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Abstract

This paper develops a DSGE model which explains variation in the nominal and real term structure along with inflation surveys and four macro variables in the UK economy. The model is estimated based on a third-order approximation to allow for time-varying term premia. We find a fall in nominal term premia during the 1990s which mainly is due to lower inflation risk premia. A structural decomposition further shows that this fall is driven by negative preference shocks, lower fixed production costs, and positive investment shocks.

Key words: Market price of risk, non-linear filtering, quantity of risk, Epstein-Zin-Weil preferences, third-order perturbation.

JEL classification: C51, E10, E32, E43, E44.

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Summary

Conventional bond prices (ie gilts) with different maturities to expiry give rise to a set of interest rates which are referred to as the nominal term structure. Similarly, the interest rates from bond prices where the pay-off is linked to inflation (real bonds) imply a real term structure. In each case, these take account of both the expected future sequence of short rates and risk premia, neither of which is directly observable. But if they can be unpacked, they potentially contain information which is of great relevance for policymakers. For instance, the nominal term structure reveals expectations of future one-period nominal interest rates and the compensation for uncertainty in interest rates with maturities beyond one period. This compensation, for the extra uncertainty in holding a nominal bond for more than one period, is called the nominal term premium. In general, expected nominal one-period interest rates are affected by changes in expected real consumption or expected inflation. Similarly, nominal term premia are affected by changes in real consumption uncertainty or inflation uncertainty. Decomposing the information content from term structure data in this way is potentially very useful for monetary policy. For example, the implications for policy to, say, an increase in nominal interest rates along the yield curve may differ according to whether it is due to higher real interest rates, higher inflation expectations or higher inflation uncertainty.

The purpose of this paper, therefore, is to decompose the information content in the two term structures. This done with the aid of a dynamic stochastic general equilibrium (DSGE) model for the UK economy. This is a many-period model that uses economic theory to tell us how the dynamic behaviour of all the agents in the economy interact in the face of random ('stochastic') shocks. A key advantage of using a DSGE model in the current setting is that it provides a consistent framework for studying the effect of monetary policy and other structural shocks on the evolution of the nominal and real term structure. In our case, to account for asset pricing, it must allow for the presence of uncertainty in computing equilibrium prices that ensure supply equals demand in all markets, which is not necessary in models that ignore asset prices. That raises some technical problems, made more complicated by the need to allow effects to vary over time, that are addressed in an efficient way in the paper.

Our model is estimated on UK data from 1992 Q3 to 2008 Q2. We find a reduction in nominal

term premia following the adoption of inflation targeting in 1992 and operational independence of the Bank of England in 1997. This is of course only one model among the many possibilities, and, as for all models, the precise estimates are subject to uncertainty. But given this caveat, in our model this fall in nominal term premia is mainly due to lower inflation risk premia. A decomposition of the 10-year inflation risk premium suggests that this fall was driven by negative shocks to the utility that households get from consumption, lower fixed production costs, positive investment shocks, and a more aggressive attitude to inflation by the Bank of England. Adopting the terminology from the finance literature, our model implies a gradual reduction in the market price of inflation risk (the amount of compensation markets require for a given quantity of inflation risk) during the 1990s. The quantity of inflation uncertainty itself is found to fall after the adoption of inflation targeting in 1992 and operational independence to the Bank of England in 1997.

1 Introduction

The nominal term structure reveals expectations to future one-period nominal interest rates and compensation for uncertainty related to interest rates with longer maturities. This compensation is typically referred to as nominal term premia. The expected nominal one-period interest rates may further be decomposed into real interest rates and expected inflation. Similarly, nominal term premia may be split into real term premia and inflation risk premia. Decomposing the information content from term structure data in this way is often very useful for monetary policy.

A large number of papers have used reduced-form term structure models to jointly study nominal and real interest rates and their term premia.[1] However, little is currently known about the structural determinants behind the dynamics of these term premia. The contribution of the present paper is to close this gap in the literature by estimating a Dynamic Stochastic General Equilibrium (DSGE) model to carry out a structural decomposition of nominal term premia, real term premia and inflation risk premia. We address this question in a New Keynesian DSGE model solved to third-order to allow for time-varying term premia. The model is estimated by non-linear filtering methods to match the nominal term structure, the real term structure, inflation surveys and four macro variables in the UK economy.

To provide the structural interpretation of term premia, we need to overcome a number of challenges. First, it is in general difficult for DSGE models to reproduce the dynamics of the nominal term structure (see for instance Haan (1995) and Rudebusch and Swanson (2008)). Matching this aspect of the data is clearly a necessary first step for a reliable decomposition of the information content in the two term structures.

Second, the mechanisms driving the nominal and real term structure impose substantial requirements on the stochastic discount factor and the DSGE model in general. Broadly speaking, the model should generate levels of future real consumption that correspond to the average expectations of investors to match the real term structure. The model is also required to generate inflation expectations in line with the expectations held by the average investor to fit the nominal term structure. Furthermore, the model should reproduce observed time series for

[1] A non-exhaustive list includes the work by Barr and Campbell (1997), Evans (1998), Evans (2003), Ang, Bekaert and Wei (2008), Christensen, Lopez and Rudebusch (2008), D'Amico, Kim and Wei (2009), Hordahl and Tristani (2010), and Joyce, Lildholdt and Sorensen (2010).

consumption and inflation along with inflation expectations from surveys. Indeed, specifying a structural model with these properties is a major challenge.

Third, the solution to DSGE models must be approximated with nonlinear terms to generate time-varying term premia, and such approximations are quite time consuming to compute. For instance, Rudebusch and Swanson (2008) report that it takes about 10 minutes to solve a third-order approximation to their benchmark model. If the model is estimated, then hundreds of thousands function evaluations are necessary and a new model solution must be computed for every evaluation.

Finally, the existing literature relies on a normality assumption or a second-order approximation of the stochastic discount factor to decompose the information content in the nominal and real term structure. We cannot apply this decomposition as our model is approximated to third-order. Hence, an extension of the current method to decompose the information embodied in the nominal and real term structure is therefore required.

In an empirical application, the suggested model is estimated on UK data after 1992 when the current inflation-targeting regime was initiated. Our focus on the UK economy is motivated by the presence of a large and liquid market for real bonds. While recognising that this is only one model among many possibilities and that estimates are subject to uncertainty, we highlight the following results. First, the model delivers in general a satisfying fit to the two term structures while simultaneously matching inflation expectations and four macro variables. The exception is the 1-quarter nominal interest rate where larger model errors are encountered. In total, we match 17 time series using just 7 structural shocks. Second, and as in much of the existing finance literature, we find a reduction in nominal term premia immediately after inflation targeting was adopted in 1992 and again after the Bank of England became independent in 1997. In our model this fall in nominal term premia is mainly due to lower inflation risk premia. Third, a decomposition of the 10-year inflation risk premium shows that the fall is driven by negative preference shocks, lower fixed production costs, positive investment shocks, and a more aggressive attitude to inflation by the Bank of England. Fourth, adopting the typical terminology from the finance literature, our model implies a gradual reduction in the market price of inflation risk during the 1990s. The quantity of inflation uncertainty is seen to fall after the adoption of inflation targeting in 1992 and operational independence to the Bank of England in 1997. Finally, our estimated model implies a 10-year nominal term premium with a standard deviation of 83 basis points. This model property is obtained by relying on a very high risk-aversion for the household through Epstein-Zin-Weil preferences.

The rest of this paper is organised as follows. Section 2 presents our New Keynesian DSGE model which we extend with the nominal and real term structure in Section 3. It is also shown in Section 3 how the information content in these term structures can be used to extract nominal term premia, real term premia and inflation risk premia. Section 4 discusses how the solution to our model is approximated by a third-order perturbation approach. We estimate the model on UK data and conduct the structural decomposition of term premia in Section 5. Concluding comments are provided in Section 6.

2 The DSGE model

This section presents a DSGE model with the same basic structure as the model by Smets and Wouters (2007). We briefly describe the behaviour of the three types of agents in this economy: i) households, ii) firms, and iii) a central bank.[2]

2.1 The households

The behaviour of the households is described by a representative agent with Epstein-Zin-Weil preferences following the work of Epstein and Zin (1989) and Weil (1990). These preferences have recently been introduced into DSGE models by Rudebusch and Swanson (2010) and Binsbergen, Fernandez-Villaverde, Koijen and Rubio-Ramirez (2010). Applying the same specification as in Rudebusch and Swanson (2010), the value function V_t for the household is given by

$$V_t \equiv \begin{cases} u_t + \beta \left(E_t \left[V_{t+1}^{1-\xi_3} \right] \right)^{\frac{1}{1-\xi_3}} & \text{for } u_t \geq 0 \\ u_t - \beta \left(E_t \left[(-V_{t+1})^{1-\xi_3} \right] \right)^{\frac{1}{1-\xi_3}} & \text{for } u_t < 0 \end{cases} \quad (1)$$

Here, E_t denotes the conditional expectation given information available at time t and $\beta \in]0, 1[$ is the household's subjective discount factor. The parameter $\xi_3 \in \mathbb{R} \backslash \{1\}$ controls the household's degree of risk-aversion, where higher values of ξ_3 imply higher levels of risk-aversion for $u_t \geq 0$,

[2]As in Ravn (1997), Nelson and Nikolov (2004), DiCecio and Nelson (2007), among others, the UK economy is here modelled as a closed-economy.

and *vice versa* for $u_t < 0$. The benefit of Epstein-Zin-Weil preferences is that they allow us to disentangle the household's risk-aversion and intertemporal elasticity of substitution which are closely linked when using standard power preferences. By allowing for a high level of risk aversion, Rudebusch and Swanson (2010) show that these preferences help an otherwise standard DSGE model match the dynamics of the 10-year nominal term premium.

We assume that the value of the periodic utility index u_t is determined by consumption c_t and labour supply h_t in a standard fashion

$$u_t \equiv \frac{d_t}{1-\xi_2}\left[\left(\frac{c_t - bc_{t-1}}{z_t^*}\right)^{1-\xi_1}(1-h_t)^{\xi_1}\right]^{1-\xi_2}, \quad \textbf{(2)}$$

where $\xi_1 \in]0, 1[$ and $\xi_2 \in]0, 1[\cup]1, \infty[$. Non-seperability between consumption and labour is introduced for two reasons. First, this assumption makes the utility index non-negative for $\xi_2 \in]0, 1[$ and non-positive for $\xi_2 \in]1, \infty[$, and the specification therefore fits nicely with **(1)**. Second, non-seperability is consistent with the presence of a balanced growth-path in the economy as shown by King, Plosser and Rebelo (1988).

