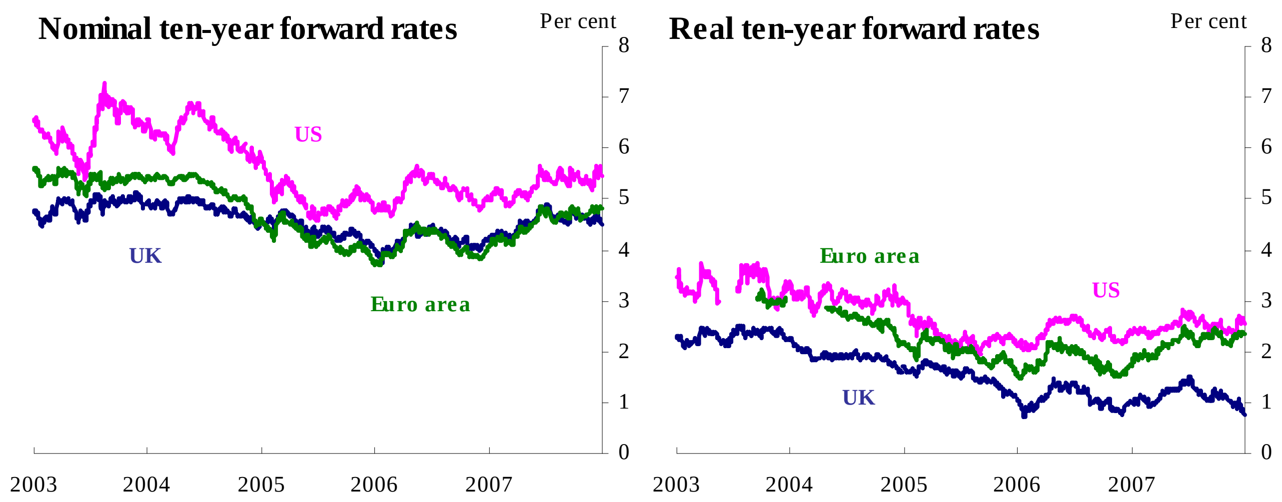
The two graphs in Chart 1 illustrate the basic facts. The graph to the left shows nominal long forward rates for the United Kingdom, United States and the euro area derived from nominal government bonds; the graph to the right shows the equivalent picture for real forward rates derived from index-linked gilts for the United Kingdom, Treasury Inflation-Protected Securities (TIPS) for the United States and inflation swaps for the euro area.[1] Two things stand out about the conundrum period: first, the high cross-country correlation between movements in international long rates during 2004 and 2005 — suggesting that the conundrum primarily reflected a global phenomenon — and, second, that the fall in nominal rates during this period largely mirrored equivalent falls in real rates — suggesting that the fall was driven by real factors, rather than by lower inflation expectations. Since their trough in early 2006, long-horizon real rates in the euro area and the United States have recovered slightly, although remaining low, while UK long real rates at the end of 2007 were close to the historical lows they reached at the beginning of 2006.

Understanding why long real rates are low matters for monetary policy makers. This is not least because different explanations have correspondingly different implications for monetary conditions. If, for example, low real rates are due to lower investor risk aversion, the response of monetary policy may differ from the scenario where they reflect expectations of weaker long-term growth. There are also implications regarding the risks of long rates reverting to more normal higher levels. For example, if low long real rates reflect a temporary rather than a permanent shock, there is a greater risk of a sharp upward adjustment in borrowing rates, which would be disruptive for the real economy.

A large number of potential explanations for the conundrum have been put forward. Some have

[1] The yield curve estimates shown were produced using a cubic spline method that imposes greater smoothness at longer maturities (see discussion in Section 3 and Anderson and Sleath (1999)). UK index-linked bonds are uprated in line with RPI inflation, which is constructed in a slightly different way to the indices used to uprate US index-linked bonds (CPI inflation) and euro-area inflation swaps (HICP inflation). To make the UK real rate series more comparable with the others, they can be adjusted upwards by about 80 basis points to reflect the estimated long-run difference between RPI and CPI inflation.

Chart 1: International forward rates



emphasised the role of saving and investment: either high global saving (the so-called Asian 'saving glut' (Bernanke (2005)) or low investment (particularly in the industrialised countries, IMF (2005)). Others have focused on looser monetary policy or 'excess liquidity'. Other explanations have related the conundrum to lower risk premia, reflecting perceptions of greater macroeconomic stability, the so-called 'Great Stability', or the so-called 'search for yield'. And search for yield itself has been seen by some as a possible consequence of excess liquidity (see King (2006)), which has depressed nominal risk-free rates and increased investors' demand for risky assets to meet their nominal return aspirations. Finally, other explanations have focused on the role of imbalances between market demand and supply, arising from either large portfolio inflows into bonds from Asian central banks[2] or strong demand from pension funds.[3] Each of these explanations has some plausibility and it is probably fair to say that no firm consensus has emerged yet on which was the most important.

The fact that the fall in international long nominal interest rates during 2004 and 2005 mainly reflected lower long real rates, as opposed to lower inflation compensation, suggests that understanding the behaviour of real rates may be particularly fruitful. This motivates the approach taken in this paper, which attempts to model the behaviour of real yields using a

[2] See, eg, Warnock and Warnock (2006) who claim that official flows mainly from East Asia had a major depressing effect on long-term US Treasury yields over this period.

[3] We do not provide a comprehensive review here, as this material has been discussed extensively elsewhere. See eg the box on 'The economics of low long-term bond yields' in the Bank of England *Inflation Report* May 2005.

standard ‘finance’ approach — the essentially affine yield curve model with latent factors. The finance approach we use has a more flexible structure than the main alternatives, like dynamic stochastic general equilibrium (DSGE) models or so-called ‘macro-factor’ yield curve models, making it potentially more robust to model misspecification. While being empirically tractable, this approach imposes a minimum of plausible theoretical restrictions, including importantly that bond prices do not permit arbitrage opportunities (there are no risk-free profits to be made), enabling us to decompose forward real rates into expectations of future short (ie risk-free) real rates and forward real term premia.

For convenience, we can classify explanations for the conundrum into three category types:

1. Expected future real short-term interest rates fell, possibly reflecting a decline in the neutral real rate of interest[4] (eg as a result of a change in the propensity to save versus planned investment).

2. There was a fall in risk premia, as normally defined in standard theoretical asset pricing models, brought about by a change in the quantity or price of risk (eg as a consequence of the Great Stability or loose global monetary policy and the associated search for yield).

3. Long-term real rates were driven lower by imbalances between demand and supply (eg arising from the preferred habitat behaviour of Asian central banks or institutional investors), with no necessary implications for either (1) or (2).

The affine models we use in this paper do not allow us to distinguish explicitly between categories (2) and (3), but both categories should be picked up in the models’ term premia estimates.[5]

Although the conundrum appears to be an international phenomenon, we focus on modelling real yields derived from UK government index-linked bonds. The existence of a long-standing and liquid market in UK index-linked bonds (see Deacon *et al* (2004)) provides an obvious source of

[4]The neutral rate can be defined as the real rate of interest that is consistent with stable inflation when the economy is growing at trend. It therefore corresponds to the notion of the long-run equilibrium real rate of interest.

[5]This is under the assumption that pure arbitrage opportunities are ruled out. This proposition has recently been shown more formally by Vayanos and Vila (2007). In their heterogenous agent model, there are two classes of investor: institutional investors, who have preferred habitats, and speculators, who maximise their expected utility by trading off the mean and variance of expected wealth. In this set-up, they show that demand imbalances are arbitraged away, so that preferred-habitat investors’ preferences are reflected in term premia.

information for us to exploit[6] and the closely integrated nature of international bond markets suggests that our conclusions are likely to hold true for other countries. In order to extract zero-coupon real yields from data on index-linked bonds, we use Bank of England estimates that have been adjusted for coupon payments and indexation lags (using the methods proposed by Evans (1997) and extended by Anderson and Sleath (1999, 2001)). Using market-derived real rates has the advantage that we do not need to model inflation separately in order to derive synthetic real rates, as other similar work for the United States and euro area is typically obliged to do, given the lack of a sufficient back run of index-linked bonds (for the United States see eg Ang, Bekaert and Wei (2007) and Kim and Wright (2005)). That said, small sample problems remain an issue for this work, because to avoid obvious structural breaks we focus our analysis on the period since October 1992, during which the United Kingdom has operated an inflation target.

We explore several model specifications, in order to test the sensitivity of our results. Recent work for the United States has suggested using survey data on the future path of interest rate expectations as a way of supplementing the available time-series data on yields (see in particular Kim and Orphanides (2005)). The availability of US survey information on long-horizon expected future policy rates and expected future inflation means that it is possible to construct a measure of short real rates expected in the long run, which can be thought of as a proxy measure of the neutral real rate of interest. Unfortunately, to our knowledge, similar survey data are not available for the United Kingdom. Instead, following simple growth theory arguments, one of the term structure models we have estimated uses data on long-run Consensus GDP forecasts as an additional information variable explaining long-horizon real rates, albeit with rather mixed success. In another model specification, we include a proxy of the real policy rate in the estimation, partly on the grounds that this may be relevant data that agents take into account in forming their future real rate expectations. Overall, however, the fact that all the models we estimate suggest reductions in term premia have played an important role in the decline in long real forward rates over the conundrum period provides some reassurance about the robustness of this aspect of our conclusions.

The paper is structured as follows. The second section sets out the basic essentially affine modelling approach, shows how we can use it to decompose real forward rates into expected

[6] Our focus on UK government index-linked bonds has the advantage that we can ignore default/credit risk and liquidity risk in our analysis.

future short rates, forward term premia and convexity effects and discusses how these models can be estimated using the Kalman filter. The following section describes our data on real rates and discusses the basic statistical properties of the data using principal components analysis. The next section discusses the empirical results for the three essentially affine term structure model specifications we estimate. We draw out the implications of each model for explaining the path of long forward real rates and assess the merits of each model against plausibility and statistical criteria. The following section interprets the results and also asks to what extent they reflect the model set-up itself. The final section provides conclusions and some suggestions for further research.

2 The essentially affine framework

2.1 Theory

The no-arbitrage affine (linear) term structure model we apply to real yields on UK index-linked government bonds is something of the industry standard in the empirical finance literature. It assumes that bond yields are driven by a small number of unobservable or latent factors and that there are no arbitrage opportunities, ie no risk-free opportunities to make money by trading across bonds of different maturities. The model belongs to the so-called 'essentially affine' class of models, as defined by Duffee (2002). Both bond yields and the market prices of risk in the model are affine functions of the underlying state variables, but the formulation of the real stochastic discount factor (SDF) used is slightly more general than the 'completely affine' models originally proposed by Duffie and Kan (1996), in that the price of risk is allowed to vary independently of interest rate volatility.

We derive the model below for the three-factor case, which is general enough to nest all the specifications we shall consider. We derive the model in discrete time, following Backus, Foresi and Telmer (1998). It is derived from three basic elements: a process driving the unobservable factors; a formulation of the real SDF that specifies the market price of risk in such a way that the model is essentially affine; and the assumption that bond prices reflect the fundamental asset pricing equation.

