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Abstract

This paper extends a popular no-arbitrage affine term structure model to model jointly bond markets and exchange rates across the United Kingdom, United States and euro area. Using a monthly data set of forward rates from 1992, we first demonstrate that two global factors account for a significant proportion in the variation of bond yields across countries. We also show that, in order to explain country-specific movements in yield curves, local factors are required. Although we implement a very general factor structure, we find that our global factors are related to global inflation and global economic activity, while local factors are closely linked to monetary policy rates. In this respect our results are similar to previous work. But an important advantage of our joint international model is that we are able to decompose interest rates into risk-free rates and risk premia. Additionally, we are able to study the implications for exchange rates. We show that while differences in risk-free rates matter, to a large extent changes in the exchange rate are determined by time-varying exchange rate risk premia.

Key words: Term structure models, exchange rates.

JEL classification: C33, E43, F31.

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Summary

Monetary policy makers routinely analyse financial market variables to extract information for policy. Of particular interest are the yields associated with government bonds of different maturities (the 'term structure of interest rates') and the exchange rates between different currencies. The term structure contains information about expectations of future short-term risk-free rates, such as Bank Rate. Longer-maturity bond yields will also reflect a 'risk premium' – a component that compensates investors for the additional risk associated with those bonds. Most previous work that estimates these risk premia has assumed that each country is a closed economy. There is, however, strong evidence that bond yields are affected by some factors that are common across countries, as well as by local factors such as domestic monetary policy. This paper presents estimates of bond risk premia that allows for a mix of common and local factors across the United Kingdom and its largest trading partners – the United States and the euro area – in the same consistent framework.

Movements in exchange rates should partly reflect differences in short-term interest rates across countries. For example, when interest rates in a 'home' country are relatively high, in the absence of any exchange rate movements investors could obtain unlimited risk-free arbitrage profits by borrowing overseas and buying home bonds. Uncovered interest parity (UIP) states that if interest rates at home are high (low) relative to overseas, investors must expect the home currency to depreciate (appreciate) in order to equalise the overall return on home and foreign bonds. But it is well documented that currencies in high interest rate countries have tended to appreciate on average. One possible explanation for this is a 'foreign exchange risk premium' that compensates investors in high interest rate currencies for some additional risk. The model estimated in this paper also provides estimates of foreign exchange risk premia for sterling, the US dollar and the euro.

The approach taken is to model bond yields and exchange rates as functions of unobserved risk factors, assuming that there are no arbitrage opportunities available from investing in foreign or domestic bonds or bonds of different maturity. The resulting model is fitted to bond and exchange rate data for the three currency areas mentioned above for the period October 1992-June 2008.

In the preferred model, bond yields in each country are driven by two 'global' factors that are common across countries and one factor that is specific to the local economy. It turns out that there is a high correlation between the two global factors and measures of global output and

inflation, while the local factor is highly correlated with the local short-term interest rate (ie the instrument of monetary policy). This is consistent with previous findings in the literature that consider only two countries.

The model estimates of expected changes in exchange rates suggest that the broad trends were expected by investors. This is consistent with foreign exchange risk being an important factor explaining deviations from UIP. The model does not fit the volatility in exchange rates observed on a month-by-month basis, but this is not surprising given the well-documented difficulty in modelling exchange rates.

1 Introduction

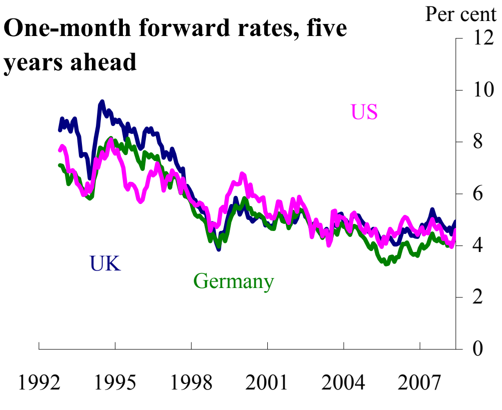
In this paper we estimate a three-country multifactor affine term structure model in order to study the link between international term structure of interest rates and exchange rates. In the first instance our interest is in analysing the drivers of comovement betwen international yield curves. In addition, we study the implications of these movements for exchange rates within a no-arbitrage framework. Our model is consistent with the notion that goverment bond yields are driven by common factors across countries. But local factors, which can be associated with the behaviour of monetary policy makers, also appear important. Because of this, our model is able to account for the high level of observed correlation of yields across countries and deviations from uncovered interest rate parity (UIP).

Motivation for this work can be drawn from monetary policy markers interest in extracting information from financial market data. Of particular interest is the term structure of nominal interest rates, which embodies information about investors' monetary policy expectations, and exchange rates, which are crucial in driving relative prices in open economies. Care needs to be taken when extracting such information, however, as observed movments in asset prices will embody compenstation for risk. In particular, forward interest rates will be affected by a time-varying term premium. Indeed, the term structure literature has shown that it is possible to model variation in a single country's bond yields as a function of a small number of factors (see, for example, Kim and Orphanides (2005)). Moreover, research has demonstrated the benefits of applying no-arbitrage restrictions on the cross-sectional and time-series behaviour of yields. In this paper we extend these insights to an open-economy setting. Our no-arbitrage model of the term structure of nominal interest rates allow us to analyse movements in expected forward interest rates and term premia in the United Kingdom, United States and euro area in a consistent framework.

Our approach also allows us to derive the implications for nominal exchange rates, enabling us to investigate deviations of nominal exchange rates from UIP. A common assumption in macro models is that interest rates and exchange rates should be related by the UIP condition, which states that expected changes in exchange rates should be equal to the differential between interest rates at home and abroad. But, as numerous studies have documented, one of the most puzzling aspects of exchange rate behaviour is the tendency for currencies with high interest rates to appreciate rather than depreciate as UIP would predict.[1] As detailed in Fama (1984), one explanation for the failure of UIP is the existence of time-varying risk premia. The model we present in this paper provides such estimates of foreign exchange risk premia for sterling, the US dollar, and the euro.

Analysing comovement between international interest rates is at the heart of this paper. Such comovement has increased markedly since the 1970s (Charts 1 and 2), which suggests that bond yields in different countries are not only driven by country-specific ('local') factors, but are also affected by international ('global') factors. More recently, researchers have documented that international yield curves exhibit common variation. In particular, empirical analysis of yield curves for Germany, the United Kingdom, the United States, and Japan by Diebold, Li and Yue (2008) provide evidence that: 'global yield factors do indeed exist and are economically important, explaining significant fractions of country yield curve dynamics, with interesting differences across countries'. In a similar way to Diebold, Li and Yue (2008) we model international yield curves in a unified framework that allows for both global and local factors. Importantly, in contrast to their work, our approach is to model international yield curves jointly in a no-arbitrage framework, which allows us to analyse whether yield curves move due to changes in expectations or due to term premia dynamics.

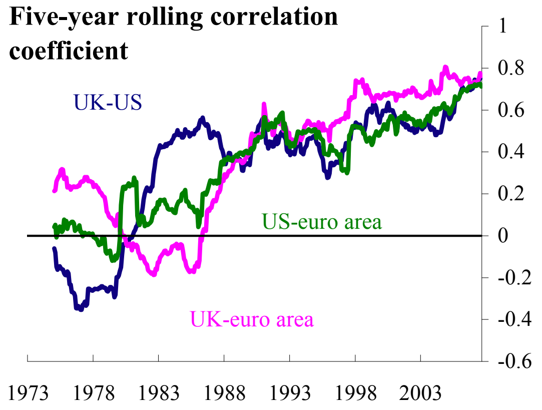
Chart 1: International one-month forward rates, five years ahead



The approach we adopt in this paper is to use a class of no-arbitrage term structure models that is sufficiently flexible to allow for time-varying risk premia: the so-called 'essentially affine' term structure models, as defined by Duffee (2002). In this setting, all bonds in an economy can be

[1]For example see: Canova and Marrinan (1995), Engel (1996) and Hodrick (1987).

Chart 2: Five-year rolling correlations between changes in five-year forwards, five years ahead



priced using a single stochastic discount factor (SDF), for which a flexible functional form is assumed. Both bond yields and the price of risk (and therefore risk premia) are driven by a time-varying set of factors (state variables), which follow some assumed time-series process. We estimate essentially affine models of the term structures in each of the three economies separately and jointly. Joint estimation across more than one economy requires us to rule out arbitrage opportunities not only across yields of different maturities but also across different economies. We assume the same functional form for SDFs in all economies, but allow its parameters to differ between economies.

Although we employ a prevailing framework in a recent international term structure literature (Dong (2007), Backus, Foresi and Telmer (2001), Chabi-Yo and Yang (2006), Diez de los Rios (2009)), we make several important contributions. First, we extend the standard two-country approach by estimating three-country models, which provides us with a consistent estimate of local and global factors, term premia and foreign exchange risk premia across the UK, US and euro area. Second, instead of assuming one model specification, we estimate a number of models, allowing different combinations of local and global factors, which allows us to choose the most appropriate model specification. And third, we incorporate the depreciation rate in our estimation.

To preview our results, we show that, in order to account for cross-country variation in yields, both local and global factors are required. Perhaps unsurprisingly, we find that local factors are closely related to monetary policy rates in the individual countries. We also show that our estimated global factors move closely with global inflation and activity measures. Our no-arbitrage approach means that we are able to decompose interest rates into risk-free rates and risk premia, with our model providing a qualitatively similar account of broad movements in international interest rates compared to the case where we model each yield curve individually. Finally our model generates deviations of exchange rates from UIP, suggesting a significant role for exchange rate risk premia.