The parameter $b \in [0, 1]$ controls the degree of external habit formation in consumption. We introduce this feature because habits in general improve the ability of DSGE models to reproduce various macroeconomic and financial moments (see for instance Campbell and Cochrane (1999), Fuhrer (2000), Christiano, Eichenbaum and Evans (2005), and Hordahl, Tristani and Vestin (2008)). It is further assumed that the household's utility from consumption is measured in deviation from the consumption trend z_t^* which we specify in the following section.[3]

Following Smets and Wouters (2007), Justiniano and Primiceri (2008), and others, we include preference shocks d_t by letting

$$\ln(d_{t+1}) = \rho_d \ln(d_t) + \epsilon_{d,t+1}, \quad \textbf{(3)}$$

where $\rho_d \in]-1, 1[$. The error terms $\left\{\epsilon_{d,t}\right\}_{t=1}^{\infty}$ are assumed to be independent and normally distributed, ie $\epsilon_{d,t} \sim \mathcal{NID}\left(0, \sigma_d^2\right)$.

The consumption good is constructed from a continuum of differentiated goods ($c_{i,t}$, $i \in [0, 1]$)

[3]This assumption is convenient as it leaves the utility index and the value function untrended.

and the aggregation function

$$c_t = \left[\int_0^1 c_{i,t}^{\frac{\eta-1}{\eta}} di\right]^{\frac{\eta}{\eta-1}}. \quad \textbf{(4)}$$

Hence, the demand for good i is

$$c_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} c_t, \quad \textbf{(5)}$$

where $P_t \equiv \left[\int_0^1 P_{i,t}^{1-\eta} di\right]^{1/(1-\eta)}$ is the nominal price index. The continuously compounded inflation rate π_t is then $e^{\pi_t} = P_t/P_{t-1}$.

The household real period-by-period budget constraint is given by

$$E_t D_{t,t+1} x_{t+1}^h + c_t + i_t/\left(e_t \Upsilon_t\right) = x_t^h/e^{\pi_t} + w_t h_t + r_t^k k_t + \phi_t. \quad \textbf{(6)}$$

Expenditures are allocated to i) state-contingent claims $\left(E_t D_{t,t+1} x_{t+1}^h\right)$, ii) consumption (c_t), and iii) investment $(i_t/(e_t \Upsilon_t))$. The variable $D_{t,t+1}$ denotes the nominal stochastic discount factor. We follow Greenwood, Hercowitz and Krusell (1997) in specifying the investment expenditures and allow for a time-varying real price of investment in terms of the consumption good $1/(e_t \Upsilon_t)$. Change in this relative price are modelled exogenously, and $1/(e_t \Upsilon_t)$ can therefore be interpreted as investment shocks. These shocks are assumed to evolve according to a stationary AR(1) process around a deterministic trend. We include the deterministic trend to allow for the common empirical property that the mean investment growth is higher than the average growth rate in consumption and output. More formally, we let

$$\ln e_{t+1} = \rho_e \ln e_t + \epsilon_{e,t+1}, \quad \textbf{(7)}$$

where $\rho_e \in]-1, 1[$, $\epsilon_{e,t} \sim \mathcal{NID}\left(0, \sigma_e^2\right)$, and $\ln \Upsilon_t = \ln \Upsilon_{t-1} + \ln \mu_{\Upsilon,ss}$.

The right-hand side of **(6)** describes the household's total wealth in period t. It consists of i) pay-off from state-contingent claims purchased in period $t-1$ $\left(x_t^h/e^{\pi_t}\right)$, ii) real labour income $(w_t h_t)$, iii) return from selling capital services to firms $\left(r_t^k k_t\right)$, and iv) dividend payments received from firms $\left(\phi_t\right)$. The latter are restricted to zero in steady state.

The household owns the capital stock k_t in the economy and therefore also makes the investment decision. When doing so, it is constrained by the law of motion for capital

$$k_{t+1} = (1-\delta) k_t + i_t \left(1 - \frac{\kappa}{2}\left(\frac{i_i}{\Upsilon_t z_t^* i_{SS}} - 1\right)^2\right), \quad \textbf{(8)}$$

where $\delta \in [0, 1]$ is depreciation and $\kappa \geq 0$. Following Christiano *et al* (2005), we allow for investment adjustment costs but in this paper relate these costs to the balanced growth-path of investment, ie $\Upsilon_t z_t^* i_{SS}$, instead of i_{t-1}.

2.2 *The firms*

There is a continuum of firms, each supplying a differentiable good $y_{i,t}$ using

$$y_{i,t} = \begin{cases} k_{i,t}^{\theta} \left(a_t z_t h_{i,t}\right)^{1-\theta} - \psi_t z_t^* & \text{if } k_{i,t}^{\theta} \left(a_t z_t h_{i,t}\right)^{1-\theta} - \psi_t z_t^* > 0 \\ 0 & \text{else} \end{cases} \quad \textbf{(9)}$$

with $\theta \in [0, 1]$ and $\psi_t \geq 0$. The variables $k_{i,t}$ and $h_{i,t}$ denote the amount of capital and labour used by firm i, respectively. Technology shocks are allowed to have a stationary component a_t and a non-stationary component z_t. We include the traditional stationary technology shocks because Hordahl *et al* (2008) and Rudebusch and Swanson (2010) find that they are important in order to generate sizable nominal term premia. The non-stationary technology shocks are primarily introduced to explain the mean growth rate of consumption and output, and a large fraction of the cyclical variation in these time series. Formally, we let

$$\ln a_{t+1} = \rho_a \ln a_t + \epsilon_{a,t+1}, \quad \textbf{(10)}$$

where $\rho_a \in]-1, 1[$ and $\epsilon_{a,t} \sim \mathcal{NID}\left(0, \sigma_a^2\right)$. The non-stationary component is given by

$$\ln\left(\frac{\mu_{z,t+1}}{\mu_{z,ss}}\right) = \rho_z \ln\left(\frac{\mu_{z,t}}{\mu_{z,ss}}\right) + \epsilon_{z,t+1} \quad \textbf{(11)}$$

where $\mu_{z,t} \equiv z_t/z_{t-1}$, $\rho_z \in]-1, 1[$, and $\epsilon_{z,t} \sim \mathcal{NID}\left(0, \sigma_z^2\right)$. The innovations $\epsilon_{a,t}$ and $\epsilon_{z,t}$ are assumed to be mutually independent and so are all other innovations in the model.

Following Altig, Christiano, Eichenbaum and Linde (2005), the value of z_t^* is defined to be equal to $\Upsilon_t^{\theta/(1-\theta)} z_t$ and z_t^* can therefore be interpreted as an overall measure of technological progress in the economy. As in Altig *et al* (2005), we scale ψ_t in **(9)** by z_t^* to ensure the presence of a balanced growth-path in the model.

Smets and Wouters (2007) document the importance of real supply shocks specified as shocks to firms' mark-up rates. However, with Calvo price contracts, these mark-up shocks prevent an exact recursive representation of the equilibrium conditions which is needed for a non-linear approximation to our model. Instead, we follow Andreasen (2011) and introduce real supply shocks by letting firms' fixed production costs be time-varying beyond the variation in z_t^*. The inclusion of these real supply shocks can be motivated by variation in firms' fixed production costs due to changes in oil prices, maintenance costs, firms' subsidies, etc. We therefore let

$$\ln\left(\frac{\psi_{t+1}}{\psi_{ss}}\right) = \rho_\psi \ln\left(\frac{\psi_t}{\psi_{ss}}\right) + \epsilon_{\psi,t+1}, \tag{12}$$

where $\rho_\psi \in]-1, 1[$ and $\epsilon_{\psi,t+1} \sim \mathcal{NID}\left(0, \sigma_\psi^2\right)$.

Firms are assumed to maximise the present value of their nominal dividend payments given by

$$div_{i,t} \equiv E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \phi_{i,t+l}, \tag{13}$$

where $\phi_{i,t}$ is real dividend payments from firm i. The firms face a number of constraints when maximizing $div_{i,t}$. The first is related to the good produced by firm i, where firms must satisfy demand in all markets. In aggregate terms this implies

$$y_t = c_t + i_t / \left(e_t \Upsilon_t\right). \tag{14}$$

The second constraint is a real budget restriction which give rise to the expression for real dividend payments from firm i in period t

$$\phi_{i,t} = \left(P_{i,t}/P_t\right) y_{i,t} - r_t^k k_{i,t} - w_t h_{i,t}. \tag{15}$$

The first term in **(15)** denotes real revenue from sales of the i'th good. The firm's expenditures are allocated to purchases of capital services $\left(r_t^k k_{i,t}\right)$ and payments to workers $\left(w_t h_{i,t}\right)$.

The third constraint introduces sticky prices following Calvo (1983). Here, a fraction $\alpha \in [0, 1[$ of randomly chosen firms cannot set the optimal nominal price of the good they produce in each period. Instead, these firms set the current prices equal to prices in the previous period, ie $P_{i,t} = P_{i,t-1}$.

