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## Abstract

This paper develops a fast method of computing arbitrary order perturbation approximations to bond prices in DSGE models. The procedure is implemented to third order where it can shorten the approximation process by more than 100 times. In a consumption-based endowment model with habits, it is further shown that a third-order perturbation solution is more accurate than the log-normal method and a procedure using consol bonds.

**Key words:** Perturbation method, DSGE models, habit model, higher-order approximation.

**JEL classification:** C63, G12.

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**Summary**

Economists have a keen interest in understanding what determines changes in attitudes to risk and how they work through the economy. This in part explains why policymakers analyse the behaviour of bond and equity prices, as these reflect peoples' preferences for risk-taking. Such analyses are often conducted using dynamic stochastic general equilibrium (DSGE) models. These models use theory to describe how all the actors in the economy behave. The word 'stochastic' indicates that there is a fundamental uncertainty pervading the economy, with different types of random disturbances affecting the dynamics of prices and quantities.

The economic relationships underlying the model uniquely determine the evolution of the interconnected system, and finding a rule which pins down that evolution is called solving the model. Unfortunately, in most cases exact solutions are unknown and therefore economists need to approximate them. This is typically done using linearisation, which often delivers very good approximations. However, this method ignores the impact of uncertainty on the transmission mechanism of shocks, and so is inadequate in an asset pricing context.

There exist many alternatives to linearisation, with 'higher-order perturbation' methods being one of them. In practice, however, there is a trade-off between accuracy and speed. In the past, this trade-off has meant that researchers studying prices of long-maturity bonds needed to rely on at most second-order perturbation approximations. This occurred because it was computationally very demanding to allow for higher-order effects, which are present in the true - though unknown - solution to any DSGE model.

The simple aim of this paper is to propose a method which speeds up the process of approximating bond prices by exploiting the relationships which they satisfy. Our method comprises two steps. In the first step, standard solution packages can be used to approximate all the variables other than bond prices. In the second step, we use the fundamental pricing equation to solve for bond prices recursively, ie using approximations to shorter-term bonds to find those for longer-term bond prices.

We show that our two step method can reduce the time it takes to solve models by more than 100

times. This is achieved with the same level of accuracy as using standard perturbation methods. The paper also compares the accuracy of bond price approximations obtained using perturbation methods to that of computationally feasible alternatives. It shows that for the models analysed third-order perturbations generate the most accurate approximations to bond yields.

## 1 Introduction

The influential paper by Mehra and Prescott (1985) highlighted issues arising when trying to simultaneously account for the dynamics of aggregate consumption and asset prices. Much work has subsequently used variants of Mehra and Prescott's (1985) consumption-based asset pricing framework to improve our understanding of the links between financial markets and the macroeconomy. A large number of models in this literature assume an exogenous consumption process as in Mehra and Prescott (1985), see for instance Campbell and Cochrane (1999) and Bansal and Yaron (2004). Another and rapidly growing strand of the literature uses dynamic stochastic general equilibrium (DSGE) models to endogenize the dynamics of consumption in an attempt to provide more detailed insights into the nature of macroeconomic risk. Important contributions are Jermann (1998), Boldrin, Christiano and Fisher (2001), and more recently Wu (2006), Uhlig (2007), De Paoli, Scott and Weeken (2007), Hordahl, Tristani and Vestin (2008), Rudebusch and Swanson (2009), Guvenen (2009), and Bekaert, Cho and Moreno (2010).

When using DSGE models to analyse asset prices, an important constraint is that closed-form solutions are in general unavailable. Accordingly, both the functions capturing state dynamics as well as those mapping state variables into asset prices need to be approximated. This leaves researchers with a challenging numerical problem which standard methods are poorly equipped to deal with. For example, the well-known log-linear approximation is inadequate as it restricts premia on risky assets to zero, contrary to existing evidence (see Campbell and Shiller (1991) or Cochrane and Piazzesi (2005)). Higher-order perturbations are the most widely used alternative (Arouba, Fernández-Villaverde and Rubio-Ramírez (2005) and Caldara, Fernandez-Villaverde, Rubio-Ramirez and Yao (2009)), but they may also become impractical when the approximated model features a yield curve.[1]

To understand why, consider a quarterly DSGE model with $n$ state variables. Assume further that we are interested in computing the 10-year interest rate from the price of a zero-coupon bond with the same maturity. This bond price is a function of $n$ state variables and to approximate it to

[1]Other alternative solution methods include value function iteration, finite elements, and Chebyshev polynomials, but these are typically considered infeasible for medium-scale DSGE models.

third order – ie by a third-order polynomial – would require computing

$$\underbrace{1}_{0^{th}\text{ Order Terms}} + \underbrace{n}_{1^{st}\text{ Order Terms}} + \underbrace{n \cdot \frac{n+1}{2}}_{2^{nd}\text{ Order Terms}} + \underbrace{n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}}_{3^{rd}\text{ Order Terms}}$$

ie a total of $n \cdot (n+1)/2 \cdot (n+2)/3$ distinct coefficients. Typically, the 10-year bond price is computed recursively along with all 40 intermediate bond prices.[2] Accordingly, in a quarterly model, for $n$ corresponding to 5, 10 or 15 years, the yield curve introduces respectively 2,240, 11,440, or 32,640 additional coefficients to be simultaneously computed. This can either make the problem too large to solve using standard solution packages or significantly increase the time required to compute the approximation.[3] While the deterioration in performance might be tolerable if the model needs to be solved once, it has the potential to make estimation or sensitivity analysis infeasible as both of them rely on repeated approximations.

The contribution of this paper is to propose a method of reducing the computational burden when approximating bond prices to arbitrary order. Matlab codes that implement the suggested method to third order are also provided. We focus on the standard case in which bond prices with maturities beyond one period do not affect the rest of the economy, but alternatives are also considered.[4] The solution we advocate splits the perturbation problem into two steps. In the first step, standard solution packages can be used to approximate the solution to a DSGE model *without* bond prices of maturity greater than one. The second step perturbs the fundamental pricing equation for bond prices up to the same order. We then exploit the information from the first step to recursively solve for the coefficients of bond prices, significantly speeding up the approximation process. On account of this structure, we refer to our method as perturbation-on-perturbation (POP). It is important to emphasise that the POP method computes *exactly* the same expressions for bond prices as the standard 'one-step' perturbation routine.

Our proposed method is closest to the one proposed in Hordahl *et al* (2008).[5] They first approximate a solution to the part of their DSGE model without bond prices to second order.

[2] Alternative, non-recursive methods involve creating many auxiliary variables which similarly complicates the approximation problem.

[3] These packages include *Dynare*, *Dynare++,* and *Perturbation AIM* (see Kamenik (2005) and Swanson, Anderson and Levin (2005), respectively), and the set of routines accompanying Schmitt-Grohé and Uribe (2004).

[4] Expressed alternatively, the assumption we rely on is that the model is such that prices of all bonds exceeding one period only appear in consumption-Euler equations.

[5] Binsbergen, Fernandez-Villaverde, Koijen and Rubio-Ramirez (2010) independently apply a related method to compute interest rates in a version of the neoclassical growth model. The method and formulas we provide are not model specific and our approach nests their procedure.

Afterwards, all bond prices are solved for using the fundamental asset pricing equations and the first approximation. Notably, we extend their work along three dimensions. Firstly, we go beyond second-order and provide third-order accurate formulas for bond prices. These third-order terms are of economic significance as they allow for time variation in risk premia. Secondly, we allow for more general transformations of variables in the model than the 'log' specification considered in Hordahl *et al* (2008). Thirdly, we consider a slightly more general set-up than in Hordahl *et al* (2008), as we do not introduce restrictions on the functional form of the stochastic discount factor.

A simulation study is used to document the reduction in computational burden achieved by using the POP method instead of the standard one-step perturbation. For the DSGE models in Rudebusch and Swanson (2008) and De Paoli *et al* (2007), the speed gains vary from between 14 and 23 times for a 10-year yield curve to between 61 and 139 times for a 20-year yield curve. As demonstrated in Andreasen (2010a), the speed gains involved are sufficient to make estimation of medium-scale DSGE models with a whole yield curve approximated to third order feasible.

We then assess accuracy of the POP method using closed-form solutions for bond prices in a consumption based model with habits (Zabczyk (2010)). Broadly in line with Arouba *et al* (2005) and Caldara *et al* (2009) we find that a third-order approximation outperforms alternative methods like the log-normal approach (Jermann (1998), Doh (2007)) and the method using consol bonds proposed in Rudebusch and Swanson (2008). We also find that the consol method gives a less accurate approximation, and we show that it may be less accurate, even at third order, than the first-order log-normal method.

The remainder of this paper is organised as follows: Section 2 describes the POP method, Section 3 documents the gains in speed (at third order), accuracy is assessed in Section 4 and Section 5 concludes.

## 2 The POP method of computing bond prices

This section presents the POP method to approximate bond prices. For parsimony, we adopt the same framework as in Schmitt-Grohé and Uribe (2004).[6] We further assume that the model can be split into two parts. The first part contains all equations in which bond prices beyond one-period maturity do *not* appear. The second part consists entirely of Euler equations for the remaining bond prices.[7] Hence, let $\mathbf{y}_t$ denote the $n_y \times 1$ vector of all non-predetermined variables *except* bond prices with a maturity exceeding one period, and let $\mathbf{x}_t$ be the $n_x \times 1$ vector of predetermined state variables. As in Schmitt-Grohé and Uribe (2004), the solution can be written as

$$\mathbf{y}_t = \mathbf{g}(\mathbf{x}_t, \sigma) \quad \textbf{(1)}$$

$$\mathbf{x}_{t+1} = \mathbf{h}(\mathbf{x}_t, \sigma) + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \quad \textbf{(2)}$$

where $\boldsymbol{\epsilon}_{t+1} \sim \mathcal{IID}(\mathbf{0}, \mathbf{I})$ is a vector of $n_\epsilon$ innovations, $\boldsymbol{\eta}$ denotes the square root of their covariance matrix, and $\sigma$ is the perturbation parameter. In the first step of the POP method, the solution **(1)**-**(2)** is approximated to $N$-th order using standard perturbation methods.

Let $P^{t,k}$ denote the price in period $t$ of a zero-coupon bond maturing in $k$ periods with a face value of one. The price of this bond satisfies the fundamental pricing equation (see Cochrane (2001))

$$P^{t,k} = E_t \left[ \mathcal{M} \times P^{t+1,k-1} \right]$$

for $k = 1, 2, ..., K$ where $\mathcal{M}$ is the stochastic discount factor. In many applications the focus is on logarithms of prices rather than their levels. To accommodate this possibility we could rewrite the equation above as

$$\exp\left(\hat{p}^{t,k}\right) = E_t \left[ \mathcal{M} \times \exp\left(\hat{p}^{t+1,k-1}\right) \right]$$

where $\hat{p}^{t,k} \equiv \log(P^{t,k})$ . More generally, since other transformations might be useful when solving DSGE models (see for example Fernandez-Villaverde and Rubio-Ramirez (2006)), we introduce an invertible transformation function $R(\cdot) \in \mathcal{C}^N$ and denote $p^{t,k} \equiv R^{-1}(P^{t,k})$. The pricing equation can then be written as

$$R\left(p^{t,k}\right) = E_t \left[ \mathcal{M} \times R\left(p^{t+1,k-1}\right) \right]. \quad \textbf{(3)}$$

[6]Extensions to the more general case in which shocks do not necessarily enter additively, as in *Dynare++* (Kamenik (2005)) or *Perturbation AIM* (Swanson *et al* (2005)), are straightforward and dealt with in the appendix. Note that when using *Dynare++* we only consider standard perturbation approximations around the deterministic steady state.

[7]This structure is standard and all macro-finance models listed in the introduction satisfy this assumption.

Setting $R(x) = x$ gives the original 'levels' specification, while letting $R(x) = \exp(x)$ corresponds to the case of a log-transformation. The gross yield-to-maturity in period $t$ of a $k$-period bond $YTM^{t,k}$ can still be computed and is given by
$YTM^{t,k} = (P^{t,k})^{-1/k} = R(p^{t,k})^{-1/k}$.