The rest of the paper is organised as follows. In Section 2 we set out our modelling framework. In Section 3 we discuss the data and the estimation procedure. In Section 4 we discuss the results. Finally, Section 5 concludes.

2 Modelling framework

In this section we discuss the details of our empirical approach. In particular, the models we estimate belong to the 'essentially affine' class of models, as defined by Duffee (2002). We start with a summary of 'essentially affine' term structure approach for a single country before turning to our extension of the model to the three-country setting. We also discuss how our model relates to other related papers in the literature discussing the related papers in the literature.

2.1 The single-country essentially affine term structure model

Since Duffee (2002), an ever-increasing number of studies have estimated so-called essentially affine term structure models for individual countries. In these models both bond yields and the market prices of risk are affine functions of underlying state variables. This formulation of the SDF is more general than in 'completely affine' models originally proposed by Duffie and Kan (1996), in that the price of risk is allowed to vary independently of interest rate volatility. This is an important development because it removes the direct link between risk premia and interest rates. In turn, this means they are better able to match the properties of the data. For example, for the United Kingdom, Joyce, Lildholdt and Sorensen (2009) jointly estimate a model of the nominal and real term structures using a combination of latent factors and observed inflation. Elsewhere, Kim and Orphanides (2005) estimate a US nominal term structure model. A universal finding in this literature is that time variation in term premia is crucial for explaining the behaviour of interest rates.

2.1.1 Model derivation

We derive the model below in discrete time, following Backus, Foresi and Telmer (1998). The underlying model for a single economy is derived from three basic elements: a process driving the unobservable factors; a formulation of the SDF that ensures the model is essentially affine; and the assumption that bond prices reflect the fundamental asset pricing equation. We then generalize the model to cover the multiple-economy case; show how we can decompose fitted yields into terms representing expectations of future short-term rates, risk premia and convexity; and derive an expression for the expected path of the exchange rate and the foreign exchange risk premium.

We start by defining a $(k \times 1)$ vector of state variables (which we refer to interchangably as factors) relevant for pricing bonds, z_t. The state vector is assumed to follow a first-order VAR,

$$z_{t+1} = \Phi z_t + \Omega^{1/2}\varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{NID}\left(0, I_k\right), \tag{1}$$

where the Ω and Φ are $(k \times k)$ matrices. In our empirical analysis we allow the off-diagonal elements of the Φ matrix to be non-zero, which enables the factors to be correlated with one another. Notice that we assume that the error terms are homoscedastic, so underlying volatility in this model is constant.

In order to get an affine form for bond yields, we assume that the nominal SDF takes the following form,

$$M_{t+1} = \exp\left(-r_t - \frac{\Lambda_t'\Lambda_t}{2} - \Lambda_t'\varepsilon_{t+1}\right), \tag{2}$$

where risk-free rate, r_t, and market prices of risk, Λ_t, are linearly related to the state vector,

$$\begin{aligned} r_t &= r + \gamma' z_t, \\ \Lambda_t &= (\lambda + \beta z_t). \end{aligned} \tag{3}$$

Thus the logarithm of the SDF is given by,

$$\ln M_{t+1} \equiv m_{t+1} = -r - \gamma' z_t - \frac{\Lambda_t'\Lambda_t}{2} - \Lambda_t'\varepsilon_{t+1}.$$

The elements in the matrix Λ_t are often described as the 'market prices of risk' associated with innovations to the SDF.[2] In this model log bond prices are affine functions of the state vector. To

[2]Note that this formulation in **(2)** is consistent with time-varying term premia in the case where some of the elements of β are non-zero. It also nests the case where term premia are constant, where all the elements of β are zero, and the case where term premia are zero, where all the elements of λ and β are zero.

see this, we first assume that the log price of a zero-coupon bond with n periods to maturity at time t is given by,

$$\ln P_t^n = p_t^n = A_n + B_n' z_t, \tag{4}$$

where B_n is a $(k \times 1)$ vector. Since $P_t^0 = 1$, it follows that,

$$A_0 = 0, \;\; B_0 = 0.$$

The fundamental asset pricing equation states that,

$$P_t^n = \mathbb{E}_t \left[M_{t+1} P_{t+1}^{n-1} \right], \tag{5}$$

so that the price of an n-period bond today is equal to the expected value of the product of the price next period and the SDF next period. On the assumption that bond prices and the SDF are jointly lognormal, we can use the property of lognormality to expand out this expression to get,

$$p_t^n = \mathbb{E}_t \left[m_{t+1} + p_{t+1}^{n-1} \right] + \frac{1}{2} \mathbb{V}ar_t \left[m_{t+1} + p_{t+1}^{n-1} \right].$$

If we now substitute in for next period's SDF and for the bond price, it is possible after some algebraic manipulation (derived in Appendix A) to show that the linear expression for bond prices must satisfy the following two recursive equations,

$$\begin{aligned} A_n &= -r + A_{n-1} - B_{n-1}' \Omega^{1/2} \lambda + \frac{B_{n-1}' \Omega B_{n-1}}{2} && (6) \\ B_n' &= -\gamma' + B_{n-1}' \left(\Phi - \Omega^{1/2} \beta \right). && (7) \end{aligned}$$

So, providing that the A_n and B_n parameters satisfy these restrictions, log bond prices are affine in the factors and the model satisfies no-arbitrage. And since log bond prices are affine, it follows that yields are also affine in the factors, with continuously compounded yields given by,

$$y_t^n = -\frac{p_t^n}{n} = -\frac{A_n}{n} - \frac{B_n'}{n} z_t.$$

In many circumstances the forward curve is of greater interest than the spot curve because it more directly conveys information about market expectations of future interest rates. Term premia and the convexity effect are, however, likely to distort the observed forward curve away from pure market expectations of future interest rates. The general relationship between spot and forward yields is,

$$f_t^n = (n+1)\, y_t^{n+1} - n y_t^n,$$

so we can write forward rates as a function of of the factors as follows,

$$f_t^n = -\left(A_{n+1} + B'_{n+1}\right) z_t + \left(A_n + B'_n\right) z_t. \quad \textbf{(8)}$$

It is worth emphasising at this point that our affine model allows us to decompose forward curves into interest rate expectations, term premia and a convexity effect, such that,

$$f_t^n = \mathbb{E}_t\left[y_{t+n}^1\right] + \phi_{t,n} + \omega_{t,n}, \quad \textbf{(9)}$$

where $\mathbb{E}_t\left[y_{t+n}^1\right]$ is the expected future one-period rate n periods ahead, $\phi_{t,n}$ is the term premium in the forward curve at maturity n, $\omega_{t,n}$ is the convexity effect at maturity n.

2.2 Extending the essentially affine approach to a multi-country setting

While a number of studies have jointly estimated completely affine term structure models across multiple economies (see, for example, Benati (2006)), relatively few have used essentially affine models. In one example, Dong (2006) jointly models the term structures in the US and Germany using a macro-factor model. We estimate joint latent factor essentially affine models for three economies (UK, US and euro area) together. Moreover, we estimate essentially affine models of the term structures in each of the three economies separately and jointly. Joint estimation across more than one economy requires us to rule out arbitrage opportunities not only across yields of different maturities but also across different economies. We assume the same functional form for the SDF but allow its parameters to differ between economies thus we allow for country-specific risk aversion to affect bond prices. The state variables driving bond yields may be a mixture of 'global' factors, that affect bond yields in more than one economy, and some may be 'local' factors, only affecting yields in a single economy.

The fundamental asset pricing equation is analogous for a foreign bond:

$$P_t^{n\star} = \mathbb{E}_t\left[M_{t+1}^{\star} P_{t+1}^{n-1\star}\right]$$

We assume that foreign bonds are priced using a foreign currency SDF:

$$\ln M_{t+1}^{\star} = m_{t+1}^{\star} = -r^{\star} - \gamma^{\star\prime} z_t - \frac{\Lambda_t^{\star\prime}\Lambda_t^{\star}}{2} - \Lambda_t^{\star\prime}\varepsilon_{t+1}$$

This specification of the foreign SDF is identical to that of the domestic SDF, with parameters specific to the foreign economy denoted with an asterisk. This specification nests all combinations of global and local factors. A factor can be specified as local by restricting the appropriate elements of γ , $\gamma^{\star}$, λ, $\lambda^{\star}$, β and $\beta^{\star}$ to zero.