2.3 The central bank

We approximate the behavior of the central bank by a rule for the one-period continuously compounded interest rate $r_{t,1}$

$$r_{t,1} = (1-\rho_r) r_{ss,1} + \rho_r r_{t-1,1} + (1-\rho_r)\left(\beta_{\pi,t}\left(\pi_t - \pi_{ss}\right) + \beta_y \ln\left(\frac{y_t}{y_{ss} z_t^*}\right)\right) + \epsilon_{r,t}, \tag{16}$$

where $\epsilon_{r,t} \sim \mathcal{NID}\left(0, \sigma_{\epsilon_r}^2\right)$, $\rho_r \geq 0$, $\beta_{\pi,t} \geq 0$, and $\beta_y \geq 0$. That is, the central bank aims to close the inflation gap $\left(\pi_t - \pi_{ss}\right)$ and the output gap $\ln\left(y_t / \left(y_{ss} z_t^*\right)\right)$, while at the same time potentially smoothing changes in the policy rate. Note that we follow Justiniano and Primiceri (2008), Rudebusch and Swanson (2010), among others, and define the output gap by output in deviation from its balanced growth-path $y_{ss}z_t^*$. An interesting feature in **(16)** is that we allow the coefficient for the inflation gap $\beta_{\pi,t}$ to be time-varying, implying that the policy response to a given inflation gap may change over time. Such changes should affect in particular all nominal interest rates and may be important for understanding their dynamics. As in Fernandez-Villaverde, Guerron-Quintana and Rubio-Ramirez (2010), $\beta_{\pi,t}$ is modelled exogenously according to

$$\ln\left(\frac{\beta_{\pi,t+1}}{\beta_\pi}\right) = \rho_{\beta_\pi} \ln\left(\frac{\beta_{\pi,t+1}}{\beta_\pi}\right) + \epsilon_{\beta_\pi,t+1} \qquad \textbf{(17)}$$

where $\rho_{\beta_\pi} \in]-1, 1[$ and $\epsilon_{\beta_\pi,t+1} \sim \mathcal{NID}\left(0, \sigma^2_{\beta_\pi}\right)$.[4] Due to rational expectations in our model, the household and firms are aware of this potential change in monetary policy and take it into account when making their decisions.[5]

3 The nominal and real term structure

This section derives the nominal and real term structure from the micro-founded stochastic discount factor and no-arbitrage arguments. Using a third-order approximation to allow for time-varying term premia, we then show how the difference between the nominal and real term structure can be decomposed into i) expected future inflation, ii) a convexity term, and iii) inflation risk premia.

3.1 Deriving the nominal and real term structure

The presence of state contingent claims imply that we can price all financial assets in the economy based on standard no-arbitrage arguments. Hence, the price of a zero-coupon bond maturing *n* periods into the future and paying 1 unit of cash at maturity is

$$P_{t,n} = E_t\left[D^{real}_{t,t+n} \prod_{j=1}^n \frac{1}{e^{\pi_{t+j}}}\right], \qquad \textbf{(18)}$$

[4] It is easy to see that shocks to $\beta_{\pi,t}$ do not have first-order effects. Accordingly, the standard requirements for a stable unique equilibrium also hold with **(16)**.

[5] Another possibility would be to let $\beta_{\pi,t}$ be regime depended as in Davig and Leeper (2007), Farmer, Waggoner and Zha (2010), and Ferman (2010). However, obtaining the non-linear model solution in this case is very challenging and we therefore prefer the specification in **(17)**.

where $D^{real}_{t,t+n} \equiv \beta\lambda_{t+n}/\lambda_t$ is the real stochastic discount factor and λ_t denotes the household's marginal value of income. In our case, it holds that

$$D^{real}_{t,t+1} = \beta\frac{d_{t+1}}{d_t}\frac{u_c\left(\frac{c_{t+1}-bc_t}{z^*_{t+1}}, 1-h_{t+1}\right)}{u_c\left(\frac{c_t-bc_{t-1}}{z^*_t}, 1-h_t\right)}\left[\frac{\left(E_t\left[(V_{t+1})^{1-\xi_3}\right]\right)^{\frac{1}{1-\xi_3}}}{V_{t+1}}\right]^{\xi_3}\frac{1}{\mu_{z^*,t+1}} \quad \textbf{(19)}$$

when $u_t \geq 0$.[6] The first term $\beta\frac{d_{t+1}}{d_t}$ is the household's subjective discount factor adjusted for preference shocks. The second term is the familiar ratio of future marginal utility of consumption to the present value of marginal utility, where the utility might depend on consumption habits and leisure. The third term (in the squared brackets) is due to the presence of Epstein-Zin-Weil preferences, ie $\xi_3 \neq 0$, and amplifies the effect of unexpected changes in the household's wealth as measured by the value function. Accordingly, expectations to all future levels of habit adjusted consumption and leisure affect bond prices when $\xi_3 \neq 0$. The last term $1/\mu_{z^*,t+1}$ is due to the presence of trends in the economy where $\mu_{z^*,t} \equiv z^*_t/z^*_{t-1}$.

With continuously compounded, it holds that $e^{-nr_{t,n}} = P_{t,n}$ where $r_{t,n}$ is the nominal interest rate in period t for a bond maturing in n periods. The nominal term structure is then derived by calculating these interest rates for different values of n.

Similarly, the price of a real zero-coupon bond maturing n periods into the future and paying 1 unit of consumption at maturity is

$$P^{real}_{t,n} = E_t\left[D^{real}_{t,t+n}\right]. \quad \textbf{(20)}$$

All real interest rates at different maturities $r^{real}_{t,n}$ are then derived from $e^{-nr^{real}_{t,n}} = P^{real}_{t,n}$.

3.2 Decomposing the difference between nominal and real interest rates

We begin by defining the term premium for a nominal bond with n periods to maturity as

$$TP_{t,n} = r_{t,n} - \frac{1}{n}E_t\left[\sum_{j=1}^{n} r_{t+j-1,1}\right] + C_t, \quad \textbf{(21)}$$

where C_t is the convexity term. A similarly definition holds for the real term premium $TP^{real}_{t,n}$ and its related convexity term C^{real}_t. For the subsequent decomposition, let $\Delta^n\hat{\lambda}_{t+n} \equiv \hat{\lambda}_{t+n} - \hat{\lambda}_t$ denote the n'th difference for the household's marginal value of income where $\hat{\lambda}_t \equiv \ln(\lambda_t/\lambda_{ss})$. We also introduce $\hat{\Pi}_{t+n} \equiv \sum_{i=1}^{n}(\pi_{t+i} - \pi_{ss})$ as accumulated inflation from period t to period

[6] for $u_t < 0$, then V_{t+1} is replaced by $-V_{t+1}$ in **(19)**.

$t+n$ when expressed in deviation from π_{ss}. Based on this parsimonious notation, Appendix A derives expressions for nominal and real interest rates accurate up to third order and thus extend the results in Hordahl *et al* (2008) from a second-order to a third-order approximation. The difference between nominal and real interest rates define break-even inflation rates which are given by

$$r_{t,n} - r_{t,n}^{real} = \frac{1}{n}\left(E_t\left[\hat{\Pi}_{t+n}\right] - C_{t,n}^{infl} + TP_{t,n}^{infl}\right). \quad \textbf{(22)}$$

The first component $E_t\left[\hat{\Pi}_{t+n}\right]$ is expected inflation until expiry of the zero-coupon bond. The second component $C_{t,n}^{infl}$ is an inflation convexity term which is given by

$$\begin{aligned} C_{t,n}^{infl} \equiv\ & \frac{1}{2}\textstyle\sum_{i=1}^{n} Var_t\left(\pi_{t+i}\right) \quad \textbf{(23)} \\ & +\frac{1}{6}\left(\left(E_t\left[\hat{\Pi}_{t+n}\right]\right)^3 - E_t\left[\left(\hat{\Pi}_{t+n}\right)^3\right] + 3E_t\left[\hat{\Pi}_{t+n}\right]E_t\left[\left(\hat{\Pi}_{t+n}\right)^2\right]\right). \end{aligned}$$

Note that $C_{t,n}^{infl}$ in a second-order approximation reduces to the familiar expression $\frac{1}{2}\sum_{i=1}^{n} Var_t\left(\pi_{t+i}\right)$. The third component $TP_{t,n}^{infl}$ in **(22)** is inflation risk premia

$$\begin{aligned} TP_{t,n}^{infl} \equiv\ & -\textstyle\sum_{i=0}^{n-1}\sum_{k=1+i}^{n} Cov_t\left(\pi_{t+1+i}, \pi_{t+k}\right) + Cov_t\left(\Delta^n\hat{\lambda}_{t+n}, \hat{\Pi}_{t+n}\right) \quad \textbf{(24)} \\ & +\frac{1}{2}Cov_t\left(\left(\Delta^n\hat{\lambda}_{t+n}\right)^2, \hat{\Pi}_{t+n}\right) - \frac{1}{2}Cov_t\left(\left(\hat{\Pi}_{t+n}\right)^2, \Delta^n\hat{\lambda}_{t+n}\right) \\ & +\frac{1}{2}\left(Cov_t\left(\Delta^n\hat{\lambda}_{t+n}, \hat{\Pi}_{t+n}\right) - 3E_t\left[\Delta^n\hat{\lambda}_{t+n}\hat{\Pi}_{t+n}\right]\right)\left(E_t\left[\Delta^n\hat{\lambda}_{t+n}\right] - E_t\left[\hat{\Pi}_{t+n}\right]\right) \end{aligned}$$

which simplifies to the first two terms in **(24)** at second order. An important implication of **(22)** is that the difference between nominal and real interest rates also in the case of a third-order approximation can be decomposed into three components which conceptually resemble the content of any term structure. As a result, the difference between the nominal and real term structure defines an inflation term structure.

A general feature of all Gaussian affine term structure models and DSGE models approximated up to second order is that inflation risk premia equal the difference between nominal and real term premia. A proof is provided in Appendix B. This also holds in our model if

$$C_{t,n}^{real} + C_{t,n}^{infl} = C_{t,n}, \quad \textbf{(25)}$$

because the expectation of one-period nominal interest rates, real interest rates, and inflation always adds up with continuous compounding. For comparability with the existing literature, we impose **(25)** such that our model also implies

$$TP_{t,n}^{infl} = TP_{t,n} - TP_{t,n}^{real}. \quad \textbf{(26)}$$

In practice, nominal term premia are computed using the approach in Rudebusch and Swanson (2010), ie $TP_{t,n}$ is the difference between $r_{t,n}$ and the yield-to-maturity on the corresponding risk neutral bond where payments are discounted by $r_{t,1}$ instead of the stochastic discount factor. A similar procedure is used to compute $TP_{t,n}^{real}$, and equation **(26)** then gives $TP_{t,n}^{infl}$.

4 A non-linear approximated model solution

It is well-established that a third-order approximation around the deterministic steady state allows for variation in risk premia as implied by the exact, but infeasible, solution to our model. However, the presence of non-stationary variables and the size of the model complicate computing a third-order approximation by the perturbation method. We deal with the issues of stationarity and the size of the model in turn.

4.1 Inducing stationarity

The presence of trends in investment shocks Υ_t and productivity shocks z_t imply that some variables are non-stationary - for instance consumption c_t, output y_t, and investment i_t. This fact must be taken into account when using the perturbation method because it only gives reliable results when the economy is close to the approximation point, and the existence of non-stationary variables clearly violate this requirement. We deal with this feature by adopting the standard procedure where all non-stationary variables are scaled by their cointegrating factor to make them stationary. For instance, c_t and y_t are scaled by $1/z_t^*$ and i_t is scaled by $1/(\Upsilon_t z_t^*)$, which implies that $C_t \equiv c_t/z_t^*$, $Y_t \equiv y_t/z_t^*$, and $I_t \equiv i_t/(\Upsilon_t z_t^*)$ are stationary. Using this equivalent representation of the model, the standard perturbation method can be applied.

