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Michael Joyce,[1] Peter Lildholdt[2] and Steffen Sorensen[3]

Abstract

This paper analyses the nominal and real interest rate term structures in the United Kingdom over the fifteen-year period that the UK monetary authorities have pursued an explicit inflation target, using a four-factor essentially affine term structure model. The model imposes no-arbitrage restrictions across nominal and real yields, enabling us to decompose nominal forward rates into expected real short rates, expected inflation, real term premia and inflation risk premia. We find that inflation risk premia and longer-term inflation expectations fell significantly when the Bank of England was made operationally independent in 1997. The 'conundrum' of unusually low long-term real rates that began in 2004 is mainly attributed by the model to a fall in real term premia, though a significant part of the fall is left unexplained. The relative inability of the model to fit long real forwards during much of this recent period may reflect strong pension fund demand for index-linked bonds. Moreover, the model decompositions suggest that these special factors affecting the index-linked market may also partly account for the contemporaneous rise in longer-horizon inflation breakeven rates.

Key words: Inflation expectations, inflation risk premia, affine term structure model.

JEL classification: C40, E43, E52.

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Summary

The nominal and real interest rates implied by government conventional and index-linked bonds of different maturities (ie the term structure of nominal and real interest rates) can potentially provide monetary policy makers with a great deal of information about financial market expectations of both future interest rates and inflation. The nominal and real term structures embody market expectations of future nominal and real interest rates respectively, while the difference between the two — the inflation term structure — embodies information about inflation expectations. Extracting this information, however, is complicated by the fact that the interest rate term structure may also reflect inflation risk premia (the compensation investors require for holding nominal bonds given the risk of unexpected inflation) and real term premia (the compensation investors require for the risk of unexpected future real interest rate movements).

In this paper we formulate and estimate a joint model of the UK nominal and real term structures, which enables us to decompose nominal forward interest rates into expected real policy (risk-free) rates, expected inflation, real term premia and inflation risk premia. The model is based on the assumption of no arbitrage, which implies that there are no risk-free profits to be made by trading combinations of nominal or real bonds. A necessary condition for this assumption to hold is that investors price nominal and real bonds consistently, so that for example the real interest rate priced into nominal bonds is the same as the real rate priced into index-linked bonds. To help identify inflation expectations, we also incorporate survey expectations of longer-term inflation, although the structure does not constrain model expectations to equal the survey expectations period by period. The model is estimated using monthly data since October 1992, to enable us to analyse the dynamics of the term structure over the period that the UK monetary authorities have had an explicit inflation target.

Our analysis suggests there has been a marked fall in both expected longer-term inflation and inflation risk premia since the Bank of England was granted operational independence for setting interest rates. Moreover, in May 1997 — the month that independence was announced — we find a significant fall in both, suggesting that this institutional change was important relative to other influences. More recently, we find that the unusually low level of long real forward interest rates since 2004 (the bond yield ‘conundrum’) reflects a decline in real term premia, although a

significant proportion remains unexplained. The relative inability of the model to fit long-dated real forwards during much of the recent period may reflect strong pension fund demand for index-linked bonds. And our analysis suggests that these special factors affecting the index-linked market may also partly explain the increase in long-term inflation forward rates since the middle of 2005, with long-term inflation expectations changing only modestly over this period, according to the model.

While more structural models are needed to analyse more carefully the economics behind the determinants of term premia and expected risk-free interest rates, our model-implied decompositions nevertheless add insights on which components have accounted for changes in short, medium and long-term forward interest rates since 1992.

1 Introduction

The nominal and real interest rate term structures implied by government conventional and index-linked bonds can potentially provide monetary policy makers with a great deal of information about financial market expectations of both future interest rates and inflation. The nominal and real term structures embody market expectations of future nominal and real interest rates respectively, while the difference between the two — the inflation term structure — embodies information about inflation expectations. Extracting this information, however, is complicated by the fact that the interest rate term structure may also reflect inflation risk premia and real term premia.

The main contribution of this paper is to estimate a joint model of the UK nominal and real term structures over the period that the United Kingdom has had an explicit inflation target, enabling us to decompose nominal forward rates into expected real policy (risk-free) rates, expected inflation, real term premia and inflation risk premia since October 1992. Although we are not the first to do so, there are surprisingly few previous papers that have estimated theoretically consistent term structure models using UK data on both index-linked and nominal bonds yields. One earlier example is a paper by Gong, Remolona and Wickens (1998), but the generalised CIR (Cox, Ingersoll and Ross (1985)) model specification they adopt is very restrictive and has been shown to fit term structure data poorly. Evans (2003) estimates an extended Vasicek (1977) model that incorporates Markov-switching regimes. But while that model allows term and inflation risk premia to vary over time according to three regimes, this set-up is still rather restrictive.[1] An unpublished paper by Risa (2001), applies a more flexible essentially affine model, similar to our own, to modelling UK data from 1983 to 1999, but does not incorporate survey information on inflation expectations as we do (see discussion below). Moreover, given his sample period, Risa does not shed much light on the impact of Bank of England independence on the term structure of interest rates and does not analyse the reasons for the period of unusually low long-term real interest rates — christened the bond yield 'conundrum' by Greenspan (2005) — that began in 2004. To our knowledge, our paper provides the first analysis of these episodes using a joint model of nominal and real yield curves.

Our proposed model is based on the so-called essentially affine class of term structure models

[1] Ang, Bekaert and Wei (2007) apply a more flexible Markov-switching model to US term structure data and inflation, but they do not include data on real bonds in their analysis.

(see eg Duffee (2002)). The approach has two main elements. First, we assume that UK nominal and real bond markets are arbitrage free, so that it is not possible to make risk-free profits from trading combinations of real and/or nominal bonds. Second, following the essentially affine term structure literature, we assume that bonds are priced by a stochastic discount factor (SDF) that takes a particularly flexible form, where the market price of risk is a linear function of the observable and unobservable factors in the model. Despite being 'reduced-form', the SDF in these models can be interpreted in the same way as the intertemporal marginal rate of substitution from a more structural macro model. A consequence of the second assumption, and the main implication of the essentially affine model, is that in our model bond prices, and thus yields, are linearly related to inflation and a small set of unobservable latent factors. We favour latent factors rather than macro factors, partly because this approach has been shown to provide a better statistical fit of term structure data and partly because by taking an agnostic approach to the underlying factors driving yields the resulting model may be less prone to misspecification.[2]

We assume that two latent factors drive movements in expected real risk-free rates and that the same two factors and two additional ones (one retail prices index (RPI) inflation, the other unobservable) drive the nominal curve and real term premia. An important feature of the model is that the same real SDF is assumed to price both real and nominal bond yields. To the extent that institutional investors have preferred habitats for index-linked bonds and demand/supply imbalances push prices away from fundamentals, this assumption may not be an accurate description of the real world. However, if the importance of demand/supply imbalances changes over time and is not a permanent feature, the ability of the model fit to various segments of the forward curve may enable us to identify the emergence of such non-fundamental or market segment-specific factors.

Joyce, Kaminska and Lildholdt (2008) apply a similar essentially affine model to the UK real term structure in isolation, in order to investigate the emergence of unusually low long real interest rates during 2004–05. Our paper extends this work by also including information from the nominal term structure and inflation. One disadvantage of applying the affine modelling framework purely to real yields is that the lack of available shorter maturity index-linked bonds means that a four-year spot yield is the shortest-maturity bond yield available over the full sample, making it difficult to identify the link between short and long-term real interest rates in

[2]Most macro-factor models of the term structure find it necessary to include additional latent factors to fit longer-maturity yields (see eg Ang and Piazzesi (2003)).

such a model. In this paper, by including nominal bond yields and inflation in the model, we are able to derive model estimates of short-maturity real rates. Moreover, estimating a joint model provides a number of advantages because it ensures consistency between the two term structures by imposing no arbitrage across them. And by modelling the dynamics of inflation expectations as a function of the information that drives real interest rates, as well as nominal rates and inflation, we use current information in the whole term structure to extract measures of expected inflation and inflation premia.

To reduce the possibility of encountering instability in term structure behaviour resulting from changes in the United Kingdom's monetary framework, we limit the sample to the period since October 1992, during which the United Kingdom has operated an inflation target. However, this means there is potentially a small sample problem. As Kim and Orphanides (2005) demonstrate, small sample bias in term structure models can lead to implausible implications for the model-implied decompositions of forward rates into expected future short rates and term premia, and in particular to low persistence in estimated expected risk-free interest rates. In their application to the US yield curve, Kim and Orphanides (2005) advocate including survey data on the future path of interest rate expectations, as a way of supplementing the available time-series data on yields. In a subsequent paper, Kim and Wright (2005) incorporate survey information on both expected policy rates and inflation into their model of the US nominal and real term structures.[3] In our paper we incorporate bi-annual Consensus survey information on expected average inflation five to ten years ahead, as an additional information variable, which helps to identify long-run inflation expectations. The implicit assumption is that the long-run inflation expectations of bond market participants will be the same as those of the economic and financial forecasters surveyed by Consensus forecasts. But we include an error term to allow long-term inflation expectations from the model to differ from those of the survey, so that expectations need only be the same on average over the sample. So, although our model incorporates survey information, the model forecasts are not always in line with the surveys. And, while the long-term survey information is only available every six months, our model has the advantage that it provides monthly estimates of expected inflation at any horizon.

The paper is structured as follows. Section 2 sets out the theoretical relationships between

[3]Kim and Wright do not incorporate market data on US index-linked bonds, so-called Treasury Inflation Protected Securities (TIPS), because of the lack of a long enough back run of data, so they effectively estimate 'virtual' real rates using a model of inflation. More recently, D'Amico, Kim and Wei (2007) have attempted to estimate a model using TIPS data. However, the sample available to them is still quite short (only nine years) and TIPS appear to contain a sizable liquidity premium until quite recently.

nominal and real yields and the SDF that obtain under no arbitrage and shows how we can use them to decompose interest rates into expected future risk-free rates and premia. We also discuss the real SDF imposed in the essentially affine term structure literature. In Section 3 we discuss the econometric methodology used to estimate the model and derive the link between bond yields and the set of variables explaining yields. In Section 4 we describe the data used for our empirical analysis. Section 5 presents the results and Section 6 concludes. Several appendices at the end of the paper explain some of the mathematical derivations and present additional tables and charts of our results.

2 Theory and the EATS model

In this section we establish the relationship between nominal and real bond prices and the real SDF in the absence of arbitrage possibilities. We show how we can decompose nominal and real interest rates into expected risk-free interest rates and term premia and then how the difference between nominal and real interest rates, ie inflation breakeven rates, can be broken down into inflation expectations, inflation risk premia and an inflation convexity effect. We go on to discuss the determinants of the SDF and of the inflation risk premium in a standard macro model that contains a specification for the utility function of a representative investor, but note the empirical limitations of this approach. Finally, we describe the modelling approach adopted in this paper, which is based on an essentially affine term structure (EATS) model.

2.1 The link between the nominal and real SDFs under no arbitrage

In an arbitrage-free environment, where all risk-free profit opportunities are eliminated, we can think of investors as pricing assets according to the fundamental asset pricing equation, ie by the discounted present value of their future pay-offs (for discussion on this see, eg, Cochrane (2005)). So the current price of a zero-coupon real bond, denoted $P_t^{n,R}$, that pays one unit of the consumption good when it matures in period $t+n$ is given by:

$$P_t^{n,R} = E_t \left[M_{t+1} \cdot M_{t+2} \cdot \ldots \cdot M_{t+n} \right] \qquad \textbf{(1)}$$

where M_{t+j} denotes the real SDF in period j. This no-arbitrage condition corresponds to the Euler condition in a representative agent macro model. In such a model, the real SDF is related to the marginal utility of the representative investor as follows:

$$M_{t+j} = \frac{\rho U'(C_{t+j})}{U'(C_{t+j-1})} \qquad \textbf{(2)}$$

where ρ is the time preference parameter of the representative agent, C_t is real consumption at time t and $U'(.)$ represents marginal utility. In macro models M_{t+j} is often referred to as the intertemporal marginal rate of substitution, as it indicates the investor's willingness to substitute consumption over time.