We first define a state vector relevant for pricing bonds, z_t, which is a 3×1 vector, containing as elements our three latent factors:

$$z_t = \begin{bmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{bmatrix}.$$

The state vector is assumed to follow a first-order VAR model

$$z_{t+1} = \Phi z_t + \Omega^{1/2}\varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim NID(0, I_3) \tag{1}$$

where the Ω and Φ matrices are given by

$$\Omega = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \Phi_{11} & 0 & 0 \\ \Phi_{21} & \Phi_{22} & 0 \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{bmatrix}$$

and

$$\varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \\ \varepsilon_{3,t+1} \end{bmatrix}.$$

In our empirical analysis, we allow the off-diagonal elements of the Φ matrix to be non-zero, which enables the factors to be correlated with each other. We assume that the error terms are homoscedastic, so volatility in this model is constant, although as mentioned above the model still allows for time-varying risk premia.

In order to get an essentially affine form for bond prices/yields, we make the real SDF take the following form:

$$M_{t+1} = \exp\left[-\left(r^* + \gamma' z_t\right) - \frac{\Lambda_t'\Omega\Lambda_t}{2} - \Lambda_t'\Omega^{1/2}\varepsilon_{t+1}\right] \tag{2}$$

where

$$\Lambda_t = (\lambda + \beta z_t) = \begin{bmatrix} \lambda_1 + \beta_{11}z_{1,t} + \beta_{12}z_{2,t} + \beta_{13}z_{3,t} \\ \lambda_2 + \beta_{21}z_{1,t} + \beta_{22}z_{2,t} + \beta_{23}z_{3,t} \\ \lambda_3 + \beta_{31}z_{1,t} + \beta_{32}z_{2,t} + \beta_{33}z_{3,t} \end{bmatrix}. \tag{3}$$

$$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}; \qquad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}; \qquad \beta = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix}.$$

The logarithm of the SDF is therefore:

$$\ln M_{t+1} = m_{t+1} = -\left(r^* + \gamma' z_t\right) - \frac{\Lambda_t' \Omega \Lambda_t}{2} - \Lambda_t' \Omega^{1/2} \varepsilon_{t+1}.$$

In a macro model the SDF would represent the intertemporal marginal rate of substitution and typically be a (negative) function of consumption growth (eg in the case of log utility). The form of the SDF here is considerably more general, but still has some of the same intuition. For example, if r^* — which can be interpreted as the real short interest rate that is expected to hold in the long run — increases the log of the SDF falls. This is what we would expect because higher interest rates will be associated with more saving, higher future consumption growth and therefore a lower intertemporal marginal rate of substitution. The presence of the z_t vector in the first term in parentheses means that the SDF is also a function of the factors included in the model. Note that in the case where the factors are latent, as here, the γ vector is normalised to unity, since it cannot be identified separately from the factors. The second term is a scalar, which ensures that the one-period short rate in the model is affine in the factors. The last term in the equation shows how risk affects the SDF. The elements in vector Λ_t represent the market prices of risk associated with shocks to the SDF from each factor. The formulation in **(3)** is consistent with time-varying term premia, in the case where some of the elements of β are non-zero. It also nests the constant term premia case, where all of the elements of β are zero, and the no term premia case, where all the elements of λ and β are zero.

Log bond prices in the model are affine functions of the state vector. To see this, we first assume that the log price of a zero-coupon bond with n periods to maturity at time t is given by

$$\ln P_t^n = p_t^n = A_n + B_n' z_t \tag{4}$$

where

$$B_n' = \begin{bmatrix} B_{n,1} & B_{n,2} & B_{n,3} \end{bmatrix}.$$

We already know $P_t^0 = 1$, so it follows that

$$A_0 = 0 \qquad B_0' = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

The fundamental asset pricing equation states that

$$P_t^n = E_t\left[M_{t+1} P_{t+1}^{n-1}\right]$$

so that the price of an $n-$period bond today is equal to the expected value of the product of its price next period and the SDF next period. On the assumption that the joint distribution of bond prices and the SDF is conditionally lognormal, we can use the property of lognormality to expand out this expression to get

$$p_t^n = E_t\left[m_{t+1} + p_{t+1}^{n-1}\right] + 1/2Var_t\left[m_{t+1} + p_{t+1}^{n-1}\right].$$

If we now substitute in for next period's SDF and for the bond price, it is possible after some algebraic manipulation (shown in Appendix A) to get to this expression

$$A_n + B_n^{'} z_t = (-r^* + A_{n-1} - B_{n-1}^{'}\Omega\lambda + \frac{B_{n-1}^{'}\Omega B_{n-1}}{2}) + (-\gamma' + B_{n-1}^{'}(\Phi - \Omega\beta))z_t. \quad \textbf{(5)}$$

Lining up the coefficients on each side of this expression, it follows that the linear expression for bond prices must satisfy the following two recursive equations

$$\begin{aligned} A_n &= -r^* + A_{n-1} - B_{n-1}^{'}\Omega\lambda + \frac{B_{n-1}^{'}\Omega B_{n-1}}{2} && \textbf{(6)} \\ B_n^{\prime} &= -\gamma' + B_{n-1}^{'}(\Phi - \Omega\beta) && \textbf{(7)} \end{aligned}$$

So provided that the A_n and $B_n^{\prime}$ parameters satisfy these restrictions, log bond prices are affine in the factors and the model satisfies no arbitrage. And since log bond prices are affine, it follows that yields are also affine in the factors, with continuously compounded yields given by

$$\begin{aligned} y_t^n &= \frac{-p_t^n}{n} \\ &= A_n^* + B_n^{*'} z_t \\ A_n^* &= -\frac{A_n}{n}, \qquad B_n^* = -\frac{B_n}{n}. \end{aligned}$$

The general relationship between spot and forward rates is

$$f_t^n = (n+1)\, y_t^{n+1} - n y_t^n$$

so forward rates are therefore given by

$$\begin{aligned} f_t^n &= (n+1)\left(A_{n+1}^* + B_{n+1}^{*'} z_t\right) - n\left(A_n^* + B_n^{*'} z_t\right) \\ &= -(A_{n+1} + B_{n+1}^{'} z_t) + (A_n + B_n^{'} z_t) \\ &= (A_n - A_{n+1}) + \left(B_n^{'} - B_{n+1}^{'}\right) z_t. \end{aligned}$$

2.2 *The forward rate model decomposition*

From a monetary policy perspective, the forward rate curve is often of greater interest than the spot curve because it more directly conveys information about market expectations of future real interest rates. But unadjusted forward rates will also incorporate term premia and a convexity effect, which we need to extract. The affine yield curve model provides a method of decomposing the real forward curve into short real interest rate expectations, real term premia and a convexity effect, such that

$$f_t^n = E_t\left[y_{t+n}^1\right] + \phi_{t,n} + \omega_{t,n} \tag{8}$$

where $E_t\left[y_{t+n}^1\right]$ is the expected future real short rate n periods ahead, $\phi_{t,n}$ is the real term premium at horizon n, and $\omega_{t,n}$ is the convexity effect at horizon n.

To compute the components of the forward curve in this equation, we follow the steps set out in Lildholdt *et al* (2007). We first define the risk-neutral forward curve, as the yield curve that would prevail if investors did not price risk (ie λ and β are equal to zero matrices) and all other parameters remain unchanged. The risk-neutral forward curve in this case can therefore be computed as

$$f_t^n\big|_{\lambda=0,\beta=0} = (A_n - A_{n+1})|_{\lambda=0,\beta=0} + \left(B_n' - B_{n+1}'\right)\big|_{\lambda=0,\beta=0} z_t \tag{9}$$

where the notation indicates that the A_ns and B_ns are computed from the recursive equations **(6)** and **(7)** with the restriction that $\lambda = 0$ and $\beta = 0$, so that

$$\begin{aligned} A_n|_{\lambda=0,\beta=0} &= -r^* + A_{n-1}|_{\lambda=0,\beta=0} + \frac{1}{2} B_{n-1}'\big|_{\lambda=0,\beta=0} \Omega\, B_{n-1}|_{\lambda=0,\beta=0} \\ B_n'\big|_{\lambda=0,\beta=0} &= -\gamma' + B_{n-1}'\big|_{\lambda=0,\beta=0} \Phi. \end{aligned}$$

However, this curve does not correspond to expectations of future real interest rates because we are not making any offsetting adjustment for convexity. We can compute term premia by subtracting this artificial forward curve, computed as if investors were risk-neutral, from the fitted forward curve (since any convexity effect affects them both equally and therefore drops out).

$$\phi_{t,n} = f_t^n - f_t^n\big|_{\lambda=0,\beta=0} \tag{10}$$

The convexity effect term from equation **(8)** is computed as the difference between the risk-neutral forward curve from equation **(9)** and a forward curve computed as if investors were risk-neutral and future bonds prices were deterministic; in other words the curve corresponding to pure expectations of future interest rates. As explained above, risk-neutrality corresponds to imposing that λ and β are zero matrices. The additional assumption that future bonds prices are deterministic is imposed by setting the variance-covariance matrix of the state variables Ω (see equation **(1)**) to a zero matrix. Consequently, the convexity effect in the forward curve is computed as

$$\omega_{t,n} = f_t^n\big|_{\lambda=0,\beta=0} - f_t^n\big|_{\lambda=0,\beta=0,\Omega=0} \quad \textbf{(11)}$$

It is easy to show by substitution from the recursive equations that the convexity effect in this model is constant over time, though varying by maturity. Note that the term structure of expected future real interest rates can be obtained by combining equations **(8)**, **(10)** and **(11)**.

$$\begin{aligned} E_t\left[y_{t+n}^1\right] &= f_t^n - \phi_{t,n} - \omega_{t,n} \\ &= f_t^n - \left(f_t^n - f_t^n\big|_{\lambda=0,\beta=0}\right) - \left(f_t^n\big|_{\lambda=0,\beta=0} - f_t^n\big|_{\lambda=0,\beta=0,\Omega=0}\right) \\ &= f_t^n\big|_{\lambda=0,\beta=0,\Omega=0} \end{aligned} \quad \textbf{(12)}$$

2.3 Estimation method

The affine term structure models we report in this paper were all estimated using maximum likelihood, using the Kalman filter to compute the log likelihood.[7]

In order to apply the Kalman filter, we first need to put the model into state-space form, which consists of a measurement equation and a state equation. In general, the measurement equation shows the assumed relationship between a vector of observed variables and a vector of underlying state variables. In our case, the observed variable vector contains data on real yields (as well as a one-month proxy real policy rate in one specification and, in another, survey information on long-run GDP growth expectations) and the state vector contains the unobserved latent variables. The theory outlined above suggests an affine relationship exists between yields and the factors that takes the form

$$y_t^n = A_n^* + B_n^{*'} z_t.$$

[7] The models were estimated using Matlab 7.0 and the optimisation procedure `fmincon` in the Optimization Toolbox.