To compute perturbation approximations to $p^{t,k}$ we exploit two facts. Firstly, the functional form of the stochastic discount factor $\mathcal{M}(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t)$ is known.[8] Secondly, since any bond price is non-predetermined, it is a function of $\mathbf{x}_t$ and $\sigma$. We denote this function by $p^k(\mathbf{x}_t, \sigma)$ where $k$ denotes the maturity of the bond. Where no ambiguity can arise, we omit the function arguments and simply write $p^{t,k}$. Using these insights and substituting **(1)** and **(2)** into **(3)**, we then define

$$F^k(\mathbf{x}, \sigma) := E\Big\{ R\left(p^k(\mathbf{x}, \sigma)\right) - R\left(p^{k-1}\left(\mathbf{h}(\mathbf{x}, \sigma) + \sigma\boldsymbol{\eta}\epsilon_{t+1}, \sigma\right)\right)$$
$$\times \mathcal{M}\left(\mathbf{g}\left(\mathbf{h}(\mathbf{x}, \sigma) + \sigma\boldsymbol{\eta}\epsilon_{t+1}, \sigma\right), \mathbf{g}(\mathbf{x}, \sigma), \mathbf{h}(\mathbf{x}, \sigma) + \sigma\boldsymbol{\eta}\epsilon_{t+1}, \mathbf{x}\right)\Big\} \quad \text{for } k = 1, 2, ..., K. \quad \textbf{(4)}$$

It follows by construction that $F^k(\mathbf{x}, \sigma) \equiv 0$ for all values of $\mathbf{x}$ and $\sigma$. Clearly, this implies that all derivatives of $F^k(\mathbf{x}, \sigma)$ must also equal zero, ie

$$F^k_{\mathbf{x}^i\sigma^j}(\mathbf{x}, \sigma) = 0 \quad \forall \mathbf{x}, \sigma, i, j \qquad \textbf{(5)}$$

where $F^k_{\mathbf{x}^i\sigma^j}(\mathbf{x}, \sigma)$ denotes the derivative of $F^k$ with respect to $\mathbf{x}$ taken $i$ times and with respect to $\sigma$ taken $j$ times. In the following subsections, we show how **(5)** together with the output from the first perturbation step **(1)** and **(2)** can be used to find derivatives of $p^k(\mathbf{x}, \sigma)$ of order up to $N$ evaluated at the deterministic steady state. These derivatives suffice to construct an $N$-th order perturbation approximation to $p^k(\mathbf{x}, \sigma)$ around the deterministic steady state.

### *2.1 Notation*

To make the subsequent formulas more transparent, we adopt the convention that indices $\alpha$ and $\gamma$ relate to elements of $\mathbf{x}$, while $\beta$ and $\phi$ correspond to elements of $\mathbf{y}$ and $\boldsymbol{\epsilon}$, respectively. Furthermore, subscripts on these indices will capture the sequence in which derivatives are being taken. For example, $\alpha_1$ corresponds to the first time a function is differentiated with respect to $\mathbf{x}$, while $\alpha_2$ is used when differentiating with respect to $\mathbf{x}$ for the second time.

[8]We assume that the variables in the first block of the model, ie $\mathbf{x}$ and $\mathbf{y}$, have also been transformed using $R(\cdot)$. Accordingly, $\mathcal{M}$ and all its derivatives are known functions of the *transformed* variables. For example, for CRRA utility and $R(x) = \exp(x)$ we would have $\mathcal{M}(c_{t+1}, c_t) = \beta \exp(-\gamma\ c_{t+1})/\exp(-\gamma\ c_t)$.

In most of the subsequent derivations we follow Schmitt-Grohé and Uribe (2004) and use the tensor notation. In particular, $[p^k_{\mathbf{x}}]_{\gamma_1}$ denotes the $\gamma_1$-th element of the $1 \times n_x$ vector of derivatives of $p^k$ with respect to $\mathbf{x}$. Similarly, the derivative of $\mathbf{h}$ with respect to $\mathbf{x}$ is an $n_x \times n_x$ matrix and $[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$ is the element of this matrix located at the intersection of the $\gamma_1$-th row and the $\alpha_1$-th column. Also, $\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} = \sum_{\gamma_1=1}^{n_x}(\partial p^{k-1}/\partial \mathbf{x}_{\gamma_1})(\partial \mathbf{h}^{\gamma_1}/\partial \mathbf{x}_{\alpha_1})$ while $\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} = \sum_{\gamma_1=1}^{n_x}\sum_{\gamma_2=1}^{n_x}(\partial^2 p^{k-1}/\partial \mathbf{x}_{\gamma_1}\partial \mathbf{x}_{\gamma_2})(\partial \mathbf{h}^{\gamma_2}/\partial \mathbf{x}_{\alpha_2})(\partial \mathbf{h}^{\gamma_1}/\partial \mathbf{x}_{\alpha_1})$ where, for instance, $\mathbf{h}^{\gamma_1}$ denotes the $\gamma_1$-th function of mapping $\mathbf{h}$ and $\mathbf{x}_{\alpha_1}$ is the $\alpha_1$-th element of vector $\mathbf{x}$.

For parsimony, we also use superscripts $t$ and $t+1$ on functions $p^k$, $\mathbf{h}$, $\mathbf{g}$, and their derivatives to indicate the arguments at which they are evaluated. When these superscripts are omitted, functions are evaluated at the deterministic steady state, ie for $(\mathbf{x}, \sigma) = (\mathbf{x}_{ss}, 0)$. For example, for $f \in \{p^k, \mathbf{g}, \mathbf{h}\}$

$$f^t := f(\mathbf{x}_t, \sigma) \qquad f^{t+1} := f(\mathbf{x}_{t+1}, \sigma) \qquad f := f(\mathbf{x}_{ss}, 0)$$

$$f^t_{\mathbf{x}} := \left(\partial f/\partial \mathbf{x}\right)|_{(\mathbf{x}_t,\sigma)} \qquad f^{t+1}_{\mathbf{x}} := \left(\partial f/\partial \mathbf{x}\right)|_{(\mathbf{x}_{t+1},\sigma)} \qquad f_{\mathbf{x}} := \left(\partial f/\partial \mathbf{x}\right)|_{(\mathbf{x}_{ss},0)}.$$

### *2.2 Finding the first-order derivatives*

To find the first-order derivatives of $p^k(\mathbf{x}, \sigma)$ with respect to $\mathbf{x}$, we start by differentiating $F^k(\mathbf{x}, \sigma)$ with respect to $\mathbf{x}$. Exploiting **(5)** we rewrite $\left[F^k_{\mathbf{x}}(\mathbf{x}_t, \sigma)\right]_{\alpha_1} = 0$ as

$$R_p\left(p^k\right)\left[p^{t,k}_{\mathbf{x}}\right]_{\alpha_1} - \left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_1} R\left(p^{t+1,k-1}\right) - \mathcal{M} R_p\left(p^{t+1,k-1}\right)\left[p^{t+1,k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} = 0 \qquad \mathbf{(6)}$$

for $\alpha_1, \gamma_1 \in \{1, 2, ..., n_x\}$. Evaluating **(6)** in the deterministic steady state gives a set of equations which determine $\left[p^k_{\mathbf{x}}\right]_{\alpha_1}$ for $\alpha_1 = 1, 2, ..., n_x$ and $k = 1, 2, ..., K$. Given the output from the first perturbation step, we now show how these derivatives can be solved recursively.

To show this and to establish the recursive argument, consider first the price of a bond with one period to maturity. The price of a maturing bond is one for all values of $(\mathbf{x}, \sigma)$, and all of its derivatives are therefore equal to zero, ie $p^{t+1,0}_{\mathbf{x}} = 0$. Accordingly, equation **(6)** evaluated at the steady state and for $k = 1$ simplifies to

$$R_p\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_1} = \left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_1}, \qquad \mathbf{(7)}$$

where we use $R\left(p^0\right) = P^0 = 1$. The value of $R_p\left(p^1\right)$ can easily be computed from its known functional form and the steady state value of $p^1$. Further, the value of $[\mathcal{M}_{\mathbf{x}}]_{\alpha_1}$ evaluated at the steady state can readily be found by differentiating $\mathcal{M}(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t)$ and exploiting equations **(1)** and **(2)**

$$[\mathcal{M}_{\mathbf{x}}]_{\alpha_1} = \left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1} \left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1} [\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} + \left[\mathcal{M}_{\mathbf{y}_t}\right]_{\beta_1} \left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1} + \left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1} [\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} + \left[\mathcal{M}_{\mathbf{x}_t}\right]_{\alpha_1}, \quad \textbf{(8)}$$

along with the fact that $\mathbf{g}_{\mathbf{x}}$, $\mathbf{h}_{\mathbf{x}}$, and all the derivatives of $\mathcal{M}$ are known in the deterministic steady state. An alternative and slightly easier way of obtaining $[\mathcal{M}_{\mathbf{x}}]_{\alpha_1}$ is to report it in the first perturbation step. However, this often comes at the cost of introducing extra variables into the state vector which slows down the first step of the POP method. Once the scalar $R_p\left(p^1\right)$ and $[\mathcal{M}_{\mathbf{x}}]_{\alpha_1}$ have been computed, the derivatives $\left[p^1_{\mathbf{x}}\right]_{\alpha_1}$ are immediately given by **(7)**.

Given that we know $\left[p^1_{\mathbf{x}}\right]_{\alpha_1}$, we can then compute the first-order terms for the remaining maturities directly from **(6)**. To do that we evaluate **(6)** in the deterministic steady state. Using $\mathcal{M} = R\left(p^1\right)$ and substituting out for $[\mathcal{M}_{\mathbf{x}}]_{\alpha_1}$ from **(7)**, we obtain the following system of equations for $p^k_{\mathbf{x}}$

$$R_p\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_1} = \left[p^1_{\mathbf{x}}\right]_{\alpha_1} R_p\left(p^1\right) R\left(p^{k-1}\right) + R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1} [\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \quad \textbf{(9)}$$

for $k = 2, 3, ..., K$ and $\alpha_1, \gamma_1 \in \{1, 2, \ldots, n_x\}$. Again, $R_p\left(p^k\right)$ is a scalar and all the terms on the right-hand side are known, which makes it straightforward to solve for $\left[p^k_{\mathbf{x}}\right]_{\alpha_1}$. In the special case of a log-transformation, the expression in **(9)** simplifies to

$$\mathbf{p}^k_{\mathbf{x}} = \mathbf{p}^1_{\mathbf{x}} + \mathbf{p}^{k-1}_{\mathbf{x}}\mathbf{h}_{\mathbf{x}}, \quad \textbf{(10)}$$

where $\mathbf{p}^k_{\mathbf{x}}$ denotes a $1 \times n_x$ vector of derivatives of $p^k$ with respect to $\mathbf{x}$. This formula reproduces the first-order expression derived in Hordahl *et al* (2008).

Expression **(9)** also suggests that the easiest way to start the recursion is to approximate $p^1$ in the first step of the POP method. This gives the derivative $\left[p^1_{\mathbf{x}}\right]_{\alpha_1}$ required to compute the right-hand side of **(9)**. This procedure does not add extra state variables to the first perturbation step and will therefore be faster than the alternative of reporting the stochastic discount factor $\mathcal{M}$ mentioned above. Moreover, if the $R$-transformed level of the one-period interest rate $ytm$ is already given in the first perturbation step, then $\left[p^1_{\mathbf{x}}\right]_{\alpha_1}$ can be computed by differentiating $p^{t,1} = R^{-1}(1/R(ytm))$. For instance, using a log-transformation it holds that $p^1_{\mathbf{x}} = -ytm^1_{\mathbf{x}}$.