The price of a zero-coupon nominal bond, $P_t^{n,N}$, is also equal to the expectation of its discounted real pay-off, but we now also have to allow for changes in the price level:

$$P_t^{n,N} = E_t\left[M_{t+1} \cdot M_{t+2} \cdot \ldots \cdot M_{t+n}\frac{Q_t}{Q_{t+n}}\right] \qquad \textbf{(3)}$$

where Q_t represents the general price level at time t. This can be rearranged to get

$$P_t^{n,N} = E_t\left[\frac{M_{t+1}Q_t}{Q_{t+1}} \cdot \frac{M_{t+2}Q_{t+1}}{Q_{t+2}} \cdot \ldots \cdot \frac{M_{t+n}Q_{t+n-1}}{Q_{t+n}}\right]. \qquad \textbf{(4)}$$

Nominal bond prices can be thought of as reflecting expectations of the nominal SDF and the link between the nominal SDF, denoted with superscript $*$, and the real SDF is given by

$$M_{t+j}^{*} = M_{t+j}\frac{Q_{t+j-1}}{Q_{t+j}}. \qquad \textbf{(5)}$$

2.2 Expectations, term premia and the inflation risk premium

We now want to decompose interest rates into their expected risk-free component and real term premia/inflation premia. Taking logs of both sides of equation **(1)**, we obtain the following relationship between the log price and the real SDF, which holds up to a second-order approximation of any distribution of the SDF:[4]

$$p_t^{n,R} = \ln E_t\left[M_{t+1} \cdot M_{t+2} \cdot \ldots \cdot M_{t+n}\right] = E_t\left[\sum_{j=1}^{n} m_{t+j}\right] + \frac{1}{2}V_t\left[\sum_{j=1}^{n} m_{t+j}\right] \qquad \textbf{(6)}$$

where $p_t^{n,R} = \ln(P_t^{n,R})$ and $m_{t+j} = \ln(M_{t+j})$. Using the relationship between yields and prices $y_t^{n,R} = -\frac{p_t^{n,R}}{n}$, where $y_t^{n,R}$ is the current real yield on a bond maturing at n, it turns out that the real one-period risk-free rate is given by $y_{t+j}^{1,R} = -E_{t+j}(m_{t+j+1}) - \frac{1}{2}V_{t+j}(m_{t+j+1})$. And the relationship between the n-period real yield, expected future real risk-free short rates and the real term premium is given by:

$$y_t^{n,R} = \frac{1}{n}\left(E_t\left[\sum_{j=1}^{n} y_{t+j-1}^{1,R}\right] - \sum_{j=2}^{n} Cov_t\left[\sum_{k=1}^{j-1} m_{t+k}, m_{t+j}\right]\right). \qquad \textbf{(7)}$$

The real yield reflects the average of current expected future one-period real risk-free interest rates and the average real forward term premium over the life of the bond. A similar expression

[4]The approximation is exact in the case where the real SDF is conditionally log-normal.

can be shown to apply for a nominal yield, $y_t^{n,N}$:

$$y_t^{n,N} = \frac{1}{n}\left(-E_t\left[\sum_{j=1}^{n}(m_{t+j} - \pi_{t+j})\right] - \frac{1}{2}V_t\left[\sum_{j=1}^{n}(m_{t+j} - \pi_{t+j})\right]\right) \tag{8}$$

where $\pi_{t+j} = \ln\left(\frac{Q_{t+j}}{Q_{t+j-1}}\right)$ is log inflation. Combining this equation with the equation for real yields, we get:

$$y_t^{n,N} = y_t^{n,R} + \frac{1}{n}\left(\underbrace{E_t\left[\sum_{j=1}^{n}\pi_{t+j}\right]}_{\text{Expected inflation}} - \underbrace{\frac{1}{2}V_t\left[\sum_{j=1}^{n}\pi_{t+j}\right]}_{\text{Inflation convexity}} + \underbrace{Cov_t\left[\sum_{j=1}^{n}m_{t+j}, \sum_{j=1}^{n}\pi_{t+j}\right]}_{\text{Inflation risk premium}}\right). \tag{9}$$

So nominal yields reflect movements in real yields, the average expected log inflation over the life of the bond, the average inflation convexity effect and the average inflation risk premium. Rearranging this expression in terms of the inflation breakeven rate brings out the point that breakevens are a function of three different terms:

$$y_t^{n,N} - y_t^{n,R} = \frac{1}{n}\left(\underbrace{E_t\left[\sum_{j=1}^{n}\pi_{t+j}\right]}_{\text{Expected inflation}} - \underbrace{\frac{1}{2}V_t\left[\sum_{j=1}^{n}\pi_{t+j}\right]}_{\text{Inflation convexity}} + \underbrace{Cov_t\left[\sum_{j=1}^{n}m_{t+j}, \sum_{j=1}^{n}\pi_{t+j}\right]}_{\text{Inflation risk premium}}\right). \tag{10}$$

The Fisher hypothesis ignores the last two terms on the right-hand side. But if the inflation convexity effect or the inflation risk premium are different from zero, it is clear that we cannot get a direct reading of inflation expectations from breakeven rates.

For many purposes, it is more useful to focus on forward interest rates, the rates implied for future time periods, rather than spot rates, which refer to average rates over a period. We can derive similar decompositions for forward rates because the difference between the n-period nominal and real spot rates is simply the average of the difference between the n one-period forward rates between t and $t+n$. Hence

$$y_t^{n,N} - y_t^{n,R} = \frac{1}{n}\left(\sum_{j=1}^{n}\left(f_t^{j,N} - f_t^{j,R}\right)\right) \tag{11}$$

where $f_t^{j,N}$ denotes the implied one-period nominal rate, j periods ahead, and $f_t^{j,R}$ is the equivalent real rate.

2.3 Interpreting the inflation risk premium

So what determines the inflation risk premium? As we noted above, the inflation risk premium is given by the conditional covariance between marginal utility and inflation. If inflation is unexpectedly high when marginal utility is also high, nominal bonds are less desirable relative to real bonds, as inflation tends to be unexpectedly high exactly in those states where nominal bonds pay off less in real terms and consumers would benefit more from additional consumption. In a representative agent model with preferences described by a simple power utility function, the inflation risk premium depends only on the real consumption-inflation trade-off. But in a more general set-up, where the representative agent has Epstein and Zin (1991) preferences, the inflation risk premium is given by:

$$\alpha_{1,t} Cov_t\left(\sum_{j=1}^{n} r^w_{t+j}, \sum_{j=1}^{n} \pi_{t+j}\right) + \alpha_{2,t} V_t\left(\sum_{j=1}^{n} \pi_{t+j}\right) + \alpha_{3,t} Cov_t\left(\sum_{j=1}^{n} \pi_{t+j}, \sum_{j=1}^{n} \Delta c_{t+j}\right) \quad \textbf{(12)}$$

where r^w_{t+j} is the real return on the wealth portfolio and Δc_{t+j} is the change in real log consumption. In this model, the inflation risk premium changes with the expected consumption-inflation trade-off, with the conditional variance of inflation and with the conditional covariance between the return on the wealth portfolio of the investor and inflation. The general point is that the interpretation of inflation risk premia is model dependent.

2.4 The real SDF in an essentially affine term structure model

If we knew how the SDF was linked to marginal utility and the dynamics of macroeconomic variables that affect marginal utility, it would be straightforward to decompose the term structure into expected risk-free rates and premia. In practice, however, representative agent models with conventional preferences have, so far, had limited success in characterising the dynamics of asset price movements (see Rudebusch and Swanson (2007) for an application to the US term structure).

An alternative approach assumes that there exists at least one real SDF that prices all bonds. But rather than specifying the underlying utility function, the dynamics of the real SDF can be summarised by a number of observable macroeconomic and/or latent factors. The affine term structure literature assumes that there is at least one real SDF that satisfies pricing equations **(1)** and **(3)**,[5] taking the form:

$$\ln M_{t+1} = m_{t+1} = -\left(\bar{r} + \gamma'(z_t - \mu)\right) - \frac{\Lambda_t' \Omega \Lambda_t}{2} - \Lambda_t' \Omega_t^{1/2} \epsilon_{t+1} \quad \textbf{(13)}$$

[5] See Duffie and Kan (1996) for a more thorough discussion of affine term structure models.

so that the log of the nominal SDF is given by

$$m^*_{t+1} = m_{t+1} - \pi_{t+1} = -\left(\overline{r} + \gamma'(z_t - \mu)\right) - \frac{\Lambda_t' \Omega \Lambda_t}{2} - \Lambda_t' \Omega_t^{1/2} \epsilon_{t+1} - \pi_{t+1} \qquad \textbf{(14)}$$

where $\overline{r}$ is the long-run level of the real risk-free rate, $(z_t - \mu)$ is an $(N \times 1)$ vector of observable and/or unobservable (latent) variables with a zero mean, ε_{t+1} is an $(N \times 1)$ vector of shocks to the observable and/or unobservable (latent) variables and $\mathbf{\Omega}_t$ is the conditional covariance matrix of these shocks, which may vary over time. The market price of risk is represented by $\Lambda_t' \mathbf{\Omega}_t^{1/2}$. One price of risk specification, the essentially affine term structure (EATS) model proposed by Duffee (2002), assumes that Λ_t is linear in the factors. If, for example, we had two factors then Λ_t would be given by:

$$\Lambda_t = \lambda + \beta(z_t - \mu) = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} z_{1,t} - \mu_1 \\ z_{2,t} - \mu_2 \end{bmatrix}. \qquad \textbf{(15)}$$

The Λ_t vector is crucial for determining the time-variation in risk premia in an essentially affine model, when the covariance matrix between the shocks, $\mathbf{\Omega}$, is constant. If $\mathbf{\Omega}$ is constant and $\beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 0$, the conditional variance of the real SDF is constant and bond risk premia, at all maturities, are constant. If, in addition, $\lambda_1 = \lambda_2 = 0$, then the conditional variance of the real SDF is zero (ie the real SDF is non-stochastic) and risk premia are zero.

The dynamics of the z_t variables determine the dynamics of the real SDF, but not all of the variables need explain the expected real SDF. So γ' is a $(1 \times N)$ vector with ones in the rows corresponding to the variables assumed to drive the expected real SDF and zeros elsewhere. The form of the SDF in this case is considerably more general than the SDF in a representative agent model but it can be given a similar interpretation in terms of marginal utility. For example, if $\overline{r}$ increases the log of the SDF falls. This is what we would expect because higher interest rates will be associated with more saving, higher future consumption growth and therefore a lower intertemporal marginal rate of substitution.

Given this specification of the real SDF, the key result of an EATS model is that expected risk-free interest rates, real term premia and inflation risk premia are all linear (affine) in the level of the N unobservable and/or observable (latent) variables. Real yields are given by

$$y_t^{n,R} = \overline{A_n} + \overline{B_n'}(z_t - \mu) \qquad \textbf{(16)}$$

and nominal bond yields are given by

$$y_t^{n,N} = \overline{A_n^*} + \overline{B_n^{*\prime}}(z_t - \mu) \qquad \textbf{(17)}$$

where $\overline{A_n}$, $\overline{A_n^*}$ are scalars, and $\overline{B_n}$ and $\overline{B_n^*}$ are $(N \times 1)$ parameter matrices that are obtained recursively, imposing no arbitrage across yields with different maturities. The recursive relationship between bonds with different maturities will depend on the underlying parameters in the SDF. The derivation of these equations is discussed in detail in Appendix A.

3 Econometric modelling

To estimate a joint model of the nominal and real term structures, we need to specify the dynamics of the factors driving the real SDF and to make an assumption about how bond market investors form their inflation expectations. In this section we discuss these assumptions and then explain how we can estimate the resulting term structure model using the Kalman filter.

3.1 Modelling the factor dynamics

The aim of our paper is to fit the nominal and real yield curves well, in order to be able to say something meaningful about the difference between the two — inflation expectations, inflation risk premia and the inflation convexity effect (though this last element turns out to be small). For this reason, we use mainly latent variables rather than macro variables, as the former are generally found to be necessary to fit the longer end of the yield curve closely. We assume that two latent factors drive movements in the expected real risk-free rates and that the same two factors and two additional ones (one inflation, the other unobservable) drive the nominal curve and real term premia.[6] The vector of state variables driving real and nominal yields is therefore given by:

$$(z_t - \mu)' = \left[\begin{array}{cccc} z_{1,t} - \mu_1 & z_{2,t} - \mu_2 & z_{3,t} - \mu_3 & z_{4,t} - \mu_4 \end{array} \right] \quad \textbf{(18)}$$

where $z_{1,t}$, $z_{2,t}$, and $z_{3,t}$ are latent factors with different time-series dynamics and the fourth factor, $z_{4,t}$, is inflation. As commonly assumed, we specify the dynamics of the factors as a first-order VAR with normally distributed errors:

$$(z_{t+1} - \mu) = \Phi(z_t - \mu) + \Omega^{1/2}\epsilon_{t+1} \qquad \epsilon_{t+1} \sim NID(0, I_4) \quad \textbf{(19)}$$

[6] Most latent or macro-factor affine models of the nominal term structure use three factors (see eg Lildholdt, Panigirtzoglou and Peacock (2007)), but since we are imposing restrictions across the real and nominal term structures we want to allow for the additional flexibility of an extra factor. In their study of the UK real term structure Joyce, Kaminska and Lildholdt (2008) find that two factors are sufficient to capture virtually all the dynamics of the real term structure. We allow shocks from each of the model's factors to affect the real pricing kernel, which introduces additional flexibility in fitting real yields.

where I_4 is a (4 × 4) identity matrix and the parameter matrices are given by:

$$\Phi = \begin{bmatrix} \Phi_{11} & 0 & 0 & 0 \\ \Phi_{21} & \Phi_{22} & 0 & 0 \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & 0 \\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} \end{bmatrix} \quad \Omega = \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2 \end{bmatrix} \quad \epsilon_{t+1} = \begin{bmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \\ \epsilon_{3,t+1} \\ \epsilon_{4,t+1} \end{bmatrix}. \quad \textbf{(20)}$$

We assume that the variance of the factor shocks is constant. This is a common assumption in the term structure literature and seems reasonable given that we estimate the term structure model over a relatively short sample. Moreover, it eases estimation considerably. The additional assumptions that Ω is diagonal and that Φ is lower triangular are necessary for the factors to be identifiable given the use of latent variables. By allowing the off-diagonal elements of the Φ matrix to be non-zero, the factors are allowed to be correlated with each other. Notice that all three latent factors feed into the determination of inflation, which is the fourth factor.