But if we assume observed yields are measured with error, we get an additional vector of measurement error terms. So, in the case where we observe k yields and assume they are each subject to measurement error, we get the following observation equation[8]

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \\ . \\ . \\ y_{k,t} \end{bmatrix} = \begin{bmatrix} A_1^* \\ A_2^* \\ A_3^* \\ . \\ . \\ A_k^* \end{bmatrix} + \begin{bmatrix} B_1^* \\ B_2^* \\ B_3^* \\ . \\ . \\ B_k^* \end{bmatrix} z_t + \begin{bmatrix} \upsilon_{1,t} \\ \upsilon_{2,t} \\ \upsilon_{3,t} \\ . \\ . \\ \upsilon_{k,t} \end{bmatrix} \tag{13}$$

where $y_{i,t}$ is the observed i-period real yield, z_t is the vector of unobservable factors, $\upsilon_{i,t}$ is the measurement error on the i-period real yield and the A_i^* and B_i^* parameters embody the theoretical no-arbitrage restrictions.

Measurement error in this context may represent a variety of things, including fitting error arising from yield curve estimation and market noise. We shall assume the errors are normally distributed, but in two variants of the model we allow the measurement error distributions to have differently sized variances for some elements of the observation vector.

The state equation in this context is the first-order VAR equation already shown above in **(1)**, namely:

$$z_{t+1} = \Phi z_t + \Omega^{1/2} \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim NID(0, I_3)$$

Given expressions for the state and measurement equations, equations **(1)** and **(13)**, we can readily apply the Kalman filter to the model to derive the prediction error decomposition of the likelihood function, which can then be maximised over different parameter values to generate maximum likelihood parameter estimates.

One of the advantages of using the Kalman filter approach is that it is possible to incorporate variables with missing observations into the observation equation (see eg Durbin and Koopman (2001)). This allows us to incorporate survey data on GDP growth expectations five to ten years ahead into the model, even though they are only available at six-monthly intervals.

[8] The usual alternative method (associated with Chen and Scott (1993)) is to assume that only some of the yields are measured with error and that a number (corresponding to the number of factors) are measured perfectly. The problem with this approach is that the choice of which yields are measured with and without error is normally rather arbitrary.

Details on the three different model specifications we estimate are discussed in Section 4 below; for completeness, the state-space representation of each of the models is shown in Appendix B. We next discuss our data.

3 The data

The UK real rate data we use in this paper are Bank of England estimates of end-of-month[9] zero-coupon real yields produced, using the method proposed by Evans (1997) and extended by Anderson and Sleath (1999, 2001), from index-linked bond yields. The (semi-annual) coupon payments and the principal payment of UK index-linked bonds are adjusted in line with movements in the retail prices index (RPI),[10] with an indexation lag of either eight months (for bonds issued prior to April 2005) or three months (for bonds issued subsequently). So to produce a real term structure, as well as accounting for coupon payments, we need to allow for the fact that the 'index-linked term structure' is actually a complicated combination of the real term structure and the nominal term structure. The Bank of England real term structure estimates account for both these issues,[11] using a smoothed cubic spline method, which results in greater flexibility in fitting the short end of the curve and less flexibility at the long end.

Although data on UK real yields are available back to the mid-1980s, we have restricted our sample period to be from October 1992 to December 2007 to avoid obvious structural breaks in the series. It seems clear, for example, that the adoption of inflation targeting in October 1992 represented a significant change in the United Kingdom's monetary policy framework and that this change is likely to have affected the term structure of real interest rates, as perceptions about how monetary policy will react to various shocks will have implications for expectations of future short real rates and for term premia. The large fall in real rates that accompanied the United Kingdom's withdrawal from the European Exchange Rate Mechanism (ERM) in September 1992 certainly suggests this was important.[12]

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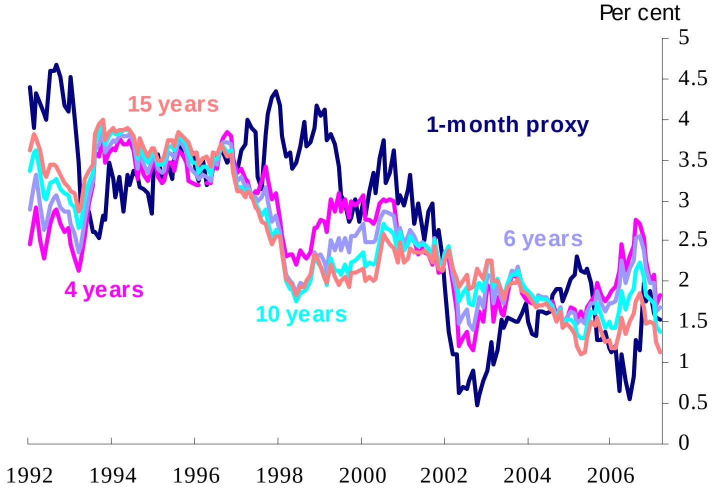
[9]The Bank of England publishes UK yield curve estimates on its website. See www.bankofengland.co.uk/statistics/yieldcurve.

[10]Since the RPI is not seasonally adjusted, real yields themselves may exhibit seasonality. We try to avoid this issue by using yields that have maturities that are multiples of a year.

[11]The method assumes that there is no indexation lag risk premium on index-linked bonds. This is probably not a bad assumption given the available empirical evidence, which suggests it is very small, see eg Risa (2001).

[12]A further reason for looking at a shorter sample is that the market for index-linked bonds has expanded rapidly since the first bond was issued in 1981, see Deacon *et al* (2004). It is unclear when the market reached a sufficient level of liquidity, but using all available data back to 1985 does not seem to be a sensible option.

Chart 2: Real spot rates



The shortest-maturity zero-coupon real spot rate we are able to derive over the full sample has a four-year maturity (because of the lack of short-maturity index-linked bonds), so we shall model zero-coupon real yields with maturities of four years, six years, ten years and fifteen years.[13] This provides the motivation for us also examining a variant of the model, which includes a proxy for the one-month real policy rate. This is calculated as the end-month policy rate less the latest current annual RPI inflation outturn (which in each case refers to the annual growth rate of the RPI in the previous month). This is obviously a proxy because the tenor of the policy rate is not monthly and we are making a reasonable but rather arbitrary assumption about inflation expectations. Nevertheless, the inclusion of a proxy one-month real rate is potentially useful, as there is otherwise a large gap in the maturity spectrum of the included real yields.

The data are displayed in Chart 2 above. The series are fairly stable for the first five years after the start of the sample period, which followed the United Kingdom's withdrawal from the ERM. But from mid-1997 to beginning of 1999 real market interest rates fell substantially. This fall seems at least partly linked to the introduction of the Minimum Funding Requirement (MFR) as part of the 1995 Pensions Act, which became effective in April 1997. This reform, which was designed to protect the solvency of pension funds, led to strong institutional demand for

[13] In principle, it should make little difference whether the model is estimated on spot rates or forward rates, provided enough maturities are included. But, in this case, the lack of data at the short end of the real curve favours using spot rates, as using only forward rates would mean that the model would be estimated using no information on the real curve below four years. The consequence of fitting the model to spot rates is that the model may do less well in fitting some forward rates, as these are strictly out of sample, in the sense that the model has not been optimised to fit them.

long-dated and index-linked bonds, driving long real forward rates down (see the May 1999 Bank of England *Inflation Report*). So this is a clear example of the type of Category 3 factor referred to in the Introduction. Downward pressure from pension fund buying was probably reinforced by the LTCM and Asian crises in Autumn 1997 and 1998, which caused a 'flight to quality' into government bonds.[14] After real rates subsequently rose, they fell back again after 2003, coinciding with a global fall in long real rates — the bond market conundrum we are trying to explain — which reached a trough at the beginning of 2006. Real rates then temporarily recovered in the first part of 2006 before falling back and then rose again in the first half of 2007, before again falling back sharply, as financial markets reacted to the sub-prime mortgage crisis in the United States with a 'flight to quality'.

Table A displays some summary statistics for our real yield data set and Table B shows yield correlations. It is apparent that our one-month proxy real rate stands slightly apart from the longer-maturity real yields derived from indexed-linked bonds. The average one-month real rate itself is higher than the other yields, more volatile and less highly correlated with market real rates than the latter are with each other. If we set aside the one-month rate, the average yield curve is very slightly downward-sloping and volatility increases with time to maturity.

Table A: Summary statistics: Oct. 1992 - Dec. 2007

	y_t^1	y_t^{48}	y_t^{72}	y_t^{120}	y_t^{180}
Mean	2.737	2.582	2.566	2.544	2.517
Std Deviation	1.093	0.694	0.698	0.763	0.844
Skewness	-0.392	-0.051	0.209	0.298	0.256
Kurtosis	-0.995	-1.065	-1.159	-1.246	-1.276

Table B: Correlation matrix: Oct. 1992 - Dec. 2007

	y_t^1	y_t^{48}	y_t^{72}	y_t^{120}	y_t^{180}
y_t^1	1	0.645	0.628	0.633	0.777
y_t^{48}		1	0.970	0.929	0.843
y_t^{72}			1	0.989	0.883
y_t^{120}				1	0.907
y_t^{180}					1

Tables C and D show the results from two principal components analyses on our data set over the sample period from October 1992 to December 2007. When we restrict the data set to the four

[14]Another relevant factor may have been the fact that the United States started issuing index-linked bonds in January 1997 (see Elsasser and Sack (2004)). It seems likely that this reduced the liquidity premium on index-linked bonds as an asset class and may also therefore have contributed to lower UK real rates.

real yields derived from index-linked bonds, the largest two principal components — which can be labelled as 'level' and 'slope' factors because their loadings on the individual yields are respectively constant across maturity and negative at short maturities and positive at long maturities — account for 99.7% of the total variation in the four real yields. But when we also include the real policy rate proxy, the explanatory power of the first two factors goes down and we need a third factor — which can be termed a 'curvature' factor because its loading on yields is positive at short and long maturities and negative at medium-term maturities — to explain a similar amount of variation in the data. This analysis suggests that we may need an extra factor in order to model all five real rates, which seems quite intuitive given that our short real policy rate is likely to be more correlated with the business cycle than the longer-maturity real yields we examine.