The derivation of the expression for the yield of foreign bonds then proceeds in a similar way for domestic bonds. The yield on an n-period foreign bond is therefore given by:

$$y_t^{n\star} = -\frac{p_t^{n\star}}{n} = -\frac{A_n^\star}{n} - \frac{B_n^{\star\prime}}{n} z_t$$

where

$$\begin{aligned} A_n^\star &= -r^\star + A_{n-1}^\star - B_{n-1}^{\star\prime}\Omega^{1/2}\lambda^\star + \frac{B_{n-1}^{\star\prime}\Omega B_{n-1}^\star}{2} && \textbf{(10)} \\ B_n^{\star\prime} &= -\gamma^{\star\prime} + B_{n-1}^{\star\prime}\left(\Phi - \Omega^{1/2}\beta^\star\right) && \textbf{(11)} \end{aligned}$$

and

$$A_0^\star = 0,\ B_0^\star = 0_{k\times 1}.$$

Forward yields on foreign bonds are given by:

$$f_t^{n\star} = -\left(A_{n+1}^\star + B_{n+1}^{\star\prime} z_t\right) + \left(A_n^\star + B_n^{\star\prime} z_t\right). \qquad \textbf{(12)}$$

We now turn to the exchange rate implications of our model. Given the SDFs implied by bond prices, it is technically straightforward to estimate changes in exchange rates in the model without additional information. The key to this is the assumption of no arbitrage across bonds denominated in different currencies. In particular, we rely on equivalence between valuing foreign currency returns using the foreign SDF *or* by domestic SDF converted to home currency. As explained above, the fundamental asset pricing equation can be written from the perspective of a foreign investor in foreign bonds as,

$$1 = \mathbb{E}_t\left[M_{t+1}^\star R_{t+1}^\star\right], \qquad \textbf{(13)}$$

where $R_{t+1}^\star$, the one-period return on an n-period foreign bond is given by,

$$R_{t+1}^\star = \frac{P_{t+1}^{n-1\star}}{P_t^{n\star}}.$$

Domestic investors can invest in foreign bonds and assuming that both markets are frictionless and arbitrage free for foreign and domestic investors, the one-period return on domestic and foreign bonds must be equal, once we have adjusted for the change in exchange rate, S_t, which is expressed as the foreign currency price of a unit of domestic currency:

$$R_{t+1} = \frac{S_{t+1}}{S_t} R_{t+1}^\star$$

Combining this with equations **(5)** and **(13)** gives:

$$\mathbb{E}_t\left[M_{t+1}^\star R_{t+1}^\star\right] = \mathbb{E}_t\left[M_{t+1}\frac{S_{t+1}}{S_t} R_{t+1}^\star\right]. \qquad \textbf{(14)}$$

Under complete markets, $M^{\star}_{t+1}$ and M_{t+1} are unique, so that the foreign SDF equals to the domestic SDF multiplied by the ratio of the exchange rates at $t+1$ and t:

$$M^{\star}_{t+1} = M_{t+1}\frac{S_{t+1}}{S_t}. \tag{15}$$

If markets are incomplete, then SDFs are not unique and the relation **(14)** does not hold for arbitrary SDFs. However, it can be shown that **(14)** remains valid if, from the set of admissible SDFs, we choose $M^{\star}_{t+1}$, M_{t+1} to be minimum-variance SDFs (see Brandt et al (2006)). Taking logs and combining equations above, we get the following expression,

$$\begin{aligned} \Delta s_t &\equiv s_{t+1} - s_t = m^{\star}_{t+1} - m_{t+1}, & \textbf{(16)} \\ &= \left(r_t - r^{\star}_t\right) - \frac{1}{2}\left(\Lambda^{\star\prime}_t\Lambda^{\star}_t - \Lambda^{'}_t\Lambda_t\right) + (\Lambda^{\star\prime}_t - \Lambda_t)\varepsilon_{t+1}, & \textbf{(17)} \end{aligned}$$

where lower case letters denote log variables. The depreciation in the exchange rate is equal to the difference between the foreign and domestic log SDFs. Note that the third term in (**(17)**) implies that even if we assume homoscedasticity of the state vector, the depreciation rate is heteroscedastic when the prices of risk are time-varying.

The expected depreciation can be obtained straightforwardly as the difference in the expected log SDFs:

$$\begin{aligned} \mathbb{E}_t\left[s_{t+1}\right] - s_t &= \mathbb{E}_t\left[m^{\star}_{t+1}\right] - \mathbb{E}_t\left[m_{t+1}\right] & \textbf{(18)} \\ &= \left(r_t - r^{\star}_t\right) - \frac{1}{2}\left(\Lambda^{\star\prime}_t\Lambda^{\star}_t - \Lambda^{\prime}_t\Lambda_t\right) & \textbf{(19)} \end{aligned}$$

Under the assumption of lognormality mentioned above, the foreign exchange risk premium, μ, is equal to half the difference in the variances of the foreign and domestic SDFs, conditional on information available at time t:

$$\begin{aligned} \mu_{t,t+1} &= \frac{\mathbb{V}ar_t\left[m^{\star}_{t+1}\right] - \mathbb{V}ar_t\left[m_{t+1}\right]}{2} & \textbf{(20)} \\ &= \frac{\Lambda^{\star\prime}_t\Lambda^{\star}_t - \Lambda^{\prime}_t\Lambda_t}{2} & \textbf{(21)} \end{aligned}$$

Clearly **(19)** implies that UIP does not hold due to time-varying foreign exchange risk premium, which is determined by the same factors as interest rate risks.

3 Estimation

In this section we describe the econometric methodology we adopt and the data used in our estimation.

3.1 Data

We start by describing the dataset used in our estimation.

3.1.1 Nominal forward rates

The bond market data used to estimate the affine term structure models in this paper are estimates of zero-coupon forward rates for the period October 1992 – May 2008 derived from UK, German and US government bonds using the smoothed cubic spline method proposed by Anderson and Sleath (2001). The forward rates we use relate to the periods one month in length starting one, three, five and ten years ahead. Additionally, we use monetary policy rates as proxies for one-period risk-free rates.

While data on UK, US and German rates are available further back, we have limited our sample period to be from October 1992 to May 2008 to avoid obvious structural breaks in the series. For instance, the adoption of inflation targeting in October 1992 represented a substantial change in the United Kingdom's monetary policy framework. This change is likely to have affected the term structure of interest rates, as perceptions about how monetary policy will react to a variety of factors will affect expectations of future short rates and term premia. This is consistent with break test evidence presented in Benati (2004). Another obvious break in the series is the creation of the euro in 1999; however, model estimation over a large number of parameters prevents further reduction of the sample period.

Time series of forward rates in different countries displayed in Chart 1. Forward rates fell at all maturities in all three economies over the sample. The larger range of falls across different maturities in the UK reflected the large falls in ten-year forward rates in the period between

October 1992 and late 1999. There is evidence of substantial covariation across forward rates from different countries, at all maturities, but particularly those longer than one year.

Forward rates of different maturity within an economy are positively correlated; unsurprisingly, this correlation is highest between yields of similar maturities, although even the one- and ten-year forward rates have correlation coefficients between 0.5 and 0.7 (Charts 3 to 5). Forward rates are also positively correlated across economies (Chart 2). UK one-year forward rates are more strongly correlated with those in the US than in Germany, whereas the opposite is true for ten-year forward rates.

Chart 3: Euro-area forward and policy rates

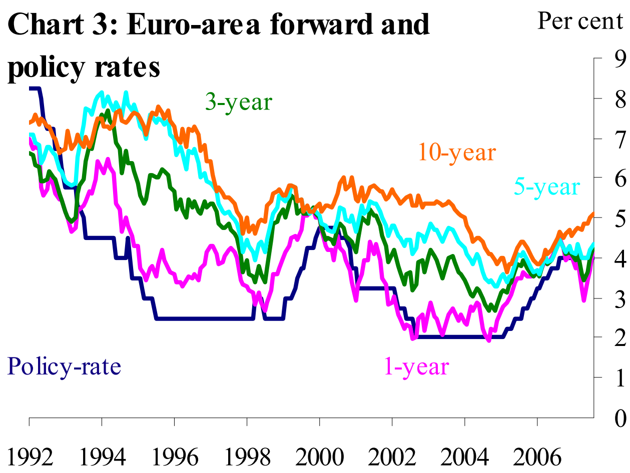


Chart 4: US forward and policy rates

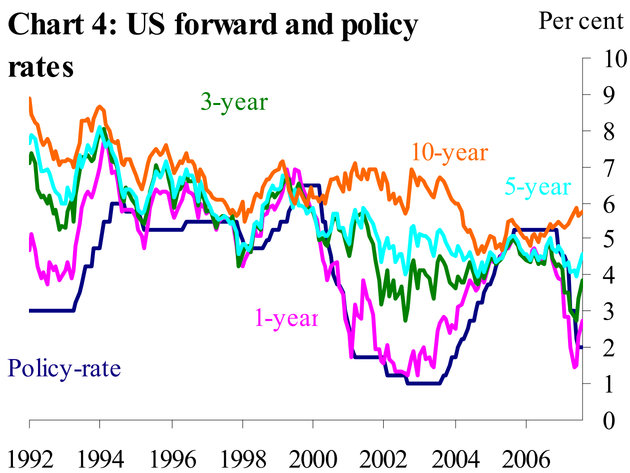
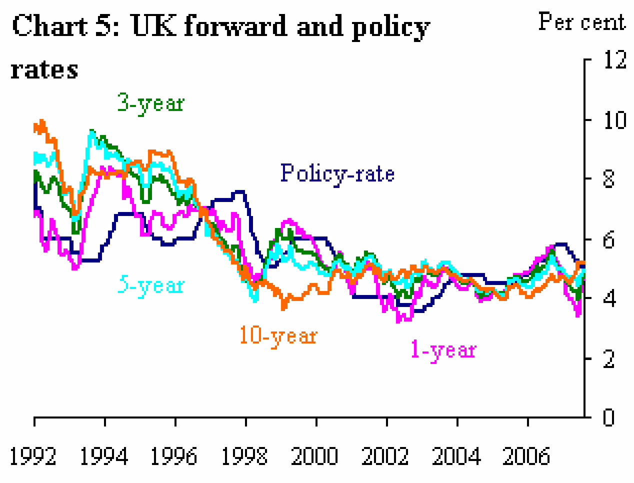


Chart 5: UK forward and policy rates



A preliminary principal components analysis suggests that three factors explain over 99% of the variation in an individual county's yield data (Table A). This confirms the finding in other papers that three factors capture effectively the variation in yields (see, for example, Duffee (2002)). But

Table A: Principal component analysis of German yields

Principal component	Proportion of total variance explained	Yield loadings:			
		y_t^{12}	y_t^{36}	y_t^{60}	y_t^{120}
1	0.92	-0.4806	-0.5162	-0.5152	-0.4870
2	0.99	-0.7019	-0.1944	0.2445	0.6401
3	1.00	-0.4913	0.5413	0.4225	-0.5358

additional factors are required in order to explain all three country yield curves jointly (Table B). So, taking our motivation from Diebold, Li and Yue (2008), we introduce a local factor to account for the variation in yields across countries. In each case, however, we use three factors to model the individual country variation yields. But because of limits to the degrees of freedom in our model we find that it is infeasible to add more than one local factor for each country.