3.2 Inflation expectations in the model

As is evident from equation **(5)**, the relevant inflation variable for pricing bonds is the one-period log change in prices. As we will be working with monthly interest rate data, the relevant inflation measure to model is therefore month-on-month inflation. Inflation in our model is assumed to follow a vector auto regression (fourth row in equation **(19)**), subject to the restriction of no arbitrage, which ensures that inflation expectations at any future maturity have to be consistent with the dynamics of the factors that determine the nominal and real interest rates at that maturity. The advantage of modelling actual inflation, rather than treating inflation as a latent factor, is the ability to identify inflation expectations and inflation risk premia such that they are consistent with observed inflation dynamics.

In the model we also include survey expectations of average RPI inflation five to ten years ahead. The motivation for including survey expectations in term structure models of this nature is cogently explained in Kim and Orphanides (2005). The main problem is that we inevitably have to estimate the term structure model over a short sample, while interest rates themselves are highly persistent.[7] This tends to lead to model estimates, which underestimate the persistence of

[7]For a discussion of the importance of accounting for time-variation in long-horizon forecasts of interest rates, see Kozicki and Tinsley (2001).

expected risk-free interest rates, and this in turn will have an impact on the decomposition of long-term interest rates into expectations and term premia.

We estimate our term structure model using monthly data from October 1992, when the UK inflation-targeting framework began, to February 2008. Since our sample period covers fifteen years of monthly data, it is quite short and contains little business cycle variation. As a result, it may be difficult for any econometric model to attribute time variation in long-term interest rates into expectations and premia. This is compounded by the fact that the EATS model implies a relationship between yields and factors that is highly non-linear in the underlying risk parameters, reflecting the restrictions that follow from the no-arbitrage condition. As a result, this class of term structure model is hard to estimate in short samples, as there are typically a number of different local maxima.

Incorporating survey information on long-horizon inflation expectations (from Consensus forecasts) into our model helps identify whether movements in breakevens are due to inflation expectations or inflation premia.[8] Although we implicitly assume that bond market investors' expectations of average inflation five to ten years ahead will be the same as those of the survey respondents, we include an error term to allow long-term inflation expectations from the model to differ from those of the survey, so expectations need only be the same on average over the model's estimation period. Moreover, the model may reject the survey information on expectations, if it is not in line with the factors that determine nominal and real yields. So importantly, although our model incorporates survey information, the model forecasts are not always in line with the surveys and may occasionally deviate substantially. And of course compared to the survey information on long-term inflation expectations, which is only available for long horizons every six months, our model has the advantage of providing monthly estimates of expected inflation at any horizon.

3.3 The Kalman filter

We estimate our model by maximum likelihood, using the Kalman filter to compute the log-likelihood function. To apply the Kalman filter, the model needs to be written in state-space

[8]Pennacchi (1991) is an early example of the use of surveys, in this case surveys on shorter-term inflation expectations, in a US term structure model. We could have included more surveys on shorter-term inflation expectations, but the more surveys included in the model, the more likely it is that the model decompositions will be driven exclusively by the surveys. To exclude that possibility, we include a relatively large number of yields, which should ensure that the estimation method does not put too much relative weight on matching the surveys, if they are not consistent with those of bond investors.

form, consisting of a state equation and a measurement equation. We have already described the state equation above, which is the first-order VAR shown in equation **(19)**, so we now turn to describe the measurement equation, which shows the relationship between the observed variables and the underlying state variables.

The vector of observed variables consists of month-on-month inflation, real and nominal bond yields and Consensus forecasts of average inflation five to ten years ahead. Inflation is relatively straightforward, since we assume it is measured without error. However, we cannot exclude the possibility that government bond yields are measured with error. Measurement error may represent a variety of things, including fitting errors arising from yield curve estimation and market noise. So we include an additional vector of error terms in the measurement equation for each yield to capture such non-fundamental factors. The Consensus survey data we use on long-term inflation expectations are only available every six months (April and October), but this presents no problem in this context because one of the advantages of using the Kalman filter is the possibility of incorporating variables with missing observations into the measurement equation (see eg Durbin and Koopman (2001)). We also include a measurement error on these expectations to allow for the possibility that bond market investors' long-term inflation expectations differ from those of Consensus forecasters.

The measurement equation is therefore given by the following expression:

$$\begin{bmatrix} \pi_t \\ y_t^{j,R} \\ y_t^{i,N} \\ E_t^C(\pi_{t+61:120}) \end{bmatrix} = \begin{bmatrix} \mu_4 \\ \overline{A_j} \\ \overline{A_i^*} \\ \mu_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ \overline{B_{j,1}} & \overline{B_{j,2}} & \overline{B_{j,3}} & \overline{B_{j,4}} \\ \overline{B_{i,1}^*} & \overline{B_{i,2}^*} & \overline{B_{i,3}^*} & \overline{B_{i,4}^*} \\ G_1 & G_2 & G_3 & G_4 \end{bmatrix} \begin{bmatrix} z_{1,t}-\mu_1 \\ z_{2,t}-\mu_2 \\ z_{3,t}-\mu_3 \\ z_{4,t}-\mu_4 \end{bmatrix} + \begin{bmatrix} 0 \\ u_{t,j} \\ u_{t,i}^* \\ u_{t,c} \end{bmatrix}$$

where π_t is monthly (seasonally-adjusted) RPI inflation, $y_t^{j,R}$ is the current observed real yield with j months to maturity, $y_t^{i,N}$ is the current observed nominal yield with i months to maturity, $E_t^C(\pi_{t+61:120})$ is Consensus forecasters' current expectation of average inflation five to ten years ahead, $u_{t,j}$ is the measurement error on a real yield with j months to maturity, $u_{t,i}^*$ is the measurement error on a nominal yield with i months to maturity and $u_{t,c}$ is the measurement error on Consensus long-term inflation expectations. The parameter vectors $\overline{A}$, $\overline{A}^*$, $\overline{B}$ and $\overline{B}^*$ embody the theoretical no-arbitrage restrictions. These are derived in Appendix A and are functions of $\overline{r}$, γ, λ, β, Ω and Φ. Appendix A also illustrates how term premia and inflation risk premia are determined by the parameters and factors in our model. The G_k parameters are determined as $G_k = \frac{1}{60} e_4' (\sum_{i=61}^{120} \Phi^i) e_k$, where e_k is a (4×1) vector with a 1 in the k^{th} row and zeros in all other rows, so they pick up the factor loadings which predict inflation.

The vector of measurement errors on real yields, nominal yields and Consensus forecasts of inflation are assumed to be independent and normally distributed, with $u_t \sim NID(\mathbf{0}, \sigma_R^2 \cdot I_{\#R})$, $u_t^* \sim NID(\mathbf{0}, \sigma_N^2 \cdot I_{\#N})$ and $u_{t,c} \sim NID(0, \sigma_c^2)$ respectively.[9] $I_{\#R}$ is an identity matrix of dimension $(\#R \times \#R)$, where $\#R$ is the number of real yields included in the estimation, and $I_{\#N}$ is the corresponding identity matrix with dimension equal to the number of nominal yields included in the model, $\#N$. We assume that the variance of measurement errors is equal across all real yields with different maturities (and equal to σ_R^2) and that the variance of the measurement errors on nominal yields with different maturities is also equal (to σ_N^2). The variance of the measurement errors on Consensus forecasts (equal to σ_c^2) is freely estimated.[10] By having no measurement error in the seasonally adjusted inflation equation, we ensure that the fourth factor is equal to seasonally adjusted inflation. Given the expressions for the state and measurement equations, we can readily apply the Kalman filter to derive the prediction error decomposition of the likelihood function, which can then be maximised over different parameter values to generate maximum likelihood parameter estimates (see eg De Jong (2000)).

4 Data

4.1 Yield curve data

The yield data we use are end-of-month zero-coupon UK nominal and real yields produced by the Bank of England.[11] The data are calculated using a so-called variable roughness penalty method (see Anderson and Sleath (1999), (2001)), which is essentially a cubic spline method with a penalty function that results in the smoothness of the curve increasing with maturity. Nominal yields are based on fitted yields from a curve estimated using general collateral (GC) repo rates and UK nominal government bonds. The real yields data are derived from index-linked and nominal bonds taking into account indexation lags, using the method proposed

[9] A special case of our model occurs when the inflation expectations of bond market investors are fundamentally different from those of Consensus forecasters, in which case we would expect an estimate of σ_c that is very large.

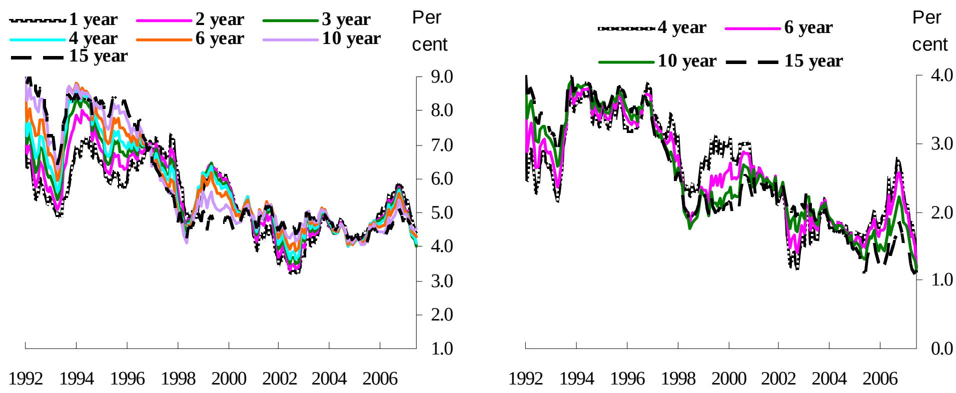
[10] We could have allowed each yield to have a different measurement error variance. But this would have resulted in an additional nine parameters to be estimated and there would have been a danger of over-fitting.

[11] The Bank of England publishes UK yield curve estimates on its external website. See www.bankofengland.co.uk/statistics/yieldcurve.

by Evans (1998) and extended by Anderson and Sleath (1999, 2001).[12]

Chart 1: Nominal and real yields

A: Nominal yields **B: Real yields**



Although the Bank of England data for nominal yields are available back to the 1970s and the real yield data back to the mid-1980s, we have restricted our sample period to end-month observations from October 1992 to February 2008 to avoid the potential structural break in term structure behaviour associated with the United Kingdom's adoption of inflation targeting in October 1992.[13] The shortest-maturity zero-coupon real spot rate we are able to derive consistently over the sample has a four-year maturity (because of the lack of short-maturity index-linked bonds), so we model zero-coupon real yields with maturities of four, six, ten and fifteen years.[14] For nominal yields, we use maturities of one, two, three, four, six, ten and fifteen years.

Yields are plotted in Chart 1. As can be seen, both nominal and real yields have varied significantly over time. Although there is a strong common trend in the yields, there are also periods where short and long-term yields move by different amounts and sometimes in different directions. Summary statistics for our interest rate data are shown in Table A below.

[12] The method implicitly assumes that there is no indexation lag risk premium on index-linked bonds.

[13] Another reason for looking at a shorter sample is that the liquidity of the UK index-linked bond market has expanded considerably since the issue of the first bond in 1981, see Deacon *et al* (2004).

[14] The lack of data at the short end of the real curve causes us to estimate the model using spot rates rather than forward rates, as otherwise the model would be estimated using no information on the real curve below four years. In our analysis (see Section 5), however, we will focus on the model's breakdown of implied forward rates.

Table A: Summary statistics for yields, Oct. 1992 - Feb. 2008											
	Real yields				Nominal yields						
	4 year	6 year	10 year	15 year	1 year	2 year	3 year	4 year	6 year	10 year	15 year
Mean	2.56	2.55	2.53	2.50	5.35	5.51	5.63	5.70	5.78	5.81	5.78
Std Dev	0.70	0.71	0.77	0.85	1.02	1.13	1.22	1.28	1.37	1.49	1.58
Skew	-0.45	-0.05	0.20	0.29	-0.03	0.20	0.39	0.51	0.63	0.73	0.76
Kurt	1.94	1.86	1.78	1.74	2.08	2.10	2.20	2.21	2.12	1.93	1.87

The UK real yield curve has been flat to very slightly downward sloping over the sample, while the nominal curve has been upward sloping. The volatility of real and nominal yields is increasing in maturity. Skewness is slightly increasing in maturity, whereas kurtosis is slightly declining. Principal components analysis, not reported here, of real and nominal yields shows that the first two principal components are able to explain virtually all the variation in real yields with the third factor explaining less than 0.2%. An equivalent principal components analysis of the nominal yields data suggests that three factors are able to explain virtually all the variation in nominal yields, with the fourth factor explaining a mere 0.02%. The main point to take from this analysis is that a relatively few independent factors can explain virtually all the variation observed in nominal and real yields.[15]

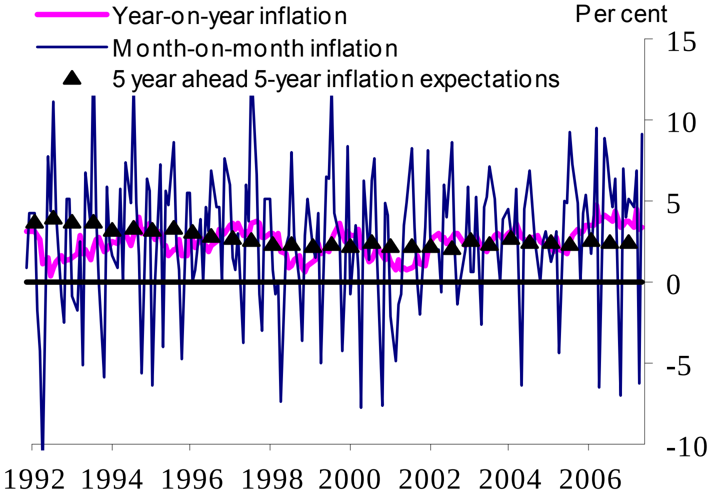
4.2 RPI inflation data

As we saw in the theory section above, the link between the nominal and real SDF is the inflation rate with the same frequency as the yield data. As we use monthly data, this means modelling monthly inflation and, as UK real rates are linked to RPI inflation, this means modelling RPI inflation. The strong seasonal pattern in RPI inflation is evident in Chart 2 below. But tests reveal that the seasonal pattern is stable and we make a simple seasonal adjustment to the data by running a regression of month-on-month log RPI changes on monthly indicator variables. Investors are assumed to form their inflation expectations using this seasonally adjusted series.[16]

[15] The principal component analysis is available upon request.