Table C: Principal component analysis of real spot yields, excluding real policy rate

		Yield loadings			
Principal component	Proportion of total variance explained	y_t^{48}	y_t^{72}	y_t^{120}	y_t^{180}
1	94.176	0.438	0.474	0.518	0.561
2	5.549	-0.752	-0.243	0.254	0.558
3	0.272	-0.457	0.623	0.374	-0.514
4	0.003	0.185	-0.574	0.726	-0.331

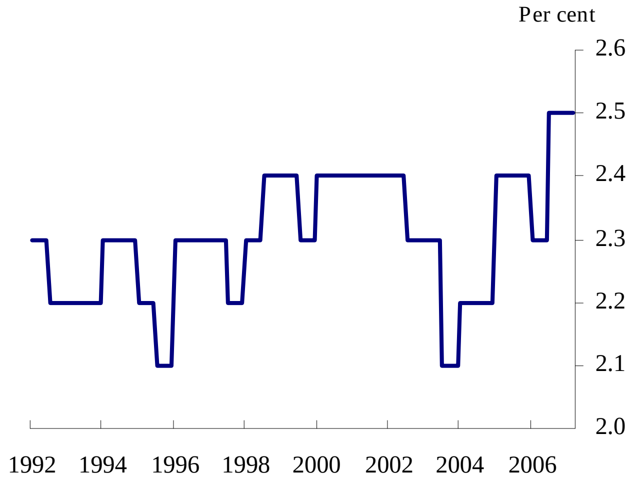
Table D: Principal component analysis of real spot yields, including real policy rate

		Yield loadings				
Principal component	Proportion of total variance explained	y_t^{1}	y_t^{48}	y_t^{72}	y_t^{120}	y_t^{180}
1	81.538	-0.547	-0.378	-0.395	-0.429	-0.467
2	14.855	0.83	-0.13	-0.266	-0.331	-0.337
3	3.483	0.096	-0.756	-0.284	0.215	0.541
4	0.123	-0.054	-0.485	0.603	0.365	-0.515
5	0.002	-0.000	0.185	-0.573	0.726	-0.332

As discussed in the introduction, some of the recent literature on term structure modelling uses survey data to supplement yields data, as a way of getting more information on the expected path of future short-term interest rates, in order to pin down the model estimates more accurately. For example, Kim and Wright (2005) use Blue Chip Financial Forecasts of future policy rates and inflation to augment their joint model of US nominal and real interest rates.[15] Unfortunately, the available survey data for the United Kingdom do not enable us to construct a survey-based measure of expected future short real rates. Instead we have used Consensus long-run forecasts

[15]See also Kim and Orphanides (2005). For an earlier example of using survey data in a term structure model see Pennachi (1991).

Chart 3: Consensus forecasts for average GDP growth five to ten years ahead



of average GDP growth five to ten years ahead to supplement our term structure model. Most theories would acknowledge GDP growth as an important driver of real risk-free rates over the long run and, although the theoretical conditions under which real rates actually equal GDP growth are very restrictive, this is often assumed to be a useful benchmark in market commentary on bond markets. As we explain further in Section 4 below, we use long-run GDP forecasts as a noisy signal of future short real rates, rather than constraining them to be equal.

The Consensus GDP growth data we use are shown in Chart 3. Compared with the real yields data, they exhibit relatively little variation, although there is some evidence of an increase at the end of the sample. Table E shows the result of including them in the principal component analysis of real yields, using a common sample based on the publication dates of the Consensus data. The loading on the real yields are quite similar to before, but the survey data loadings are noticeably different to the loadings on real yields and the large weight of the survey data on the third principal component suggests that the survey contains additional information not contained in real yields.

4 Empirical results

On the basis of the initial data analysis in Section 3, we estimated three different term structure model specifications. In our **baseline model** specification, we estimated an essentially affine model with two latent factors, using data on real spot yields with maturities of four, six, ten and

Table E: Principal component analysis of real spot yields and survey data

Principal component	Proportion of total variance explained	Yield loadings y_t^{48}	y_t^{72}	y_t^{120}	y_t^{180}	$g_t^{5\to10}$
1	93.699	0.421	0.466	0.523	0.576	-0.026
2	5.677	-0.766	-0.243	0.249	0.524	-0.130
3	0.359	0.248	-0.210	-0.214	0.141	-0.910
4	0.261	-0.370	0.595	0.309	-0.510	-0.390
5	100.000	-0.192	0.570	-0.723	0.337	0.038

fifteen years. In a variant of this model, described below as the **survey model**, we supplemented the baseline specification with survey information on long-horizon GDP growth. This involved including the survey data as an additional element in the observation equation, which was allowed to have a measurement error distribution with a different variance to the real yields. Since the principal components analysis suggested that the survey data had a large weight on the third principal component, we also included an additional latent factor in the model specification. As a final variant, we estimated a **policy rate model**, which included the proxy measure of the one-month real policy rate (discussed in Section 3 above), as well as the four longer-maturity real yields. Given the results of the principal components analysis, this model also included three latent factors and we allowed the measurement error of the real policy rate to have a different variance. However, since the estimation results produced a very large measurement error on the policy rate proxy (effectively discarding its information content and suggesting that it behaves rather differently to the real yields included in the model), we only present results for a model in which we constrained the measurement error variance of the policy rate proxy to be the same as for the included market rates. All the model specifications reported were first estimated in a general form and then tested down to produce a preferred specification. We now discuss the estimation results for these models.

4.1 Baseline model (two factors, four yields)

To recap, this model is a basic two-factor version of the yield curve model set out in Section 2, fitted to four, six, ten and fifteen year real spot yields.

The estimation results from a tested-down version of the model are shown in Table F. As well as showing the parameter estimates and *t*-ratios (based on outer-product estimates of the standard errors), the table shows the value of the log-likelihood function of the model and the associated

Akaike, Schwartz and Hannan-Quinn information criteria statistics. It is worth noting that where particular parameters are reported that do not appear statistically significant according to the reported t-statistics, this implies that the data rejected imposing a zero restriction according to a conventional likelihood ratio test.

Working down from the top of the table, the expected long-run real short rate in the model (the r^* parameter) is 2.4%. This tells us where the model expects short rates to get to in the very long run, rather than where they should be today. The two latent factors in the model are both highly persistent, judged by the size of the Φ_{11} and Φ_{22} coefficients. However, it is obviously difficult to interpret what the factors represent, since they are latent. The market price of risk parameter estimates are difficult to interpret directly for the same reason. The λ_i parameters indicate the estimated average price of risk (since the factors are by construction mean zero) but the signs are dependent on what the factors themselves represent. Similarly, the β_{ij} parameters give the loadings of the factors in the SDF and so their meaning is also dependent on what the factors are picking up. The key point to note is that in testing down the model, we were unable to restrict all the price of risk β parameters to zero, so the results are consistent with time-varying term premia. Notice finally that the estimate of the standard deviation of the measurement error (the η_1 parameter) is quite small, only 5 basis points.

Table F: Baseline model

	Estimate	t-stat
$r^* \cdot 12$	0.0235	3.16
Φ_{11}	0.9824	77.12
Φ_{22}	0.9902	73.18
Φ_{21}	-0.0046	-0.49
$\sigma_1 \cdot 1000$	0.2975	4.43
$\sigma_2 \cdot 1000$	0.1418	1.18
$\lambda_1/1000$		
$\lambda_2/1000$	-0.0186	-0.19
$\beta_{11}/120$	2924	2.52
$\beta_{21}/120$		
$\beta_{12}/120$	1785	0.97
$\beta_{22}/120$	-4060.3	-1.70
$\eta_1 \cdot 1200$	0.0531	16.02

$Log - lik$	AIC	HQ	SC
5864.2	-64.0	-63.9	-63.8

Note: $Log - lik$=log-likelihood value; AIC=Akaike information criterion; HQ=Hannan-Quinn information criterion; SC=Schwartz information criterion

The easiest way to assess the model is to examine how it decomposes actual forward rates into expected future short rates, term premia and the residual component (the part unexplained by the model).[16] This is an out-of-sample comparison, to the extent that forward rates are not included in the model estimation.

Chart 4 shows this decomposition for the ten-year forward rate, which is the horizon arguably of most relevance for the bond market conundrum we are trying to explain (we discuss the decomposition at shorter horizons below). From the small size of the unexplained component, it is clear that the model fits the forward rate data quite closely, apart from the period in the late 1990s when the MFR was introduced (see earlier discussion in Section 3). From the decomposition, the main result that emerges is that the ten-year real forward term premium exhibits much larger swings than the expected short rate over the sample. The estimated term premium averages 100 basis points in the period up to 1997,[17] before falling to a low of -75 basis points in early 2006. Not surprisingly, given these large swings, the fall in the term premium explains the larger part of the fall in the real forward rate over the conundrum period (which in the case of the United Kingdom extends back to the beginning of 2004). But the expected short rate ten years ahead, which might be thought of as a proxy for the neutral real rate of interest, also fell over the period (the model implies it fell 40 basis points between the end of 2003 and January 2006, less than half the fall in implied term premium) and shows a downward trend over most of the sample.

The pink and green dashed horizontal lines in the chart represent the model-implied long-run values of the ten-year ahead expected future short rate and the ten-year forward term premium respectively. As is apparent from the chart, this would suggest that the ten-year forward term premium was expected to rise by some 80 basis points from the value it reached at the beginning of 2006, while the expected future short rate was expected to increase by around 50 basis points.

To further assess the plausibility of the baseline model, Chart 5 presents the same forward rate decomposition at three and five year horizons (in the left and right-hand panels respectively). As might have been expected, the model estimates suggest movements in expected future short rates at short horizons are more volatile than at longer horizons.

[16]Convexity effects implied by the models are not shown separately as they are small and do not vary over time.

[17]Though note that the estimation method has a recursive element, so estimates in the first few years will be based on relatively little data and therefore may be more unreliable.

Chart 4: Decomposition of the ten-year real forward rate from the baseline model

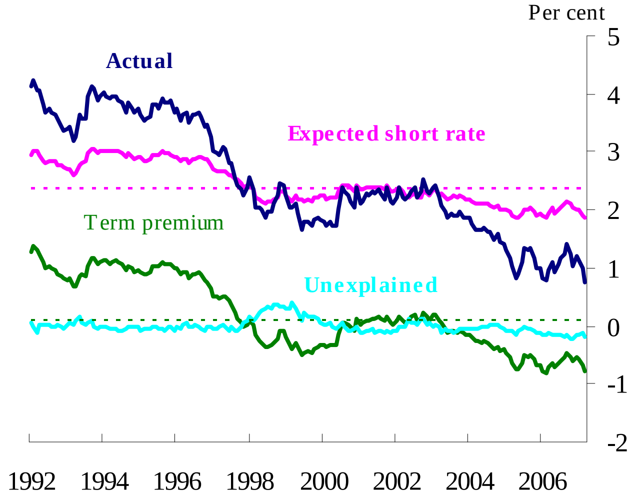
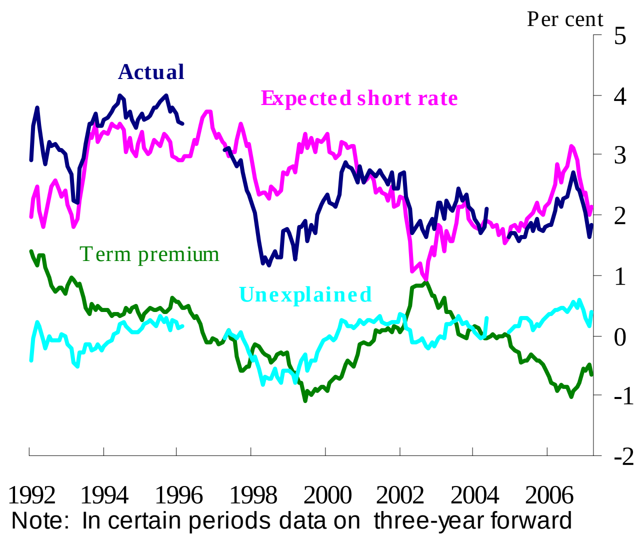
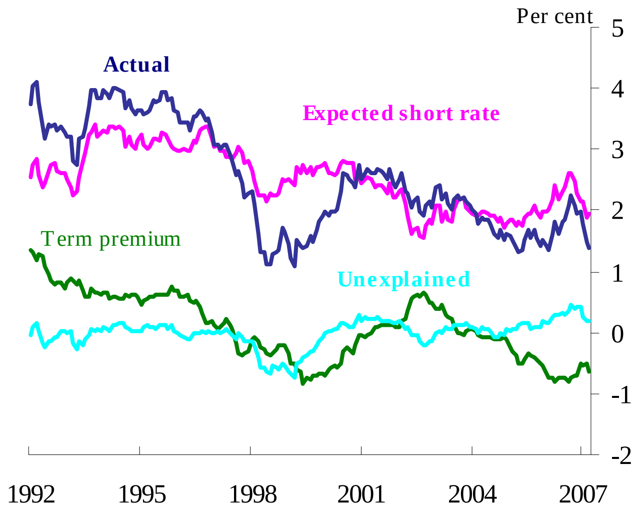


Chart 5: Decompositions of three and five-year forward rates from the baseline model





Note: In certain periods data on three-year forward rates are not available due to the absence of short-maturity bonds.