The first-order derivatives of bond prices with respect to $\sigma$ are found in a similar way.[9] That is,

[9] We know from Schmitt-Grohé and Uribe (2004) that these derivatives are zero. Nevertheless, we solve for these terms to make subsequent derivations of higher-order derivatives more transparent.

we exploit the fact that the derivative of $F^k(\mathbf{x},\sigma)$ with respect to $\sigma$ evaluated at the deterministic steady state equals zero, ie

$$F^k_\sigma(\mathbf{x}_{ss},0) = E_t\left\{R_p\left(p^k\right)\left[p^k_\sigma\right] - [\mathcal{M}_\sigma]\,R\left(p^{k-1}\right) - \mathcal{M}R_p\left(p^{k-1}\right)\left(\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left([\mathbf{h}_\sigma]^{\gamma_1}+\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)+\left[p^{k-1}_\sigma\right]\right)\right\} = 0. \quad \mathbf{(11)}$$

For the one-period bond, this reduces to

$$E_t\left\{R_p\left(p^1\right)\left[p^1_\sigma\right]\right\} = E_t\mathcal{M}_\sigma \quad \mathbf{(12)}$$

as $R\left(p^0\right)=1$, $E_t\left[\boldsymbol{\epsilon}_{t+1}\right]=0$, and all the derivatives of $p^0$ are zero. The fact that $E_t\mathcal{M}_\sigma = 0$ implies $\left[p^1_\sigma\right]=0$. Moreover, $\mathbf{h}_{\boldsymbol{\sigma}}=0$ and this suffices to show that $p^k_\sigma = 0$ for $k=2,3,...,K$.

### *2.3 Second-order terms*

This section shows how to compute all second-order terms for bond prices. The procedure is similar to that used to compute all first-order derivatives of bond prices. In particular, we use terms computed in the previous section, output from the first step of the POP method, and second-order derivatives of $F^k(\mathbf{x},\sigma)$ evaluated in the deterministic steady state.

Starting with second-order derivatives with respect to the state vector, we obtain

$$\begin{aligned}\left[F^k_{\mathbf{xx}}(\mathbf{x}_{ss},0)\right]_{\alpha_1,\alpha_2} = {} & R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_2}\left[p^k_{\mathbf{x}}\right]_{\alpha_1} + R_p\left(p^k\right)\left[p^k_{\mathbf{xx}}\right]_{\alpha_1\alpha_2} \\ & - [\mathcal{M}_{\mathbf{xx}}]_{\alpha_1\alpha_2}\,R\left(p^{k-1}\right) - [\mathcal{M}_{\mathbf{x}}]_{\alpha_1}\,R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \\ & - [\mathcal{M}_{\mathbf{x}}]_{\alpha_2}\,R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\ & - \mathcal{M}R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\ & - \mathcal{M}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\ & - \mathcal{M}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2} = 0\end{aligned} \quad \mathbf{(13)}$$

for $\alpha_1,\alpha_2,\gamma_1,\gamma_2 = 1,2,...,n_x$. To solve for second-order bond price derivatives, we consider the case where the price of the one-period bond is approximated in the first step of the POP method and focus on computing $p^k_{\mathbf{xx}}$ given $p^{k-1}_{\mathbf{xx}}$.[10] To evaluate the right-hand side of equation **(13)** we need expressions for $\mathcal{M}$, $\mathcal{M}_{\mathbf{x}}$, and $\mathcal{M}_{\mathbf{xx}}$. The value of $\mathcal{M}$ equals $R(p^1)$ and $\mathcal{M}_{\mathbf{x}}$ is given

[10] Along the lines discussed in Section 2.2 for $\mathcal{M}_{\mathbf{x}}$, the value of $\mathcal{M}_{\mathbf{xx}}$ can also be computed by second-order differentiation of $\mathcal{M}$, or $\mathcal{M}_{\mathbf{xx}}$ may be reported directly in the first step of the POP method.

by equation **(7)**. The expression for $\mathcal{M}_{\mathbf{xx}}$ can be computed from equation **(13)** when $k = 1$

$$[\mathcal{M}_{\mathbf{xx}}]_{\alpha_1\alpha_2} = R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1,\alpha_2} + R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}, \tag{14}$$

as all derivatives of $p^0(\mathbf{x}, \sigma)$ are zero. Exploiting these findings in equation **(13)** gives

$$\begin{aligned} R_p\left(p^k\right)\left[p^k_{\mathbf{xx}}\right]_{\alpha_1,\alpha_2} &= -R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_2}\left[p^k_{\mathbf{x}}\right]_{\alpha_1} \\ &+ \left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1,\alpha_2} + R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}\right) R\left(p^{k-1}\right) \\ &+ \left[p^1_{\mathbf{x}}\right]_{\alpha_1} R_p\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2} \\ &+ \left[p^1_{\mathbf{x}}\right]_{\alpha_2} R_p\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\ &+ R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\ &+ R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\ &+ R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2} \end{aligned} \tag{15}$$

for $k = 2, 3, ..., K$ and for $\alpha_1, \alpha_2, \gamma_1, \gamma_2 = 1, 2, ..., n_x$. For a log-transformation, the formula in **(15)** simplifies to

$$\mathbf{p}^k_{\mathbf{xx}} = \mathbf{p}^1_{\mathbf{xx}} + \mathbf{h}'_{\mathbf{x}}\mathbf{p}^{k-1}_{\mathbf{xx}}\mathbf{h}_{\mathbf{x}} + \textstyle\sum_{\gamma_1=1}^{n_x} p^{k-1}_{\mathbf{x}}(\gamma_1)\mathbf{h}_{\mathbf{xx}}(\gamma_1, :, :). \tag{16}$$

Here, we have adopted the notation used in Hordahl *et al* (2008) to clearly demonstrate that equation **(15)** nests their second-order expression. Using this notation, $A(\gamma_1, \gamma_2, \ldots, \gamma_N)$ denotes an element on the intersection of dimensions $\gamma_1$, $\gamma_2$, and $\gamma_N$ in matrix **A** and colons refer to entire dimensions. For example, $\mathbf{h}_{\mathbf{xx}}(\gamma_1, :, :)$ is an $n_x \times n_x$ matrix of second-order derivatives of the $\gamma_1$-th mapping of **h** evaluated at the steady state, and $\mathbf{p}^k_{\mathbf{xx}}$ is the $n_x \times n_x$ matrix of second-order derivatives of $p^k$ with respect to **x**.

To find $p^k_{\sigma\sigma}$, we differentiate $F^k(\mathbf{x}, \sigma)$ twice with respect to $\sigma$ and evaluate the expression in the deterministic steady state. Since this derivative is equal to zero, we get

$$
\begin{aligned}
\left[F_{\sigma\sigma}\left(\mathbf{x}_{ss},0\right)\right] \; = \; & E_t\Big\{-R_p\left(p^k\right)\left[p^k_{\sigma\sigma}\right]+\left[\mathcal{M}_{\sigma\sigma}\right]R\left(p^{k-1}\right) \\
& +\left[\mathcal{M}_{\sigma}\right]R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\epsilon_{t+1}\right]^{\phi_2} \\
& +\left[\mathcal{M}_{\sigma}\right]R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\epsilon_{t+1}\right]^{\phi_1} \\
& +\mathcal{M}R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\epsilon_{t+1}\right]^{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\epsilon_{t+1}\right]^{\phi_1} \\
& +\mathcal{M}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\epsilon_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\epsilon_{t+1}\right]^{\phi_1} \\
& +\mathcal{M}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}+\mathcal{M}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\sigma\sigma}\right]\Big\}=0
\end{aligned} \tag{17}
$$

where $\gamma_1, \gamma_2 = 1, 2, ..., n_x$ and $\phi_1, \phi_2 = 1, 2, ..., n_\epsilon$. To simplify equation **(17)** we have relied on the fact that the terms $\mathbf{h}_\sigma$, $p^k_{\boldsymbol{\sigma}}$, and $p^k_{\mathbf{x}\boldsymbol{\sigma}}$ are known to be zero (Schmitt-Grohé and Uribe (2004)). Again, the important thing to observe is that equation **(17)** allows us to solve for $p^k_{\sigma\sigma}$. To show this, we first differentiate $\mathcal{M}$ with respect to $\sigma$ to get

$$
\left[\mathcal{M}_{\sigma}\right]=\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\epsilon_{t+1}\right]^{\phi_2}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\epsilon_{t+1}\right]^{\phi_2}. \tag{18}
$$

To find an expression for $E_t\left[\mathcal{M}_{\sigma\sigma}\right]$ we consider the case in which $p^1_{\sigma\sigma}$ is reported in the first step of the POP method. Evaluating equation **(17)** at $k=1$ and exploiting the fact that all derivatives of $p^0(\mathbf{x},\sigma)$ are zero gives

$$
E_t\left[\mathcal{M}_{\sigma\sigma}\right]=\left[p^1_{\sigma\sigma}\right]R_p\left(p^1\right). \tag{19}
$$

Combining the results in **(18)** and **(19)** to evaluate **(17)** we get

$$
\begin{aligned}
R_p\left(p^k\right)\left[p^k_{\sigma\sigma}\right] \; = \; & \left[p^1_{\sigma\sigma}\right]R_p\left(p^1\right)R\left(p^{k-1}\right) \\
& +2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1} \\
& +2\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1} \\
& +R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2} \\
& +R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2} \\
& +R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1} \\
& +R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\sigma\sigma}\right]
\end{aligned} \tag{20}
$$

As discussed previously, the derivatives of the stochastic discount factor $\mathcal{M}_{\mathbf{y}_{t+1}}$ and $\mathcal{M}_{\mathbf{x}_{t+1}}$ are straightforward to compute from the known functional form of $\mathcal{M}$. Applying a log-transformation makes it possible to simplify the above formula to

$$p^k_{\sigma\sigma} = p^1_{\sigma\sigma} + p^{k-1}_{\sigma\sigma} + \mathbf{p}^{k-1}_{\mathbf{x}}\mathbf{h}_{\sigma\sigma} + \text{trace}\left(\boldsymbol{\eta}'\mathbf{p}^{k-1}_{\mathbf{xx}}\boldsymbol{\eta}\right) + \mathbf{p}^{k-1}_{\mathbf{x}}\boldsymbol{\eta}\boldsymbol{\eta}'\left(\mathbf{p}^{k-1}_{\mathbf{x}}\right)' + 2\left(\boldsymbol{\lambda}_{\mathbf{x}} - \boldsymbol{\pi}_{\mathbf{x}}\right)\boldsymbol{\eta}\boldsymbol{\eta}'\left(\mathbf{p}^{k-1}_{\mathbf{x}}\right)', \quad \mathbf{(21)}$$

when $\mathcal{M} = \beta\lambda_{t+1}/(\lambda_t\pi_{t+1})$. Here, $\beta$ is the discount factor, $\lambda_t$ denotes the marginal utility of consumption, and $\pi_t$ stands for the inflation rate. We use $\boldsymbol{\lambda}_{\mathbf{x}}$ and $\boldsymbol{\pi}_{\mathbf{x}}$ to denote $1 \times n_x$ matrices of first-order derivatives for $\lambda_t$ and $\pi_t$ with respect to $\mathbf{x}$ in the steady state, respectively. In this special case, formula **(20)** reproduces the second-order expression derived in Hordahl *et al* (2008).