[16] In their term structure model for the euro area, Hördahl and Tristani (2007) also model seasonally adjusted inflation rates.

Chart 2: RPI inflation and Consensus long-term forecasts



4.3 Survey data

Twice a year Consensus ask their panel of economic and financial forecasters for their estimates for the underlying rate of UK RPI inflation out to ten years ahead. The data we include, which refer to forecasts for average expected underlying RPI inflation five to ten years ahead, are shown in Chart 2. The data refer to RPI expectations before 1997 and to RPIX (RPI excluding mortgage interest payments) expectations post-1997. Although there is potentially a problem with the break in the series, RPI and RPIX inflation tend to follow each other at medium to long-term frequencies and any wedge between them in the long run is likely to be small, probably of the order of 0.1 percentage points. Moreover, as already described in Section 3.3 above, we allow for the possibility that long-term inflation expectations differ from those of the surveys.

As the chart shows, long-term survey expectations have declined over the past fifteen years and they have exhibited much less volatility than monthly and annual inflation rates.

5 Results

5.1 Model parameter estimates

As we have four factors and allow all risk prices to depend on each of these factors, the market price of risk has 20 parameters (see equation **(15)**). We started by estimating a completely general model, where all the parameters in the β matrix and λ vector were left unrestricted. We then used a general to specific method, using likelihood ratio tests, to test down until we could no longer reject the joint significance of the parameters.[17] The estimated parameters of our preferred model are shown in Table C below.

Table C: Model estimates

$$\begin{bmatrix} \bar{r} \cdot 12 \\ \mu_4 \cdot 12 \end{bmatrix} = \begin{bmatrix} \underset{(4.03)}{0.0222} \\ \underset{(3.45)}{0.0300} \end{bmatrix} \qquad \Phi = \begin{bmatrix} \underset{(220.81)}{0.98} & 0 & 0 & 0 \\ \underset{(2.11)}{0.002} & \underset{(373.04)}{0.99} & 0 & 0 \\ \underset{(1.85)}{0.003} & \underset{(1.19)}{-0.01} & \underset{(1417.87)}{0.998} & 0 \\ \underset{(3.40)}{0.28} & \underset{(2.03)}{-1.71} & \underset{(1.11)}{-0.22} & \underset{(1.44)}{0.11} \end{bmatrix}$$

$$\lambda^{*} = \begin{bmatrix} \underset{(0.71)}{-1.180} \\ \underset{(0.98)}{1.414} \\ \underset{(0.16)}{-0.011} \\ \underset{(1.15)}{-0.083} \end{bmatrix} \qquad \beta^{*} = \begin{bmatrix} 0 & 0 & \underset{(1.11)}{4800} & \underset{(1.50)}{2655} \\ 0 & 0 & 0 & \underset{(1.63)}{9814} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \underset{(0.80)}{157} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma^{*}_{R} \\ \sigma^{*}_{N} \\ \sigma^{*}_{C} \end{bmatrix} = \begin{bmatrix} \underset{(22.63)}{0.103} \\ \underset{(17.11)}{0.148} \\ \underset{(6.79)}{0.195} \end{bmatrix} \qquad \Omega^{*} = \begin{bmatrix} \underset{(12.12)}{0.15} & 0 & 0 & 0 \\ 0 & \underset{(2.45)}{0.05} & 0 & 0 \\ 0 & 0 & \underset{(1.11)}{0.25} & 0 \\ 0 & 0 & 0 & \underset{(15.54)}{1.88} \end{bmatrix}$$

t-statistics in brackets

$\lambda^{*} = \lambda/1000$, $\beta^{*} = \beta/120$, $\Omega^{*} = 1000 \cdot \Omega^{1/2}$, $\sigma^{*}_{k} = 1200 \cdot \sigma_{k}$ for $k = N, R, C$

Given the reduced-form nature of the model, it is difficult to give a meaningful structural interpretation to most of the estimated parameters. But the key point to bring out is that risk premia are found to display significant time variation, as can be seen from the estimated parameters in the β matrix. From the parameter estimates it seems evident that the first, second and fourth risk prices change over time and, in fact, we cannot reject a joint test of time variation in these risk prices. Interestingly, we find that it is the two additional nominal factors that

[17]Note that where particular parameters do not appear statistically significant according to the reported t-statistics, this implies that the data rejected imposing a zero restriction according to a conventional likelihood ratio test.

determine all the time variation in the market price of risk.[18] As found by other studies, the latent factors are all quite persistent — the persistence of the third factor being particularly strong. From the estimates of Φ, it is also noteworthy that inflation is forecastable in-sample. Perhaps surprisingly, it is not the lag of inflation which is strongly significant, but the lag of the first and second factors. Finally, it is worth mentioning that the variance of the measurement error of the Consensus survey data on inflation expectations is estimated relatively precisely, so we cannot reject the inclusion of the survey data. We will shed further light on the estimated parameters below.

To judge the performance of the model, Table F in Appendix B includes a number of summary statistics of the in-sample fit of the model. This table reveals that the model fits the data reasonably well. The average measurement error is close to zero, with a relatively low standard deviation. While there is some serial autocorrelation in the measurement errors, it is comparable to similar models (many studies do not report these autocorrelation statistics, but De Jong (2000) reports first-order autocorrelations of around 0.25 for a three-factor model that is fitted to US nominal yields only).

5.2 *Model forward rate decompositions*

Chart 3 shows the model-implied decompositions of the instantaneous forward rate, four years and ten years ahead, including the part of forward rates that cannot be explained by the model (indicated by the unexplained lines).[19] The nominal forward rate decompositions in the first panel show that the model fits nominal forwards well, with the residual lines quite close to zero for most of the sample period. This is particularly impressive, as the model was optimised to fit spot interest rates rather than forward rates *per se*, so in this sense these forward-rate residuals are out of sample. Over the sample as a whole, nominal forward rates at medium and long-term horizons have fallen, with most of the fall occurring in the period from 1997 to 1998. The model mainly attributes this fall to lower term premia, although longer-horizon expected risk-free nominal interest rates have also fallen over the period.

[18] Balfoussia and Wickens (2007) also find that inflation strongly outperforms any real variables in explaining the SDF in an empirical application to US data. But as they use a slightly different set-up, with time-varying covariances between bond returns and inflation it is difficult to make an exact comparison.

[19] By instantaneous rate, we mean the one-month rate. The ten-year instantaneous forward rate is the one-month rate implicit ten years in the future. Appendix C discusses the method used to back out the forward curve of expected real interest rates, expected inflation and term premia from the model. We include convexity effects in our term and inflation risk premia estimates, as the latter are constant over time and small (eg less than 12 basis points in absolute terms in the fifteen-year nominal forward rate).

If we look at the second and third panels of the chart we see the corresponding decompositions for real and inflation forwards. Again the model fits these forward rates well, although there are a few periods where there is a more pronounced deterioration in the model's ability to fit the data (see discussion below). Medium to long-term inflation expectations have declined over the sample and have remained at relatively low and stable levels since 1997–98. Long-horizon expected real risk-free rates have varied less over the sample, with an average level of 2.1% (about 2.9% after adjusting for an estimate of the long-run wedge between RPI and CPI inflation) with the lowest value being 1.7% and the highest level around 2.3%. It is apparent that the fall in nominal premia up to 1997-98 shown in the first panel is accounted for by both lower real term premia and lower inflation risk premia, although it is falls in the former that have dominated the decline in nominal premia. Most of the variation since 1998 in nominal term premia is also attributed to changing real term premia, while inflation risk premia have been quite stable.

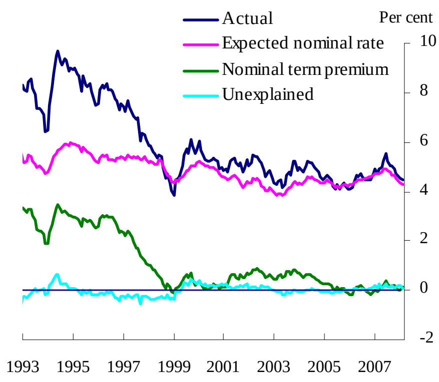
We note that although the model fits well over most of the sample, it tends to do less well in periods that coincide with severe turbulence in financial markets such as the LTCM crisis in 1998, the troubles of IT stocks in the early 2000s and more recent market disruption since the middle of 2007 triggered by the sub-prime mortgage crisis in the United States. One potential explanation for this may be the assumption that the variance of the shocks to the factors is constant over the model's estimation period.

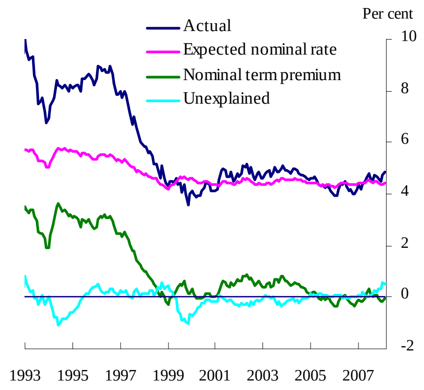
The factors themselves are shown in Chart 8 in Appendix E. Panels A, B and C reveal that, although the three latent factors are highly persistent, their time-series dynamics are rather different. More revealing perhaps are the impulse responses, contained in Chart 9, which enable us to analyse the impact on forward rates of a one standard deviation shock to each of the factors. The loadings suggest that the first factor affects the slope of both the real and nominal forward curves, with its impact coming through its effect on expected real risk-free rates and inflation. The second factor primarily affects the level of the real forward curve, with the effect coming through expected real risk-free rates (declining with horizon) and real term premia (where the effect increases with horizon), although it also impacts on nominal forward rates mainly through expected inflation. The third factor affects the curvature of both the real and nominal forward curves, with its impact coming entirely through expected inflation and inflation risk premia and real term premia. Combining the results from these charts and Chart 3 suggests that this factor helps explain the fall in expected inflation, the real term premium and the inflation risk premium in the middle of the 1990s. The impulse responses also reveal that RPI inflation

Chart 3: Decomposition of four-year and ten-year forward rates

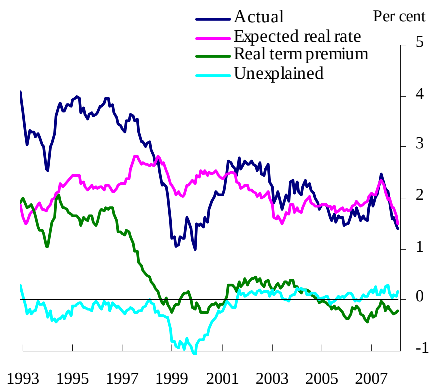
Four-year forward rates | **Ten-year forward rates**

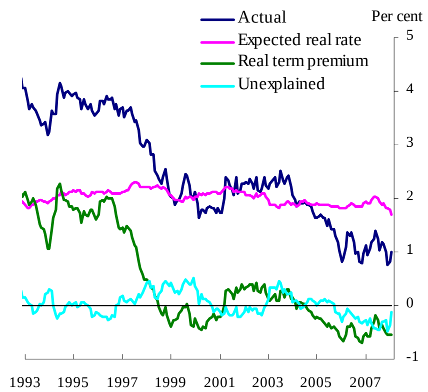
Nominal



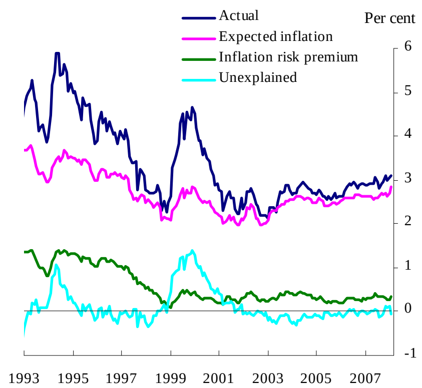


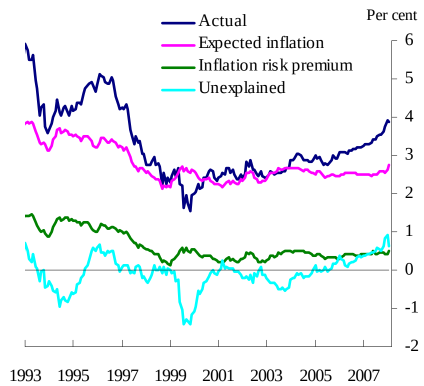
Real





Inflation





shocks have their strongest positive impact on very short-term forward rates, through their impact on short-term inflation expectations. Interestingly, inflation shocks do not impact on medium or long-term real and nominal forward rates, which is consistent with what we would expect to see under a credible monetary policy framework. As was apparent from the estimated parameters of the model (Table C above), inflation shocks push down on the short-term real term premium, perhaps indicating increasing demand for real indexed-linked bonds in periods of relatively large inflation shocks. But importantly the impulse responses show that this fall in real term premia is relatively small and concentrated in horizons out to 24 months.