4.2 Survey model (three factors, four yields and long-term GDP growth forecasts)

This model is the three-factor version of the model from Section 2 extended to include survey data (which adds another line to the measurement equation, see Appendix B). The basic idea here is that we fit expectations of short real rates five to ten years ahead to survey forecasts of UK GDP growth five to ten years ahead, taken from Consensus forecasts. Since the survey forecasts are only sampled every six months, expectations from the model are not constrained by this method to be equal to the survey values even on average during intervening months. Of course, as already discussed, it is not clear from a theoretical point of view that expected short real rates at medium to long-horizons should be equated to expected real GDP growth, although this is often assumed to be a useful benchmark in market commentary. Unfortunately, there are no UK survey data — as there are for the United States — that refer to expectations of future short interest rates which, in combination with survey expectations of future inflation, could be used to derive a more direct proxy for the neutral real rate of interest.

The estimation results from a parsimonious version of the model are reported in Table G. The expected long-run real short rate in the model is 2.2%, a little lower than in the baseline model. Like the baseline model, the latent factors are highly persistent judged by the size of their autoregressive coefficients. In terms of the market price of risk parameter estimates, the main point to note again is that we are unable to restrict all the price of risk β parameters to zero, so the results are consistent with time-varying term premia. Finally, the estimate of the standard deviation of the measurement error of real yields is very small (less than 1 basis point), though the measurement error on the survey is a lot larger (41 basis points). Intuitively, the maximum likelihood estimates put less weight on fitting the survey data, though the measurement error is not unreasonably large.

The implied decomposition of the ten-year real forward rate into the expected future short real rate, the term premium and the unexplained component is shown in Chart 6. Although the model does not constrain expected future short rates to be equal to the survey GDP forecasts period by period, the results suggest that the model is highly sensitive to the inclusion of the survey data. Average long-run GDP growth is 2.3% over the sample, according to the Consensus forecasts, and this may help account for the fact that the expected long-run real short rate in this model is lower than in the other estimated models. Moreover, the low variability of the GDP survey forecasts (the standard deviation is as low as 0.1%) may explain why the model-implied expected

Table G: Survey model results

	Estimate	t-stat
$r^* \cdot 12$	0.0219	20.9
Φ_{11}	0.9402	117.3
Φ_{22}	0.9615	67.6
Φ_{33}	0.9837	178
Φ_{21}	-0.0098	-1.4
Φ_{31}	-0.0001	-0.04
Φ_{32}	0.0013	0.64
$\sigma_1 \cdot 1000$	0.3469	8.3
$\sigma_2 \cdot 1000$	0.1182	2.9
$\sigma_3 \cdot 1000$	0.0414	0.8
$\lambda_1/1000$		
$\lambda_2/1000$		
$\lambda_3/1000$	0.1248	0.6
$\beta_{11}/120$		
$\beta_{21}/120$		
$\beta_{31}/120$	-2345.9	-0.25
$\beta_{12}/120$		
$\beta_{22}/120$	-23209	-1.97
$\beta_{32}/120$		
$\beta_{13}/120$	22520	0.97
$\beta_{23}/120$	16195	0.98
$\beta_{33}/120$		
$\eta_1 \cdot 1200$	0.0081	17.1
$\eta_1 \cdot 1200$	0.41	3.1

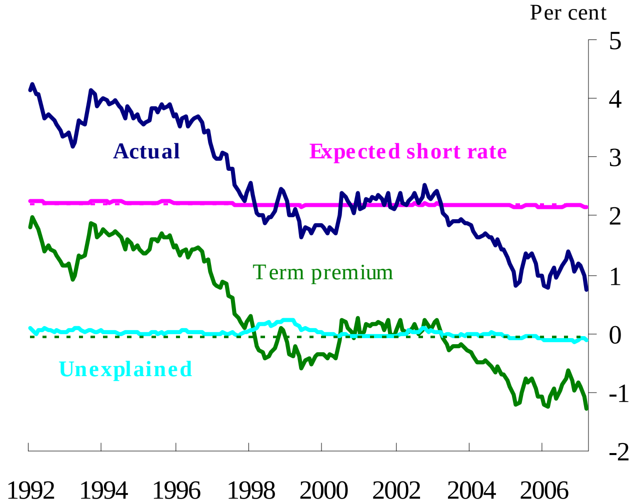
$Log-lik$	AIC	HQ	SC
6600.2	-71.9	-71.8	-71.6

Note: $Log-lik$=log-likelihood value; AIC=Akaike information criterion; HQ=Hannan-Quinn information criterion; SC=Schwartz information criterion

future real rate ten-years ahead is almost flat at its steady-state value. With all variation in the expected future short rate removed, the decomposition attributes all movements in forward rates over the period to the forward term premium. This seems implausible on *a priori* grounds and, given that the restriction we have imposed between real rates and growth is rather unclear theoretically, suggests placing less weight on this model.

For completeness, Chart 7 presents the decomposition from this model at three and five-year horizons into expected future short rates, forward premia and the unexplained component (in the left and right-hand graphs respectively). At these horizons, the unexplained component of forward rates is much larger and exhibits considerable autocorrelation, suggesting the model may be misspecified in some way. The decomposition also shows the variability in expected future short rates is somewhat larger, as we might expect, though much less than for the baseline model,

Chart 6: Decomposition of the ten-year real forward rate from the survey model



implying a much larger role for changing forward premia in explaining forward rates. This pattern of largely time-invariant expectations of future short real rates also seems rather implausible *a priori* and suggests that this version of the model may be overly constrained in imposing that future short real rates equal GDP growth on average.

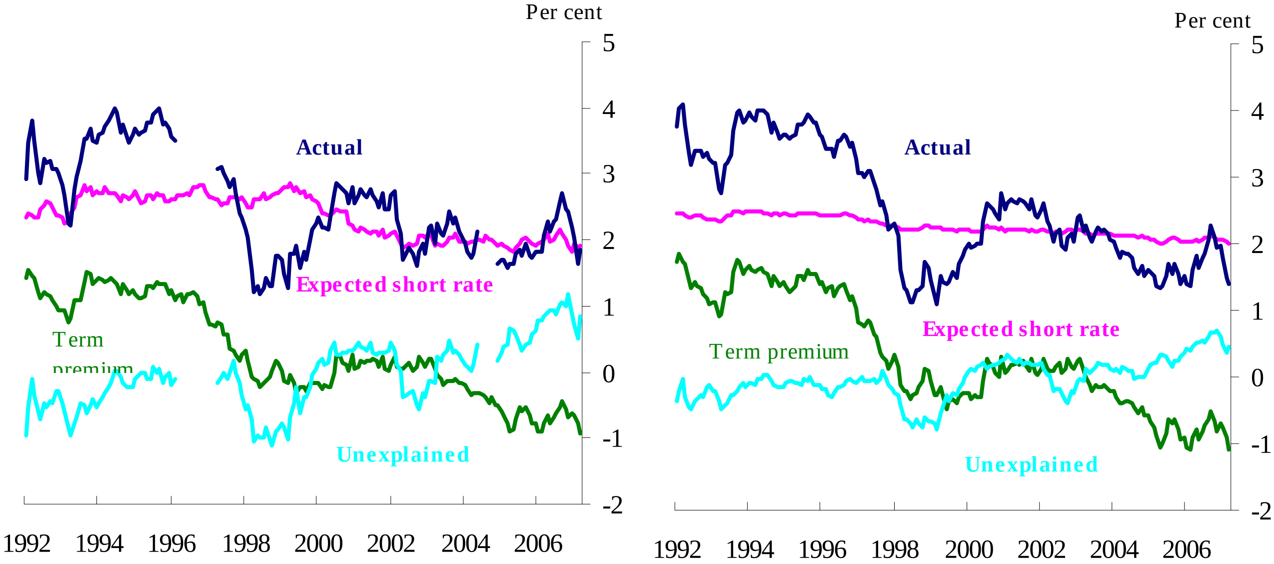
4.3 Policy rate model (three factors, four yields and proxy one-month policy rate)

This variant of the model is fitted to the one-month policy rate proxy as well as to real yields and allows for three factors.

The empirical results for the model are shown in Table H. In this case, the expected long-run short rate parameter is 2.8%, slightly higher than the baseline and survey models. The three latent factors are again all highly persistent judged by the size of the autoregressive coefficients. The key point to note is that in testing down the model, we were again unable to restrict all the price of risk β parameters to zero, so the results are consistent with time-varying term premia.

We illustrate the model's implied decomposition of the ten-year real forward rate into the expected future short rate, the term premium and the unexplained component in Chart 8. In contrast to the baseline and survey model results, movements in the implied forward term premium are now more similar to those of the expected short rate, and exhibit slightly less

Chart 7: Decompositions of three and five-year forward rates from the survey model



Note: In certain periods data on three-year forward rates are not available due to the absence of short-maturity bonds.

variation. The estimated term premium from the model averaged about 35 basis points in the period up to 1997, before becoming negative over the rest of the sample. It also seems that the effect of including the real policy rate proxy as one of the observed variables in the model is to place more weight on the downward trend in the real policy rate over the sample. Although even for this model, the larger part of the fall in the long forward rate during the conundrum period is attributed to a lower term premium (the estimated term premium fell by 75 basis points between the end of 2003 and the beginning of 2006, while the expected future short real rate fell by 40 basis points). Again the pink and green dashed horizontal lines in the chart represent the model implied long-run values of the ten-year ahead expected future short rate and ten-year forward term premium respectively. In this case, the model suggests that the fall in the forward premium during the conundrum period brought it back more closely in line with its long-run expected level. In contrast, the decline in the expected future short rate over the conundrum period brought it about 60 basis points below its expected long-run level.