4.2 A third-order perturbation approximation

It is fairly standard in the considered model to derive i) market clearings conditions, ii) first-order conditions for the representative household, iii) first-order conditions for the firms, and iv) recursive equations for the nominal and real term structure. The exact solution to the system is given by

$$\mathbf{y}_t = \mathbf{g}(\mathbf{x}_t, \sigma) \tag{27}$$

$$\mathbf{x}_{t+1} = \mathbf{h}(\mathbf{x}_t, \sigma) + \sigma\eta\epsilon_{t+1} \tag{28}$$

where σ is the perturbation parameter. The vector $\mathbf{y}_t$ with dimension $n_y \times 1$ contains all control variables and the state vector $\mathbf{x}_t$ with dimension $n_x \times 1$ contains all the pre-determined variables. Both $\mathbf{y}_t$ and $\mathbf{x}_t$ are expressed in deviation from the steady state. The functions $\mathbf{g}(\mathbf{x}_t, \sigma)$ and $\mathbf{h}(\mathbf{x}_t, \sigma)$ are unknown but can be approximated by polynomials in $(\mathbf{x}_t, \sigma)$ around the deterministic steady state as shown by Judd and Guu (1997). We use the codes by Schmitt-Grohé and Uribe (2004) to compute the first-order and second-order derivatives of $\mathbf{g}(\mathbf{x}_t, \sigma)$ and $\mathbf{h}(\mathbf{x}_t, \sigma)$. The third-order derivatives are computed using the codes accompanying Andreasen (2010a). Based on the work by Kim, Kim, Schaumburg and Sims (2008) and Andreasen, Fernandez-Villaverde and Rubio-Ramirez (2011), we apply the pruning method when setting up the state space system for the approximated model solution.

Although the perturbation method is computationally fast compared to many other approximation methods, the size of the model makes it numerically challenging to find non-linear terms for $\mathbf{g}(\mathbf{x}_t, \sigma)$ and $\mathbf{h}(\mathbf{x}_t, \sigma)$.[7] However, this numerical problem has recently been made easier to solve by Andreasen and Zabczyk (2010) who develop an efficient method for computing bond prices in DSGE models. They propose a two-step perturbation method where the output from the first perturbation step is used as input in a second perturbation step. In the first step, the DSGE model without bond prices beyond one period is solved up to any desired order by the standard perturbation method. The second step then perturbates the fundamental pricing equation for bond prices up to the same order. Andreasen and Zabczyk (2010) then show that derivatives of bond prices can be solved for in a recursive manner, given the output from the first perturbation step. Only simple summations are needed to compute these bond prices which therefore can be computed almost instantaneously. As emphasised by Andreasen and Zabczyk (2010), this 'perturbation on perturbation' (POP) method gives *exactly* the same expression for bond prices as standard perturbation where all bond prices are solved simultaneously with other variables in the model.

When taking the model to the data, we also use long-term inflation expectations. However, including the 10-year inflation expectations in a quarterly model adds 40 additional control

[7] Our model has 11 state variables and 10 control variables without the two term structures. In a quarterly model and a maximum maturity of 10-years, the two term structures add $2 \times 40 = 80$ control variables to the system. Hence, we have $n_y = 90$. To compute the second-order terms $\mathbf{g}_{\mathbf{xx}}$ and $\mathbf{h}_{\mathbf{xx}}$, we need to solve a linear system with a dimension of $(n_y + n_x)\, n_x (n_x + 1)/2 = 6,666$. Solving this system typically requires a lot of computer memory and is very time consuming even though this system is sparse. The numerical problem to find the third-order terms $\mathbf{g}_{\mathbf{xxx}}$ and $\mathbf{h}_{\mathbf{xxx}}$ is even more challenging because it requires solving a linear system with a dimension of $(n_y + n_x)\, n_x (n_x + (n_x - 1)(n_x - 2)/6) = 28,888$.

variables to the system. We want to avoid such a large extension of the system for numerical reasons, and we therefore show in Appendix C that expected values of any control variable in DSGE models solved up to third order can also be computed by the POP method in a fast and straightforward manner.[8]

As a result, the POP method enables us to solve the model to third order in just 5 seconds on a standard desktop computer and therefore makes estimation feasible.

5 An application to the UK economy

This section estimates our DSGE model on UK data. We begin by describing the data and our estimation methodology in Section 5.1 and 5.2, respectively. The estimation results are reported in Section 5.3, and the model's ability to fit the data is examined in Section 5.4. Impulse response functions and a variance decomposition are presented in Section 5.5, before we conduct a structural decomposition of term premia in Section 5.6.

5.1 UK data

The model is estimated using nominal interest rates, real interest rates, inflation surveys, and four macro series. We next describe each of these data sources in turn. The nominal term structure is represented by the 1-quarter, 1-year, 3-year, 5-year, 7-year, and 10-year constant maturity interest rates on government bonds. All these interest rates are measured at the end of the quarter and expressed in annual terms. The data is available from the Bank of England's homepage, except the 1-quarter rate where we use the implied rate from a 3-month Treasury bill.[9]

The real term structure is represented by the 3-year, 5-year, 7-year, and 10-year constant maturity rates on index-linked government bonds as provided by the Bank of England.[10] The interest rates are from end of quarter and expressed in annual terms. We also note that the series for the 3-year real interest rate is incomplete with missing values from 1995 Q4 to 1996 Q4 and from 2005 Q1

[8] These formulas may also be of useful when computing impulse response functions.

[9] The study by Lildholdt, Panigirtzoglou and Peacock (2007) adopt a similar approach and verifies that this 3-month interest rate is in line with the data from the Bank of England.

[10] The 1-quarter and 1-year real interest rates are not available in this data set.

to 2005 Q2. The next section discusses how to account for these missing values in our estimation procedure.

Data on inflation surveys is available from Consensus Forecasts which provides inflation expectations for RPI at different horizons.[11] We focus on expected inflation 1 and 3 years ahead, along with long term expected inflation from 5 to 10 years into the future (ie the 5-year 5-year forward inflation expectation).[12]

The last group of variables consists of four macro variables. The first series is the inflation rate in the retail price index (RPI) which is used instead of the more familiar consumer price index (CPI) because our real interest rates are derived from bonds index-linked to the RPI. The remaining macro variables are the real growth rates in consumption, investment, and GDP.[13]

5.2 Our estimation methodology

Let the vector $\mathbf{y}_t^{obs}$ contain the 17 data series which are used for the estimation. We allow for measurement errors in $\mathbf{y}_t^{obs}$ and assume that these errors $\mathbf{w}_t$ are of the form $\mathbf{w}_t \sim \mathcal{NID}\left(\mathbf{0}, \mathbf{R_w}\right)$ where $\mathbf{R_w}$ is a diagonal matrix. To economize on the number of free parameters in $\mathbf{R_w}$, it is further assumed that 20% of the variation in the series for inflation and the three real growth rates are due to measurement errors.[14] Expressed in annual terms, this implies measurement errors with a standard deviation of: i) 17 basis points for inflation, ii) 24 basis points for consumption growth, iii) 76 basis points for investment growth, and iv) 24 basis points for output growth. This assumption can be considered as restricting the model, in a probabilistic sense, to match the four macro series in our sample period.

The size of the measurement errors in the two term structures and the inflation surveys are left as free parameters in order to assess the model's ability to fit these variables. We adopt the

[11] Only inflation expectations on RPIX (that is RPI excluding mortgage interest payments) are available from 1997 and onwards. However, the difference between RPI and RPIX is in general small.

[12] The setup of the Consensus survey implies that inflation expectations with a horisont of 11 and 13 quarters are used to approximate the inflation expectations at a horisont of 3 years.

[13] The growth rate in consumption is calculated from real final consumption expenditures. We use the series for real gross fixed capital formation to calculate the growth rate in investment. The growth rate in GDP is calculated from real GDP. These variables are seasonal adjusted and downloaded from Datastream. All series are computed as annual growth rates and expressed in per capita based on the total population in the UK.

[14] An and Schorfheide (2007) use a similar assumption.

following parsimonious specification for the variance of the measurement errors along the nominal term structure (Var_n)

$$\ln \sqrt{Var_n} = \gamma_1 + \gamma_2 n + \gamma_3 n^2, \qquad \textbf{(29)}$$

where n denotes the maturity of the interest rates. For the variance of the measurement errors in the real term structure (Var_n^{real}), we let

$$\ln \sqrt{Var_n^{real}} = \ln \gamma_4 + \ln \sqrt{Var_n}. \qquad \textbf{(30)}$$

Finally, the standard deviations of measurement errors for inflation expectations at 1 year $\left(\sigma_{\pi_4^e}\right)$, 3 years $\left(\sigma_{\pi_{12}^e}\right)$, and 5 to 10 years $\left(\sigma_{\pi_{long}^e}\right)$ are estimated as free parameters.

The set of structural parameters in our model is partitioned into two groups. The first group contains coefficients which are hard to identify in aggregated macro time series and are therefore determined based on standard calibration arguments. The second group consists of all the remaining parameters which are estimated. We emphasise that this partitioning of the structural parameters is standard practice when taking large DSGE models to the data (see Christiano *et al* (2005), Smets and Wouters (2007), Justiniano and Primiceri (2008), among others).

We now describe how parameters in the first group are determined. For firms' production function, we set θ to a standard value of 0.36 (see for instance Ravn (1997)). For a given value of θ, the average real growth rate in consumption (0.0219) and average investment growth (0.0278) can be used to find the deterministic trends in technology shocks $\mu_{z,ss}$ and investment shocks $\mu_{\Upsilon,ss}$. This follows from the fact that the mean of consumption and investment growth are given by

$$E\left[4 \ln \mu_{c,t}\right] = 4 \ln \left(\mu_{\Upsilon,ss}^{\frac{\theta}{1-\theta}} \mu_{z,ss}\right) \qquad \textbf{(31)}$$

$$E\left[4 \ln \mu_{i,ss}\right] = 4 \ln \left(\mu_{\Upsilon,ss}^{\frac{1}{1-\theta}} \mu_{z,ss}\right). \qquad \textbf{(32)}$$

This calibration implies $\mu_{z,ss} = 1.0046$ and $\mu_{\Upsilon,ss} = 1.0015$. The depreciation rate is set to $\delta = 0.025$, and the parameter η controlling firms' steady state mark-up is calibrated to 4.33 based on an assumption of a 30% mark-up. Finally, the steady state inflation rate π_{ss} is calibrated to match the average inflation rate in the sample using the non-linear calibration technique outlined in Andreasen (2011).