Table B: Principle component analysis of international yields

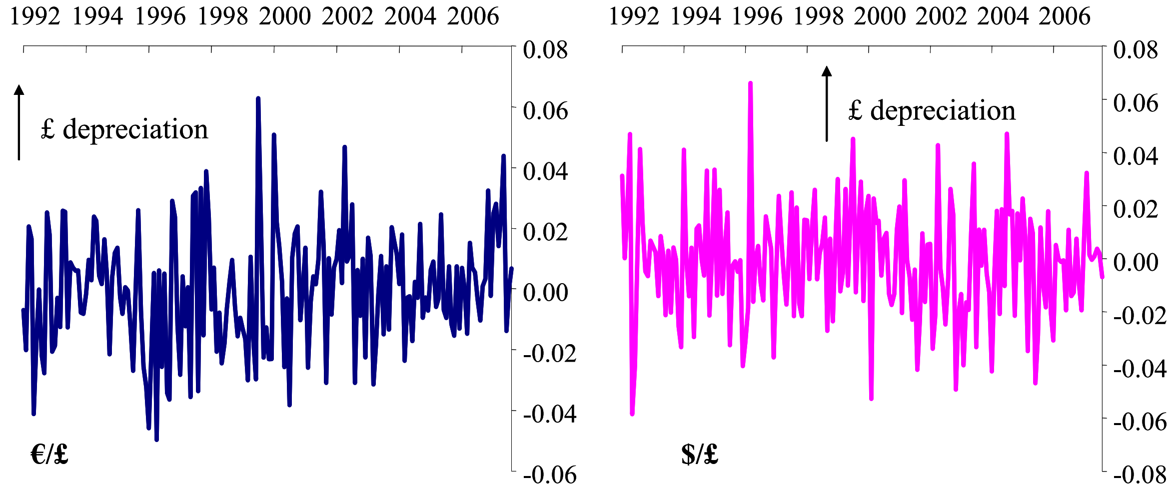
Principal component	Proportion of total variance explained in:		
	UK and US	UK and Germany	UK, US and Germany
1	0.85	0.85	0.80
2	0.96	0.94	0.92
3	0.99	0.99	0.97
4	1.00	1.00	0.99
5	1.00	1.00	1.00

3.1.2 Exchange rates

The exchange rate data we use in this paper are end-of-month data on the sterling-dollar and sterling-euro rates. We use a synthetic series for the sterling–euro exchange for the period before the creation of the euro in 1999. This was obtained by geometrically weighting the bilateral exchange rates of the eleven original euro-area countries using weights based on the country shares of trade with countries outside the euro area.

Chart 6 shows that exchange rate data are much volatile than interest rates and less persistent. The extent of this difference is shown more formally in Table C. The table reports sample moments for the depreciation rate, the short rate and the forward premium. The table illustrates the fundamental difficulty in linking movements in exchange rates and interest rates: exchanges

Chart 6: Exchange rate data



rates are an order of magnitude more volatile than interest rates and exhibit little or no autocorrelation. This underscores the need for a model in which the price of risk is allowed to vary independently of interest rate volatility.

Table C: Sample moments

Currency	Mean	Std deviation	Autocorrelation
Depreciation rate of sterling, Δs_{t+1}			
versus euro	-0.14	23.7	0.05
versus dollar	-0.70	27.8	-0.03
Short rate, r_t			
sterling	5.15	1.04	0.933*
euro	3.89	1.98	0.915*
dollar	4.02	1.67	0.986*
Forward premium, $f_t - s_t = r_t - r_t^\star$			
versus euro	1.25	1.89	0.874*
versus dollar	1.12	1.14	0.935*

All data are expressed as annualised percentages. Data are end-month values. '*' indicates significance at the 5% level.

3.2 Estimation method

Our estimation method follows naturally from the state-space representation of the model. Indeed, the state equation in this context is the first-order VAR:

$$z_{t+1} = \Phi z_t + \Omega^{1/2}\varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim \mathcal{NID}\left(0, I_k\right).$$

The observation equation is given by **(8)**,

$$f_t^n = A_n - A_{n+1} + (B_n' - B_{n+1}')z_t,$$

which captures the relationship between the factors and forward rates. However, when using more maturities than factors, this exact relation cannot be satisfied by yields of all maturities. Hence, some kind of measurement error is required. We assume that the measurement errors have zero mean and are uncorrelated:

$$f_t^{n\star} = A_n^\star - A_{n+1}^\star + (B_n^{\star\prime} - B_{n+1}^{\star\prime})z_t + \upsilon_t^n,\ \upsilon_t^n \sim \mathcal{NID}\left(0, \mathbf{H}\right)$$

The full state-space form of the model with measurement errors is then

$$\begin{aligned} z_{t+1} &= \Phi z_t + \Omega^{1/2}\varepsilon_{t+1},\ \ \varepsilon_{t+1} \sim \mathcal{NID}\left(0, I\right) \\ F_t &= A + B' z_t + V_t,\ V_t \sim \mathcal{NID}\left(0, H\right),\ H \equiv diag(\mathbf{H}), \end{aligned}$$

where F_t is a stacked vector of observable forward rates in both home and foreign countries, and A, B' are stacked vector and matrix of corresponding factor loadings. Given this state-space set-up, the most convenient way to estimate the parameters is by maximum likelihood based on the Kalman filter.

As noted above, because the exchange rate is given as a function of the two stochastic discount factors, the assumption of no-arbitrage implies that the exchange rate itself does not provide any independent information that can be exploited at the estimation stage. However, the estimation procedure is subject to several issues. In particular, the likelihood function has an extremely complicated form and has to be optimised over 40-50 parameters, making it difficult to find a global maximum. In this context it is possible that exchange rate might provide us more precise estimates of unobservable market prices of risk, and hence risk premia. Moreover, if markets are incomplete, the choice of SDFs satisfying **(14)** is not, in general, unique. If markets are incomplete, then incorporating the exchange rate at the estimation stage may help choose from the admissible set of SDFs.

The difficulty with incorporating the depreciation rate into the vector of observed variables is that the measurement equation becomes non-linear. Indeed, the depreciation rate is given by,

$$\begin{aligned}
\Delta s_t &= m_t^\star - m_t \\
&= r_{t-1} + \frac{1}{2}\Lambda_{t-1}'\Lambda_{t-1} + \Lambda_{t-1}'\varepsilon_t - r_{t-1}^\star - \frac{1}{2}\Lambda_{t-1}^{\star\prime}\Lambda_{t-1}^\star - \Lambda_{t-1}^{\star\prime}\varepsilon_t \\
&= \delta - \delta^\star + \frac{1}{2}(\lambda'\lambda - \lambda^{\star\prime}\lambda^\star) + \left(\lambda' - \lambda^{\star\prime}\right) z_t \\
&\quad + \left(\gamma - \gamma^\star + \lambda'\beta - \lambda^{\star\prime}\beta^\star + \left(\lambda^{\star\prime} - \lambda'\right)\Phi\right) z_{t-1} + \frac{1}{2} z_{t-1}'\beta'\beta z_{t-1} \\
&\quad - \frac{1}{2} z_{t-1}'\beta^{\star\prime}\beta^\star z_{t-1} + z_{t-1}'\left(\beta^{\star\prime} - \beta'\right)\Phi z_{t-1} + z_{t-1}'\left(\beta' - \beta^{\star\prime}\right) z_t, \qquad \textbf{(22)}
\end{aligned}$$

which is a non-linear function of $\left[z_{t-1}\ z_t\right]$. In this case we can still use maximum likelihood method, based on extended Kalman filter, linearizing the observation equation for the depreciation rate. (The formulae for estimation are derived in Appendix B.)

As we already discussed in the theory section, the model implies that exchange rate risk is explained by the same factors that drive interest rate risk. This assumption is rather strong and is usually statistically rejected (see Bekaert, Wei and Xing (2007) and Wu (2007)). To account for this, we introduce an additional variable ξ_t, assuming it to be orthogonal to interest rate risk, such that

$$\Delta s_t = r_{t-1} + \frac{1}{2}\Lambda_{t-1}'\Lambda_{t-1} + \Lambda_{t-1}'\varepsilon_t - r_{t-1}^\star - \frac{1}{2}\Lambda_{t-1}^{\star\prime}\Lambda_{t-1}^\star - \Lambda_{t-1}^{\star\prime}\varepsilon_t + \xi_t,$$

where we assume $\xi_t \sim \mathcal{NID}(0, \mathbf{F})$. In practice, this assumption allows us to treat ξ_t as a measurement error variable in the observation equation, such that the model can be written in the standard state-space form (see **(B-1)** in Appendix B).

4 Empirical results

In this section we discuss the results. We start by explaining which models we estimate and present some parameter estimates. One feature of these models is that it is difficult to provide economic interpretations for many of the parameters, particularly those determining the price of risk. Because of this we devote more attention to interpreting the estimated factors. We also discuss the decompositions provided by the models between expected policy rates and risk premia. And finally we consider the exchange rate implications of our models.