As a further plausibility check on the properties of the model, Chart 9 shows the decomposition it implies for three and five-year forward rates. The estimates from the policy rate model suggest that expected short rates exhibit more variation at shorter maturities, although this is less marked

Table H: Real policy rate model

	Estimate	t-stat
$r^* \cdot 12$	0.0281	2.41
Φ_{11}	0.9862	76.7
Φ_{22}	0.9962	775.8
Φ_{33}	0.9744	63.02
Φ_{21}	−0.0001	−53.2
Φ_{31}	−0.0176	−1.86
Φ_{32}	−1.1825	−2.47
$\sigma_1 \cdot 1000$	0.1863	11.8
$\sigma_2 \cdot 1000$	0.0012	28.0
$\sigma_3 \cdot 1000$	0.1029	4.4
$\lambda_1/1000$	3.2772	1.04
$\lambda_2/1000$	84.25	2.4
$\lambda_3/1000$	3.6501	1.06
$\beta_{11}/120$	14615	3.49
$\beta_{21}/120$		
$\beta_{31}/120$	14021	2.13
$\beta_{12}/120$		
$\beta_{22}/120$		
$\beta_{32}/120$		
$\beta_{13}/120$	−20881	−2.24
$\beta_{23}/120$		
$\beta_{33}/120$	−26548	−1.82
$\eta_1 \cdot 1200$	0.0474	20.4

$Log-lik$	AIC	HQ	SC
7196.6	−78.5	−78.3	−78.1

Note: $Log - lik$=log-likelihood value; AIC=Akaike information criterion; HQ=Hannan-Quinn information criterion; SC=Schwartz information criterion

than in the case of the baseline model.

4.4 Comparing the models

While a comparison of the broad plausibility of the forward rate decompositions from the models tends to favour the baseline and policy rate models, what do statistical criteria suggest? A comparison based on statistical criteria is more difficult because the favoured baseline and policy rate models are not nested, so likelihood ratio tests are not strictly valid.[18] Informal comparisons based on information criteria, however, tend to marginally favour the policy rate model.

Another means of comparison is through the properties of the models' in-sample and

[18]This is because some of the model parameters will not be identified under the null hypothesis.

Chart 8: Decomposition of ten-year real forward rates from the policy rate model

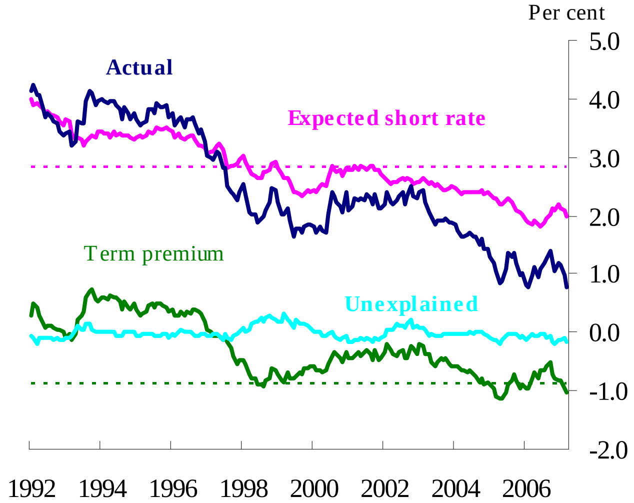
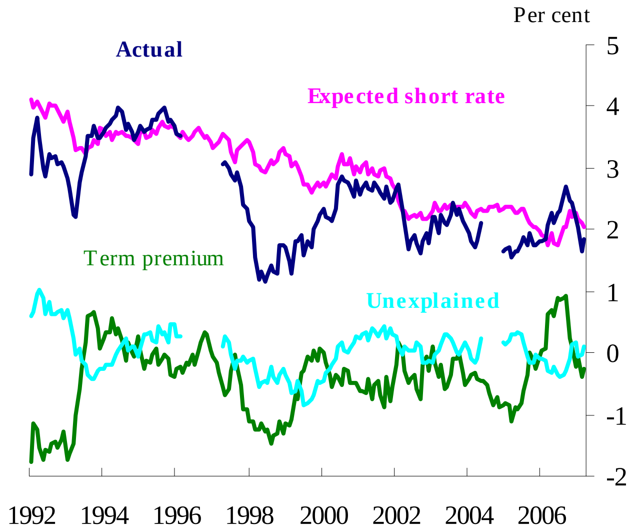
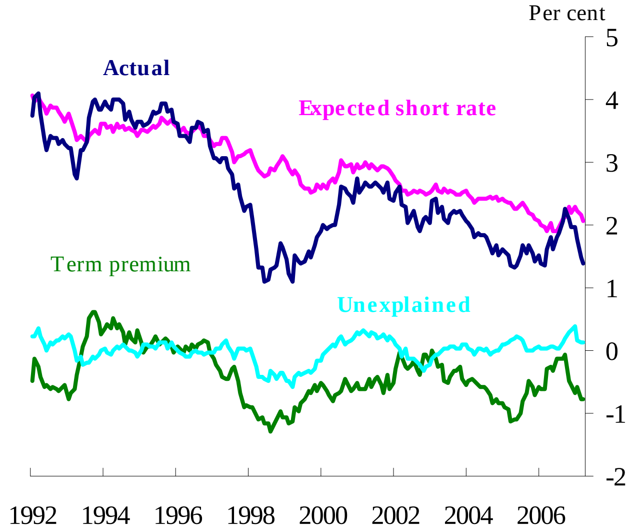


Chart 9: Decompositions of three and five-year forward rates from the policy rate model



Note: In certain periods data on three-year forward rates are not available due to the absence of short-maturity bonds.



out-of-sample residuals. As shown in Table I, in-sample residuals on the included spot yields are all close to mean zero across all three models, in the worst-fitting case just over 2 basis points, and do not exhibit any serious serial correlation. So it is difficult to discriminate between the models on these grounds. We have already illustrated the ability of the models to fit forward rates (which were not included directly in the model estimation) in the earlier section on model results and this is formalised in Table J, which provides a number of summary statistics for these out-of-sample errors for each of the three models. Again it is quite difficult to discriminate decisively between the models on these grounds, although the survey model tends to perform slightly more poorly than the other models.

Table I: In-sample model residuals (percentage points)

		Maturity					
		1month	4yrs	6yrs	10yrs	15yrs	Survey
Baseline model							
	Mean		0.0016	-0.0012	-0.0043	-0.0065	
	Standard deviation		0.1938	0.1664	0.1605	0.1706	
	ρ_1		0.1031	0.1027	0.1418	0.2080	
	ρ_{12}		0.0526	0.0502	0.0155	0.0208	
Survey model							
	Mean		0.0159	0.0147	0.0122	0.0107	-0.0246
	Standard deviation		0.1855	0.1749	0.1687	0.1718	0.4217
	ρ_1		0.0490	0.1006	0.1477	0.1532	-
	ρ_{12}		0.0760	0.0377	0.0186	0.0520	-
Policy rate model							
	Mean	-0.0205	-0.0091	0.0119	0.0055	0.0061	
	Standard deviation	0.2892	0.2153	0.1965	0.1948	0.2031	
	ρ_1	-0.1218	0.1330	0.0970	0.1405	0.1792	
	ρ_{12}	-0.2067	0.0063	0.0192	-0.0033	-0.0111	

5 Interpretation

One clear finding of our results across all the models we estimate is the importance of movements in estimated real term premia in explaining movements in real rates. This is contrary to what appears to be the conventional wisdom that real term premia are small and negligible. Indeed, many papers simply ignore the presence of real term premia altogether (for a recent example, see Ang, Bekaert and Wei (2007)).

Another important finding, common to all the estimated model specifications, is that our term

Table J: Out-of-sample model diagnostics: forward rates (percentage points)

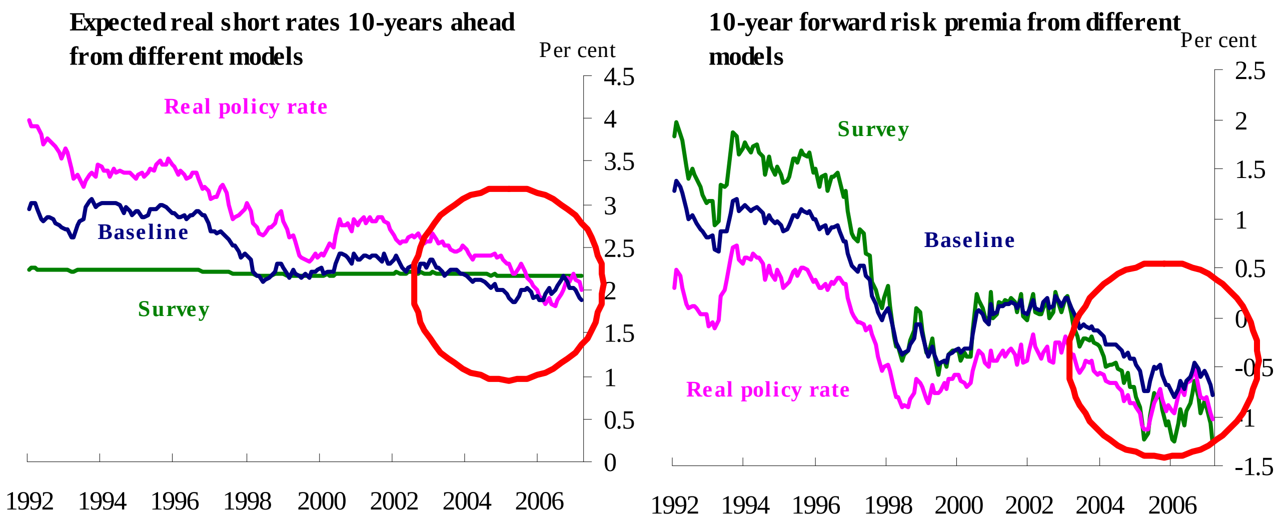
		Maturity			
		3yrs	5yrs	10yrs	15yrs
Baseline model					
	Mean	-0.0084	0.0273	-0.0197	-0.0745
	Standard deviation	0.3066	0.1064	0.1795	0.1632
	ρ_1	0.9115	0.9102	0.9553	0.9488
	ρ_{12}	0.2697	0.2657	0.2698	0.3040
	MAE	0.2541	0.0941	0.1450	0.1306
Survey model					
	Mean	0.0332	0.0574	-0.0315	-0.1030
	Standard deviation	0.5392	0.0585	0.2081	0.0775
	ρ_1	0.9510	0.9663	0.9582	0.9587
	ρ_{12}	0.4100	0.0573	0.1571	0.0723
	MAE	0.4422	0.0670	0.1623	0.1035
Policy rate model					
	Mean	0.0060	0.0210	-0.0128	-0.0893
	Standard deviation	0.2540	0.1706	0.2155	0.1363
	ρ_1	0.8263	0.9326	0.9637	0.9316
	ρ_{12}	-0.2035	0.2640	0.3610	0.1937
	MAE	0.2046	0.1265	0.1684	0.1216

premia estimates appear to have been negative over much of the sample period since the late 1990s.[19] Negative term premia are of course quite consistent with finance theory and may indicate that for some investors long-maturity index-linked bonds are seen as providing a form of 'insurance'. However, the emergence of negative term premia in the late 1990s seems likely to have reflected the impact of various accounting and regulatory changes that have caused pension funds to match their assets more closely to their liabilities by switching into long-maturity conventional and index-linked bonds (see McGrath and Windle (2006)). Indeed, the timing of the move to negative term premia suggested by the model decompositions seems to broadly match the introduction of the MFR in 1997, which market commentary suggests had a significant impact on UK pension fund asset allocation.[20] Whether or not negative term premia reflect genuine investor risk perceptions/preferences (Category 2) or the impact of regulatory pressures (Category 3) clearly matters for how we interpret subsequent movements in our model-based

[19] Interestingly, our results for the United Kingdom contrast with those for the United States where real term premia estimates appear to be positive (see eg Kim and Wright (2005)). This may be an indication of the greater relative importance of institutional investors in the UK market, though it might also reflect the fact that most US studies of the real term structure use synthetic real rates constructed by combining inflation data with nominal yields, rather than modelling real yields from index-linked instruments.