All the remaining parameters are estimated by quasi maximum likelihood (QML) based on the

Central Difference Kalman Filter (CDKF) developed by Norgaard, Poulsen and Ravn (2000). This filter extends the standard Kalman filter to non-linear and non-normal state-space systems where the non-linear moments in the filtering equations are approximated at least up to second-order accuracy. Andreasen (2010c) shows that this QML estimator can be expected to be consistent and asymptotically normal for DSGE models approximated up to third-order. The main advantage of this QML estimator is that it is very fast to compute even for models solved by third-order approximations. This is convenient in our case with 17 observables and a fairly large state vector.[15]

The presence of missing values for some inflation surveys and the 3-year real interest rate complicate the evaluation of the quasi log-likelihood function, and the existing algorithm for the CDKF no longer applies. It is, however, straightforward to show that the standard method to deal with missing observations in the Kalman filter (see for instance Durbin and Koopman (2001)) also applies to the CDKF. That is, we only need to adjust the dimension of the Kalman gain in the CDKF and the one-step ahead prediction density for the observables to match the available data points in each period. All other steps in the CDKF are unaffected by the presence of missing values and are as given in Norgaard *et al* (2000).

5.3 The estimated structural parameters

The model is estimated on data from 1992 Q3 to 2008 Q2. The starting date of 1992 Q3 is chosen for two reasons. First, the United Kingdom introduced monetary policy with inflation targeting in this quarter, and this is the key assumption underlying our interest rate rule in **(16)**. Second, Bianchi, Mumtaz and Surico (2009) find evidence of a regime change for the interaction between the macroeconomy and the nominal term structure in 1992 Q3.

The estimated structural parameters and their standard errors are reported in Table A.[16] The household is seen to place a reasonable weight on leisure in the utility index ($\xi_1 = 0.63$) and displays a moderate degree of habit formation ($b = 0.29$). With $\xi_2 = 2.39$, these estimates imply an intertemporal elasticity of substitution of 0.47 in steady state. The parameter related to the

[15] An alternative to the CDKF is to use particle filtering and approximate the likelihood function as in Fernández-Villaverde and Rubio-Ramírez (2007). Unfortunately, particle filtering is computationally infeasible for our model, given its dimension and relative high approximation order.

[16] The optimization of the quasi log-likelihood function is done with a modified version of the CMA-ES routine which Andreasen (2010b) shows can optimise likelihood functions for DSGE models.

Epstein-Zin-Weil preferences ξ_3 is estimated to be -183, and this gives strong preferences for early resolution of uncertainty. A measure for the level of relative risk aversion is $(\xi_2 + \xi_3 (1 - \xi_2))/(1 - b)$ according to Swanson (2010). We therefore find a very high relative risk aversion of 336, and our results are in this sense similar to those of Rudebusch and Swanson (2010).

In relation to the firms, we find sizeable adjustment costs in investment ($\kappa = 5.40$) and fairly sticky prices as $\alpha = 0.79$. The latter implies that the average firm approximately change its prices once every year. The central bank focuses mostly on stabilizing inflation $\left(\beta_\pi = 1.49\right)$ compared to output $\left(\beta_y = 0.05\right)$. In doing so it assigns a large weight to smoothing changes in the policy rate as $\rho_r = 0.88$. Both findings are standard for the UK economy (DiCecio and Nelson (2007), Harrison and Oomen (2010)).

Given the estimated values, our non-linear calibration of the steady state inflation implies $\pi_{ss} = 1.0157$. This is the value of π_{ss} which ensures that the model reproduces the mean level of RPI inflation (1.0069 expressed in quarterly terms) when accounting for household's precautionary saving motive.

5.4 Model fit

Chart 1 shows historical time series (the black lines) and model-implied time series (the red lines) for all 17 variables. Starting with the nominal term structure, some differences are observed between data and model-implied series for the 1-quarter and 1-year rates. A better fit is obtained for all other nominal interest rates where the model closely matches the historical time series. Inflation expectations are shown in the third row of Chart 1, and we see that the model is successful in matching the gradual fall in these expectations during the 1990's. This is particularly evident for the long-term inflation expectations as measured by the 5-year 5-year forward inflation expectations.

The model is also largely able to fit the real term structure as shown by the second part of Chart 1. Notable errors are only visible at the 3-year maturity around 2000 where the real rate is predicted to be slightly lower than what is observed in the data. The reduced form affine model by Joyce *et al* (2010) experiences similar problems, and the authors attribute it to i) the opening of a real bond market in the US and ii) the introduction of the Minimum Funding Requirements in the UK which increased the demand for real bonds among UK pension funds. We also note that the two periods of missing observations for the 3-year real interest rate are well accounted for by our model as the reappearance of this rate does not induce abnormal model errors.

The remaining figures show that the model at the same time is able to reproduce the dynamics of RPI inflation and the three real growth rates.

In summary, our model delivers in general a satisfying fit to the data and should therefore serve as a useful framework for a structural decomposition and interpretation of term premia. In this context it should also be noted that we fit 17 time series with just 7 structural shocks, whereas the norm in much empirical macro is to use at least the same number of shocks as the number of observables (see for instance Smets and Wouters (2007) and Justiniano and Primiceri (2008)).

Table A: Estimation results

The standard errors are computed from the variance of the score function which is pre- and post multiplied by the inverse of the Hessian matrix. Given these estimates, the non-linear calibration implies $\pi_{ss} = 1.0157$.

Label		Estimates	SE
Discount factor	β	0.9992	0.0002
Habit formation	b	0.2876	0.0398
Preference	ξ_1	0.6289	0.0475
Preference	ξ_2	2.3931	0.2567
Preference	ξ_3	-183.1	24.77
Adj costs for investment	κ	5.4033	0.3137
Price stickiness	α	0.7925	0.0083
Interest rate rule	ρ_r	0.8838	0.0115
Interest rate rule	β_π	1.4895	0.0820
Interest rate rule	β_y	0.0505	0.0104
Non-stationary technology shocks	ρ_z	0.5289	0.0814
Preference shocks	ρ_d	0.9635	0.0023
Firms' fixed costs	ρ_ψ	0.9969	0.0016
Stationary technology. shocks	ρ_a	0.8450	0.0149
Stationary investment shock	ρ_e	0.9870	0.0013
Central bank's reaction to inflation	ρ_{β_π}	0.9711	0.0046
Std. for non-stationary technology shocks	σ_z	0.0053	0.0006
Std. for preference shocks	σ_d	0.0227	0.0039
Std. for shocks to firms' fixed costs	σ_ψ	0.0198	0.0038
Std. for stationary technology. shocks	σ_a	0.0097	0.0017
Std. for investment shocks	σ_e	0.0203	0.0017
Std. for Central bank's reaction to inflation	σ_{β_π}	0.0280	0.0053
Std. for shocks to the interest rate rule	σ_{ϵ_r}	0.0019	0.0003
Std. for measurement errors in 4-quarters inflation expectations	$\sigma_{\pi^e_4}$	0.0041	0.0005
Std. for measurement errors in 12-quarters inflation expectations	$\sigma_{\pi^e_{12}}$	0.0033	0.0005
Std. for measurement errors in long inflation expectations	$\sigma_{\pi^e_{long}}$	0.0010	0.0002
Parameter for measurement error in nominal yield curve	γ_1	-4.7867	0.1280
Parameter for measurement error in nominal yield curve	γ_2	-0.1835	0.0087
Parameter for measurement error in nominal yield curve	γ_3	0.0033	0.0002
Parameter for measurement error in real yield curve	γ_4	1.4051	0.0674

Chart 1: Historical model fit

The historical time series are denoted by black lines and the model-implied series evaluated at the estimated states are denoted by red lines. The numbers in parentheses are the correlation between the historical series and the model-implied series.