4.1 Parameter estimates

We estimate a range of models with forward rate data over the period from October 1992 to June 2008. In each case forward rates for each economy are described by three factors. We start by considering individual country models for the United Kingdom, United States and euro area. When we come to estimate models across countries these models will be used as a benchmark. By this we mean that they will be used to evaluate the extent to which modelling multiple yield curves and the exchange rate impairs our ability to explain the yields for an individual country. Table D presents estimates of UK, US and euro-area three-factor models. In particular, it shows that the state variables in all three models are highly persistent and exhibit long-run short rates close to the unconditional means of the series (Table C).

As a second step we estimate three-country models. Initially we estimated a three-country global-factor model. We found, however, that we were unable to satisfactorily fit the yields across countries using three global factors alone. In particular, the RMSE on the fitted forward rates was around 42 basis points compared with a range of 11 basis points-13 basis points for the single-country models. Moreover, the model assigned implausible amounts of variation to expected risk-free rates and term premia. Taking this as corroborative evidence for the need to use local factors, we estimated a version of the three-country model, with forward rates driven by two global and one local factors. Table E presents the parameter estimates. As with the single country models the estimates show that the factors driving yields are highly persistent. As the parameter estimates themselves provide little information about the model's ability to match the key features of interest rates and exchange rates, we move on to discussing the interpretation of our estimated local and global factors. The residuals from this version of the model lie in a narrow band around observed forward rates (around $\pm$ 50 basis points), with a RMSE error of 27 basis points, indicating that moving from isolated single-country models to a global model we still fit forward rates reasonably well.

Table D: Parameter estimates for single-country models

Parameter	UK model	US model	Euro model
Φ_{11}	0.9703	0.9973	0.9989
Φ_{21}	0.0401	0.0869	0.0059
Φ_{22}	0.978	0.8704	0.9494
Φ_{31}	-0.5265	0.5833	-0.0322
Φ_{32}	0.1223	-1.2947	-0.4855
Φ_{33}	0.9239	0.971	0.9606
$\Omega_{11} \times 10^9$	3.3961	0.00394	3.5743
$\Omega_{22} \times 10^9$	3.9718	1.4817	3.0391
$\Omega_{33} \times 10^9$	2.3918	13.733	13.975
$r \times 1200$	5.76	4.56	3.48
λ_1	1531.3	-11251.9	1245.36
λ_2	-6.9815	12.039	1.4097
λ_3	39.3553	-13.037	-2.2652
β_{11}	0	20332705	1419741
β_{12}	0	0.0984	0
β_{13}	7.478	0	0
β_{21}	18160687	-48674397	-269660
β_{22}	-7693065	0	0
β_{23}	1398032	0	0
β_{31}	-4.5643	135216258	5592836
β_{32}	-4.5617	0	0
β_{33}	0	0	0
$h \times 10^8$	1.7593	1.7907	1.3364

4.2 Commonality in factors

In this section we examine whether allowing for global and local factors helps to explain movements in international interest rates. First, we find striking evidence of commonality in the factors from the individual country models. Charts 7–9 plot the estimated factors from single-country models. It is immediately obvious that the dynamics of the factors are highly correlated; *prima facie* evidence of the existence of global factors. The third factors from single-country models, however, appear to be less correlated across countries, suggesting that there is a clear role for local factors in explaining the variation in yields.

Table E: Parameter estimates for three-country model.

Parameter	Estimate	Parameter	Estimate	Parameter	Estimate
Φ_{11}	0.995	λ_{12}	−0.0113	β^2_{22}	0.0284
Φ_{21}	0.00004	λ_{13}	−0.0033	β^2_{23}	1.337
Φ_{22}	0.9671	λ_{21}	−0.0458	β^2_{31}	0.3457
Φ_{31}	0.0758	λ_{22}	−0.0002	β^2_{32}	−80.896
Φ_{32}	−0.4959	λ_{23}	−0.0026	β^2_{33}	−0.3905
Φ_{33}	−0.9239	λ_{31}	−0.0594	β^3_{11}	−14.53
Φ_{41}	0.0377	λ_{32}	−0.0019	β^3_{12}	2.033
Φ_{42}	−0.2793	λ_{33}	−0.0032	β^3_{13}	23.2362
Φ_{44}	0.9544	β^1_{11}	−0.3872	β^3_{21}	−6.859
Φ_{51}	0.0284	β^1_{12}	−436.28	β^3_{22}	−0.705
Φ_{52}	−0.2196	β^1_{13}	−8.401	β^3_{23}	11.75
Φ_{55}	0.9562	β^1_{21}	−2.256	β^3_{31}	−5.573
$\Omega_{11} \times 10^5$	5.0	β^1_{22}	−110.845	β^3_{32}	−5.584
$\Omega_{22} \times 10^5$	3.19	β^1_{23}	−2.2689	β^3_{33}	−0.4251
$\Omega_{33} \times 10^5$	2.18	β^1_{31}	−0.2725	γ_{21}	−0.0016
$\Omega_{44} \times 10^5$	1.94	β^1_{32}	−43.936	γ_{22}	0.0261
$\Omega_{55} \times 10^5$	2.08	β^1_{33}	3.8132	γ_{31}	0.14
$r_{UK} \times 1200$	4.8	β^2_{11}	−12.038	γ_{32}	−2.5315
$r_{euro} \times 1200$	4.08	β^2_{12}	−257.08	$h \times 10^8$	5.83
$r_{US} \times 1200$	4.2	β^2_{13}	−10.833	$h^{€} \times 10^4$	2.70
λ_{11}	−0.0343	β^2_{21}	−1.3121	$h^{\$} \times 10^4$	5.41

Chart 7: First factors from single-country models

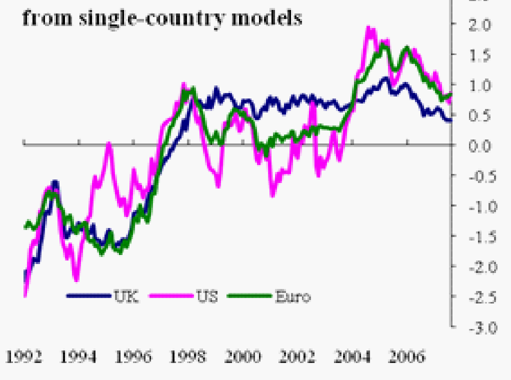


Chart 8: Second factors from single-country models

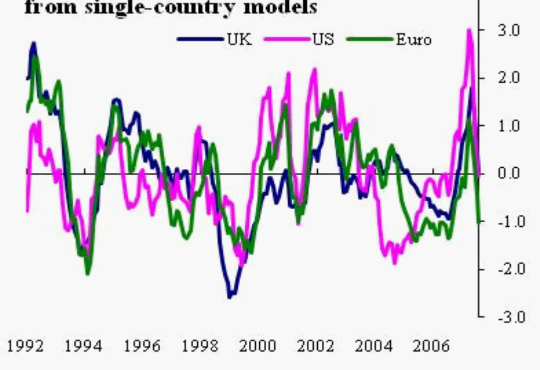
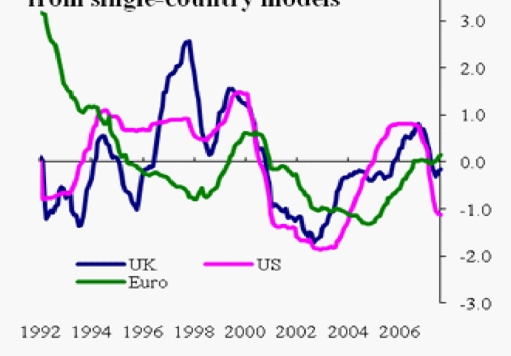


Chart 9: Third factors from single-country models



We then extend the single-country frameworks to a setting where we jointly estimate a model for three country's yield curves and the associated exchange rates. Initially we restricted the model to only allow for movements in interest rates to depend on global factors. As mentioned above, these models did not provide a satisfactory fit to yields across countries (in terms of mean square error of the pricing errors). Indeed the best-performing models according to the log likelihood criterion are the models allowing for two global and one local factors.[3] This fits in with the evidence from the individual country models as well as our preliminary principal component analysis. The factors implied by this model are shown in Charts 10-13. Comparing Charts 7-9 it is worth noting that the global factors appear to be highly correlated with first two factors from single-country models, while local factors have the same pattern as third factors from those models.