[20] More recently, the Pensions Act 2004, which became effective in December 2005, introduced a new Pensions Regulator with powers to require pension fund trustees and sponsors to address issues of underfunding. Another factor that may also have influenced pension fund behaviour has been the 'FRS 17' accounting standard, which became effective from the start of 2005, and has meant that pension scheme deficits/surpluses need to be measured at market value and included on company balance sheets. Both these factors are thought to have increased pension fund demand for longer-duration nominal and real gilts, as assets which provide a better match for their liabilities. See discussion in the 'Markets and Operations' article of the *Bank of England Quarterly Bulletin*, Spring 2006.

Chart 10: Accounting for the conundrum: model comparisons



term premia estimates.

In terms of understanding the fall in long real rates over the conundrum period, all the estimated models suggest that falls in UK long real rates to a large degree reflected larger negative real term premia (see Chart 10), though the extent to which this is true varies with the precise model specification used and there is some evidence that financial market expectations of long-horizon short real rates have also declined.

If we assume that negative premia are a reflection of the constraints on institutional investors then the subsequent fall in term premia over the conundrum period could be consistent with lower risk premia, and therefore also consistent with the general rise in asset prices and the compression of corporate bond spreads that occurred during the same period. There are several more or less plausible explanations for why term premia might have declined, including financial innovation resulting from more highly integrated capital markets (although the precise timing is more difficult to explain), greater macroeconomic stability (here again the timing is difficult to explain), and the search for yield/excess liquidity hypothesis. It may also be consistent with a greater impact from imbalances between demand and supply, arising from preferred habitat behaviour by pension funds or Asian central banks (what we have called in the Introduction Category 3 explanations). However, the fact that the decline in long rates over the past few years has been a global phenomenon suggests that this cannot have been the primary cause, as portfolio flow explanations, eg the buying up of US Treasuries by Asian central banks, tend to be country specific and difficult to reconcile with the global nature of the conundrum. This inclines us to put more weight on the search for yield/excess liquidity explanation for the compression of premia. But the fact that our models provide some evidence that long-horizon expected risk-free rates — a proxy neutral rate of interest estimate — have fallen suggests that explanations related to changes in the balance of investment and saving may also have been at work.

The more recent decline in real rates and real term premia in the second half of 2007 seems likely to be the consequence of a 'flight to quality' from risky assets, as this has been a time of greater uncertainty and volatility in financial markets, originally triggered by the sub-prime crisis in the United States and its impact on money and credit markets. Taking our results at face value would suggest that there are clear upside risks to real rates going forward, as real rates remained well below the long-run equilibrium levels implied by our models, though the models do not of course provide clear guidance on the likely timing of any adjustment.

One possible concern about taking our results too literally is that they may be a reflection of the affine model set-up. In particular, the model assumes a stationary structure in which the short rate is constrained to be mean-reverting (see eg Kozicki and Tinsley (2001)). It is possible to show from **(12)** above that

$$E_t\left[y^1_{t+n}\right] = r^* + \gamma'\Phi^n z_t.$$

Given that the model is stationary (the eigenvalues of Φ lie inside the unit circle), as n gets larger the expected short rate approaches a constant neutral rate, r^*. If expected short rates are constrained in this way, is it the case that movements in interest rates are inevitably attributed to movements in term premia, if the model is to fit the data? In fact, it is possible to show that this is an empirical question. The amount of persistence in the models we have estimated suggests that, in practice, variation in expected future short rates can be equally important. This has already been shown graphically by the policy rate model, in that movements in long forward rates are attributed more equally to movements in expected short rates and forward term premia.

Another concern about the stationarity assumption in these models is that they will not be able to handle a structural change in the neutral rate (ie permanent changes in r^*), but this problem is hardly unique to affine term structure models and would beset most other approaches. A more serious issue is probably that the model is estimated over a relatively short period, which may mean that it is subject to small sample problems.

6 Conclusions

This paper describes the results of applying a standard 'finance' approach — the essentially affine term structure model with latent factors — to modelling the UK real term structure derived from government index-linked bonds, over the period since October 1992 when the United Kingdom has operated an inflation target.

One key finding that emerges across all the models estimated is that time-varying term premia appear to be extremely important in explaining movements in long real forward rates over time. This contradicts the conventional wisdom that real term premia are small and insignificant and therefore can safely be ignored.

Another important finding, common to all the estimated model specifications, is that our term premia estimates appear to have been negative over much of the sample period since the late 1990s. We have argued that this is likely to reflect the impact of various accounting and regulatory changes in the United Kingdom that have encouraged pension funds to match their assets more closely to their liabilities, by switching into long-maturity conventional and index-linked bonds. The importance of this Category 3 explanation for the behaviour of term premia after the 1990s needs to be borne in mind when interpreting more recent downward moves in term premia.

In terms of understanding the fall in long rates over the conundrum period during 2004 and 2005, all the estimated models suggest that falls in UK long real rates have to a significant degree reflected reductions in real term premia, though the extent to which this is true varies with the precise model specification used. The importance of the reduction in term premia might indicate the influence of changing institutional investor behaviour, but the fact that the decline in long rates was a global phenomenon suggests to us that this is unlikely to have been the primary cause. This leads us to the conclusion that excess liquidity and search for yield were more important in explaining the compression of real term premia. But since our models also suggest that there is some evidence that perceptions of the neutral rate of interest may have fallen, we cannot rule out the possibility that changes in the balance of investment and saving may also have had an impact. In terms of the recent conjuncture, all our model specifications would suggest that there is a risk real rates may rise in the future, since in all of the models forward premia or expected future short rates are below their long-run expected levels.

In interpreting our results, we noted that there are a number of caveats with the framework that also need to be borne in mind. The models are unsuited to dealing with structural change in the neutral rate since they are stationary and they may suffer from small sample problems, given the relatively short span of the available data. This may help explain the sensitivity of the results to the specification used. Of course, these caveats would apply to most of the other available empirical approaches.

In terms of further research, one obvious extension to the models presented here would be to incorporate information from both the nominal and real term structures into the same essentially affine framework. By including nominal yields and a model of inflation, this would provide a more systematic way of getting information on the short end of the real term structure and, by imposing a consistent risk-pricing structure across nominal and real bonds, such a joint model would enable us in principle to extract market expectations of future nominal, real and inflation rates and of nominal, real and inflation premia.

Appendix A: Model derivations

In this section we show the derivation of the recursive equations **(6)** and **(7)**.

For convenience we first restate the three basic equations in the model. The state variables are assumed to follow a first-order VAR model

$$z_{t+1} = \Phi z_t + \Omega^{1/2}\varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim NID(0, I_3).$$

The logarithm of the SDF is

$$m_{t+1} = -\left(r^* + \gamma' z_t\right) - \frac{\Lambda_t'\Omega\Lambda_t}{2} - \Lambda_t'\Omega^{1/2}\varepsilon_{t+1}.$$

Finally, we assume that log bond prices are affine in the state vector, so that the log price of a zero-coupon bond with n period to maturity at time t is given by

$$\ln P_t^n = p_t^n = A_n + B_n' z_t$$

where

$$B_n' = \begin{bmatrix} B_{n,1} & B_{n,2} & B_{n,3} \end{bmatrix}.$$

To derive the recursive equations consistent with the model, we begin with the fundamental asset pricing equation which states that

$$P_t^n = E_t\left[M_{t+1}P_{t+1}^{n-1}\right].$$

Recall that if X is lognormal then

$$E(X) = \exp(E(\ln X) + 1/2 Var(\ln X)).$$

So if we assume that bond prices and the log stochastic discount factor are jointly conditional lognormal we can then write

$$p_t^n = E_t\left[m_{t+1} + p_{t+1}^{n-1}\right] + 1/2 Var_t\left[m_{t+1} + p_{t+1}^{n-1}\right] \qquad \textbf{(A-1)}$$

where lower case denotes that we have taken the natural logarithm. The proof now proceeds by substituting in for log bond prices and the log SDF and then manipulating this expression. To see this more clearly, we expand out the first and second terms separately before combining them. So first taking the expectation term and substituting in for the bond price and the SDF we have the following expression:

$$E_t\left[m_{t+1}+p_{t+1}^{n-1}\right]=E_t\left[\left(-\left(r^*+\gamma' z_t\right)-\frac{\Lambda_t'\Omega\Lambda_t}{2}-\Lambda_t'\Omega^{1/2}\epsilon_{t+1}\right)+\left(A_{n-1}+B_{n-1}^{'}z_{t+1}\right)\right] \tag{A-2}$$

If we now substitute in for z_{t+1}, take through the expectations operator and rearrange we get

$$\begin{aligned}E_t\left[m_{t+1}+p_{t+1}^{n-1}\right] &= E_t\left[\left(-\left(r^*+\gamma' z_t\right)-\frac{\Lambda_t'\Omega\Lambda_t}{2}-\Lambda_t'\Omega^{1/2}\epsilon_{t+1}\right)+A_{n-1}+B_{n-1}^{'}\left(\Phi z_t+\Omega^{1/2}\varepsilon_{t+1}\right)\right]\\ &= \left(-\left(r^*+\gamma' z_t\right)-\frac{\Lambda_t'\Omega\Lambda_t}{2}-\Lambda_t'\Omega^{1/2}E_t\varepsilon_{t+1}\right)+A_{n-1}+B_{n-1}^{'}\left(\Phi z_t+E_t\Omega^{1/2}\varepsilon_{t+1}\right)\\ &= -\left(r^*+\gamma' z_t\right)-\frac{\Lambda_t'\Omega\Lambda_t}{2}+A_{n-1}+B_{n-1}^{'}\Phi z_t.\end{aligned}$$