To quantify more precisely the main drivers of nominal forwards, real forwards and implied inflation breakevens at different horizons, Table D contains a simple variance decomposition breakdown over the full sample period.[20] From the results in the first panel, we can see that expected real risk-free rates and expected inflation account for about 60% of the variance of nominal one-year forward rates. While the contribution of expected real risk-free rates is broadly equal to the contribution of expected inflation at one-year horizons, it tails off at medium to long horizons. In contrast, expected inflation accounts for about 25% of the variance in both four-year and ten-year nominal forward rates. Real term and inflation risk premia explain more than half of the variance in medium to long-term nominal forward rates, with real term premia accounting for roughly twice as much as inflation risk premia. In terms of monthly movements (Panel 2), the model accounts for about three quarters of the variance of nominal forward rates.

Turning to the decomposition for real forward rates (Panel 3), we first need to note that the decomposition of the variance of one-year rates has to be based on model-implied rates, as we do not have data for one-year real (or inflation) forwards. Nevertheless, it is interesting that real term premia account for as much as 17% of the variance, suggesting that we need to adjust for time-varying real term premia even at quite short horizons if we wish to measure real policy rate expectations.[21] At medium and long-term horizons, expected real risk-free rates make little contribution and real term premia dominate. The decomposition of breakevens (Panel 5) suggests that as much as a third of the variance of one-year rates is attributable to inflation risk premia (although again note that this is based on model-implied rates, so there is no residual category). At medium and longer-term horizons inflation premia remain important but over 40%

[20] We explain the variance decomposition in Appendix D.

[21] This point is also relevant if we want to use the model estimates of the short end of the real curve to produce a measure of the monetary stance.

of the variance is attributed to inflation expectations. In terms of the variance of month-to-month movements (Panels 4 and 6), we find that the model is more successful in explaining real forward rates than in explaining inflation forward rates.

Table D: Variance decompositions (per cent)

$f_t^{n,N}$	$E_t(y_{t+n}^{1,R})$	$E_t(\pi_{t+n})$	$\varphi_{r,t}^n$	$\varphi_{\pi,t}^n$	$u_t^{n,N}$
n=12	28.40	30.40	16.00	17.40	7.80
n=48	3.70	25.10	45.20	26.60	-0.60
n=120	1.70	24.90	48.50	22.30	2.60
$\Delta f_t^{n,N}$	$\Delta E_t(y_{t+n}^{1,R})$	$\Delta E_t(\pi_{t+n})$	$\Delta\varphi_{r,t}^n$	$\Delta\varphi_{\pi,t}^n$	$\Delta u_t^{n,N}$
n=12	32.40	27.50	5.80	7.30	27.00
n=48	7.40	20.40	28.10	18.50	25.60
n=120	0.70	20.50	37.00	17.90	23.90
$f_t^{n,R}$	$E_t(y_{t+n}^{1,R})$		$\varphi_{r,t}^n$		$u_t^{n,R}$
n=12	82.80		17.20		
n=48	10.00		81.70		8.30
n=120	5.20		91.00		3.80
$\Delta f_t^{n,R}$	$\Delta E_t(y_{t+n}^{1,R})$		$\Delta\varphi_{r,t}^n$		$\Delta u_t^{n,R}$
n=12	89.40		10.60		
n=48	29.90		36.10		34.00
n=120	8.80		64.80		26.40
BE_t^n	$E_t(y_{t+n}^{1,R})$	$E_t(\pi_{t+n})$	$\varphi_{r,t}^n$	$\varphi_{\pi,t}^n$	$u_t^{n,BE}$
n=12		66.40		33.60	
n=48		43.00		39.20	17.80
n=120		44.10		28.30	27.60
ΔBE_t^n	$\Delta E_t(y_{t+n}^{1,R})$	$\Delta E_t(\pi_{t+n})$	$\Delta\varphi_{r,t}^n$	$\Delta\varphi_{\pi,t}^n$	$\Delta u_t^{n,BE}$
n=12		81.00		19.00	
n=48		31.50		19.90	48.60
n=120		22.60		17.50	59.90

$\varphi_{r,t}^n$ denotes the real term premium; $\varphi_{\pi,t}^n$ denotes the inflation risk premium.

The term structure decompositions in Chart 3 suggest that there are two periods in particular, which merit more analysis. The first is the 1997–98 period, following the granting of operational independence to the Bank of England in setting interest rates, which we consider in Section 5.2.1 below. The second is the period since 2004, which accompanied the emergence of the bond market conundrum, where long-horizon real rates fell to historically low levels. From the model decompositions shown in the second panel of Chart 3, it is clear that this accompanied both a fall in real term premia (to negative levels) and an increase in the unexplained component of long-horizon real rates, as the model overpredicted real forward rates. We discuss these issues further in Section 5.2.2 below.

5.2.1 The impact of Bank of England independence

We have already shown that forward rates fell sharply around the time that the Bank of England was granted operational independence in setting interest rates. Charts 4A and 4B show the forward-rate decompositions from the model out to fifteen years, if we average over the years in the sample before and after Bank independence. Chart 4A shows that average expected inflation is lower across all maturities post independence. Chart 4B shows that the average level of the forward curve of inflation risk premia is lower as well, with the fall at ten-year horizons of the order of 70 basis points. In contrast, while average expected short to medium-term real risk-free rates have been higher since Bank independence, mainly due to the higher level of expected real risk-free rates between 1997 and 2001, little has happened to the level of expected long-horizon real risk-free rates.[22] But the average level of real term premia has fallen significantly and has, on average, been negative since Bank independence.

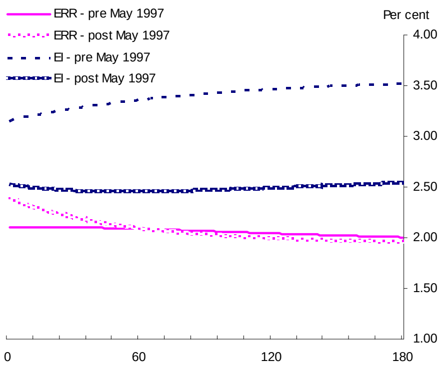
The large fall in the level of real term premia is unlikely to have been mainly related to the independence of the Bank and could reflect a number of other events that affected bond markets over this period. One important contributory factor is likely to have been the introduction of the Minimum Funding Requirement, part of the 1995 Pensions Act, which became effective in April 1997 (see the May 1999 Bank of England *Inflation Report*). This regulatory reform, designed to protect the solvency of pension funds, led to increased pension fund demand for index-linked bonds, which seems to have compressed measured real term premia. And this may have been reinforced by the LTCM and Asian crises in Autumn 1997 and 1998, which caused a ‘flight to

[22] This is perhaps not surprising, as one would not expect the announcement of Bank independence to cause a permanent reassessment of the level of the neutral real interest rate.

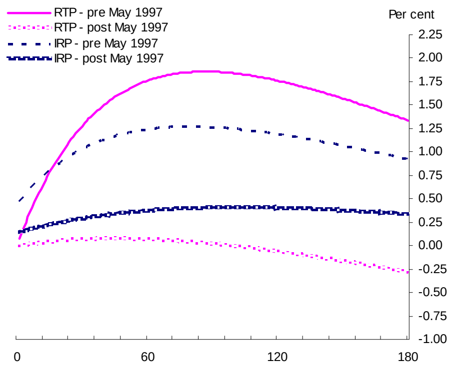
quality' into government bonds. At around the same time, there was also an expansion of the global index-linked bond market, with the first issuance of US Treasury Inflation Protected Securities (TIPS) in January 1997 (see Elsasser and Sack (2004)), which may have reduced the liquidity premium attached to index-linked bonds as an asset class. Perhaps unsurprisingly, our model has more difficulty in explaining real forward rates over this period.

Chart 4: Forward curves of expectations and real term/inflation risk premia

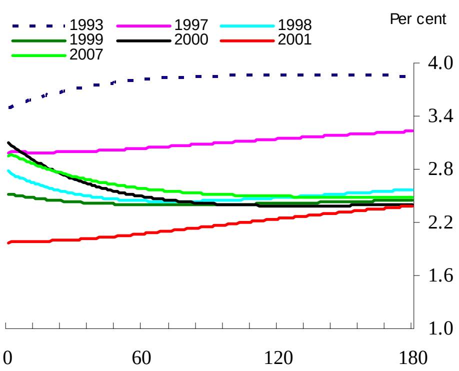
A: Expected real risk-free rates (ERR) and expected inflation (EI)



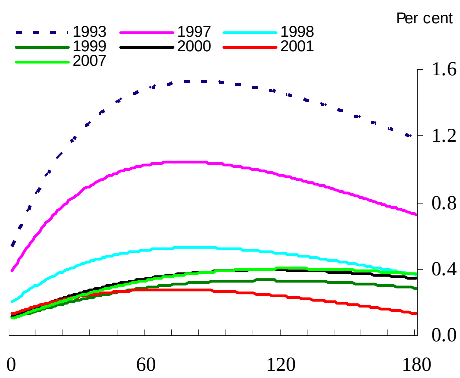
B: Inflation risk premia (IRP) and real term premia (RTP)



C: Forward inflation expectations



D: Forward inflation risk premia



The curves in panels C and D refer to the end of April in each year. We excluded years where the changes were relatively small.

To put the changes surrounding Bank independence in context, Charts 4C and 4D plot forward curves for inflation expectations and inflation risk premia at the end of April for selective years between 1993 and 2007.[23] A few interesting features stand out. First, while the level of the expected inflation curve shifted downwards between 1993 and 1999, the largest annual fall occurred in the year following the independence of the Bank of England. Since then, short-term

[23] End of April is chosen as the announcement of Bank of England independence came in early May 1997.

inflation expectations have occasionally risen, including in the first half of 2007, but long-term inflation expectations have remained anchored at lower levels. Chart 4D shows that the inflation risk premia forward curve has also fallen, the largest fall again occurring in the year after Bank independence. Variance decompositions, similar to those reported in Table D above for the whole sample, pre and post Bank independence reveal that expected real risk-free rates explain a larger proportion of the variation in nominal forward rates post independence.[24]

Focusing on the impact at the long end of the term structure, Chart 5 plots monthly changes in the model-implied decomposition of ten year ahead forward rates. It is evident that expected inflation (Panel A) and the inflation risk premium (Panel B) both fell significantly in May 1997. The fall was larger than two standard deviations of the average change over the sample back to 1992. And long-term inflation expectations continued to fall in the months after the announcement. But while the change was significant, it is important to note that there were several other occasions over the whole sample, where similar (or larger) falls in the inflation components occurred.

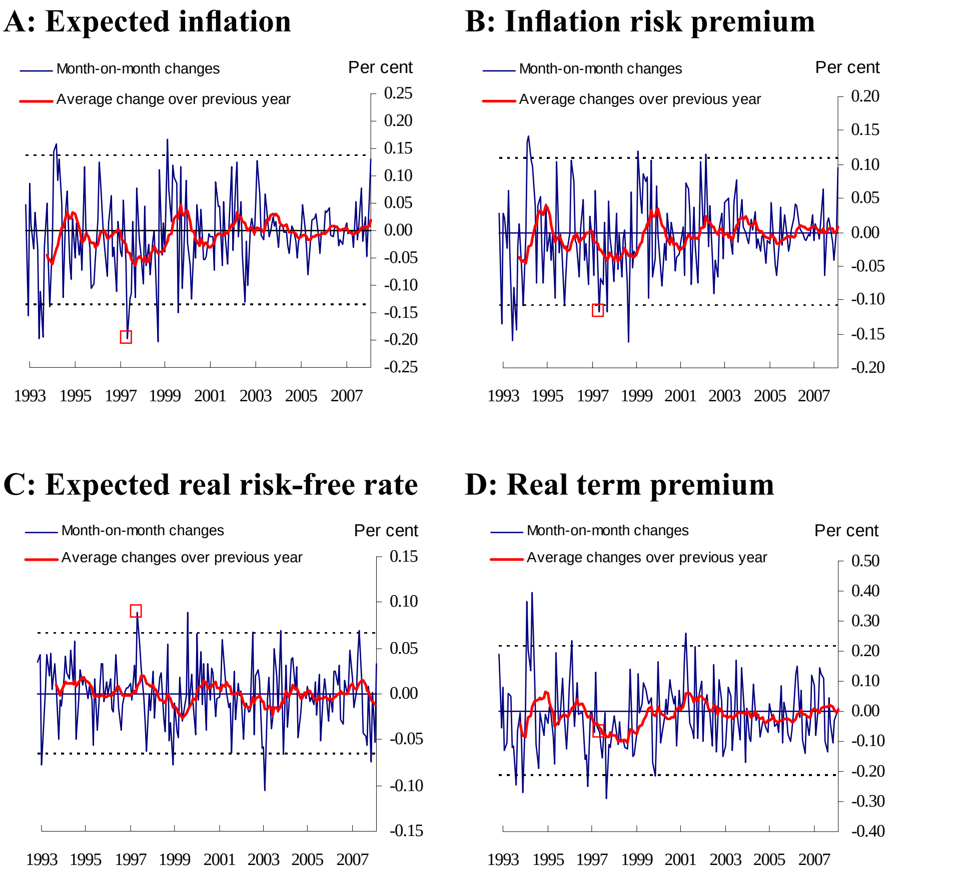
Turning to real forward rate movements, Panel C in the chart shows that the ten year ahead expected real risk-free interest rate, which can be thought of as a proxy for the neutral real rate, increased significantly in the month of the independence announcement. While such an increase is hard to justify, it may just have been an overreaction by investors, as expected rates fell very sharply in June and July reversing the large increase in May. Finally, Panel D of the chart shows that the announcement of Bank independence did not lead to any significant change in the real term premium on the month. Much of the fall in the real term premium occurs later and is more likely to be associated with events described earlier.