### *2.4 Higher-order approximations*

The method described in the previous two subsections naturally generalises to perturbation approximations of order higher than two. Third-order terms are of significant economic interest because they allow for time-varying risk premia. We therefore provide explicit formulas for $\mathbf{p}^k_{\mathbf{xxx}}$, $\mathbf{p}^k_{\mathbf{x}\sigma\sigma}$, and $\mathbf{p}^k_{\boldsymbol{\sigma\sigma\sigma}}$, with the proof of $\mathbf{p}^k_{\mathbf{xx}\boldsymbol{\sigma}} = \mathbf{0}$ provided in Andreasen (2010b). In the interest of space, we only report simpler expressions for the log-transformation case in the body of the text and refer to the appendix for the general solutions corresponding to arbitrary $R(\cdot)$. As derived in the appendix

$$\begin{aligned} p^k_{\mathbf{xxx}}(\alpha_1,\alpha_2,\alpha_3) &= p^1_{\mathbf{xxx}}(\alpha_1,\alpha_2,\alpha_3) \\ &+ \textstyle\sum_{\gamma_1=1}^{n_x}\mathbf{h}_{\mathbf{x}}(\gamma_1,\alpha_1)\,\mathbf{h}_{\mathbf{x}}(:,\alpha_2)'\,\mathbf{p}^{k-1}_{\mathbf{xxx}}(\gamma_1,:,:)\,\mathbf{h}_{\mathbf{x}}(:,\alpha_3) \\ &+ \mathbf{h}_{\mathbf{x}}(:,\alpha_1)'\,\mathbf{p}^{k-1}_{\mathbf{xx}}\mathbf{h}_{\mathbf{xx}}(:,\alpha_2,\alpha_3) \\ &+ \mathbf{h}_{\mathbf{xx}}(:,\alpha_1,\alpha_3)'\,\mathbf{p}^{k-1}_{\mathbf{xx}}\mathbf{h}_{\mathbf{x}}(:,\alpha_2) \\ &+ \mathbf{h}_{\mathbf{xx}}(:,\alpha_1,\alpha_2)'\,\mathbf{p}^{k-1}_{\mathbf{xx}}\mathbf{h}_{\mathbf{x}}(:,\alpha_3) \\ &+ \mathbf{p}^{k-1}_{\mathbf{x}}\mathbf{h}_{\mathbf{xxx}}(:,\alpha_1,\alpha_2,\alpha_3) \end{aligned} \quad \mathbf{(22)}$$

for $k = 2, 3, ..., K$ and $\alpha_1, \alpha_2, \alpha_3 = 1, 2, \ldots, n_x$. The notation follows that in equation **(16)**. When $\mathcal{M} = \beta\lambda_{t+1}/(\lambda_t\pi_{t+1})$, the general formulas for $\mathbf{p}^k_{\sigma\sigma\mathbf{x}}$ and $\mathbf{p}^k_{\sigma\sigma\sigma}$ reported in the appendix simplify to

$$\begin{aligned}
\mathbf{p}^k_{\sigma\sigma\mathbf{x}} &= \mathbf{p}^1_{\sigma\sigma\mathbf{x}} - 2\left(\boldsymbol{\lambda}_\mathbf{x} - \boldsymbol{\pi}_\mathbf{x}\right)\boldsymbol{\eta}\boldsymbol{\eta}'\left(\mathbf{p}^{k-1}_\mathbf{x}\right)'\mathbf{p}^1_\mathbf{x} \\
&+2\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}\boldsymbol{\eta}'\left(\boldsymbol{\lambda}'_\mathbf{x}\boldsymbol{\lambda}_\mathbf{x} - \boldsymbol{\lambda}'_\mathbf{x}\boldsymbol{\pi}_\mathbf{x} - \boldsymbol{\pi}'_\mathbf{x}\boldsymbol{\lambda}_\mathbf{x} + \boldsymbol{\pi}'_\mathbf{x}\boldsymbol{\pi}_\mathbf{x} + \mathbf{g}^\lambda_\mathbf{xx} - \mathbf{g}^\pi_\mathbf{xx} + \mathbf{p}^{k-1}_\mathbf{xx}\right)\mathbf{h}_\mathbf{x} \\
&+2\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}\boldsymbol{\eta}'\left(\boldsymbol{\lambda}'_\mathbf{x}\boldsymbol{\pi}_\mathbf{x} - \boldsymbol{\lambda}'_\mathbf{x}\boldsymbol{\lambda}_\mathbf{x}\right) \\
&+2\left(\boldsymbol{\lambda}_\mathbf{x} - \boldsymbol{\pi}_\mathbf{x}\right)\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{p}^{k-1}_\mathbf{xx}\mathbf{h}_\mathbf{x} \\
&+\textstyle\sum_{\gamma_1=1}^{n_x}\boldsymbol{\eta}\left(\gamma_1,:\right)\boldsymbol{\eta}'\mathbf{p}^{k-1}_\mathbf{xxx}\left(\gamma_1,:,:\right)\mathbf{h}_\mathbf{x} \\
&+\mathbf{h}'_{\sigma\sigma}\mathbf{p}^{k-1}_\mathbf{xx}\mathbf{h}_\mathbf{x} + \mathbf{p}^{k-1}_\mathbf{x}\mathbf{h}_{\sigma\sigma\mathbf{x}} + \mathbf{p}^{k-1}_{\sigma\sigma\mathbf{x}}\mathbf{h}_\mathbf{x}
\end{aligned} \quad \textbf{(23)}$$

and

$$\begin{aligned}
p^k_{\sigma\sigma\sigma} &= p^1_{\sigma\sigma\sigma} + \mathbf{p}^{k-1}_\mathbf{x}\mathbf{h}_{\sigma\sigma\sigma} + p^{k-1}_{\sigma\sigma\sigma} \\
&+\textstyle\sum_{\phi_1=1}^{n_e} 3\left(\boldsymbol{\eta}\left(:,\phi_1\right)m^3\left(\epsilon_{t+1}\left(\phi_1\right)\right)\right)'\left(\boldsymbol{\lambda}'_\mathbf{x}\boldsymbol{\lambda}_\mathbf{x} - \boldsymbol{\lambda}'_\mathbf{x}\boldsymbol{\pi}_\mathbf{x} - \boldsymbol{\pi}'_\mathbf{x}\boldsymbol{\lambda}_\mathbf{x} + \boldsymbol{\pi}'_\mathbf{x}\boldsymbol{\pi}_\mathbf{x}\right)\boldsymbol{\eta}\left(:,\phi_1\right)\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}\left(:,\phi_1\right) \\
&+\textstyle\sum_{\phi_1=1}^{n_e} 3\left(\boldsymbol{\eta}\left(:,\phi_1\right)m^3\left(\epsilon_{t+1}\left(\phi_1\right)\right)\right)'\left(\boldsymbol{\lambda}_\mathbf{xx} - \boldsymbol{\pi}_\mathbf{xx}\right)\boldsymbol{\eta}\left(:.\phi_1\right)\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}\left(:,\phi_1\right) \\
&+\textstyle\sum_{\phi_1=1}^{n_e} 3\left(\boldsymbol{\lambda}_\mathbf{x} - \boldsymbol{\pi}_\mathbf{x}\right)\mathbf{g}_\mathbf{x}\boldsymbol{\eta}\left(:,\phi_1\right)m^3\left(\epsilon_{t+1}\left(\phi_1\right)\right)\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}\left(:,\phi_1\right)\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}\left(:,\phi_1\right) \\
&+\textstyle\sum_{\phi_1=1}^{n_e} 3\left(\boldsymbol{\lambda}_\mathbf{x} - \boldsymbol{\pi}_\mathbf{x}\right)\boldsymbol{\eta}\left(:,\phi_1\right)m^3\left(\epsilon_{t+1}\left(\phi_1\right)\right)\boldsymbol{\eta}\left(:,\phi_2\right)'\mathbf{p}^{k-1}_\mathbf{xx}\boldsymbol{\eta}\left(:,\phi_3\right) \\
&+\textstyle\sum_{\phi_1=1}^{n_e} \mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}\left(:,\phi_1\right)\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}\left(:,\phi_1\right)\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}\left(:,\boldsymbol{\phi}_1\right)m^3\left(\epsilon_{t+1}\left(\phi_1\right)\right) \\
&+\textstyle\sum_{\phi_1=1}^{n_e} 3\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}\left(:,\phi_1\right)\left(\boldsymbol{\eta}\left(:,\phi_1\right)m^3\left(\epsilon_{t+1}\left(\phi_1\right)\right)\right)'\mathbf{p}^{k-1}_\mathbf{xx}\boldsymbol{\eta}\left(:,\phi_1\right) \\
&+\textstyle\sum_{\phi_1=1}^{n_e} \boldsymbol{\eta}\left(:,\phi_1\right)'\mathbf{p}^{k-1}_\mathbf{xxx}\left(\gamma_1,:,:\right)\boldsymbol{\eta}\left(:,\phi_1\right)\boldsymbol{\eta}\left(\gamma_1,\phi_1\right)m^3\left(\epsilon_{t+1}\left(\phi_1\right)\right)
\end{aligned} \quad \textbf{(24)}$$

for $k = 2, 3, ..., K$. Here, $m^3\left(\epsilon_{t+1}\left(\phi_1\right)\right)$ denotes the third moment of $\boldsymbol{\epsilon}_{t+1}\left(\phi_1\right)$ for $\phi_1 = 1, 2, ..., n_\epsilon$. Notably, formulas **(22)-(24)** extend the results in Hordahl *et al* (2008) to third-order perturbation approximations.

All the formulas derived in this paper are implemented in Matlab and the codes are publicly available to facilitate their use.

### *2.5 Extensions*

The set-up considered above assumes that the model can be split into two distinct parts: one containing all equations in which bond prices beyond one-period maturity do *not* appear and another consisting entirely of Euler equations for the remaining bond prices. However, the POP method may still be useful if this condition does not hold. To see how, consider the case in which one is interested in the dynamics of the 10-year yield curve but it is only possible to separate out bond price Euler equations of maturity greater than 5 years. To then apply our method, the model including bond prices of maturity up to 20 quarters (5 years) needs to be solved in the first step of the POP method. This gives all derivatives of bond prices for $k \leq 20$. The remaining derivatives, for bond prices with maturities between 5 and 10 years, ie $k \in \{21, 22, \ldots 40\}$, can then be computed in the second step by starting the recursions derived in this paper at $k = 20$.

Andreasen (2010a) presents another extension in which the expected value of future non-predetermined variables are computed in the second perturbation step, making it possible to efficiently solve for expected future short interest rates, inflation rates, etc. We also illustrate in the appendix that the POP method can be extended to the case in which no restrictions are imposed on the way shocks enter, as considered in *Dynare++* or *Perturbation AIM* (Kamenik (2005) and Swanson *et al* (2005)). Furthermore, the scalar example presented in the appendix is derived using *Mathematica* and the underlying codes may be used to derive bond price approximations of orders higher than three.

## 3 Evaluating the computational gain

This section assesses the speed of the POP method and compares it to that of standard one-step perturbation. Clearly, both the absolute and relative performance of POP will depend on a number of factors. Those we focus on in this paper include the maximum maturity of bonds in the yield curve and the number of state variables in the model.

To illustrate the role played by the maximum bond maturity, we report results corresponding to nominal yield curves of maturities ranging from 5 to 20 years.[11] The relevance of the number of state variables is shown by reporting computing times for the DSGE model by Rudebusch and Swanson (2008) with 9 states as well as a version of the model by De Paoli *et al* (2007) with 15 state variables. Both models are approximated to third order using *Dynare++*.[12]

[11] The case of the 20-year yield curve is seldom considered in the literature, with the 10-year yield curve being the benchmark. However, from a computational perspective, approximating the 20-year yield curve is equivalent to: i) computing jointly the 10-year nominal and real yield curves, or ii) computing the 10-year yield curve and the corresponding term premia.

[12] Our implementation of the POP method does not exploit multi-threading, and we therefore do not use it in *Dynare++*. Note however, that by using Matlab to code our routine we are already sacrificing performance relative to the more efficient *C++* implementation of *Dynare++*.

The absolute computing times are reported in seconds, while the relative computational gain from the POP method is measured as

$$\text{Speed gain} = \frac{\text{Computing time using the one-step perturbation method}}{\text{Computing time using the POP method}}.$$

Table A reports Monte Carlo estimates of the time and the speed gain based on 20 replications for a third order approximation. The POP method turns out to be 23 times faster than the one-step perturbation method for the model by Rudebusch and Swanson (2008) with a 10-year yield curve. This number increases to 139 with a 20-year yield curve. The computational gains from the POP method in the model by De Paoli *et al* (2007) are somewhat lower, with the corresponding figures equal to 14 and 61, respectively. As shown in Andreasen (2010a), the speed gains involved are sufficient to make estimation of medium-scale DSGE models with a whole yield curve approximated to third order feasible.

**Table A: Gain in computing speed from the POP method**

This table compares the computing time of the one-step perturbation method to that of the POP method. The reported numbers are averages from 20 Monte Carlo replications for third-order approximations. Both DSGE models are solved in Dynare++ and bond prices from the POP method are implemented in Matlab. All computations are done on an Intel Core 2 Duo P7350 PC with 3.0 GB of RAM running Windows Vista.

| | 5-year | 10-year | 15-year | 20-year |
|---|---|---|---|---|
| **Rudebusch and Swanson (2008)** | | | | |
| One-step perturbation method (seconds) | 2.43 | 10.47 | 32.46 | 68.87 |
| POP method (seconds) | 0.43 | 0.45 | 0.48 | 0.50 |
| **Speed gain** | **5.61** | **23.09** | **68.11** | **138.55** |
| | | | | |
| **De Paoli** ***et al*** **(2007)** | | | | |
| One-step perturbation method (seconds) | 7.18 | 24.53 | 59.37 | 111.86 |
| POP method (seconds) | 1.72 | 1.75 | 1.79 | 1.83 |
| **Speed gain** | **4.18** | **14.02** | **33.11** | **61.09** |

## 4 Comparing solution accuracy

Our proposed POP method is faster to execute than traditional one-step perturbation, but there are other approximation methods which have become popular, in part due to their speed. This section compares the accuracy of the POP method to that of three well-known alternatives. In doing so, we add to the results of Arouba *et al* (2005) and Caldara *et al* (2009) by examining the accuracy of different interest rate approximations.