Now turning to the second term in **(A-1)** and using the same substitutions we have

$$Var_t\left[m_{t+1}+p_{t+1}^{n-1}\right]=Var_t\left[\left(-\left(r^*+\gamma' z_t\right)-\frac{\Lambda_t'\Omega\Lambda_t}{2}-\Lambda_t'\Omega^{1/2}\epsilon_{t+1}\right)+\left(A_{n-1}+B_{n-1}^{'}z_{t+1}\right)\right].$$

We again substitute in for z_{t+1} and this time drop the constant terms which can be ignored.

$$\begin{aligned}Var_t\left[m_{t+1}+p_{t+1}^{n-1}\right] &= Var_t\left[\left(-\left(r^*+\gamma' z_t\right)-\frac{\Lambda_t'\Omega\Lambda_t}{2}-\Lambda_t'\Omega^{1/2}\epsilon_{t+1}\right)+A_{n-1}+B_{n-1}^{'}\left(\Phi z_t+\Omega^{1/2}\varepsilon_{t+1}\right)\right]\\ &= Var_t\left[-\left(\Lambda_t'\Omega^{1/2}-B_{n-1}^{'}\Omega^{1/2}\right)\varepsilon_{t+1}\right].\end{aligned}$$

We now expand out the right-hand side of this expression

$$\begin{aligned}Var_t\left[m_{t+1}+p_{t+1}^{n-1}\right] &= \left(\Lambda_t'\Omega^{1/2}-B_{n-1}^{'}\Omega^{1/2}\right)\left(\Lambda_t'\Omega^{1/2}-B_{n-1}^{'}\Omega^{1/2}\right)'\\ &= \Lambda_t'\Omega\Lambda_t-\Lambda_t'\Omega B_{n-1}-B_{n-1}^{'}\Omega\Lambda_t+B_{n-1}^{'}\Omega B_{n-1}\\ &= \Lambda_t'\Omega\Lambda_t-2B_{n-1}^{'}\Omega\Lambda_t+B_{n-1}^{'}\Omega B_{n-1}\end{aligned}$$

where the last line uses the fact that $B_{n-1}^{'}\Omega\Lambda_t$ is a scalar. We can now combine both these expressions to get an expression for the log bond price, and then rearranging terms obtain

$$\begin{aligned} p_t^n &= \left(-\left(r^* + \gamma' z_t\right) - \frac{\Lambda_t' \Omega \Lambda_t}{2} + A_{n-1} + B_{n-1}^{'} \Phi z_t\right) + \left(\frac{\Lambda_t' \Omega \Lambda_t - 2B_{n-1}^{'} \Omega \Lambda_t + B_{n-1}^{'} \Omega B_{n-1}}{2}\right) \\ &= -\left(r^* + \gamma' z_t\right) - \frac{\Lambda_t' \Omega \Lambda_t}{2} + A_{n-1} + B_{n-1}^{'} \Phi z_t + \frac{\Lambda_t' \Omega \Lambda_t}{2} - B_{n-1}^{'} \Omega \Lambda_t + \frac{B_{n-1}^{'} \Omega B_{n-1}}{2} \\ &= -\left(r^* + \gamma' z_t\right) + A_{n-1} + B_{n-1}^{'} \Phi z_t - B_{n-1}^{'} \Omega \Lambda_t + \frac{B_{n-1}^{'} \Omega B_{n-1}}{2}. \end{aligned}$$

We now substitute in for Λ_t from **(3)** and collect terms.

$$\begin{aligned} p_t^n &= -\left(r^* + \gamma' z_t\right) + A_{n-1} + B_{n-1}^{'} \Phi z_t - B_{n-1}^{'} \Omega \left(\lambda + \beta z_t\right) + \frac{B_{n-1}^{'} \Omega B_{n-1}}{2} \\ &= \left(r^* + A_{n-1} - B_{n-1}^{'} \Omega \lambda + \frac{B_{n-1}^{'} \Omega B_{n-1}}{2}\right) + \left(-\gamma' + B_{n-1}^{'} \left(\Phi - \Omega \beta\right)\right) z_t \end{aligned}$$

Substituting for p_t^n on the left-hand side using **(4)** we get the following expression, which is shown as **(5)** in the text

$$A_n + B_n^{'} z_t = \left(-r^* + A_{n-1} - B_{n-1}^{'} \Omega \lambda + \frac{B_{n-1}^{'} \Omega B_{n-1}}{2}\right) + \left(-\gamma' + B_{n-1}^{'} \left(\Phi - \Omega \beta\right)\right) z_t.$$

From this expression we immediately get the two recursive equations **(6)** and **(7)**:

$$\begin{aligned} A_n &= -r^* + A_{n-1} - B_{n-1}^{'} \Omega \lambda + \frac{B_{n-1}^{'} \Omega B_{n-1}}{2} \\ B_n' &= -\gamma' + B_{n-1}^{'} \left(\Phi - \Omega \beta\right). \end{aligned}$$

Since we know $P_t^0 = 1$, we can start up these recursions with

$$A_0 = 0 \qquad B_0^{'} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

Appendix B: State-space representations

We report three model specifications in the paper. We give the state-space representations of each of them below.

Baseline model (two factors, four yields)

This model is a two-factor version of the yield curve model set out in Section 2, fitted to four, six, ten and fifteen-year real spot yields.

The observation equation is

$$\begin{bmatrix} y_{48,t} \\ y_{72,t} \\ y_{120,t} \\ y_{180,t} \end{bmatrix} = \begin{bmatrix} A^*_{48} \\ A^*_{72} \\ A^*_{120} \\ A^*_{180} \end{bmatrix} + \begin{bmatrix} B^{*\prime}_{48} \\ B^{*\prime}_{72} \\ B^{*\prime}_{120} \\ B^{*\prime}_{180} \end{bmatrix} z_t + \begin{bmatrix} \upsilon_{48,t} \\ \upsilon_{72,t} \\ \upsilon_{120,t} \\ \upsilon_{180,t} \end{bmatrix}$$

$$\begin{bmatrix} \upsilon_{48,t} \\ \upsilon_{72,t} \\ \upsilon_{120,t} \\ \upsilon_{180,t} \end{bmatrix} \sim NID\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \eta_1^2 & 0 & 0 & 0 \\ 0 & \eta_1^2 & 0 & 0 \\ 0 & 0 & \eta_1^2 & 0 \\ 0 & 0 & 0 & \eta_1^2 \end{bmatrix} \right)$$

and the state equation is

$$z_{t+1} = \Phi z_t + \Omega^{1/2} \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim NID(0, I_2)$$

where $z_t = \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix}$.

Survey model (three factors, four yields and long-term GDP growth forecasts)

This is the three-factor model from Section 2 extended to include survey data. The basic idea here is to relate model expectations about average real policy rates five to ten years ahead to survey expectations of GDP growth five to ten years ahead, denoted by $g_t^{5\to10}$, assuming that they are on average equal but not imposing that they are equal in any particular time period.

Remember that real policy rate expectations are

$$
\begin{aligned}
E_t\left[y^1_{t+n}\right] &= f^n_t - \phi_{t,n} - \omega_{t,n} \\
&= f^n_t - \left(f^n_t - f^n_t\big|_{\lambda=0,\beta=0}\right) - \left(f^n_t\big|_{\lambda=0,\beta=0} - f^n_t\big|_{\lambda=0,\beta=0,\Omega=0}\right) \\
&= f^n_t\big|_{\lambda=0,\beta=0,\Omega=0} && \textbf{(B-1)} \\
&= \left(A_n - A_{n+1}\right)\big|_{\lambda=0,\beta=0,\Omega=0} + \left(B'_n - B'_{n+1}\right)\big|_{\lambda=0,\beta=0,\Omega=0} z_t && \textbf{(B-2)}
\end{aligned}
$$

using equations **(9)** and **(12)**. Now define

$$
\begin{aligned}
A_{survey} &= \frac{\sum_{n=60}^{119}\left(A_n - A_{n+1}\right)\big|_{\lambda=0,\beta=0,\Omega=0}}{60} \\
B'_{survey} &= \frac{\sum_{n=60}^{119}\left(B'_n - B'_{n+1}\right)\big|_{\lambda=0,\beta=0,\Omega=0}}{60}
\end{aligned}
$$

The measurement equation for the model is

$$
\begin{bmatrix} g_t^{5\to10} \\ y_{48,t} \\ y_{72,t} \\ y_{120,t} \\ y_{180,t} \end{bmatrix} = \begin{bmatrix} A_{survey} \\ A^*_{48} \\ A^*_{72} \\ A^*_{120} \\ A^*_{180} \end{bmatrix} + \begin{bmatrix} B'_{survey} \\ B^{*\prime}_{48} \\ B^{*\prime}_{72} \\ B^{*\prime}_{120} \\ B^{*\prime}_{180} \end{bmatrix} z_t + \begin{bmatrix} \upsilon_{survey,t} \\ \upsilon_{48,t} \\ \upsilon_{72,t} \\ \upsilon_{120,t} \\ \upsilon_{180,t} \end{bmatrix}
$$

$$
\begin{bmatrix} \upsilon_{survey,t} \\ \upsilon_{48,t} \\ \upsilon_{72,t} \\ \upsilon_{120,t} \\ \upsilon_{180,t} \end{bmatrix} \sim NID\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \eta_2^2 & 0 & 0 & 0 & 0 \\ 0 & \eta_1^2 & 0 & 0 & 0 \\ 0 & 0 & \eta_1^2 & 0 & 0 \\ 0 & 0 & 0 & \eta_1^2 & 0 \\ 0 & 0 & 0 & 0 & \eta_1^2 \end{bmatrix}\right)
$$

and state equation in this case is

$$z_{t+1} = \Phi z_t + \Omega^{1/2}\varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim NID(0, I_3)$$

where

$$z_t = \begin{bmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{bmatrix}.$$

Policy rate model (three factors, four yields and proxy one-month policy rate)

This is a three-factor model version of the model from Section 2 fitted to the one-month real rate proxy and to four, six, ten, and fifteen-year real spot yields. Consequently, the observation equation is

$$\begin{bmatrix} y_{1,t} \\ y_{48,t} \\ y_{72,t} \\ y_{120,t} \\ y_{180,t} \end{bmatrix} = \begin{bmatrix} A_1^* \\ A_{48}^* \\ A_{72}^* \\ A_{120}^* \\ A_{180}^* \end{bmatrix} + \begin{bmatrix} B_1^{*\prime} \\ B_{48}^{*\prime} \\ B_{72}^{*\prime} \\ B_{120}^{*\prime} \\ B_{180}^{*\prime} \end{bmatrix} z_t + \begin{bmatrix} v_{1,t} \\ v_{48,t} \\ v_{72,t} \\ v_{120,t} \\ v_{180,t} \end{bmatrix}$$

$$\begin{bmatrix} v_{1,t} \\ v_{48,t} \\ v_{72,t} \\ v_{120,t} \\ v_{180,t} \end{bmatrix} \sim NID\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_2^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_2^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_2^2 \end{bmatrix} \right)$$

and the state equation is as before.

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