Chart 10: Global factors

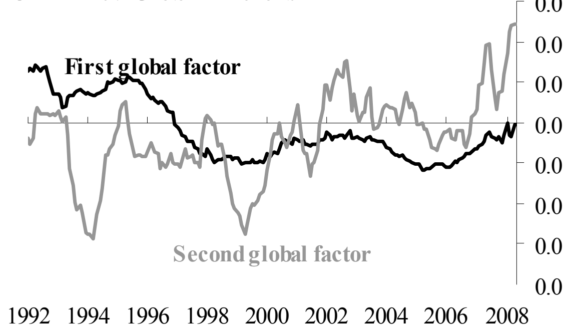


Chart 11: Local UK factor

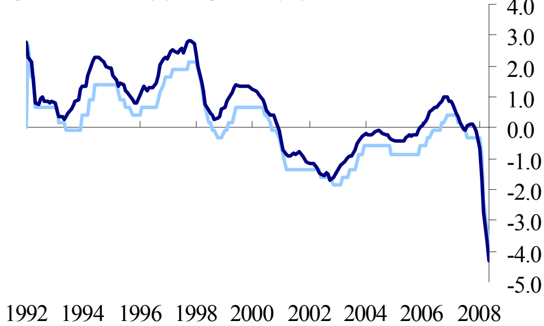


Chart 12: Local US factor

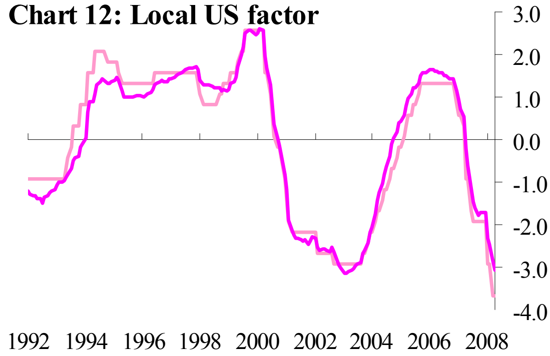
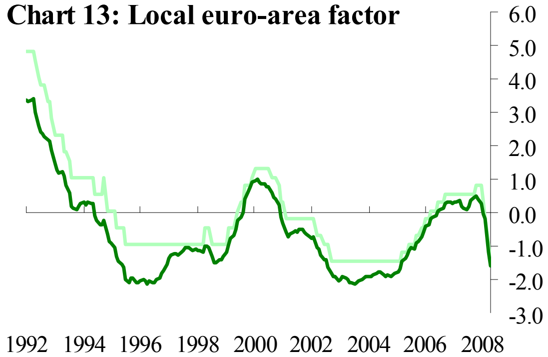


Chart 13: Local euro-area factor



Dark solid lines represent factors, while pale coloured lines stand for coresponding normalised monetary policy rates

[3]Perhaps one obvious alternative here is to allow for more than one country-specific factor. But such a set-up would imply estimating models in which movements in interest rates were driven by as many as seven latent factors. As noted earlier, we found that we were unable to estimate models with this number of latent factors.

Charts 11 to 13 show that the local factors are highly correlated with country-specific monetary policy rates, which confirms that the global factor only model would be misspecified in the environment of independent monetary policies across different economies. Additional evidence of the close link between local factors and policy rates comes from the factor loading analysis (see Chart 14). Loadings on the local factors are decreasing with horizon, which confirms that local factors have a close link with local monetary policy and little explanatory power at longer maturities. Long-term maturity rates are mostly explained by the global factors (Chart 15), which have an important economic interpretation too. Indeed, following the previous literature studying the relationship between yield factors and macroeconomic factors (see Ang and Piazzesi (2003) and Christensen, Diebold and Rudebusch (2008)), we find that our extracted global factors have a striking correlation with global inflation and global economic activity (Charts 16-17).

Chart 14: Spot yield loadings on the local factor

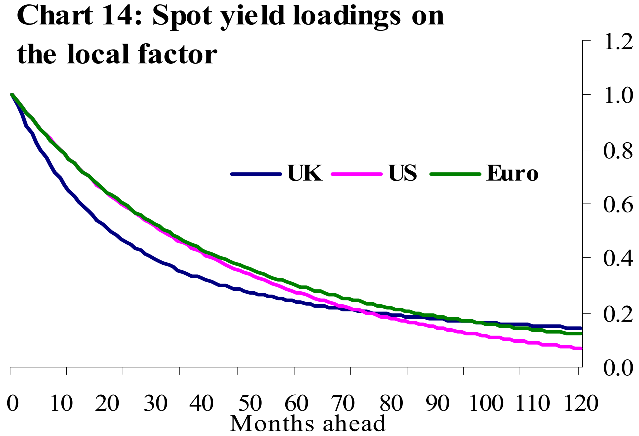


Chart 15: Spot yield loadings on the global factor

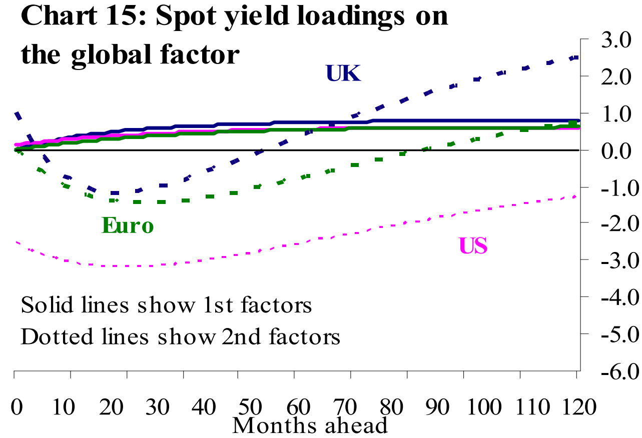


Chart 16: First global factor from the three-country model and inflation

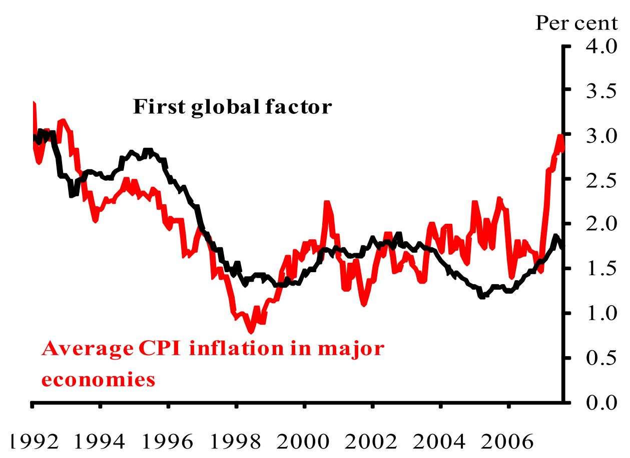
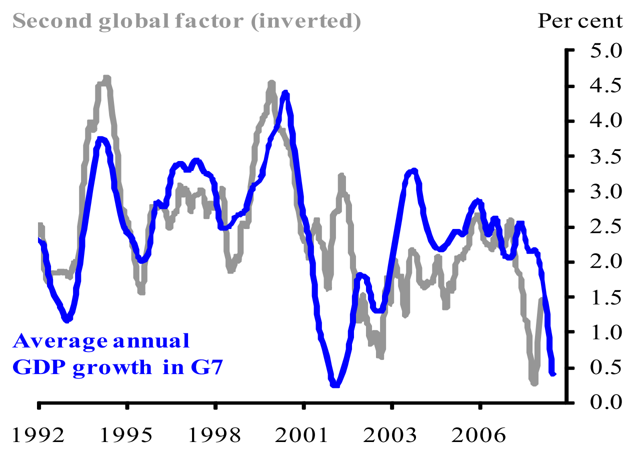


Chart 17: Second global factor from the three-country model and international activity



4.3 Implied decompositions into risk-free rates and term premia

Charts 18-20 show implied decompositions of ten-year forward rates from our three-country model (driven by two global and one local factors) and our benchmark single-country models. The main point to note about these decompositions is that they provide a qualitatively similar account of broad movements in interest rates: expected risk-free rates and term premia have fallen in all economies since the late 1990s. UK term premia are very low, became negative by some measures in 1998, and remaining low thereafter. This accounts for the downward sloping UK curve. By contrast, US and euro-area term premia have remained positive in all our models.

Chart 18: UK ten-year forward decomposition

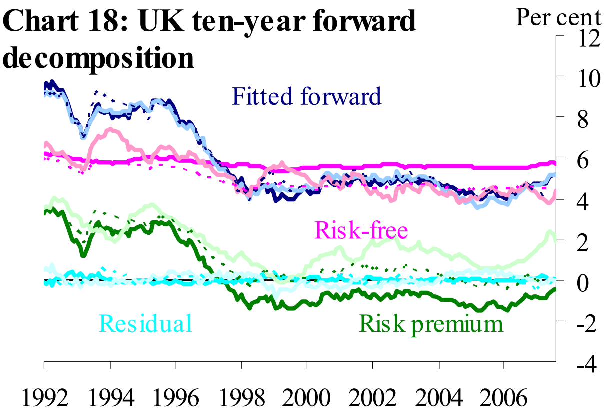


Chart 19: Euro-area ten-year forward decomposition

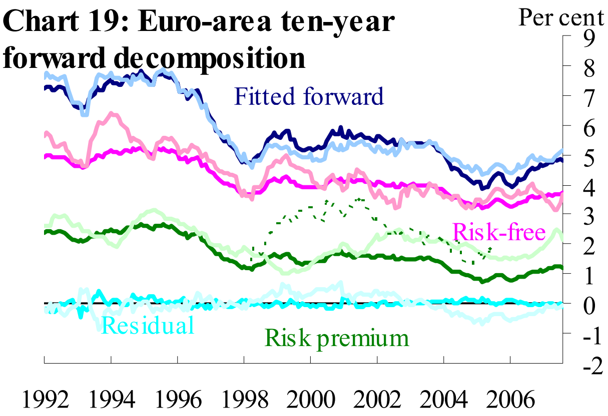
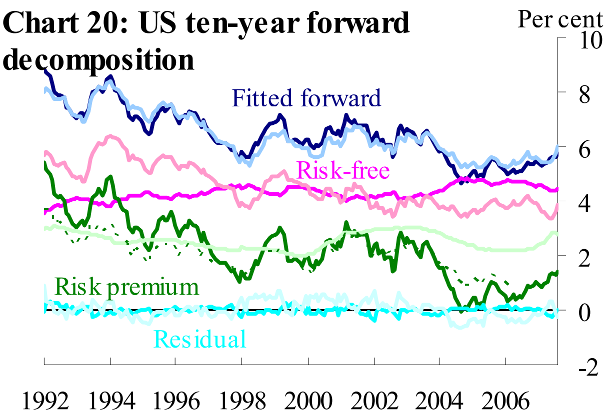


Chart 20: US ten-year forward decomposition



Dark lines show single-country model; pale lines show three-country model; dotted lines represent decompositions from: Joyce *et al* (2009) for the United Kingdom; Hordahl and Tristani (2007) for the euro area; and Kim and Orphanides (2005) for the United States.