The first alternative considered is the first-order log-normal method proposed by Jermann (1998). Its accuracy is potentially compromised by the fact that it only includes some second-order terms in bond price approximations. The next alternative is the second-order log-normal method in Doh (2007), which extends Jermann's approach by combining second-order perturbation approximations with bond prices derived from the log-normal assumption. This approach is subject to similar type of criticisms as Jermann's method, because it only includes some third and fourth-order terms (see Andreasen (2009)). The final alternative considered is the 'consol' method proposed in Rudebusch and Swanson (2008) where consol bonds are used to compute yields.[13] Its accuracy may be adversely affected by the fact that consol bonds and zero-coupon bonds have very different cash flows, and matching the first-order concept of duration potentially allows for different higher-order properties of these bond prices.

To assess the accuracy of the aforementioned methods, we use expressions for zero-coupon bond prices in a consumption endowment model with external habits derived in Zabczyk (2010). We use a habit-based set-up because it has become a standard ingredient of many consumption-based asset pricing models. For example, De Paoli *et al* (2007), Hordahl *et al* (2008), Bekaert *et al* (2010), among others, use habits when studying the properties of bond prices in DSGE models. Furthermore, while our closed-form solution makes it straightforward to assess accuracy, the habit model is at the same time sufficiently flexible to match several salient features of the data.

We proceed by briefly introducing the habit model in Section 4.1. Section 4.2 then compares the approximation accuracy of the POP method to that of the three alternatives mentioned above.

### *4.1 The consumption endowment model with habits*

We consider a representative agent with the standard utility function

$$U_0 = \sum_{t=0}^{\infty} \beta^t E_0 \left[ \frac{(C_t - hC_{t-1})^{1-\gamma} - 1}{1-\gamma} \right],$$

where $C_t$ is consumption and $h \in [0, 1]$ controls the degree of external habit formation. Consumption growth is defined as $x_t := \ln (C_t/C_{t-1})$, and $x_t$ is assumed to follow an AR(1) process, ie

$$x_t = (1-\rho)\,\mu + \rho x_{t-1} + \xi_t \qquad \textbf{(25)}$$

[13] As shown by Rudebusch and Swanson (2008), prices of consols satisfy simple recursive relationships which make them easy to approximate.

where $\xi_t \sim \mathcal{NID}\left(0, \sigma_\xi^2\right)$. This implies the following expression for the stochastic discount factor

$$\mathcal{M}_{t+1} := \beta \frac{(C_{t+1} - hC_t)^{-\gamma}}{(C_t - hC_{t-1})^{-\gamma}} = \left(\frac{1 - h\exp\{-x_{t+1}\}}{1 - h\exp\{-x_t\}}\right)^{-\gamma} \exp\{-\gamma\, x_{t+1}\},$$

and a closed-form solution for zero-coupon bond prices of the form (Zabczyk (2010))

$$P_t^k = \left(1 - h\exp(-x_t)\right)^{\gamma} \beta^k \exp\left\{-\gamma\left(k\mu + (x_t - \mu)\frac{\rho(1-\rho^k)}{(1-\rho)}\right)\right\}$$
$$\times \sum_{n=0}^{+\infty} \begin{pmatrix} -\gamma \\ n \end{pmatrix} (-h)^n \exp\left\{-n\mu - n(x_t - \mu)\rho^k\right\} \prod_{j=0}^{k} \mathcal{L}_\xi\left(-\gamma\frac{(1-\rho^j)}{(1-\rho)} - n\frac{\rho^j - 0^j}{\rho}\right).$$

Here, $\mathcal{L}_\xi$ is the Laplace transform of $\xi$, and $\begin{pmatrix} \alpha \\ n \end{pmatrix}$ denotes a generalised binomial coefficient, ie

$$\begin{pmatrix} \alpha \\ n \end{pmatrix} := \prod_{k=1}^{n} (\alpha - k + 1)/k, \quad \text{for } n > 0 \text{ and } \begin{pmatrix} \alpha \\ 0 \end{pmatrix} := 1,$$

where $\alpha \in \mathbb{R}$ and $n \in \mathcal{N}$.[14]

The model is calibrated as follows. We let $\beta = 0.9995$ and set $h = 0.7$ based on the findings in Christiano, Eichenbaum and Evans (2005) and Smets and Wouters (2007). The coefficients in the consumption process in **(25)** are determined from an OLS regression for US non-durable consumption in the period 1947 Q1 to 2009 Q2. This implies $\mu = 0.0062$, $\rho = 0.0633$, and $\sigma_\xi = 6.4379 \times 10^{-5}$. Two values are considered for the curvature parameter $\gamma$. Our first choice is to let $\gamma = 1$ which corresponds to standard log-preferences. Our second choice is to set $\gamma$ to 5 and serves to explore the effects of stronger non-linearities in the model.

### *4.2 The accuracy of various approximation methods*

Chart 1 plots the benchmark 10-year interest rate as a function of consumption growth when $\gamma = 1$. The solid red line represents the exact solution and the other lines correspond to various approximation methods. Approximated solutions from one-step perturbation and the POP method are identical and are referred to as 'perturbation method' throughout this section. Furthermore, under our calibration, the functions from second-order perturbations are

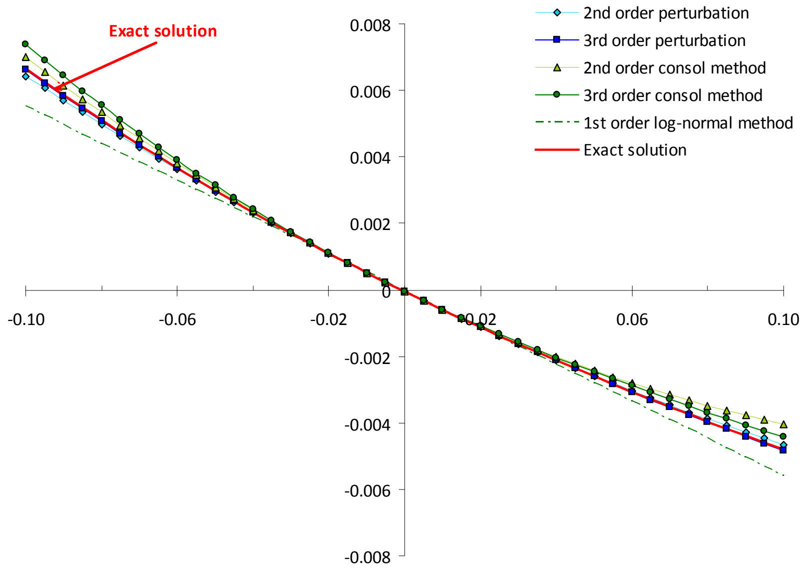
[14] Note that this model can be fairly easily extended to account for persistent habits and exogenous inflation (eg along the lines of Binsbergen *et al* (2010)). The latter would make it straightforward to price nominal bonds and analyse the properties of the nominal yield curve. However, since this would come at the cost of introducing extra state variables, we leave it for possible extensions.

indistinguishable from those implied by the second-order log-normal method, and we therefore only plot the former.

We first note that for the range of consumption growth considered, the third-order perturbation method delivers an approximation which is hard to distinguish from the exact solution. The second-order perturbation method is also quite accurate, with small deviations from the exact solution only visible for consumption growth deviations exceeding $\pm 0.08$. The consol method generates larger approximation errors with clearer deviations from the exact solution. Importantly, for $\gamma = 1$, all of these methods are more accurate than the first-order log-normal method.

**Chart 1: The function for the 10-year interest rate: $\gamma$ =1**

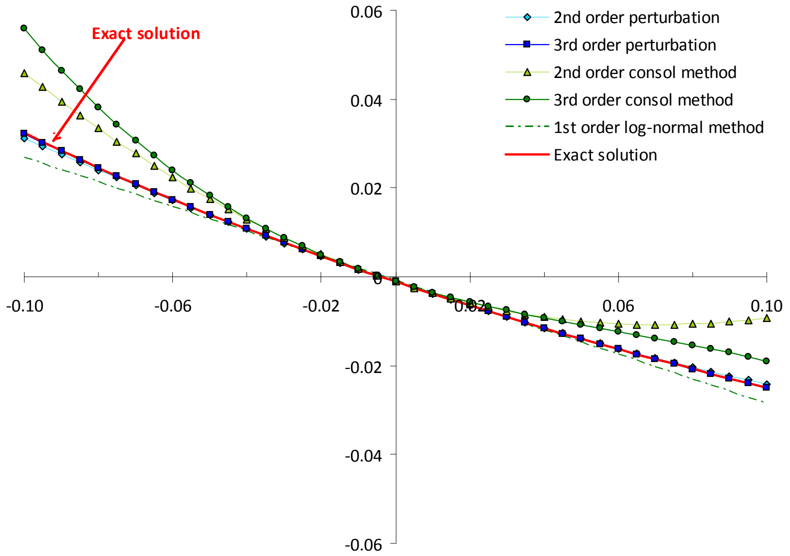
The x-axis reports consumption growth in deviation from the deterministic steady state. The y-axis reports the value of the 10-year interest rate in deviation from the deterministic steady state and expressed in quarterly terms.



In Chart 2, we turn to the case of stronger non-linearities with $\gamma = 5$. Again, the third-order perturbation approximation is almost indistinguishable from the exact solution. Chart 2 also shows that the second-order perturbation approximation is fairly accurate. The consol method, on the other hand, does worse than the first-order log-normal method. It is also interesting to note that moving from a second-order to a third-order approximation in the consol method does not improve its accuracy.

**Chart 2: The function for the 10-year interest rate: $\gamma$ =5**

The x-axis reports consumption growth in deviation from the deterministic steady state. The y-axis reports the value of the 10-year interest rate in deviation from the deterministic steady state and expressed in quarterly terms.




These observations are also confirmed by Table B which reports the root mean squared errors implied by the approximations in Charts 1 and 2. For our model and the chosen calibration we also see that third-order perturbation clearly outperforms all the alternative methods.

**Table B: Approximation accuracy for the 10-year interest rate**

The root mean sqaured errors for the approximations are computed for consumption growth at the following points: -0.1, -0.095, ..., 0.095 , 0.1. The figures in the table are multiplied by 100.

| | $\gamma = 1$ | $\gamma = 5$ |
|---|---|---|
| 2nd order perturbation | 0.007 | 0.037 |
| 3rd order perturbation | 0.001 | 0.007 |
| 2nd order consol method | 0.028 | 0.692 |
| 3rd order consol method | 0.027 | 0.769 |
| 1st order log-normal method | 0.045 | 0.212 |
| 2nd order log-normal method | 0.007 | 0.037 |

To evaluate the impact of the approximation errors reported above, we now focus on the first four moments of the benchmark 10-year interest rate. These moments are compared in Tables C and D which correspond to $\gamma = 1$ and $\gamma = 5$, respectively. We see that both versions of the perturbation method and the log-normal method reproduce the correct annualised mean. However, the consol method overestimates it by approximately 20 basis points when $\gamma = 1$ and by 24 basis points when $\gamma = 5$. The consol method also underestimates the standard deviation by more than all the other methods. Values of skewness and kurtosis are closely matched by the two perturbation methods and the second-order log-normal method, but not by the consol and first-order log-normal formulas.