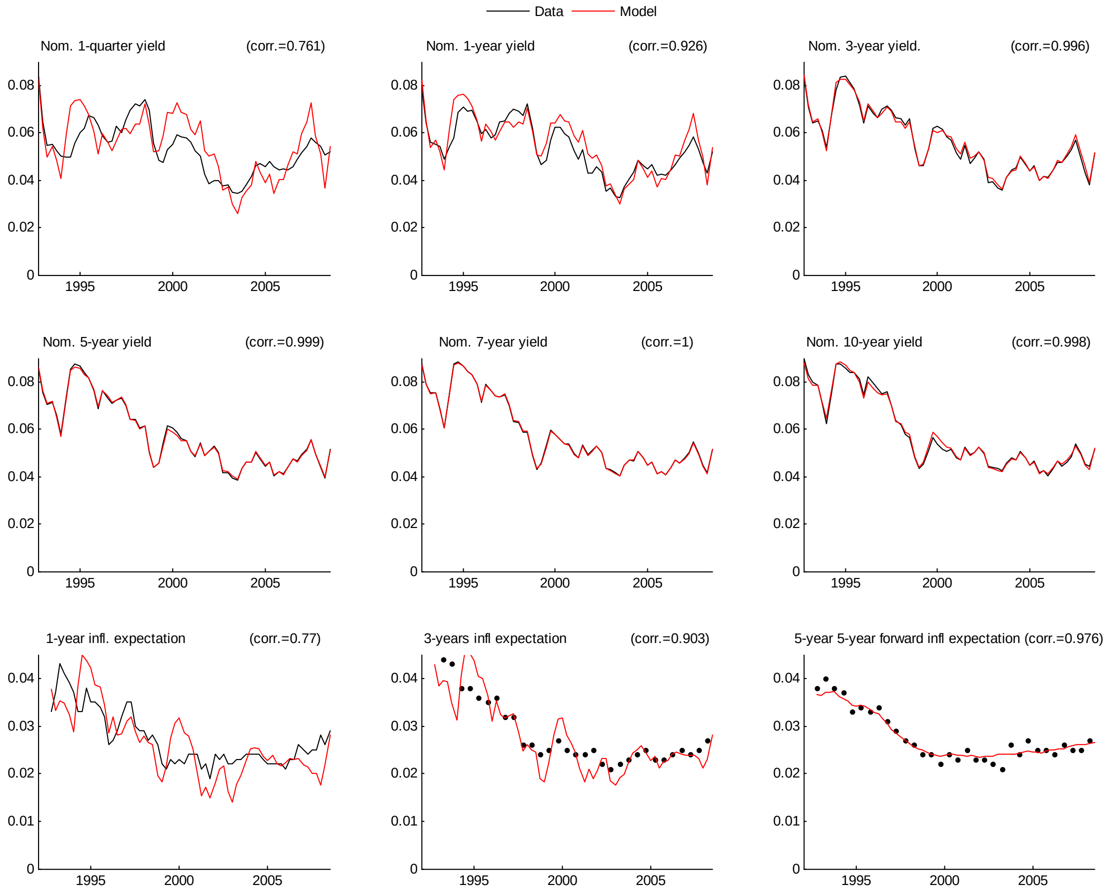
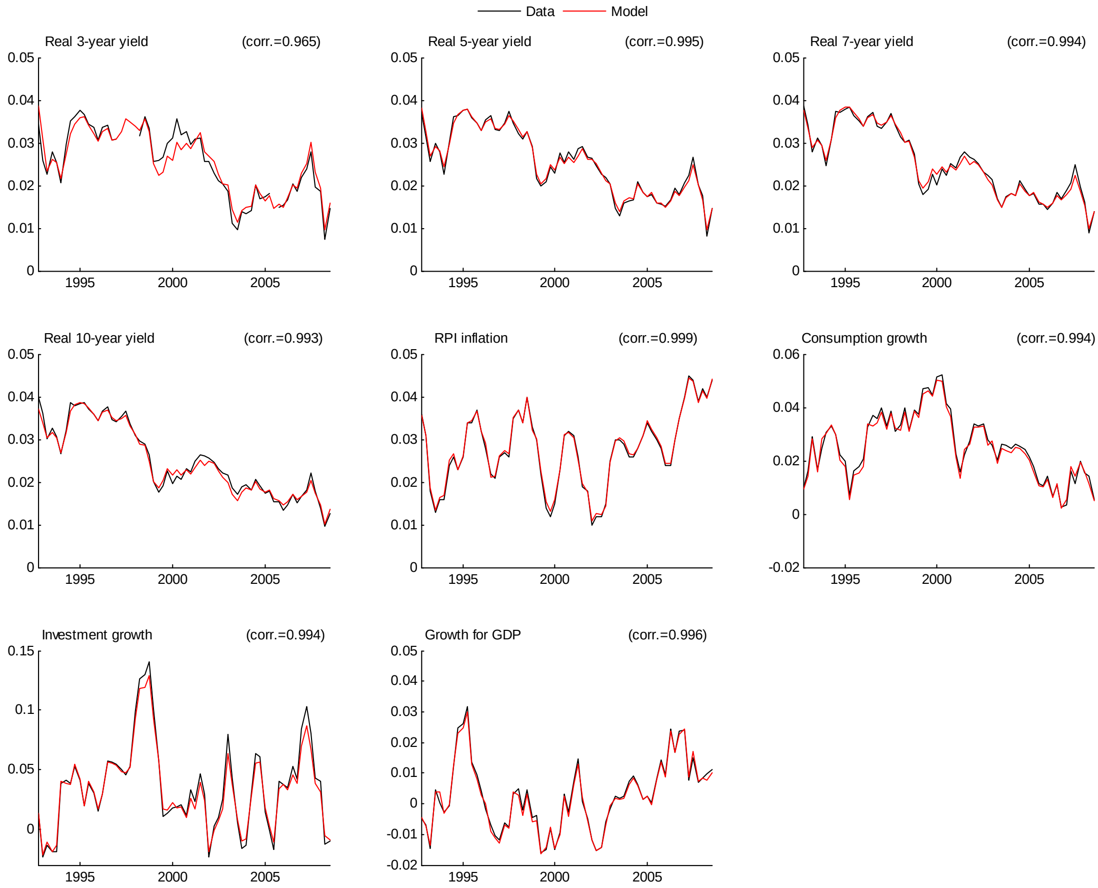


Chart 1: Continued



5.5 Analyzing the model

Before we turn to the decomposition and interpretation of the various term premia, it is instructive to see how the shocks affect the model and which shocks are important for matching the data. We deal with each of these issues in turn.

5.5.1 *Impulse response functions*

The impulse response functions for the 7 shocks are shown in Chart 2 where each row shows responses to the same shock for a selected number of variables.[17] Given the topic of the paper, our focus in this subsection is devoted to explaining the economics behind impulse response functions for term premia.

We start by considering the effects of a positive shock to firms' fixed costs (ψ_t) in the first row of Chart 2. The increased production costs lower firms' dividend payments to the household which effectively see consumption fall due to a negative wealth effect. The household tries to off-set some of the reduction in consumption by increasing its labor supply (not shown). Through the sticky prices this increased activity in firms raises inflation. The response of the central bank is to increase the short nominal interest rate, and this leads to a higher 10-year nominal interest rate as shown in top left figure of Chart 2. The rise in inflation is higher than the rise in nominal interest rates and this generates lower real interest rates as shown in the top figure, second to the left. Hence, the value of the 10-year real bond increases as consumption falls, and this bond can therefore be used as a hedge by the household to generate a more smooth consumption profile. It is thus desirable for the household to buy the 10-year real bond, and the 10-year real term premia is therefore positively correlated with consumption. In contrast, the value of the 10-year nominal bond falls as consumption decreases, and this asset therefore makes it more difficult for the household to generate a smooth consumption profile. Accordingly, the household requires a premium for holding the 10-year nominal bond, and we therefore see an increase in the 10-year nominal term premium as $TP_{t,40} = TP_{t,40}^{infl} + TP_{t,40}^{real}$.

The same logic can be used to explain the variation in term premia following other shocks. In the interest of space, we simply emphasize two features in relation to the remaining impulse response functions. Firstly, the different responses following a non-stationary technology shock (z_t) compared to a stationary technology shock (a_t) relate mainly to the positive wealth effects following an increase in z_t as noted by Rudebusch and Swanson (2010). That is, a rise in z_t generates a desire for the household to work less and enjoy more leisure. To maintain the required production level, firms therefore have to increase the wage level to partly off-set this effect, and this raises production costs which in return leads into higher inflation and higher

[17] The reported impulse response functions are computed by simulation to account for non-linearities in the model solution.

nominal interest rates. This wealth effect is less pronounced after a stationary technology shock which therefore lowers inflation and all nominal interest rates.

Secondly, Chart 2 also displays responses to a temporary change in monetary policy to a more aggressive response to inflation, ie a rise in $\beta_{\pi,t}$. This change in policy lowers the variation in inflation and other macro variables which in turn reduces the risk faced by the household. The response of the risk averse household is therefore to lower its amount of precautionary saving which causes a small boom and higher inflation in the economy.[18] The central bank reacts by increasing the short interest rate, while we see a small fall in the 10-year nominal interest rate. Hence, the household experiences an increase in consumption and a rise in the price of the 10-year nominal bond, and this asset therefore makes it more difficult for the household to maintain a smooth consumption profile. As a result, the household requires compensation for holding this bond, and the 10-year nominal term premium is therefore negatively correlated with consumption. This explains why our model generates a reduction in nominal term premia following a more aggressive reaction to inflation by the central bank.

[18]This effect is not present in a second-order approximation to the model where the degree of precausionary saving is constant.

Chart 2: Impulse response functions

Each row shows responses to a positive one standard deviation shock for a number of variables. All responses are in deviation from the steady state, except for term premia which are reported in annualized basis points. The order of the structural shocks is: shock to firms' fixed costs (ψ_t), non-stationary technology shock (z_t), stationary technology shock (a_t), preference shock (d_t), investment shock (e_t), shock to the central bank's inflation reaction ($\beta_{\pi,t}$), monetary policy shock ($\epsilon_{r,t}$).