4.4 Exchange rate implications

The international no-arbitrage model allows us to study the decomposition of exchange rate movements into risk free and risk premia. Indeed Charts 21 and 22 show the model-implied depreciation rate from our three-country model and the risk-free component (UIP path). While deviations from UIP are relatively small, changes in expected exchange rates differ substantially from the differences in risk-free rates. Throughout the sample period, UIP path has been indicating the depreciation of sterling. In contrast, sterling has – on average – tended to appreciate over this period. By contrast our model shows prolonged periods in which the expected sterling weakening is tempered with periods of sterling strength. As the model suggests, the dynamics of exchange rates has been determined to a large extent by time-varying exchange rate risk premia. Nevertheless, even though our model captures broad movements in exchange rates, it clearly fails to match exchange rate volatility.

Chart 21: €/£ depreciation rate results

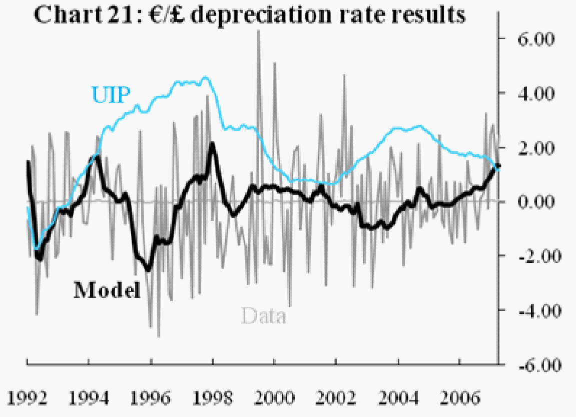
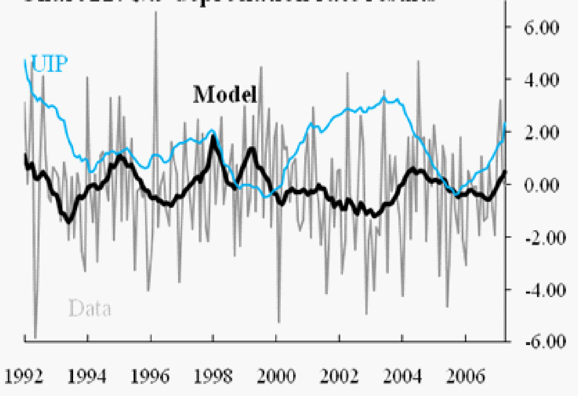


Chart 22: $/£ depreciation rate results



5 Conclusion

We find that two global factors account for a significant proportion of the variation in bond yields across countries. Moreover, we also demonstrate that, in order to explain country-specific movements in yield curves, local factors are required. In turn, these local factors are closely linked to monetary policy. Although these conclusions are consistent with the findings of Diebold, Li and Yue (2008), we have taken the additional step of estimating these factors in our joint international model under the assumption of no-arbitrage. This means that we are able to decompose interest rates into risk-free rates and risk premia. We show that even when jointly estimating a model across all three countries, incorporating the exchange rate, we are still able to provide a qualitatively similar account of broad movements in international interest rates compared to the case where we model each yield curve individually. A further advantage of our approach is that we are able to study the implications for foreign exchange rates. Here we show that, while the differences in risk-free rate matters, to a large extent changes in the exchange rate are determined by exchange rate risk premia.

Appendix A: Derivation of recursive bond pricing coefficients

In this section, we show the derivation of the recursive equations [7] and [8] shown in the text above. For convenience we re-state the three basic equations of the model. The state variables driving yields are assumed to follow a first-order VAR process:

$$z_{t+1} = \Phi z_t + \Omega^{1/2}\varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{NID}(0, I)$$

The logarithm of the nominal SDF is:

$$m_{t+1} = -r - \gamma' z_t - \frac{\Lambda_t' \Lambda_t}{2} - \Lambda_t^{'} \varepsilon_{t+1}$$

Finally, we assume that log bond prices are affine in the state vector, so that the log price of a zero-coupon bond with n periods to maturity at time t is given by:

$$\log P_t^n = p_t^n = A_n + B_n' z_t$$

where:

$$B_n' = \begin{bmatrix} B_{n,1} & B_{n,2} & B_{n,2} \end{bmatrix}$$

To derive the recursive equations, we begin with the fundamental asset pricing equation, which states that:

$$P_t^n = \mathbb{E}_t \left[M_{t+1} P_{t+1}^{n-1} \right]$$

If we assume that the expression within the expectations term on the right-hand side of this equation is jointly lognormally distributed, we can then write:

$$p_t^n = \mathbb{E}_t \left[m_{t+1} + p_{t+1}^{n-1} \right] + \frac{1}{2} \mathbb{V}ar_t \left[m_{t+1} + p_{t+1}^{n-1} \right]$$

where lower case letters denote that we have taken the natural logarithm. The proof now proceeds by substituting in for log bond prices and the log SDF and manipulating this expression. First, if we substitute for m_{t+1} and p_{t+1}^{n-1} in the expectations term from the previous expression:

$$\mathbb{E}_t \left[m_{t+1} \right] + \mathbb{E}_t \left[p_{t+1}^{n-1} \right] = \mathbb{E}_t \left[-r - \gamma' z_t - \frac{\Lambda_t' \Lambda_t}{2} - \Lambda_t^{'} \varepsilon_{t+1} + A_{n-1} + B_{n-1}' z_{t+1} \right]$$

If we now substitute in for z_{t+1} take through the expectations operator and use the fact that $\mathbb{E}_t \varepsilon_{t+1} = 0$, we get:

$$\begin{aligned} \mathbb{E}_t \left[m_{t+1} \right] + \mathbb{E}_t \left[p_{t+1}^{n-1} \right] &= \mathbb{E}_t \left[-r - \gamma' z_t - \frac{\Lambda_t' \Lambda_t}{2} - \Lambda_t^{'} \varepsilon_{t+1} + A_{n-1} + B_{n-1}' \left(\Phi z_t + \Omega^{1/2} \varepsilon_{t+1} \right) \right] \\ &= -r - \gamma' z_t - \frac{\Lambda_t' \Lambda_t}{2} + A_{n-1} + B_{n-1}' \Phi z_t \end{aligned}$$

Second,

$$\mathbb{V}ar_t\left[m_{t+1}+p_{t+1}^{n-1}\right]=\mathbb{V}ar_t\left[-r-\gamma' z_t-\frac{\Lambda_t'\Lambda_t}{2}-\Lambda_t^{'}\varepsilon_{t+1}+A_{n-1}+B_{n-1}' z_{t+1}\right]$$

If we now substitute in for z_{t+1} as before and drop the constant terms (since these have zero variance and can be ignored), we obtain:

$$\begin{aligned}\mathbb{V}ar_t\left[m_{t+1}+p_{t+1}^{n-1}\right] &= \mathbb{V}ar_t\left[-\Lambda_t^{'}\varepsilon_{t+1}+B_{n-1}'\left(\Phi z_t+\Omega^{1/2}\varepsilon_{t+1}\right)\right]\\ &= \mathbb{V}ar_t\left[-\left(\Lambda_t^{'}-B_{n-1}'\Omega^{1/2}\right)\varepsilon_{t+1}\right]\end{aligned}$$

Expanding out the right-hand side of this expression, we obtain:

$$\begin{aligned}\mathbb{V}ar_t\left[m_{t+1}+p_{t+1}^{n-1}\right] &= \left(\Lambda_t^{'}-B_{n-1}'\Omega^{1/2}\right)\left(\Lambda_t^{'}-B_{n-1}'\Omega^{1/2}\right)'\\ &= \Lambda_t'\Lambda_t-\Lambda_t'\Omega^{1/2}B_{n-1}'-B_{n-1}'\Omega^{1/2}\Lambda_t+B_{n-1}'\Omega B_{n-1}\\ &= \Lambda_t'\Lambda_t-2B_{n-1}'\Omega^{1/2}\Lambda_t+B_{n-1}'\Omega B_{n-1}\end{aligned}$$

where the last line uses the fact that $B_{n-1}'\Omega^{1/2}\Lambda_t'$ is a scalar. Now we can obtain an expression for the log bond price:

$$\begin{aligned}p_t^n &= \left(-r-\gamma' z_t-\frac{\Lambda_t'\Lambda_t}{2}+A_{n-1}+B_{n-1}'\Phi z_t\right)+\frac{1}{2}\left(\Lambda_t'\Lambda_t-2B_{n-1}'\Omega^{1/2}\Lambda_t+B_{n-1}'\Omega B_{n-1}\right)\\ &= -r-\gamma' z_t+A_{n-1}+B_{n-1}'\Phi z_t-B_{n-1}'\Omega^{1/2}\Lambda_t+\frac{B_{n-1}'\Omega B_{n-1}}{2}\end{aligned}$$

We now substitute in for Λ_t and collect terms:

$$\begin{aligned}p_t^n &= -r-\gamma' z_t+A_{n-1}+B_{n-1}'\Phi z_t-B_{n-1}'\Omega^{1/2}\left(\lambda+\beta z_t\right)+\frac{B_{n-1}'\Omega B_{n-1}}{2}\\ &= \left(-r+A_{n-1}-B_{n-1}'\Omega^{1/2}\lambda+\frac{B_{n-1}'\Omega B_{n-1}}{2}\right)+\left(-\gamma'+B_{n-1}'\left(\Phi-\Omega^{1/2}\beta\right)\right)z_t\end{aligned}$$

Finally, substituting in for p_t^n on the left-hand side , we get the following expression:

$$A_n + B_n' z_t = \left(-r + A_{n-1} - B_{n-1}' \Omega^{1/2} \lambda + \frac{B_{n-1}' \Omega B_{n-1}}{2} \right) + \left(-\gamma' + B_{n-1}' \left(\Phi - \Omega^{1/2} \beta \right) \right) z_t$$

From this equation, we get the two recursive equations in the text:

$$\begin{aligned} A_n &= -r + A_{n-1} - B_{n-1}' \Omega^{1/2} \lambda + \frac{B_{n-1}' \Omega B_{n-1}}{2} \\ B_n' &= -\gamma' + B_{n-1}' \left(\Phi - \Omega^{1/2} \beta \right) \end{aligned}$$

Since we know that $P_t^0 = 1$, we can start up these recursions with:

$$A_0 = 0, \quad B_0' = \mathbf{0}$$

Appendix B: The extended Kalman filter

ATSM with depreciation rate included as observable variable can be presented in the state-space form:

$$\begin{aligned} x_t &= \Gamma x_{t-1} + \Theta \epsilon_t \\ Y_t &= f(x_t) + \rho_t, \end{aligned} \quad \textbf{(B-1)}$$

where f is continuos and differentiable.