**Table C: Moments for the 10-year interest rate: $\gamma$=1**

The 10-year interest rate is expressed in annual terms. All moments are computed based on a simulated time series of 1,000,000 observations.

| | Mean | Standard deviation | Skewness | Kurtosis |
|---|---|---|---|---|
| 2nd order perturbation | 2.6724 | 0.1786 | 0.0815 | 3.0065 |
| 3rd order perturbation | 2.6724 | 0.1787 | 0.0816 | 3.0113 |
| 2nd order consol method | 2.8740 | 0.1775 | 0.1337 | 3.0214 |
| 3rd order consol method | 2.8740 | 0.1774 | 0.1341 | 3.0332 |
| 1st order log-normal method | 2.6758 | 0.1786 | 0.0000 | 2.9972 |
| 2nd order log-normal method | 2.6724 | 0.1786 | 0.0816 | 3.0060 |
| Exact solution | 2.6724 | 0.1787 | 0.0818 | 3.0110 |

**Table D: Moments for the 10-year interest rate: $\gamma$ =5**

The 10-year interest rate is expressed in annual terms. All moments are computed based on a simulated time series of 1,000,000 observations.

| | Mean | Standard deviation | Skewness | Kurtosis |
|---|---|---|---|---|
| 2nd order perturbation | 12.2038 | 0.8930 | 0.0815 | 3.0065 |
| 3rd order perturbation | 12.2038 | 0.8935 | 0.0816 | 3.0113 |
| 2nd order consol method | 12.4429 | 0.8899 | 0.3402 | 3.1520 |
| 3rd order consol method | 12.4429 | 0.8571 | 0.3589 | 3.2342 |
| 1st order log-normal method | 12.2906 | 0.8928 | 0.0000 | 2.9972 |
| 2nd order log-normal method | 12.2039 | 0.8929 | 0.0816 | 3.0060 |
| Exact solution | 12.2035 | 0.8935 | 0.0818 | 3.0110 |

Summarising, third-order perturbations – and hence the POP method – approximate the 10-year interest rate most accurately in the examples considered. The precision of the second-order log-normal method is very similar to that of the second-order perturbation method, and both outperform the first-order log-normal approximation. We also find that the consol method gives a less accurate approximation, and we show that it may be less accurate, even at third order, than the first-order log-normal formula.

## 5 Conclusion

This paper proposes a new method of computing bond price approximations. The approach is applicable to a wide class of DSGE models and uses the perturbation principle sequentially. While the final formulae for bond prices *exactly* match those derived using the standard one-step perturbation method, a simulation study documents that execution times can be more than 100 times shorter. In general, the exact improvement in speed depends on the maturity of the approximated yield curve and the number of state variables in the DSGE model.

The paper also assesses the accuracy of the POP/perturbation method implemented up to third order in a consumption endowment model with habits. Our results show that the third-order approximation to the 10-year interest rate is more accurate than those of popular alternatives and can be hard to distinguish from the true solution. It is also shown that interest rates approximated from prices of consol bonds can be less precise, even at third order, than those computed using the first-order log-normal approach.

## Appendix A: Third-order terms for bond prices

This appendix derives the third-order terms for bond prices in the framework of Schmitt-Grohé and Uribe (2004).

### *A.1 Derivative of $p^k$ with respect to $(\mathbf{x}, \mathbf{x}, \mathbf{x})$*

Applying the chain rule to the definition of $F^k$ one can show that $\left[F_{\mathbf{xxx}}\left(\mathbf{x}_{ss}, 0\right)\right]_{\alpha_1\alpha_2\alpha_3} = 0$ equals

$$
\begin{aligned}
&R_p\left(p^k\right)\left[p^k_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3} = -R_{ppp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\mathbf{x}}\right]_{\alpha_2}\left[p^k_{\mathbf{x}}\right]_{\alpha_1} - R_{pp}\left(p^k\right)\left[p^k_{\mathbf{xx}}\right]_{\alpha_2\alpha_3}\left[p^k_{\mathbf{x}}\right]_{\alpha_1} \\
&-R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_2}\left[p^k_{\mathbf{xx}}\right]_{\alpha_1\alpha_3} - R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\mathbf{xx}}\right]_{\alpha_1\alpha_2} + \left[\mathcal{M}_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3} R\left(p^{k-1}\right) \\
&+\left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1,\alpha_2} + R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} \\
&+\left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1\alpha_3} + R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_1} R_p\left(p^1\right)/R\left(p^0\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_1} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_1} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3} \\
&+\left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_2,\alpha_3} + R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_2} R_p\left(p^1\right)/R\left(p^0\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_2} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_2} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_3} R_p\left(p^1\right)/R\left(p^0\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_{ppp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_3} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1} + R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_3} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}
\end{aligned}
$$

$$+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{xx}}]_{\alpha_1\alpha_2}^{\gamma_1}$$
$$+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]_{\alpha_3}^{\gamma_3}[\mathbf{h}_{\mathbf{xx}}]_{\alpha_1\alpha_2}^{\gamma_1}+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{xxx}}]_{\alpha_1\alpha_2\alpha_3}^{\gamma_1}$$

Note that we can also eliminate $[\mathcal{M}_{\mathbf{xxx}}]_{\alpha_1\alpha_2\alpha_3}$ from this expression. Again, the trick is to observe that for $k=1$ we have $P^0=1$ for all values of $(\mathbf{x}_t,\sigma)$ and so all derivatives have to equal zero. Thus

$$R_p\left(p^1\right)\left[p_{\mathbf{xxx}}^1\right]_{\alpha_1\alpha_2\alpha_3}=-R_{ppp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_3}\left[p_{\mathbf{x}}^1\right]_{\alpha_2}\left[p_{\mathbf{x}}^1\right]_{\alpha_1}-R_{pp}\left(p^1\right)\left[p_{\mathbf{xx}}^1\right]_{\alpha_2\alpha_3}\left[p_{\mathbf{x}}^1\right]_{\alpha_1}$$
$$-R_{pp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_2}\left[p_{\mathbf{xx}}^1\right]_{\alpha_1\alpha_3}-R_{pp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_3}\left[p_{\mathbf{xx}}^1\right]_{\alpha_1\alpha_2}+[\mathcal{M}_{\mathbf{xxx}}]_{\alpha_1\alpha_2\alpha_3}R\left(p^0\right)$$

$\Updownarrow$

$$(R_p\left(p^1\right)\left[p_{\mathbf{xxx}}^1\right]_{\alpha_1\alpha_2\alpha_3}+R_{ppp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_3}\left[p_{\mathbf{x}}^1\right]_{\alpha_2}\left[p_{\mathbf{x}}^1\right]_{\alpha_1}+R_{pp}\left(p^1\right)\left[p_{\mathbf{xx}}^1\right]_{\alpha_2\alpha_3}\left[p_{\mathbf{x}}^1\right]_{\alpha_1}$$
$$R_{pp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_2}\left[p_{\mathbf{xx}}^1\right]_{\alpha_1\alpha_3}+R_{pp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_3}\left[p_{\mathbf{xx}}^1\right]_{\alpha_1\alpha_2})=[\mathcal{M}_{\mathbf{xxx}}]_{\alpha_1\alpha_2\alpha_3}\text{ because }R\left(p^0\right)=1.$$

Thus we get for $k>1$

$$R_p\left(p^k\right)\left[p_{\mathbf{xxx}}^k\right]_{\alpha_1\alpha_2\alpha_3}=-R_{ppp}\left(p^k\right)\left[p_{\mathbf{x}}^k\right]_{\alpha_3}\left[p_{\mathbf{x}}^k\right]_{\alpha_2}\left[p_{\mathbf{x}}^k\right]_{\alpha_1}-R_{pp}\left(p^k\right)\left[p_{\mathbf{x}}^k\right]_{\alpha_2\alpha_3}\left[p_{\mathbf{x}}^k\right]_{\alpha_1}$$
$$-R_{pp}\left(p^k\right)\left[p_{\mathbf{x}}^k\right]_{\alpha_2}\left[p_{\mathbf{x}}^k\right]_{\alpha_1\alpha_3}-R_{pp}\left(p^k\right)\left[p^k\right]_{\alpha_3}\left[p_{\mathbf{xx}}^k\right]_{\alpha_1\alpha_2}$$
$$+(R_p\left(p^1\right)\left[p_{\mathbf{xxx}}^1\right]_{\alpha_1\alpha_2\alpha_3}+R_{ppp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_3}\left[p_{\mathbf{x}}^1\right]_{\alpha_2}\left[p_{\mathbf{x}}^1\right]_{\alpha_1}+R_{pp}\left(p^1\right)\left[p_{\mathbf{xx}}^1\right]_{\alpha_2\alpha_3}\left[p_{\mathbf{x}}^1\right]_{\alpha_1}$$
$$+R_{pp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_2}\left[p_{\mathbf{xx}}^1\right]_{\alpha_1\alpha_3}+R_{pp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_3}\left[p_{\mathbf{xx}}^1\right]_{\alpha_1\alpha_2})R\left(p^{k-1}\right)$$
$$+\left(R_p\left(p^1\right)\left[p_{\mathbf{xx}}^1\right]_{\alpha_1,\alpha_2}+R_{pp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_2}\left[p_{\mathbf{x}}^1\right]_{\alpha_1}\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]_{\alpha_3}^{\gamma_3}$$
$$+\left(R_p\left(p^1\right)\left[p_{\mathbf{xx}}^1\right]_{\alpha_1\alpha_3}+R_{pp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_3}\left[p_{\mathbf{x}}^1\right]_{\alpha_1}\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]_{\alpha_2}^{\gamma_2}$$
$$+\left[p_{\mathbf{x}}^1\right]_{\alpha_1}R_p\left(p^1\right)/R\left(p^0\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]_{\alpha_2}^{\gamma_2}$$
$$+\left[p_{\mathbf{x}}^1\right]_{\alpha_1}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]_{\alpha_3}^{\gamma_3}[\mathbf{h}_{\mathbf{x}}]_{\alpha_2}^{\gamma_2}$$
$$+\left[p_{\mathbf{x}}^1\right]_{\alpha_1}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\mathbf{h}_{\mathbf{xx}}]_{\alpha_2\alpha_3}^{\gamma_2}$$
$$+\left(R_p\left(p^1\right)\left[p_{\mathbf{xx}}^1\right]_{\alpha_2,\alpha_3}+R_{pp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_3}\left[p_{\mathbf{x}}^1\right]_{\alpha_2}\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]_{\alpha_1}^{\gamma_1}$$
$$+\left[p_{\mathbf{x}}^1\right]_{\alpha_2}R_p\left(p^1\right)/R\left(p^0\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]_{\alpha_1}^{\gamma_1}$$
$$+\left[p_{\mathbf{x}}^1\right]_{\alpha_2}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]_{\alpha_3}^{\gamma_3}[\mathbf{h}_{\mathbf{x}}]_{\alpha_1}^{\gamma_1}$$
$$+\left[p_{\mathbf{x}}^1\right]_{\alpha_2}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{xx}}]_{\alpha_1\alpha_3}^{\gamma_1}$$
$$+\left[p_{\mathbf{x}}^1\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]_{\alpha_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]_{\alpha_1}^{\gamma_1}$$
$$+R\left(p^1\right)R_{ppp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]_{\alpha_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]_{\alpha_1}^{\gamma_1}$$
$$+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]_{\alpha_3}^{\gamma_3}[\mathbf{h}_{\mathbf{x}}]_{\alpha_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]_{\alpha_1}^{\gamma_1}$$

$$
\begin{aligned}
&+R\left(p^{1}\right) R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]_{\alpha_2\alpha_3}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\\
&+R\left(p^{1}\right) R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\\
&+R\left(p^{1}\right) R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]_{\alpha_1\alpha_3}^{\gamma_1}\\
&+\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3} R_{p}\left(p^{1}\right)/R\left(p^{0}\right) R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\\
&+R\left(p^{1}\right) R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\\
&+R\left(p^{1}\right) R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xxx}}^{k-1}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}+R\left(p^{1}\right) R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]_{\alpha_2\alpha_3}^{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\\
&+R\left(p^{1}\right) R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]_{\alpha_1\alpha_3}^{\gamma_1}\\
&+\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3} R_{p}\left(p^{1}\right)/R\left(p^{0}\right) R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}^{\gamma_1}\\
&+R\left(p^{1}\right) R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}^{\gamma_1}\\
&+R\left(p^{1}\right) R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\mathbf{h}_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}^{\gamma_1}+R\left(p^{1}\right) R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3}^{\gamma_1}
\end{aligned}
$$

With a log-transformation $R\left(p^{t,k}\right)=M^{k}$, $R_{p}\left(p^{t,k}\right)=M^{k}$, $R_{pp}\left(p^{t,k}\right)=M^{k}$, and $R_{ppp}\left(p^{t,k}\right)=M^{k}$ in the deterministic steady state. Using the expressions for first and second-order derivatives of bond prices derived above, we get, after simplifying, the expression stated in the body of the text.