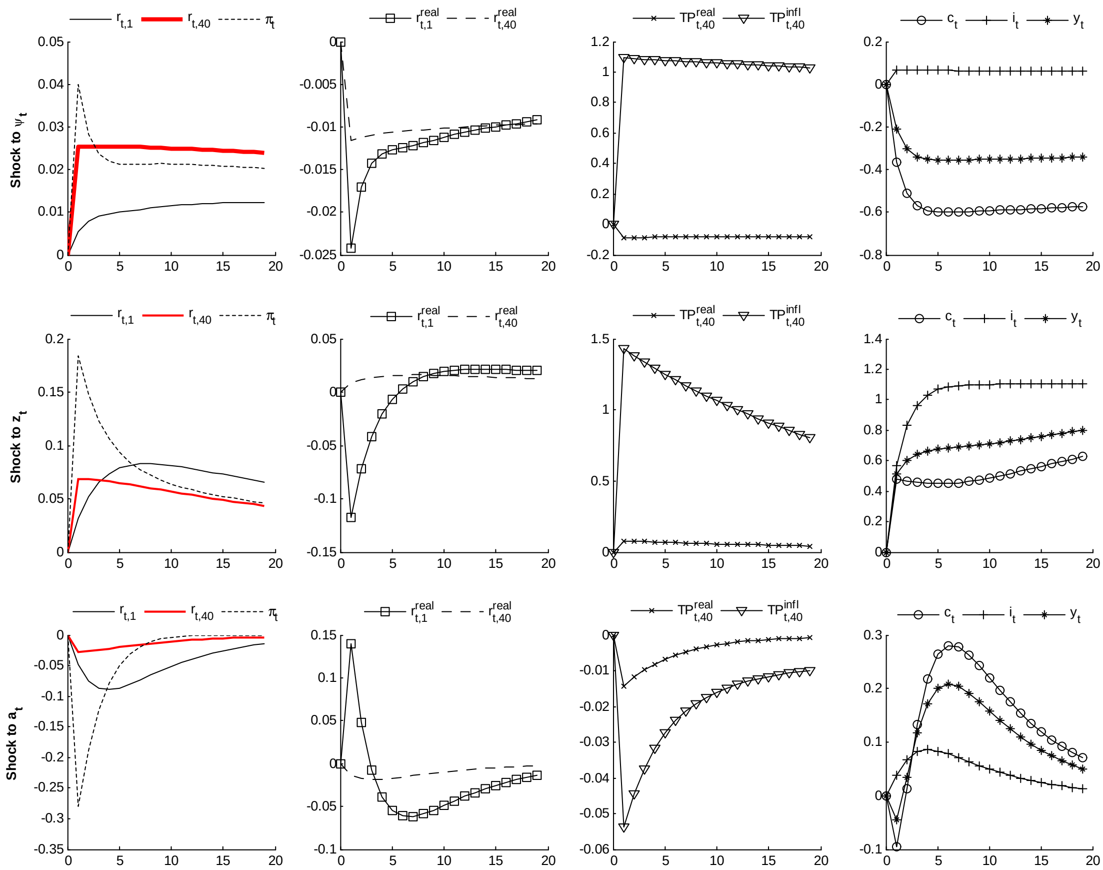
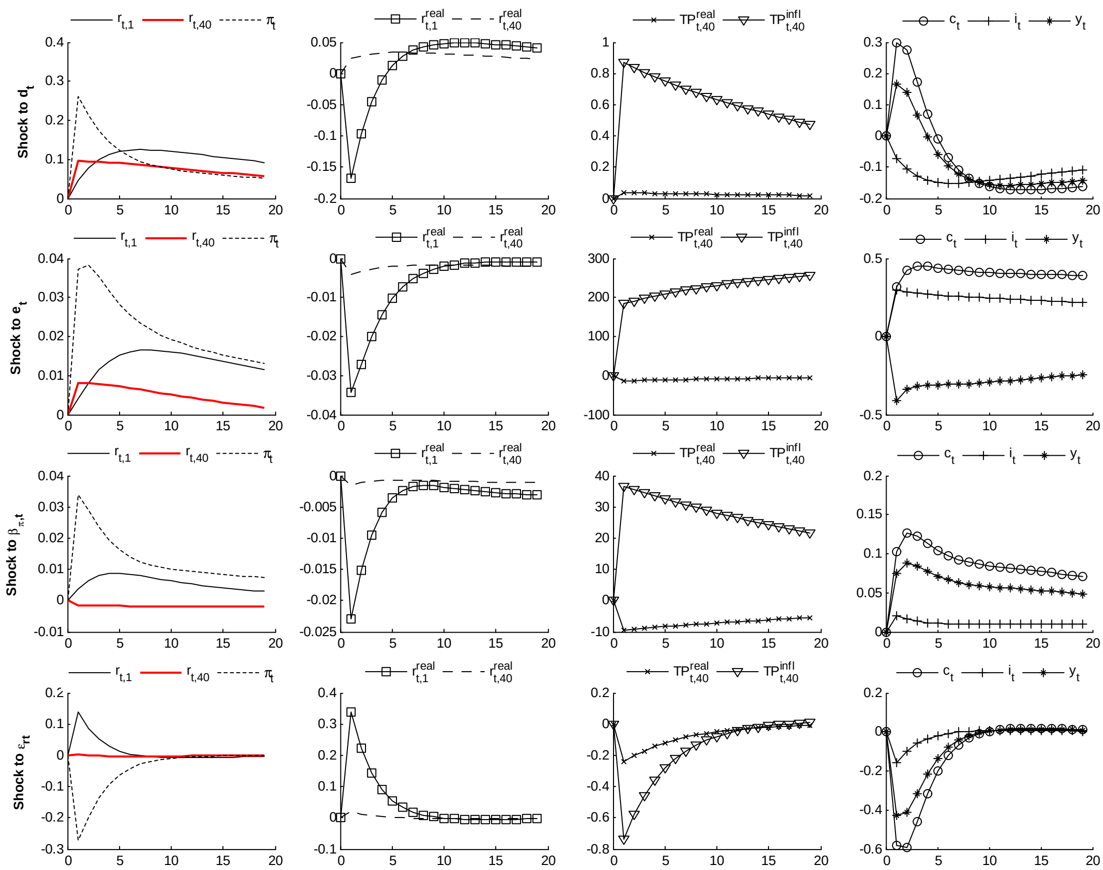


Chart 1: Continued



5.5.2 *Variance decomposition*

In linearized DSGE models, the importance of various shocks for different dimensions of the data are usually addressed by a shock or variance decomposition. However, the non-linear terms in our model solution complicates such a decomposition because the structural shocks enter in a non-linear fashion. We overcome this problem by linearizing the non-linear model solution around the estimated states to get a locally linear state space system. Based on this approximation, the standard method for a variance decomposition can then be applied.[19] Results from the variance decomposition are provided in Table B.

[19] We refer to the paper's technical appendix for additional details.

Starting with a decomposition of the short-term dynamics (1-step ahead), we see that the nominal term structure is mainly explained by preference shocks, the two technology shocks, and monetary policy shocks. It is interesting to note that monetary policy is the key driver for variation in the real term structure which is possible due to the presence of sticky prices in the model. Given the large impact of monetary policy on real interest rates, we also see that monetary policy explains about 30% of the short-term variation in consumption and output growth. As for inflation, its dynamics is equally determined by preference shocks, stationary technology shocks, and monetary policy.

The long-term dynamics of the data are examined by increasing the forecast horizon for the decomposition to 8 periods. We see that preference shocks are the key driver for the nominal term structure, and the two types of technology shocks now explain a smaller fraction of the variation in the data. The real term structure is explained by several disturbances; only non-stationary technology and investment shocks appear unimportant. Investment shocks are unsurprisingly a key determinant for investment growth, but also for long-term inflation expectations which almost exclusively are explained by this shock.

5.6 A structural decomposition and interpretation of term premia

The aim of this section is to use the estimated model for a structural decomposition and interpretation of term premia. As a starting point for this analysis, we begin in Section 5.6.1 by displaying model-implied estimates of nominal and real term premia along with inflation risk premia. The following sections then conduct two decompositions of term premia.

5.6.1 Historical estimates of term premia

The estimated time series for nominal term premia at different maturities are displayed in the top figure of Chart 3.[20] We first note that nominal term premia fall immediately after the introduction of inflation-targeting in 1992 Q3. At the 10-year maturity, the premium drops from 110 to 60 basis points between 1992 Q3 and 1993 Q4. During the next three years, nominal term premia is seen to gradually increase and is in 1997 Q1 close to the level at the start of our sample. Following the operational independence of the Bank of England in 1997 Q2, nominal term premia fall yet again and reach an all-time low of 40 basis points at the 10-year maturity in 1998 Q3. This corresponds to a total fall of 60 basis points after the Bank of England became operational independent. With the exception of the period from 1999-2000, nominal term premia then remained at the new low level until 2006. These findings are broadly similar to results in Joyce *et al* (2010) based on a reduced form affine term structure model. After 2006, we once again see an increase in nominal term premia which at the end of our sample again have returned to the level of 1992 Q3.

The following two figures in Chart 3 show estimated series for real term premia and inflation risk premia to explore which of the two components drive nominal term premia.[21] We first note that real premia have a fairly low and stable level throughout the estimation period and therefore do not contribute much to the variation in nominal premia. The work by Ang *et al* (2008) draws the same conclusion for the United States, whereas Joyce *et al* (2010) find more variation in real term premia in the United Kingdom. Inflation risk premia, on the other hand, are very volatile and display broadly the same pattern as nominal term premia. Note in particular how inflation

[20] All risk premia in this paper are for the corresponding spot interest rates.

[21] We have not offered confidence intervals for the term premia. In principle we could use asymptotic results to devise a Monte Carlo experiment but in practice the small sample properties are unknown, and an investigation is infeasible. However, the diagnostics reported above demonstrate that in general the model is a good fit to the data, suggesting our results are robust.

risk premia display a sharp fall after the adoption of inflation targeting in 1992 Q3 and operational independence to the Bank of England in 1997 Q2. Similarly, the increase in nominal term premia after 2006 is due to a fairly sharp increase in inflation risk premia.

Table B: Variance decomposition

Summing the contributions of the various shocks might not exactly equal 100 due to rounding errors.

	Non-stationary tech. shocks	Preference shocks	Shocks to firms' fixed costs	Stationary tech. shocks	Investment shocks	Shocks to CB's inflation raction	Interest rate shocks	Measure-ment errors
1-step								
$r_{t,1}$	2	4	0	4	0	0	35	55
$r_{t,4}$	8	20	0	14	1	2	20	35
$r_{t,12}$	20	48	0	15	1	3	5	7
$r_{t,20}$	25	57	0	10	1	3	2	2
$r_{t,28}$	27	60	0	7	1	2	1	2
$r_{t,40}$	28	59	1	4	1	1	1	6
$r^{real}_{t,12}$	3	1	1	3	1	2	62	27
$r^{real}_{t,20}$	1	2	5	7	1	2	61	21
$r^{real}_{t,28}$	0	9	8	8	1	2	50	22
$r^{real}_{t,40}$	1	12	6	4	0	1	22	54
π_t	13	27	0	26	1	3	29	1
$\triangle c_t$	30	17	5	1	10	4	33	0
$\triangle i_t$	47	1	1	0	35	1	11	4
$\triangle y_t$	44	9	2	0	11	3	30	0
$\pi^e_{t,4}$	17	31	0	3	2	9	7	31
$\pi^e_{t,12}$	18	26	0	0	2	5	0	49
$\pi^e_{t,long}$	1	1	1	0	21	0	0	76
8-steps								
$r_{t,1}$	1	42	0	14	1	4	15	23
$r_{t,4}$	2	66	0	16	2	6	3	4
$r_{t,12}$	2	79	0	9	3	6	0	0
$r_{t,20}$	3	85	0	5	2	5	0	0
$r_{t,28}$	3	88	0	3	2	3	0	0
$r_{t,40}$	3	89	1	2	1	2	0	0
$r^{real}_{t,12}$	2	10	16	9	9	23	22	9
$r^{real}_{t,20}$	1	11	37	15	6	13	14	4
$r^{real}_{t,28}$	0	31	38	10	3	8	7	3
$r^{real}_{t,40}$	0	45	32	5	2	5	3	8
π_t	2	62	0	22	2	9	3	0
$\triangle c_t$	6	41	14	1	27	10	3	0
$\triangle i_t$	9	2	3	0	84	1	1	0
$\triangle y_t$	10	29	6	0	42	9	3	0
$\pi^e_{t,4}$	3	65	0	2	6	21	1	2
$\pi^e_{t,12}$	4	68	1	0	6	16	0	5
$\pi^e_{t,long}$	0	3	6	0	77	1	0	13

Chart 3: Historical time series of risk premia

Another observation from Chart 3 is that nominal term premia are quite volatile. Indeed, the standard deviation for the 10-year nominal term premium is 83 basis points. We emphasise that it is usually difficult to obtain such a sizable and time-varying nominal term premium in DSGE models without compromising the model's ability to fit the macroeconomy as argued by Rudebusch and Swanson (2008). We match the macro economy as shown by Chart 1, and the flexible nominal term premia is obtained by relying on a very high level of risk-aversion through the Epstein-Zin-Weil preferences.