An important question is whether the changes surrounding independence constitute a structural break in the data. This is difficult to test formally, as the sample over which the model is estimated is very short. Also since our model is not structural and contains latent factors, which we might expect to pick up any structural changes, it is hard to assess the extent to which the various relationships (eg the relationship between inflation expectations, inflation risk premia and the factors) have changed. But from the shocks to the three latent factors and actual RPI inflation, there is little to suggest that there has been a structural change in the relation between the factors explaining yields. While the absolute value of the shocks to the factors are relatively

[24] The variance decompositions are available upon request.

high in the months after April 1997, they are not abnormally high relative to shocks over the full sample period.[25] There is therefore little indication of a break in the relation between month-on-month inflation and the latent factors.

Chart 5: Monthly changes in ten year ahead forward rate components



Dashed lines indicates +/- 2 standard deviations; a square indicates May 1997.

5.2.2 The bond yield conundrum

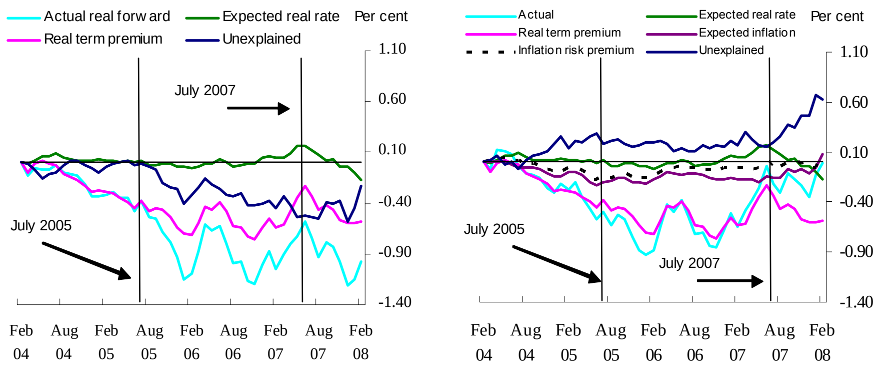
The more recent bond market conundrum refers to the large fall in international medium to long-term yields which began in the middle of 2004 (see Chart 3). Joyce, Kaminska and Lildholdt (2008) estimate an essentially affine real term structure model, which allows a decomposition of the UK real yield curve into expected future real risk-free rates and real term premia. Based on a number of different models, they conclude that the main reason for the large fall in long-term real forward rates over this period was a fall in real term premia. Our model of

[25] Charts are available upon request.

the nominal and real term structure adds additional structure to their analysis and allows us to reinterpret the reasons for the large fall in long-term real forward rates.

Chart 6: Cumulative changes in ten-year forward rates

A: Real **B: Nominal**



In Chart 6 we plot cumulative changes in the model-implied components of ten-year nominal and real forward rates since early 2004. It is useful to distinguish between three different periods: before July 2005, between July 2005 and July 2007 and after July 2007. Consistent with the findings of Joyce, Kaminska and Lildholdt (2008), we find that a large fall in the real term premium accounts for most of the fall in the ten-year real forward rate up to July 2005. But in contrast to their results, we find that our model cannot account for much of the fall in the real long-term forward rate between July 2005 and July 2007. Over this period, the more negative contribution from the unexplained part of the model indicates that the actual ten-year real forward rate became lower relative to the real forward rate predicted by the model. But over the same period there is no deterioration in the model's fit to the nominal rate. Our model therefore suggests that there were factors that were pushing down on either expected real risk-free rates or the real term premia embodied in the real yields, which did not affect the expected real risk-free rates and the real term premia embodied in nominal forward rates. Obviously this could be a symptom of model misspecification and the fact we have assumed that bond markets are not segmented at different maturities. There are reasons to believe that the longer end of the index-linked bond market may have become more segmented during these years, as a result of a number of regulatory developments.[26] By encouraging pension funds to match their long-term

[26]For instance, in April 2004 the Pension Act 2004 came into force and the Pension Protection Fund (PPF) and pensions regulator were launched. On 1 January 2005 FRS17 (the 'Retirement benefits' standards) became mandatory in the United Kingdom and on 11 July 2005 the risk-based levy consultation document was published by the PPF. Perhaps the most important event was the confirmation of the

liabilities, these changes may have led to an increase in institutional demand for long-maturity index-linked bonds (see also McGrath and Windle (2006)). So on this interpretation, the large negative residual on long-term real forwards might reflect the fact that institutional investors have been forced to buy and hold longer-term index-linked bonds for regulatory reasons, even if their price exceeded the price consistent with fundamentals at the time.

Since the start of the financial market turbulence in July 2007, however, the story has reversed somewhat. While the unexplained part of the real forward rate has become smaller in absolute terms, the unexplained part of the nominal rate has become larger. In other words, while actual long-term real forward rates are more in line with the model predictions, the opposite is true for nominal bonds. But one should take care with drawing strong conclusions based on the model since July 2007. As we noted in Chart 3, our model has particular problems in fitting bond yields during periods of severe financial market turbulence, as the model focuses on capturing the longer-term trends in the data.

Concluding on our model-based reasons for the fall in real rates, we find that, of the 100 basis points or so fall in the ten-year real forward rate over the period since the beginning of 2004, around 60 basis points is accounted for by a lower real term premium and about 20 basis points by a lower expected real risk-free rate, leaving 20 basis points unexplained. So long-term real rates remain slightly lower than our model-implied long-term real rate. The inability of the model to explain the large fall in long-term real forward rates during July 2005 to July 2007 also accounts for the increase in implied forward inflation rates over this period (see also Chart 3). Although our model-implied long-term inflation expectations did pick up slightly in late 2007 to early 2008, during July 2005 to July 2007 both inflation expectations and inflation risk premia were broadly stable.

Obviously we need to be very careful when making structural interpretations of the unexplained part of the model and the reasons for the changes in the various components of forward rates. To give a more meaningful interpretation of the dynamics of interest rates, we would need a more formal general equilibrium model, containing structural relationships between identified factors. To address the issue of potential segmentation of the long end of the real yield curve we have described, however, such a model would also need to incorporate heterogeneous agents.

replacement of the Minimum Funding Requirement by the new Department for Work and Pension's regulations in December 2005, requiring firms to undertake evaluations. See the box on 'Pension fund valuation and liability driven investment strategies' in the Spring 2006 *Bank of England Quarterly Bulletin* for further discussion.

Needless to say such models are not yet well developed (for an attempt to incorporate bond market heterogeneity into a partial equilibrium model, see eg Vayanos and Vila (2007)).

5.3 The impact of including survey information in the model

To what extent does the inclusion of long-term surveys of inflation expectations affect our results?[27] As argued above, the inclusion of surveys should help to distinguish correctly between the dynamics of long-term inflation expectations and inflation risk premia, given the short sample over which the model is estimated and the use of latent factors. In fact, when we exclude the survey information, the estimated model parameters turn out to be broadly similar, with the important exception being that the coefficients determining the degree of time variation in the market price of risk are more significant and the lag of the third factor is insignificant in explaining inflation expectations (for details, see Table E in Appendix B).[28] The main reason for this, as discussed above, is that the term structure model does not reject the surveys and hence attributes a much larger proportion of the variance in long-term forward implied inflation rates to inflation expectations than the model without surveys. But our conclusions related to the bond yield conundrum still hold when surveys are not included (see decompositions in Chart 10 in Appendix E). Moreover, it is not possible to discriminate between the two models on the basis of in-sample fit, as they both fit the data equally well.

The main difference between the models is the extent to which they suggest that long-term inflation expectations have changed significantly over time (see Chart 7). Using the model without surveys, we would conclude that the large fall in long-term implied forward inflation rates in 1997-98 was solely due to a fall in inflation risk premia, and that long-term inflation expectations have been higher, on average, post Bank independence.

While this result highlights the sensitivity of our findings to different modelling assumptions, there are a number of reasons to favour the model including surveys. First, our priors would suggest it is implausible that long-term inflation expectations were unaffected, or even raised, by the announcement of Bank independence. All the survey measures of inflation expectations fell sharply after Bank independence and it would be surprising if expectations of bond market investors were that different. Second, since the model has to fit a relatively large number of

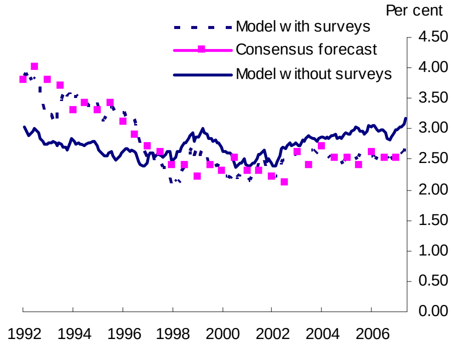
[27] All the results from the model without surveys are available on request.

[28] The choice of which parameters to include in the market price of risk was based on a general to specific approach.

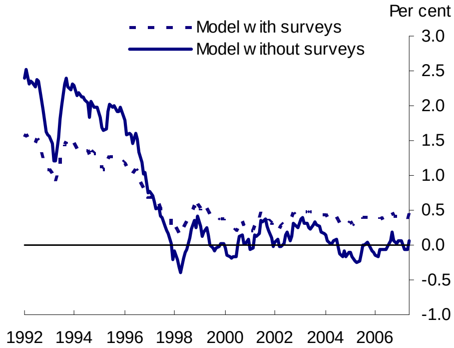
yields (we include eleven yields and RPI inflation), there is no reason to think that the estimates would give a disproportionate weight to fitting the survey data, if the dynamics of these were inconsistent with the factors that determine real and nominal yields. Indeed we can see a number of occasions where the model including surveys implies inflation expectations that are quite different from those of the surveys. For these reasons we favour the model we have described throughout this paper, which has been estimated using surveys of long-term inflation expectations.

Chart 7: Inflation expectations and the inflation risk premium

A: Expected inflation five to ten years ahead



B: Inflation risk premium on five-year inflation rates, five years forward



6 Conclusions

In this paper, we developed a joint, essentially affine, model of the UK real and nominal term structures, which allows us to decompose forward rates into expected real risk-free rates, expected inflation, real term premia and inflation risk premia. To our knowledge, this is the first study to estimate an essentially affine no-arbitrage model of this nature for the United Kingdom over the period since October 1992, when UK monetary policy adopted an explicit inflation target.

The model set-up implies that the market price of risk is linear in a small set of latent factors and observable RPI inflation. The advantage of using latent factors is the ability of this approach to fit nominal and real yields well, at both long and short horizons. However, due to the use of latent factors and small sample problems, we also argued that it is important to include survey measures of long-term inflation expectations as additional information in the model, in order to identify correctly the dynamics of long-term expectations and term premia. Importantly, since we allow long-term inflation expectations to equal survey expectations with an error, we also effectively allow the term structure model to reject the surveys, if they are inconsistent with the information implicit in the term structure and the factors driving nominal and real yields. And, by including a large number of yields, we do not force the model to give a disproportionate weight to fitting the surveys.

An advantage of the joint model is its ability to decompose nominal rates into their various components using market data on both nominal and index-linked bonds. We find that expected real risk-free interest rates and inflation explain around 60% of the variation in one-year nominal forward rates, but as the horizon increases inflation risk premia and, in particular, real term premia explain a much larger fraction of the variation.

We used the model to analyse the impact of Bank of England independence on the term structure of interest rates and the large fall in UK long-term real interest rates since 2004. We found that longer-term inflation risk premia and inflation expectations embodied in the term structure have been lower since the Bank of England was granted operational independence for setting interest rates in May 1997. Moreover, in the month that independence was announced, we find there was a significant fall in both.

More recently, we find that the conundrum of unusually low long-term real rates is mainly attributed by the model to a fall in real term premia, although a significant part of the fall is left unexplained, particularly over the period between July 2005 and July 2007. The relative inability of the model to fit long horizon real forwards during this period may be a consequence of strong pension fund demand for index-linked bonds arising from recent regulatory changes. Moreover, the model decompositions suggest that these special factors affecting the index-linked market may also partly explain the rise in longer-horizon inflation breakeven rates over the same period.