### *A.2 Derivative of $p^k$ with respect to $(\sigma, \sigma, \mathbf{x})$*

It is possible to show that $\left[F_{\sigma\sigma\mathbf{x}}\left(\mathbf{x}_{ss}, 0\right)\right]_{\alpha_3}=0$ implies

$$
\begin{aligned}
&E_t\{-R_{pp}\left(p^{k}\right)\left[p_{\mathbf{x}}^{k}\right]_{\alpha_3}\left[p_{\sigma\sigma}^{k}\right]-R_{p}\left(p^{k}\right)\left[p_{\sigma\sigma\mathbf{x}}^{k}\right]_{\alpha_3}\\
&+\left[\mathcal{M}_{\sigma\sigma\mathbf{x}}\right] R\left(p^{k-1}\right)+\left[\mathcal{M}_{\sigma\sigma}\right] R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\\
&+2\left[\mathcal{M}_{\sigma\mathbf{x}}\right]_{\alpha_3} R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\epsilon_{t+1}\right]^{\phi_2}\\
&+2\left[\mathcal{M}_{\sigma}\right] R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\epsilon_{t+1}\right]^{\phi_2}\\
&+2\left[\mathcal{M}_{\sigma}\right] R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\epsilon_{t+1}\right]^{\phi_2}\\
&+\left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_3} R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\epsilon_{t+1}\right]^{\phi_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\epsilon_{t+1}\right]^{\phi_1}\\
&+\mathcal{M} R_{ppp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\epsilon_{t+1}\right]^{\phi_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\epsilon_{t+1}\right]^{\phi_1}\\
&+\mathcal{M} R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\epsilon_{t+1}\right]^{\phi_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\epsilon_{t+1}\right]^{\phi_1}\\
&+\mathcal{M} R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\epsilon_{t+1}\right]^{\phi_2}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\epsilon_{t+1}\right]^{\phi_1}\\
&+\left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_3} R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\epsilon_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\epsilon_{t+1}\right]^{\phi_1}\\
&+\mathcal{M} R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\epsilon_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\epsilon_{t+1}\right]^{\phi_1}\\
&+\mathcal{M} R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xxx}}^{k-1}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\epsilon_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\epsilon_{t+1}\right]^{\phi_1}\\
&+\left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_3} R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}+\mathcal{M} R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}
\end{aligned}
$$

$$+\mathcal{M}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}+\mathcal{M}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}_{\sigma\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3}$$
$$+[\mathcal{M}_{\mathbf{x}}]_{\alpha_3}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\sigma\sigma}\right]+\mathcal{M}R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\sigma\sigma}\right]$$
$$+\mathcal{M}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\sigma\sigma\mathbf{x}}\right]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}\}=0$$

$$\Updownarrow$$

$$R_p\left(p^k\right)\left[p^k_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}=-R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\sigma\sigma}\right]+E_t\left[\mathcal{M}_{\sigma\sigma\mathbf{x}}\right]R\left(p^{k-1}\right)$$
$$+\left[p^1_{\sigma\sigma}\right]R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}$$
$$+2E_t\left([\mathcal{M}_{\sigma\mathbf{x}}]_{\alpha_3}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)$$
$$+2E_t\left([\mathcal{M}_{\sigma}]R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)$$
$$+2E_t\left([\mathcal{M}_{\sigma}]R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)$$
$$+\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$
$$+R\left(p^1\right)R_{ppp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$
$$+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$
$$+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$
$$+\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$
$$+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$
$$+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$
$$+\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}$$
$$+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}$$
$$+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}_{\sigma\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3}+\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\sigma\sigma}\right]$$
$$+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\sigma\sigma}\right]+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\sigma\sigma\mathbf{x}}\right]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}$$

where we have used

$$\mathcal{M}=R\left(p^1\right)\ [\mathcal{M}_{\mathbf{x}}]_{\alpha_3}=\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)\ E_t\left[\mathcal{M}_{\sigma\sigma}\right]=\left[p^1_{\sigma\sigma}\right]R_p\left(p^1\right)/R\left(p^0\right).$$

We now compute the terms with derivatives of $\sigma$. Here we recall that

$$[\mathcal{M}_{\sigma}]\equiv(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}).$$

So

$$
\begin{aligned}
&2E_t\left([\mathcal{M}_\sigma]\,R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&=2E_t\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_x}\right]_{\gamma_1}^{\beta_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&+2E_t\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&=2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_x}\right]_{\gamma_1}^{\beta_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}[\mathbf{I}]_{\phi_1}^{\phi_2}\\
&+2\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}[\mathbf{I}]_{\phi_1}^{\phi_2}
\end{aligned}
$$

and

$$
\begin{aligned}
&2E_t\left([\mathcal{M}_\sigma]\,R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&=2E_t\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_x}\right]_{\gamma_1}^{\beta_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&+2E_t\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&=2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_x}\right]_{\gamma_1}^{\beta_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}[\mathbf{I}]_{\phi_1}^{\phi_2}\\
&+2E_t\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}[\mathbf{I}]_{\phi_1}^{\phi_2}
\end{aligned}
$$

To compute the $[\mathcal{M}_{\sigma\mathbf{x}}]_{\alpha_3}$ term, we need to find an expression for $[\mathcal{M}_{\sigma\mathbf{x}}]_{\alpha_3}$. Hence

$$
\begin{aligned}
&[\mathcal{M}_{\sigma\mathbf{x}}]_{\alpha_3}=\\
&\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]_{\gamma_3}^{\beta_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]_{\alpha_3}^{\beta_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]_{\beta_1\alpha_3}\right)\\
&\times\left[\mathbf{g_x}\right]_{\gamma_1}^{\beta_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}+\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_{xx}}\right]_{\gamma_1\gamma_3}^{\beta_1}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\\
&+\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}\left[\mathbf{g_x}\right]_{\gamma_3}^{\beta_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_3}\left[\mathbf{g_x}\right]_{\alpha_3}^{\beta_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]_{\gamma_1\alpha_3}\right)\\
&\times[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}
\end{aligned}
$$

So

$$
\begin{aligned}
&2E_t\left([\mathcal{M}_{\sigma\mathbf{x}}]_{\alpha_3}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)=\\
&2E_t\{\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]_{\gamma_3}^{\beta_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]_{\alpha_3}^{\beta_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]_{\beta_1\alpha_3}\right)\\
&\times\left[\mathbf{g_x}\right]_{\gamma_1}^{\beta_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\\
&+\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_{xx}}\right]_{\gamma_1\gamma_3}^{\beta_1}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\\
&+\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}\left[\mathbf{g_x}\right]_{\gamma_3}^{\beta_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_3}\left[\mathbf{g_x}\right]_{\alpha_3}^{\beta_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]_{\gamma_1\alpha_3}\right)\\
&\times[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\}=\\
&2\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]_{\gamma_3}^{\beta_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]_{\alpha_3}^{\beta_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]_{\beta_1\alpha_3}\right)\\
&\times\left[\mathbf{g_x}\right]_{\gamma_1}^{\beta_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}[\mathbf{I}]_{\phi_1}^{\phi_2}
\end{aligned}
$$

$$+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$
$$+2\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]_{\gamma_1\alpha_3}\right)$$
$$\times\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$

We finally note that $E_t\left(\left[\mathcal{M}_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}\right)$ can be solved for and then substituted out by exploiting the fact that for $k=1$ we have $P^0=1$ for all values of $(\mathbf{x}_t,\sigma)$ and so all derivatives have to equal zero. Thus

$$R_p\left(p^1\right)\left[p^1_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}=-R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\sigma\sigma}\right]+E_t\left(\left[\mathcal{M}_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}\right)R\left(p^0\right)$$

$$\Updownarrow$$

$$\left(R_p\left(p^1\right)\left[p^1_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\sigma\sigma}\right]\right)/R\left(p^0\right)=E_t\left(\left[\mathcal{M}_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}\right)$$

So for $k>1$ we get

$$R_p\left(p^k\right)\left[p^k_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}=-R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\sigma\sigma}\right]$$
$$+\left(R_p\left(p^1\right)\left[p^1_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\sigma\sigma}\right]\right)R\left(p^{k-1}\right)/R\left(p^0\right)$$
$$+\left[p^1_{\sigma\sigma}\right]R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}$$
$$+2\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]_{\beta_1\alpha_3}\right)$$
$$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$
$$+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$
$$+2\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]_{\gamma_1\alpha_3}\right)$$
$$\times\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$
$$+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$
$$+2\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$
$$+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$
$$+2E_t\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$
$$+\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$
$$+R\left(p^1\right)R_{ppp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$
$$+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$
$$+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$
$$+\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$

$$+R\left(p^{1}\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}[\mathbf{I}]_{\phi_2}^{\phi_1}$$
$$+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xxx}}^{k-1}\right]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}[\mathbf{I}]_{\phi_2}^{\phi_1}$$
$$+\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3}R_{p}\left(p^{1}\right)/R\left(p^{0}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}$$
$$+R\left(p^{1}\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}$$
$$+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\sigma\sigma\mathbf{x}}]_{\alpha_3}^{\gamma_1}+\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3}R_{p}\left(p^{1}\right)/R\left(p^{0}\right)R_{p}\left(p^{k-1}\right)\left[p_{\sigma\sigma}^{k-1}\right]$$
$$+R\left(p^{1}\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}\left[p_{\sigma\sigma}^{k-1}\right]+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\sigma\sigma\mathbf{x}}^{k-1}\right]_{\gamma_3}[\mathbf{h_x}]_{\alpha_3}^{\gamma_3}$$

For a logarithm transformation $R\left(p^{t,k}\right)=M^{k}$, $R_{p}\left(p^{t,k}\right)=M^{k}$, $R_{pp}\left(p^{t,k}\right)=M^{k}$, and $R_{ppp}\left(p^{t,k}\right)=M^{k}$. Using the expressions for first and second order derivatives of bond prices derived above, we get, after simplifying,

$$p_{\sigma\sigma\mathbf{x}}^{k}(1,\alpha_3)=-2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{y}_{t+1}}\mathbf{g_x}\boldsymbol{\eta}\boldsymbol{\eta}'\left(p_{\mathbf{x}}^{k-1}\right)'p_{\mathbf{x}}^{1}(1,\alpha_3)-2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{x}_{t+1}}\boldsymbol{\eta}\boldsymbol{\eta}'\left(p_{\mathbf{x}}^{k-1}\right)'p_{\mathbf{x}:}^{1}(1,\alpha_3)$$
$$+p_{\sigma\sigma\mathbf{x}}^{1}(1,\alpha_3)+2\mathcal{M}^{-1}p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'(\mathbf{g_x})'$$
$$\times\left(\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\mathbf{g_x}\mathbf{h_x}(:,\alpha_3)+\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\mathbf{g_x}(:,\alpha_3)+\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\mathbf{h_x}(:,\alpha_3)+\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}(:,\alpha_3)\right)$$
$$+\sum_{\beta_1=1}^{n_y}2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{y}_{t+1}}\left(1,\beta_1\right)p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{g_{xx}}\left(\beta_1,:,:\right)\mathbf{h_x}(:,\alpha_3)+2\mathcal{M}^{-1}p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'$$
$$\times\left(\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\mathbf{g_x}\mathbf{h_x}(:,\alpha_3)+\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\mathbf{g_x}(:,\alpha_3)+\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\mathbf{h_x}(:,\alpha_3)+\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}(:,\alpha_3)\right)$$
$$+2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{y}_{t+1}}\mathbf{g_x}\boldsymbol{\eta}\boldsymbol{\eta}'p_{\mathbf{xx}}^{k-1}\mathbf{h_x}(:,\alpha_3)+2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{x}_{t+1}}\boldsymbol{\eta}\boldsymbol{\eta}'p_{\mathbf{xx}}^{k-1}\mathbf{h_x}(:,\alpha_3)+p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'p_{\mathbf{xx}}^{k-1}\mathbf{h_x}(:,\alpha_3)$$
$$+p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'p_{\mathbf{xx}}^{k-1}\mathbf{h_x}(:,\alpha_3)+\sum_{\gamma_1=1}^{n_x}\boldsymbol{\eta}\left(\gamma_1,:\right)\boldsymbol{\eta}'p_{\mathbf{xxx}}^{k-1}\left(\gamma_1,:,:\right)\mathbf{h_x}(:,\alpha_3)+(\mathbf{h}_{\sigma\sigma})'p_{\mathbf{xx}}^{k-1}\mathbf{h_x}(:,\alpha_k)$$
$$+p_{\mathbf{x}}^{k-1}\mathbf{h}_{\sigma\sigma\mathbf{x}}(:,\alpha_3)+p_{\sigma\sigma\mathbf{x}}^{k-1}\mathbf{h_x}(:,\alpha_3)$$