Proof. Let's denote the old state vector by z_t. Since depreciation rate is a function of both, current and past z_t, we have to change the system and introduce z_{t-1} into a new state vector

$$x_t = \begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix}$$

Then the state equation becomes

$$\begin{aligned} x_t &= \begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix} = \begin{pmatrix} \Phi & 0 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} z_{t-1} \\ z_{t-2} \end{bmatrix} + \begin{pmatrix} \Omega^{1/2} & 0 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix} \\ &= \Gamma x_{t-1} + \Theta \epsilon_t \end{aligned}$$

In the extended ATSM the space equation is a system of observation equations for yields and a depreciation rate. Yields are linear functions of z_t and hence of x_t

$$\begin{aligned} y_t &= A + B z_t + \upsilon_t \\ &= A + \begin{pmatrix} B & 0 \end{pmatrix} x_t + \upsilon_t \\ &= A + \bar{B} x_t + \upsilon_t \end{aligned}$$

Depreciation rate is given by

$$
\begin{aligned}
\Delta s_t &= m_t^* - m_t = r_{t-1} + \frac{1}{2}\Lambda_{t-1}'\Lambda_{t-1} + \Lambda_{t-1}'\varepsilon_t - r_{t-1}^* - \frac{1}{2}\Lambda_{t-1}^{*\prime}\Lambda_{t-1}^* - \Lambda_{t-1}^{*\prime}\varepsilon_t \\
&= \delta + \gamma z_{t-1} + \frac{1}{2}(\lambda + \beta z_{t-1})'(\lambda + \beta z_{t-1}) + (\lambda + \beta z_{t-1})'\Omega^{-\frac{1}{2}}(z_t - \Phi z_{t-1}) \\
&\quad -\delta^* - \gamma^* z_{t-1} - \frac{1}{2}(\lambda^* + \beta^* z_{t-1})'(\lambda^* + \beta^* z_{t-1}) - (\lambda^* + \beta^* z_{t-1})'\Omega^{-\frac{1}{2}}(z_t - \Phi z_{t-1}) \\
&= \delta - \delta^* + \frac{1}{2}(\lambda'\lambda - \lambda^{*\prime}\lambda^*) \\
&\quad + \left(\lambda' - \lambda^{*\prime}\right)\Omega^{-\frac{1}{2}} z_t + \left(\gamma - \gamma^* + \lambda'\beta - \lambda^{*\prime}\beta^* + \left(\lambda^{*\prime} - \lambda'\right)\Omega^{-\frac{1}{2}}\Phi\right) z_{t-1} \\
&\quad + \frac{1}{2}z_{t-1}'\beta'\beta z_{t-1} - \frac{1}{2}z_{t-1}'\beta^{*\prime}\beta^* z_{t-1} + z_{t-1}'\left(\beta^{*\prime} - \beta'\right)\Omega^{-\frac{1}{2}}\Phi z_{t-1} + z_{t-1}'\left(\beta' - \beta^{*\prime}\right)\Omega^{-\frac{1}{2}} z_t \\
&\equiv a + b_{(1x6)} x_t + x_t' C x_t,
\end{aligned}
$$

where

$$
\begin{aligned}
a &= \delta - \delta^* + \frac{1}{2}(\lambda'\lambda - \lambda^{*\prime}\lambda^*) \\
b &= \begin{pmatrix} \left(\lambda' - \lambda^{*\prime}\right)\Omega^{-\frac{1}{2}} \\ \gamma - \gamma^* + \lambda'\beta - \lambda^{*\prime}\beta^* + \left(\lambda^{*\prime} - \lambda'\right)\Phi\Omega^{-1/2} \end{pmatrix}' \\
C &= \begin{pmatrix} 0 & 0 \\ \left(\beta' - \beta^{*\prime}\right)\Omega^{-1/2} & \frac{1}{2}\left(\beta'\beta - \beta^{*\prime}\beta^*\right) + \left(\beta^{*\prime} - \beta'\right)\Omega^{-1/2}\Phi \end{pmatrix}
\end{aligned}
$$

The observation equations for yields and depreciation rate can be combined:

$$
\begin{aligned}
Y_{t(m+1)x1} &\equiv \begin{pmatrix} y_{t(mx1)} \\ \Delta s_t \end{pmatrix} = \begin{pmatrix} A_{(mx1)} \\ a \end{pmatrix} + \begin{pmatrix} \bar{B}_{(mx6)} x_t \\ b_{(1x6)} x_t \end{pmatrix} + \begin{pmatrix} 0_{(mx1)} \\ x_t' C_{(6x6)} x_t \end{pmatrix} + \begin{pmatrix} \upsilon_t \\ \xi_t \end{pmatrix} \\
&\equiv f(x_t) + \rho_t,
\end{aligned}
$$

so that the state-space system is given by:

$$
\begin{aligned}
x_t &= \Gamma x_{t-1} + \Theta^{1/2}\epsilon_t \\
Y_t &= f(x_t) + \rho_t,
\end{aligned}
$$

where $\rho_t \sim N(0, R)$. Although the state equation is linear in x_t, the observation equation is non-linear: $f(x_t)$ is quadratic, and hence continuous and differentiable.

In the case of non-linearities in the state or space equations, the extended Kalman filter must be used. The extended Kalman filter, when only the measurement equation is non-linear, is obtained by linearizing $f(x_t)$ around the conditional mean $\hat{x}_{t|t-1}$:

$$f(x_t) = f(\hat{x}_{t|t-1}) + H_t \times (x_t - \hat{x}_{t|t-1}),$$

where $H_t = \frac{\partial}{\partial x_t'} f(x_t)|_{x_t=\hat{x}_{t|t-1}}$. The prediction step equations are the same as before:

$$
\begin{aligned}
\hat{x}_{t|t-1} &= \Gamma x_{t-1} \\
P_{t|t-1} &= \Gamma P_{t-1}\Gamma' + \Theta
\end{aligned}
$$

The update step equation under the extended Kalman Filter is modified to account for the linearization:

$$
\begin{aligned}
\hat{x}_t &= \hat{x}_{t|t-1} + P_{t|t-1}H_t'F_t^{-1}\upsilon_t, \\
P_t &= P_{t|t-1} - P_{t|t-1}H_t'F_t^{-1}H_t P_{t|t-1}
\end{aligned}
$$

where

$$
\begin{aligned}
\upsilon_t &= Y_t - f(\hat{x}_{t|t-1}), \\
F_t &= H_t P_{t|t-1}H_t' + R,
\end{aligned}
$$

Proposition 1 In the case of our model, the measurement equation for the extended Kalman filter takes the form

$$Y_t = f(\hat{x}_{t|t-1}) + H_t\,(x_t - \Gamma x_{t-1}) + \rho_t,$$

where

$$\begin{aligned}
H_t &= \begin{pmatrix} \bar{B}_{(m\times 6)} \\ (b' + (C + C')\Gamma x_{t-1})' \end{pmatrix} \\
f(\hat{x}_{t|t-1}) &= \begin{pmatrix} A_{(m\times 1)} + \bar{B}_{(m\times 6)}\Gamma x_{t-1} \\ a + b_{(1\times 6)}\Gamma x_{t-1} + x'_{t-1}\Gamma' C_{(6\times 6)}\Gamma x_{t-1} \end{pmatrix}
\end{aligned}$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial x'_t} f_s|_{x_t=\hat{x}_{t|t-1}} &= \left(b' + (C + C')x_t|_{x_t=\hat{x}_{t|t-1}}\right)' = \left(b' + (C + C')\Gamma x_{t-1}\right)' \\
\frac{\partial}{\partial x'_t} f_y|_{x_t=\hat{x}_{t|t-1}} &= \bar{B}
\end{aligned}$$

hence

$$f(x_t) = \begin{pmatrix} A_{(m\times 1)} + \bar{B}_{(m\times 6)}\Gamma x_{t-1} \\ a + b_{(1\times 6)}\Gamma x_{t-1} + x'_{t-1}\Gamma' C_{(6\times 6)}\Gamma x_{t-1} \end{pmatrix} + \begin{pmatrix} \bar{B}_{(m\times 6)} \\ (b' + (C + C')\Gamma x_{t-1})' \end{pmatrix} (x_t - \Gamma x_{t-1})$$

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