Appendix A: Yields and no arbitrage

Real yields and no arbitrage

Starting from the fundamental asset pricing equation, the price of a real bond price with n periods to maturity is given by:

$$P_t^{n,R} = E_t\left[M_{t+1}P_{t+1}^{n-1,R}\right]. \qquad \textbf{(A-1)}$$

Taking logarithms of both sides of this equation:

$$p_t^{n,R} = E_t\left[m_{t+1} + p_{t+1}^{n-1,R}\right] + \frac{1}{2}Var_t\left[m_{t+1} + p_{t+1}^{n-1,R}\right] \qquad \textbf{(A-2)}$$

where lower-case letters denote logs. With the assumed real SDF (equation **(13)**), the real bond price will be given by:

$$p_t^{n,R} = A_n + B_n'(z_t - \mu),$$

where $B_n = \begin{bmatrix} B_{n,1} & B_{n,2} & B_{n,3} & B_{n,4} \end{bmatrix}$ is a (4×1) vector, $(z_t - \mu)$ is a (4×1) vector and A_n is a scalar. If we now substitute in for next period's SDF and for the bond price, it is possible after some algebraic manipulation to get to this expression, where A_n and B_n' are defined recursively by

$$A_n = -\overline{r} + A_{n-1} - B_{n-1}'\Omega\lambda + \frac{B_{n-1}'\Omega B_{n-1}}{2} \qquad \textbf{(A-3)}$$

$$B_n' = -\gamma' + B_{n-1}'(\Phi - \Omega\beta) \qquad \textbf{(A-4)}$$

where λ and β are defined in equation **(15)** and are matrices of dimensions (4×1) and (4×4) respectively. The price of a bond maturing today is $P_t^{0,R} = 1$, so the initial condition is simply:

$$A_0 = 0 \qquad B_0' = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \textbf{(A-5)}$$

This yields the recursive relationship between yields of different maturities obeying the no-arbitrage restriction. Real yields, continuously compounded, are given by:

$$y_t^{n,R} = -\frac{p_t^{n,R}}{n} = -\frac{A_n}{n} - \frac{B_n'}{n}(z_t - \mu) = \overline{A_n} + \overline{B_n}'(z_t - \mu). \qquad \textbf{(A-6)}$$

The one-period real rate, for instance, is therefore given by:

$$y_t^{1,R} = \overline{r} + \gamma'(z_t - \mu). \qquad \textbf{(A-7)}$$

Nominal yields and no arbitrage

The relation between the price of a nominal bond today with n periods to maturity and the price of this bond in the next period when it has $n-1$ periods to maturity is given by:

$$P_t^{n,N} = E_t\left[M_{t+1}P_{t+1}^{n-1,N}\frac{Q_t}{Q_{t+1}}\right]. \quad \textbf{(A-8)}$$

Taking logarithms of both sides of this equation we get:

$$p_t^{n,N} = E_t\left[m_{t+1} + p_{t+1}^{n-1,N} - \pi_{t+1}\right] + 1/2Var_t\left[m_{t+1} + p_{t+1}^{n-1,N} - \pi_{t+1}\right] \quad \textbf{(A-9)}$$

where lower-case letters denote logs. Using $\pi_{t+1} = E_t\left[z_{4,t+1}\right] + \sigma_4\epsilon_{4,t+1}$ and $E_t\left[z_{4,t+1}\right] = E_t\left[\pi_{t+1}\right] = \mu_4 + \Phi_{[4,:]}(z_t - \mu)$, the nominal SDF is given by:

$$m_{t+1}^* = -(\bar{r} + \mu_4) - \left(\gamma' + \Phi_{[4.:]}\right)(z_t - \mu) - \frac{\Lambda_t'\Omega\Delta_t}{2} - \Lambda_t^{*'}\Omega^{1/2}\epsilon_{t+1},$$

where $\Phi_{[4,:].}$ refers to the fourth row of Φ and

$$\Lambda_t^* = \lambda^* + \beta z_t = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 + 1 \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} \end{bmatrix}(z_t - \mu).$$

The log of nominal bond prices is linear in the factors:

$$p_t^{n,N} = A_n^* + B_n^{*'}(z_t - \mu) \quad \textbf{(A-10)}$$

where

$$A_n^* = -\bar{r} - \mu_4 + A_{n-1}^* - B_{n-1}^{*'}\Omega\lambda^* + \frac{B_{n-1}^{*\prime}\Omega B_{n-1}^*}{2} + \frac{\sigma_4^2}{2} + \sigma_4^2\lambda_4 \quad \textbf{(A-11)}$$

and

$$B_n^{*\prime} = -\left(\gamma' + \Phi_{[4,:]}\right) + B_{n-1}^{*'}\left(\Phi - \Omega\beta\right) + e_4'\Omega\beta. \quad \textbf{(A-12)}$$

Using $P_t^{0,N} = 1$, ie the price of a nominal bond maturing today has a price equal to one, the initial condition is given by:

$$A_0^* = 0 \qquad B_0^{*\prime} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}. \quad \textbf{(A-13)}$$

Nominal yields, continuously compounded, are therefore linear in the demeaned factors and given by

$$y_t^{n,N} = -\frac{p_t^{n,N}}{n} = -\frac{A_n^*}{n} - \frac{B_n^{*\prime}}{n}(z_t - \mu) = \overline{A_n^*} + \overline{B_n^{*\prime}}(z_t - \mu). \quad \textbf{(A-14)}$$

The one-period nominal rate is given by:

$$y_t^{1,N} = y_t^{1,R} + \mu_4 + \Phi_{[4,:]}(z_t - \mu) - \frac{\sigma_4^2}{2} - \sigma_4^2\lambda_4 + e_4'\Omega\beta(z_t - \mu). \quad \textbf{(A-15)}$$

Appendix B: Tables

Table E: Model excluding Consensus inflation surveys

$$\begin{bmatrix} \bar{r}\cdot 12 & \mu_4\cdot 12 \end{bmatrix} = \begin{bmatrix} \underset{(3.81)}{0.023} & \underset{(5.03)}{0.028} \end{bmatrix} \qquad \begin{bmatrix} \sigma^*_R & \sigma^*_N \end{bmatrix} = \begin{bmatrix} \underset{(24.26)}{0.10} & \underset{(17.07)}{0.15} \end{bmatrix}$$

$$\Phi = \begin{bmatrix} \underset{(239.59)}{0.98} & 0 & 0 & 0 \\ \underset{(1.89)}{0.002} & \underset{(226.82)}{0.99} & 0 & 0 \\ \underset{(0.06)}{-0.002} & \underset{(0.89)}{-0.10} & \underset{(678.36)}{0.998} & 0 \\ \underset{(3.40)}{0.34} & \underset{(1.81)}{-2.03} & \underset{(0.49)}{-0.004} & \underset{(1.10)}{0.09} \end{bmatrix} \qquad \lambda^* = \begin{bmatrix} \underset{(0.63)}{-1.195} \\ \underset{(1.60)}{1.232} \\ \underset{(0.51)}{-0.006} \\ \underset{(0.63)}{-0.095} \end{bmatrix}$$

$$\Omega^* = \begin{bmatrix} \underset{(12.00)}{0.149} & 0 & 0 & 0 \\ 0 & \underset{(2.30)}{0.041} & 0 & 0 \\ 0 & 0 & \underset{(3.34)}{3.127} & 0 \\ 0 & 0 & 0 & \underset{(16.21)}{1.863} \end{bmatrix} \qquad \beta^* = \begin{bmatrix} 0 & 0 & \underset{(3.41)}{414.85} & \underset{(1.10)}{2430.0} \\ 0 & 0 & 0 & \underset{(1.13)}{12398} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \underset{(1.60)}{36.41} & 0 \end{bmatrix}$$

t-statistics in brackets.

$\lambda^* = \lambda/1000$, $\beta^* = \beta/120$, $\Omega^* = 1000 \cdot \Omega^{1/2}$, $\sigma^*_k = 1200 \cdot \sigma_k$ for $k = N, R, C$

Table F: Measurement error statistics

	Mean	Std dev	ρ_1	ρ_{12}
$y_t^{1,N}$	0.036	0.206	0.458	0.114
$y_t^{2,N}$	-0.027	0.160	0.263	0.058
$y_t^{3,N}$	-0.047	0.136	0.142	0.017
$y_t^{4,N}$	0.013	0.149	0.343	0.090
$y_t^{6,N}$	-0.052	0.270	0.443	0.142
$y_t^{10,N}$	0.003	0.250	0.311	-0.172
$y_t^{15,N}$	0.022	0.278	0.449	-0.202
$y_t^{4,R}$	0.017	0.290	0.496	-0.162
$y_t^{6,R}$	-0.016	0.280	0.457	-0.076
$y_t^{10,R}$	-0.060	0.237	0.241	-0.005
$y_t^{15,R}$	-0.024	0.246	0.345	-0.048
Survey	0.016	0.265	0.239	n/a

Appendix C: Decomposition of forward rates

Real and nominal forward rates are defined as:

$$f_t^{n,R} = (n+1)y_t^{n+1,R} - ny_t^{n,R} = (A_n - A_{n+1}) + (B_n - B_{n+1})'(z_t - \mu) \quad \textbf{(C-1)}$$

$$f_t^{n,N} = (n+1)y_t^{n+1,N} - ny_t^{n,N} = \left(A_n^* - A_{n+1}^*\right) + (B_n^* - B_{n+1}^*)'(z_t - \mu). \quad \textbf{(C-2)}$$

The ATSM described above allows us to decompose the nominal and real forward curve into interest rate expectations, term premia and a convexity effect, such that:

$$f_t^{n,R} = E_t\left[y_{t+n}^{1,R}\right] + \phi_{t,n}^R + \omega_{t,n}^R \quad \textbf{(C-3)}$$

where $E_t\left[y_{t+n}^{1,R}\right]$ is the expected future one-period real risk-free short rate n periods ahead, $\phi_{t,n}^R$ is the real term premium in the forward curve at maturity n, and $\omega_{t,n}^R$ is the convexity effect at maturity n which is constant for our set-up. We can obtain a similar expression for the nominal forward rate:

$$f_t^{n,N} = E_t\left[y_{t+n}^{1,N}\right] + \phi_{t,n}^N + \omega_{t,n}^N = E_t\left[y_{t+n}^{1,R}\right] + E_t\left[\pi_{t+n}\right] + \phi_{t,n}^N + \omega_{t,n}^N \quad \textbf{(C-4)}$$

where $E_t\left[y_{t+n}^{1,N}\right]$ is the expected future risk-free nominal short rate n periods ahead (ie including inflation expectations), $\phi_{t,n}^N$ is the nominal term premium in the forward curve at maturity n, and $\omega_{t,n}^N$ is the nominal convexity effect at maturity n which is also constant for our set-up. Combining the two we obtain an expression for the forward breakeven rate:

$$f_t^{n,N} - f_t^{n,R} = E_t\left[\pi_{t+n}\right] + \phi_{t,n}^N - \phi_{t,n}^R + \omega_{t,n}^N - \omega_{t,n}^R = E_t\left[\pi_{t+n}\right] + \phi_{t,n}^\pi + \omega_{t,n}^\pi \quad \textbf{(C-5)}$$

where $\phi_{t,n}^\pi$ is the forward inflation risk premium and $\omega_{t,n}^\pi$ is the forward inflation convexity effect. To compute the components of the forward curve in this equation, we follow the steps set out in Lildholdt *et al* (2007). The risk-neutral (ie λ and β are equal to zero matrices) real forward curve can be computed as

$$f_t^{n,R}\Big|_{\lambda=0,\beta=0} = (A_n - A_{n+1})\big|_{\lambda=0,\beta=0} + (B_n - B_{n+1})'\Big|_{\lambda=0,\beta=0}(z_t - \mu) \quad \textbf{(C-6)}$$

and the risk-neutral nominal curve can be computed as

$$f_t^{n,N}\Big|_{\lambda^*=0,\beta^*=0} = \left(A_n^* - A_{n+1}^*\right)\Big|_{\lambda^*=0,\beta^*=0} + (B_n^* - B_{n+1}^*)'\Big|_{\lambda^*=0,\beta^*=0}(z_t - \mu) \quad \textbf{(C-7)}$$

where the notation indicates that the A_n, A_n^*, B_n and B_n^* are computed from the recursions in equations **(A-3)**, **(A-11)** and **(A-12)** with the restriction that $\lambda = \lambda^* = 0_{(4x1)}$ and $\beta = 0_{(4x4)}$, where $0_{(4x1)}$ is a (4×1) vector of zeros. The real forward term premium is given by:

$$\phi_{t,n}^{R} = f_t^{R,n} - f_t^{R,n}\big|_{\lambda=0,\beta=0} \quad \textbf{(C-8)}$$

and the nominal forward term premium by

$$\phi_{t,n}^{N} = f_t^{N,n} - f_t^{N,n}\big|_{\lambda^*=0,\beta^*=0} . \quad \textbf{(C-9)}$$

The inflation risk premium can thus be computed as

$$\phi_{t,n}^{\pi} = \phi_{t,n}^{N} - \phi_{t,n}^{R} = f_t^{N,n} - f_t^{R,n} - f_t^{N,n}\big|_{\lambda^*=0,\beta^*=0} + f_t^{R,n}\big|_{\lambda=0,\beta=0} . \quad \textbf{(C-10)}$$

The convexity effect term is computed as the difference between the risk-neutral forward curve and a forward curve computed as if investors were risk-neutral and future bonds prices were deterministic, in other words the curve corresponding to pure expectations of future interest rates. The convexity effect in the real forward curve is computed as

$$\omega_{t,n}^{R} = f_t^{R,n}\big|_{\lambda=0,\beta=0} - f_t^{R,n}\big|_{\lambda=0,\beta=0,\Omega=0} \quad \textbf{(C-11)}$$

and the convexity effect in the nominal forward curve as

$$\omega_{t,n}^{N} = f_t^{N,n}\big|_{\lambda^*=0,\beta^*=0} - f_t^{N,n}\big|_{\lambda^*=0,\beta^*=0,\Omega=0} . \quad \textbf{(C-12)}$$

It is relatively easy to show, by substituting in from the recursive equations, that the convexity effect in this model is constant over time, though varying by maturity. The term structure of expected future real risk-free interest rates can be obtained as:

$$E_t\left[y_{t+n}^{1,R}\right] = f_t^{n,R} - \phi_{t,n}^{R} - \omega_{t,n}^{R}.$$