### *A.3 Derivative of* $p^k$ *with respect to* $(\sigma,\sigma,\sigma)$

It is possible to show that $F_{\sigma\sigma\sigma}(\mathbf{x}_{ss},0)=0$ implies

$$[F_{\sigma\sigma\sigma}(\mathbf{x}_{ss},0)]=E_t\{-R_p\left(p^k\right)\left[p_{\sigma\sigma\sigma}^{k}\right]+[\mathcal{M}_{\sigma\sigma\sigma}]R\left(p^{k-1}\right)$$
$$+3[\mathcal{M}_{\sigma\sigma}]R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}$$
$$+3[\mathcal{M}_{\sigma}]R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}$$
$$+3[\mathcal{M}_{\sigma}]R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}$$
$$+R\left(p^{1}\right)R_{ppp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+3R\left(p^{1}\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xxx}}^{k-1}\right]_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\sigma\sigma\sigma}]^{\gamma_1}+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\sigma\sigma\sigma}^{k-1}\right]\}=0$$

We next use the expression for $[\mathcal{M}_\sigma]$ found previously. We also have from differentiation of $\mathcal{M}$ that

$$[\mathcal{M}_{\sigma\sigma}] =$$
$$([\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}]_{\beta_1\beta_3}[\mathbf{g_x}]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\epsilon_{t+1}]^{\phi_3}+[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}]_{\beta_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\epsilon_{t+1}]^{\phi_3})[\mathbf{g_x}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\epsilon_{t+1}]^{\phi_1}$$
$$+[\mathcal{M}_{\mathbf{y}_{t+1}}]_{\beta_1}[\mathbf{g_{xx}}]^{\beta_1}_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\epsilon_{t+1}]^{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\epsilon_{t+1}]^{\phi_1}+[\mathcal{M}_{\mathbf{y}_{t+1}}]_{\beta_1}[\mathbf{g_x}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}$$
$$+[\mathcal{M}_{\mathbf{y}_{t+1}}]_{\beta_1}[\mathbf{g}_{\sigma\sigma}]^{\beta_1}+[\mathcal{M}_{\mathbf{y_t}}]_{\beta_1}[\mathbf{g}_{\sigma\sigma}]^{\beta_1}$$
$$+([\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}]_{\gamma_1\beta_3}[\mathbf{g_x}]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\epsilon_{t+1}]^{\phi_3}+[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}]_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\epsilon_{t+1}]^{\phi_3})[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\epsilon_{t+1}]^{\phi_1}$$
$$+[\mathcal{M}_{\mathbf{x}_{t+1}}]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}$$

For $[\mathcal{M}_{\sigma\sigma\sigma}]$, we exploit the fact $P^0 = 1$ for all values of $(\mathbf{x}_t, \sigma)$ and so all derivatives have to equal zero. Thus $R_p\left(p^1\right)\left[p^1_{\sigma\sigma\sigma}\right] = E_t\left\{[\mathcal{M}_{\sigma\sigma\sigma}]\right\}$.

To evaluate the expectations in the term for $[F_{\sigma\sigma\sigma}(\mathbf{x}_{ss}, 0)]$, we define

$$\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3} = \begin{cases} m^3(\epsilon_{t+1}(\phi_1)) & \text{if } \phi_1 = \phi_2 = \phi_3 \\ 0 & \text{otherwise} \end{cases}$$

where $m^3(\epsilon_{t+1})$ denotes the third moment of $\epsilon_{t+1}(\phi_1)$ for $\phi_1 = 1, 2, ..., n_\epsilon$. Notice that $\mathbf{m}^3(\epsilon_{t+1})$ is a $n_\epsilon \times n_\epsilon \times n_\epsilon$ matrix. Following some simplifications we finally get

$$R_p\left(p^k\right)\left[p^k_{\sigma\sigma\sigma}\right] = +R_p\left(p^1\right)\left[p^1_{\sigma\sigma\sigma}\right]R\left(p^{k-1}\right)$$
$$+3\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}[\mathbf{g_x}]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\mathbf{g_x}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+6\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\mathbf{g_x}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}[\mathbf{g_{xx}}]^{\beta_1}_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}[\mathbf{g_x}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\right)\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}$$
$$+3\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}[\mathbf{g_x}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\right)\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}$$
$$+R\left(p^1\right)R_{ppp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}_{\sigma\sigma\sigma}]^{\gamma_1}+R\left(p^1\right)R_p\left(p^{k-1}\right)p^{k-1}_{\sigma\sigma\sigma}$$

For a logarithm transformation, it is straightforward to show that

$$
\begin{aligned}
p^k_{\sigma\sigma\sigma} &= p^1_{\sigma\sigma\sigma} + 3\mathcal{M}^{-1}\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3\left(\epsilon_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
&+6\mathcal{M}^{-1}\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3\left(\epsilon_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
&+3\mathcal{M}^{-1}\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3\left(\epsilon_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
&+3\mathcal{M}^{-1}\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3\left(\epsilon_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
&+3\mathcal{M}^{-1}\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\right)\left[\mathbf{m}^3\left(\epsilon_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \\
&+3\mathcal{M}^{-1}\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\right)\left[\mathbf{m}^3\left(\epsilon_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2} \\
&+\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\epsilon_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
&+3\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\epsilon_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3} \\
&+\left[p^{k-1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\epsilon_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}+\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\sigma}\right]^{\gamma_1}+p^{k-1}_{\sigma\sigma\sigma}
\end{aligned}
$$

## Appendix B: The POP method when perturbing the state variables and the innovations

Consider the class of DSGE models with the following set of equilibrium and market clearing conditions:

$$E_t\left[f\left(z_{t+1}, z_t, z_{t-1}, \sigma u_t\right)\right] = 0$$

where $\sigma$ is the perturbation parameter. The vector $z_t$ contains all the endogenous variables and $u_t$ is the vector of disturbances with the property $u_t \sim \mathcal{IID}\left(0, \Sigma\right)$. The general solution is given by $z_t = g\left(z_{t-1}, u_t, \sigma\right)$.

When the approximation is done in levels, the fundamental pricing equation implies

$$P^n(z_{t-1}, u_t, \sigma) = E_t\left[\mathcal{M}(g_t\left(z_{t-1}, u_t, \sigma\right), u_{t+1}, \sigma)P^{n-1}(g_t\left(z_{t-1}, u_t, \sigma\right), u_{t+1}, \sigma)\right].$$

The recursive solution for a third-order approximation of bond prices is then given by

$$\begin{aligned}
P^n = {} & \underbrace{\mathcal{M}p^{n-1}}_{P^n} + u_t \underbrace{g_u\mathcal{A}_1^{n-1}}_{P_u^n} + z_t \underbrace{g_z\mathcal{A}_1^{n-1}}_{P_z^n} \\
& + \tfrac{1}{2}u_t^2 \underbrace{\left(g_{uu}\mathcal{A}_1^{n-1} + g_u^2\mathcal{A}_2^{n-1}\right)}_{P_{uu}^n} + z_t u_t \underbrace{\left(g_{zu}\mathcal{A}_1^{n-1} + g_u g_z\mathcal{A}_2^{n-1}\right)}_{P_{zu}^n} + \tfrac{1}{2}z_t^2 \underbrace{\left(g_{zz}\mathcal{A}_1^{n-1} + g_z^2\mathcal{A}_2^{n-1}\right)}_{P_{zz}^n} \\
& + \tfrac{1}{2}\sigma^2 \underbrace{\left(p^{n-1}\left(\Sigma\mathcal{M}_{uu} + \mathcal{M}_{\sigma\sigma}\right) + 2\Sigma\mathcal{M}_u p_u^{n-1} + \mathcal{M}\left(\Sigma p_{uu}^{n-1} + p_{\sigma\sigma}^{n-1}\right) + g_{\sigma\sigma}\mathcal{A}_1^{n-1}\right)}_{P_{\sigma\sigma}^n} \\
& + \tfrac{1}{6}\sigma^3 m^3 \underbrace{\left(\mathcal{M}p_{uuu}^{n-1} + 3p_{uu}^{n-1}\mathcal{M}_u + 3p_u^{n-1}\mathcal{M}_{uu} + p^{n-1}\mathcal{M}_{uuu}\right)}_{P_{\sigma\sigma\sigma}^n} \\
& + \tfrac{1}{6}z_t^3 \underbrace{\left(g_{zzz}\mathcal{A}_1^{n-1} + 3g_z g_{zz}\mathcal{A}_2^{n-1} + g_z^3\mathcal{A}_3^{n-1}\right)}_{P_{zzz}^n} + \tfrac{1}{6}u_t^3 \underbrace{\left(g_{uuu}\mathcal{A}_1^{n-1} + 3g_u g_{uu}\mathcal{A}_2^{n-1} + g_u^3\mathcal{A}_3^{n-1}\right)}_{P_{uuu}^n} \\
& + \tfrac{1}{2}z_t^2 u_t \underbrace{\left(g_{zzu}\mathcal{A}_1^{n-1} + \left(2g_z g_{zu} + g_u g_{zz}\right)\mathcal{A}_2^{n-1} + g_u g_z^2\mathcal{A}_3^{n-1}\right)}_{P_{zzu}^n} \\
& + \tfrac{1}{2}z_t u_t^2 \underbrace{\left(g_{zuu}\mathcal{A}_1^{n-1} + \left(2g_u g_{zu} + g_z g_{uu}\right)\mathcal{A}_2^{n-1} + g_z g_u^2\mathcal{A}_3^{n-1}\right)}_{P_{zuu}^n} \\
& + \tfrac{1}{2}\sigma^2 u_t \underbrace{\left(g_{u\sigma\sigma}\mathcal{A}_1^{n-1} + g_u\mathcal{A}_4^{n-1}\right)}_{P_{\sigma\sigma u}^n} + \tfrac{1}{2}\sigma^2 z_t \underbrace{\left(g_{z\sigma\sigma}\mathcal{A}_1^{n-1} + g_z\mathcal{A}_4^{n-1}\right)}_{P_{\sigma\sigma z}^n}
\end{aligned}$$

where

$$
\begin{aligned}
&\mathcal{A}_1^{n-1} \equiv P^{n-1}\mathcal{M}_z + \mathcal{M}P_z^{n-1} \; \mathcal{A}_2^{n-1} \equiv P^{n-1}\mathcal{M}_{zz} + 2\mathcal{M}_z P_z^{n-1} + \mathcal{M}P_{zz}^{n-1} \\
&\mathcal{A}_3^{n-1} \equiv P^{n-1}\mathcal{M}_{zzz} + 3\mathcal{M}_{zz}P_z^{n-1} + 3\mathcal{M}_z P_{zz}^{n-1} + \mathcal{M}P_{zzz}^{n-1} \\
&\mathcal{A}_4^{n-1} \equiv P^{n-1}\left(\Sigma\mathcal{M}_{zuu} + \mathcal{M}_{z\sigma\sigma}\right) + \mathcal{M}\left(\Sigma P_{zuu}^{n-1} + P_{z\sigma\sigma}^{n-1}\right) \\
&+\Sigma\left(2\mathcal{M}_{zu}P_u^{n-1} + \mathcal{M}_z P_{uu}^{n-1} + \mathcal{M}_{uu}P_z^{n-1} + 2\mathcal{M}_u P_{zu}^{n-1}\right) \\
&+\mathcal{M}_{\sigma\sigma}P_z^{n-1} + P_{\sigma\sigma}^{n-1}\mathcal{M}_z + g_{\sigma\sigma}\mathcal{A}_2^{n-1}
\end{aligned}
$$

These formulas are derived using *Mathematica* codes and could easily be extended to higher approximation orders.

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