And future nominal risk-free interest rates is given by:

$$E_t\left[y_{t+n}^{1,N}\right] = f_t^{n,N} - \phi_{t,n}^{N} - \omega_{t,n}^{N}.$$

Finally, we can back out implied inflation expectations:

$$E_t\left[\pi_{t+n}\right] = E_t\left[y_{t+n}^{1,N}\right] - E_t\left[y_{t+n}^{1,R}\right] . \quad \textbf{(C-13)}$$

Appendix D: Variance decompositions

The variance of nominal forward rates at different maturities can be attributed to five covariance terms:

$$C(f_t^{n,N}, E_t(y_{t+n}^{1,R})) + C(f_t^{n,N}, E_t(\pi_{t+n})) + C(f_t^{n,N}, \varphi_{r,t}^n) + C(f_t^{n,N}, \varphi_{\pi,t}^n) + C(f_t^{n,N}, u_t^{n,N})$$

where $C(f_t^{n,N}, E_t(y_{t+n}^{1,R}))$ is the covariance between the nominal forward rate and the expected n periods ahead real risk-free rate; $C(f_t^{n,N}, E_t(\pi_{t+n}))$ is the covariance between the nominal forward rate and future expected inflation; $C(f_t^{n,N}, \varphi_{r,t}^n)$ is the covariance between the nominal forward rate and the real forward term premium; $C(f_t^{n,N}, \varphi_{\pi,t}^n)$ is the covariance between the nominal forward rate and the inflation risk premium; and $C(f_t^{n,N}, u_t^{n,N})$ is the covariance between the nominal forward rate and the part of the nominal forward rate unexplained by the model. Dividing through by the variance of the nominal forward rate, we obtain:

$$1 = \frac{C(f_t^{n,N}, E_t(y_{t+n}^{1,R}))}{V(f_t^{n,N})} + \frac{C(f_t^{n,N}, E_t(\pi_{t+n}))}{V(f_t^{n,N})} + \frac{C(f_t^{n,N}, \varphi_{r,t}^n)}{V(f_t^{n,N})} + \frac{C(f_t^{n,N}, \varphi_{\pi,t}^n)}{V(f_t^{n,N})} + \frac{C(f_t^{n,N}, u_t^{n,N})}{V(f_t^{n,N})}.$$

A similar variance decomposition can be used for real forward rates and forward inflation breakevens. The variation in real forward rates, $f_t^{n,R}$, can be decomposed into three terms.

$$1 = \frac{C(f_t^{n,R}, E_t(y_{t+n}^{1,R}))}{V(f_t^{n,R})} + \frac{C(f_t^{n,R}, \varphi_{r,t}^n)}{V(f_t^{n,R})} + \frac{C(f_t^{n,R}, u_t^{n,R})}{V(f_t^{n,R})}. \quad \textbf{(D-1)}$$

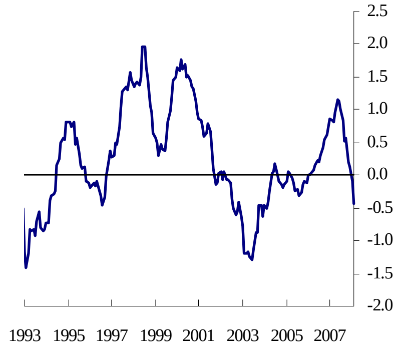
And for inflation breakevens, $\mathrm{BE}_t^n = f_t^{n,N} - f_t^{n,R}$, we have a similar expression:

$$1 = \frac{C(BE_t^n, E_t(\pi_{t+n}))}{V(BE_t^n)} + \frac{C(BE_t^n, \varphi_{\pi,t}^n)}{V(BE_t^n)} + \frac{C(BE_t^n, u_t^{n,BE})}{V(BE_t^n)}. \quad \textbf{(D-2)}$$

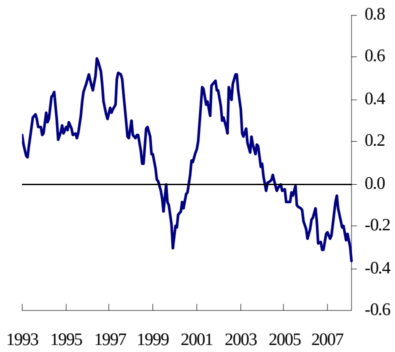
Appendix E: Charts

Chart 8: Factors driving the yield curve (multiplied by 1200)

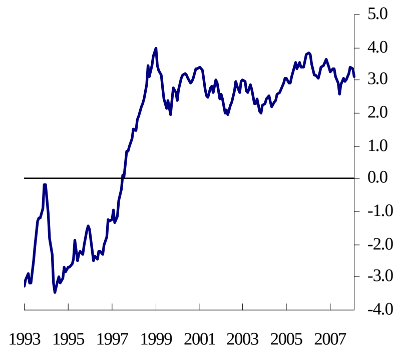
A: Factor 1 (demeaned)



B: Factor 2



C: Factor 3



D: Factor 4 (demeaned inflation)

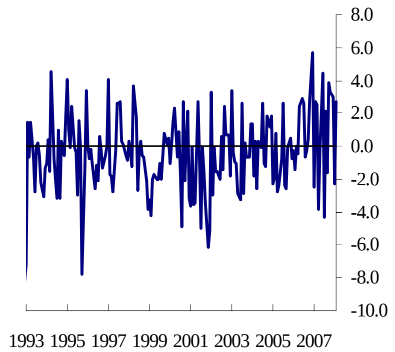
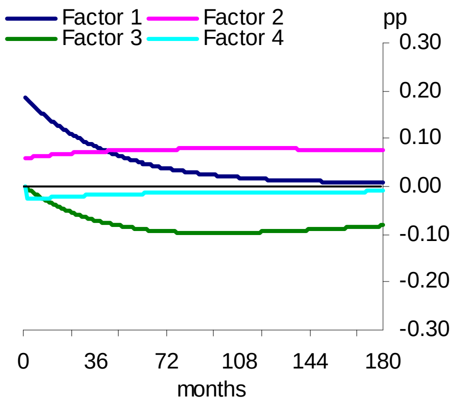
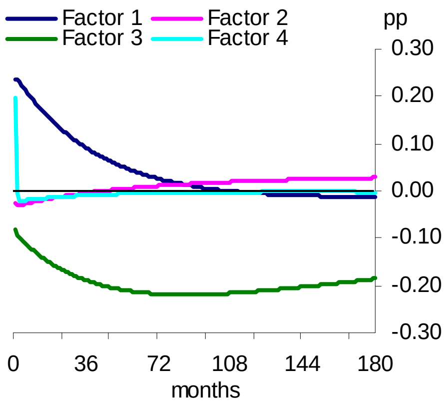


Chart 9: Factor loadings on forward rates

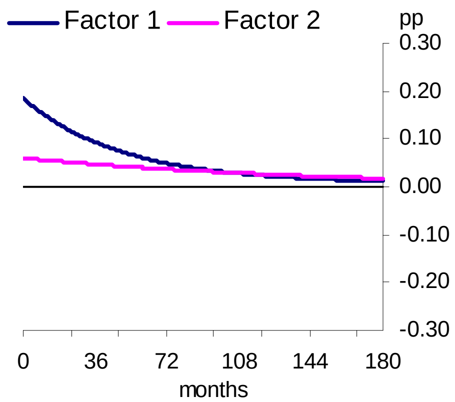
A: Impact on real forward rates



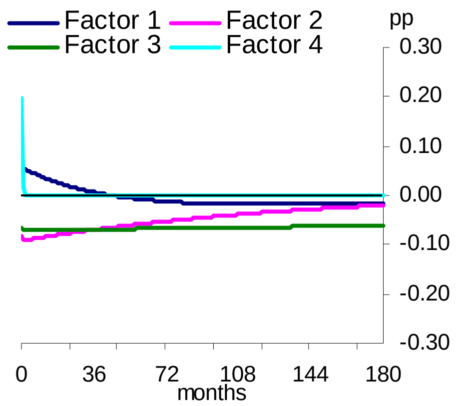
B: Impact on nominal forward rates



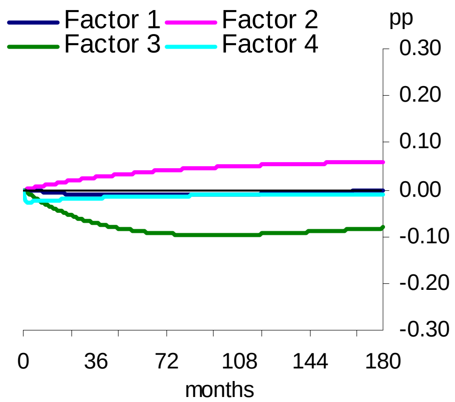
C: Impact on expected real risk-free rates



D: Impact on expected inflation



E: Impact on real term premia



F: Impact on inflation risk premia

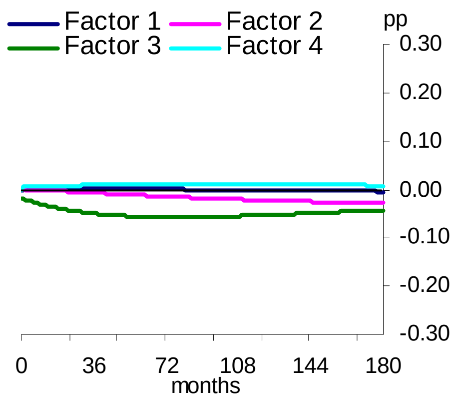
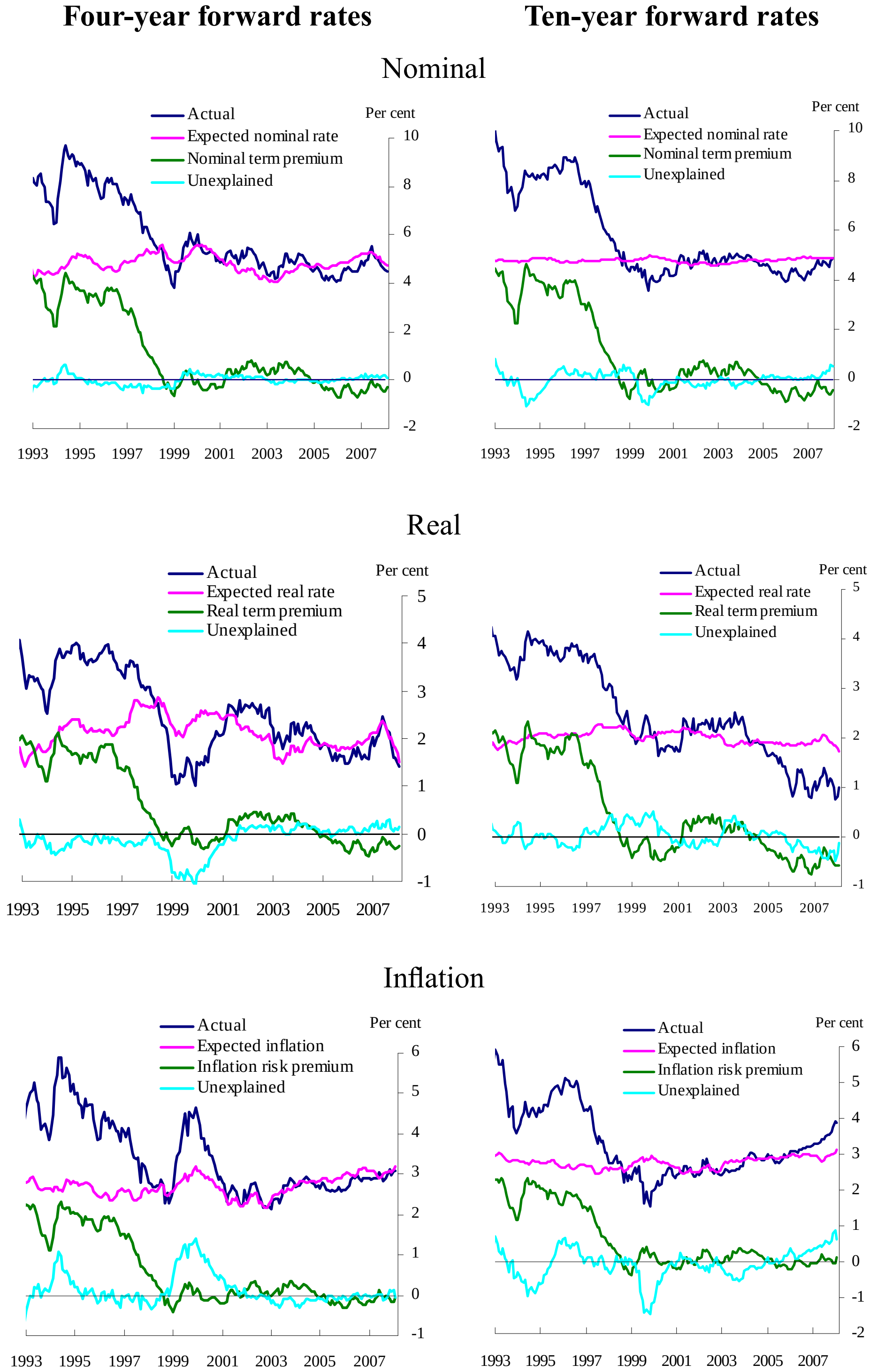


Chart 10: Decomposition of four-year and ten-year forward rates



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