Given that most of the variation in nominal term premia relates to inflation risk, we focus on inflation risk premia for the remaining part of this section. In the interest of space, we consider the 10-year maturity but the subsequent decompositions could easily be done for other risk premia and at different maturities.

5.6.2 Structural decomposition of the 10-year inflation risk premium

The structural foundation of our model allow us to decompose risk premia further, because we can assess how much of the variation is generated by each of the structural shocks. Such a decomposition is interesting from an economic perspective as it reveals which structural shocks are most important for term premia, ie which shocks the household requires most compensation for when buying bonds.

The effects of the individual structural shocks on the 10-year inflation risk premium are examined by considering the evolution of this premium when only one shock is present. Such counter-factuals allow us to explore what the inflation risk premium would have been if the UK economy only had experienced that particular shock. Results from these counter-factuals are shown in Chart 4. Here we note that four shocks account for most of the dynamics of the 10-year inflation risk premium, namely i) preference shocks, ii) shocks to firms' fixed costs, iii) investment shocks, and iv) shocks to the central bank's inflation reaction. The time series for the structural shocks are plotted in Chart 5.

Combining Chart 4 and 5, we draw the following conclusions. Firstly, a sequence of negative preference shocks from 1997 to 1999 reduces the inflation risk premium by nearly 80 basis points. Some of this reduction is off-set by positive preference shocks from 1999-2004. Overall, preference shocks account for a fall of 50 basis points in the inflation risk premium from 1992 Q3 to 2008 Q2.

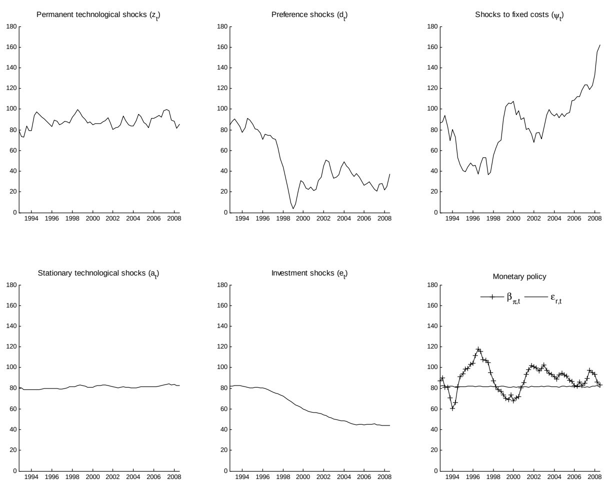
Secondly, a reduction in firms' fixed production costs in the beginning of the 1990's accounts for a reduction of 40 basis points in the inflation risk premium. The picture is reversed after 1998 where higher production costs generate an increase in the inflation risk premium of about 60 basis points around 2000. Even more striking is the effect from these shocks after 2005 where they generate a further increase in the inflation risk premium of 60 basis points. As a result, these shocks account for a rise in the inflation risk premium of nearly 80 basis points from 1992 Q3 to the end of our sample. We also note that these shocks explain the recent increase in inflation risk after 2006.

Thirdly, a sequence of positive investment shocks from 1995 to 2005 lowers inflation risk gradually by about 40 basis points. Recall that these shocks enter in the household's budget constraint as $i_t / (e_t \Upsilon_t)$ and thereby lower the real cost of investing during this period.

Fourthly, changes in the central bank's reaction to inflation during the 1990's also account for some of the variation in the inflation risk premium, although the level of that premium ends the period where it starts.

Chart 4: Structural decomposition of the 10-year inflation risk premium

All counter-factuals are shown in annualized basis points.

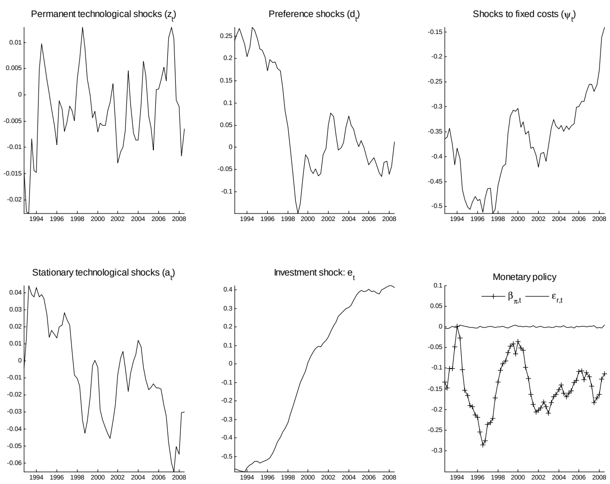


5.6.3 Market price of nominal risk versus quantity of the 10-year inflation risk premium

The 10-year inflation risk premium can also be decomposed based on the standard finance terminology where this premium is expressed in terms of "market price of nominal risk" and "quantity of inflation risk". The first term reveals the required compensation for carrying additional inflation risk when buying nominal bonds instead of real bonds, whereas the second

Chart 5: Estimated structural shocks

All shocks are shown in percentage deviation from steady state.



term illustrates the uncertainty linked to future inflation. We use the standard measure for the market price of nominal risk $\sqrt{Var_t\left[D_{t,t+1}\right]}/E_t\left[D_{t,t+1}\right]$ where $D_{t,t+1}$ is the nominal stochastic discount factor (see Hansen and Jagannathan (1991), Cogley and Sargent (2008), among others). This means that the inflation risk premium can be decomposed as

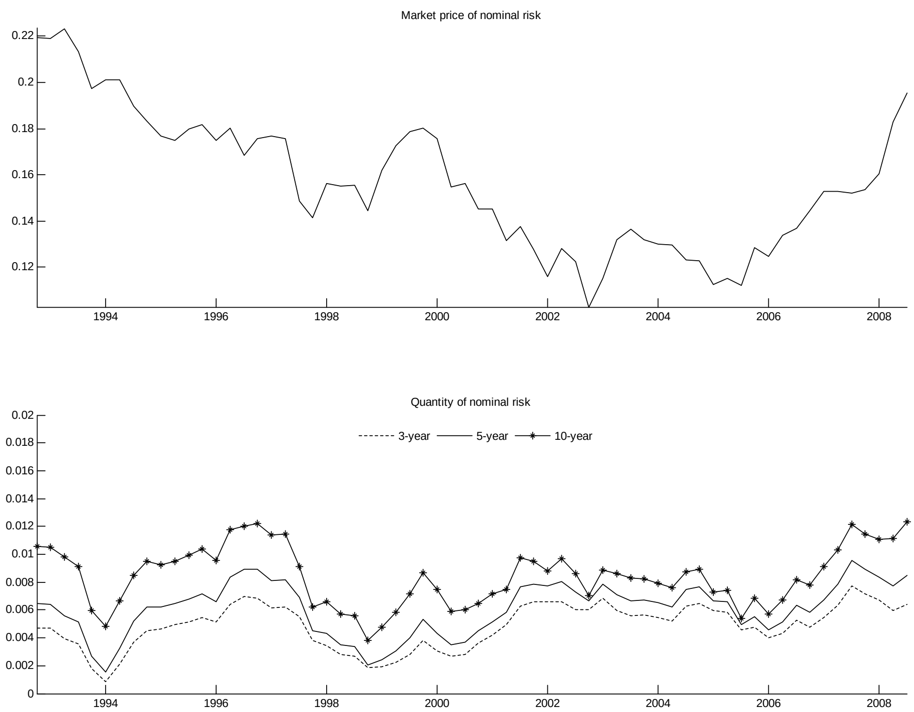
$$TP_{t,n}^{infl} \equiv \underbrace{\frac{\sqrt{Var_t\left[D_{t,t+1}\right]}}{E_t\left[D_{t,t+1}\right]}}_{\text{Market price of nom. risk}} \times \underbrace{\frac{E_t\left[D_{t,t+1}\right]TP_{t,n}^{infl}}{\sqrt{Var_t\left(D_{t,t+1}\right)}}}_{\text{quantity of inflation risk}},$$

where the second term defines the quantity of inflation risk at maturity n. Note here that the time-variation in the quantity of inflation risk is endogenously generated by the model and does not stem from our shock specifications which all have constant conditional second moments. In other words, our model does not rely on shocks with time-dependent second moments (ie stochastic volatility or GARCH effects) to generate time-variation in the quantity of inflation risk.

The top figure in Chart 6 shows a more or less gradual fall in the market price of nominal risk from 1992 to 2005. This finding is similar to the results in the affine and homoscedastic reduced-form model by Joyce *et al* (2010) where all variation in risk premia is due to changes in the market price of risk. A repricing of risk occurs after 2005 where the market price of risk increases steadily and is close to the level in the mid 1990's by the end of our sample.

The time series for the quantity of inflation risk is provided in the bottom figure of Chart 6. It is interesting that the quantity of risk is broadly constant during the sample period, and suggests that the fall in inflation risk premia from 1992 to 2005 mainly stems from a lower market price of risk. We conjecture that this is a natural interpretation in terms of the "search for yield" in the first part of the sample, with the market increasing the price of risk after 2005.

Chart 6: The market price of nominal risk and the quantity of inflation risk



6 Conclusion

This paper develops a DSGE model which explains variation in the nominal and real term structure along with key macro variables for the UK economy. The proposed model belongs to the New Keynesian tradition and incorporates Epstein-Zin-Weil preferences to generate sizable and time-varying term premia when solved up to third order. With the exception of the 1-quarter nominal interest rate, our model delivers a satisfying fit to the two term structures while simultaneously matching inflation expectations and four macro variables. We find a fall in nominal term premia after the introduction of inflation targeting in 1992 and operational independence to the Bank of England in 1997. This fall relates mainly to a reduction in inflation risk premia. A structural decomposition shows that this fall is driven by negative preference shocks, lower fixed production costs, positive investment shocks, and a more aggressive response to inflation by the Bank of England. We also find a gradual reduction in the market price of inflation risk during the 1990's before rising after 2005. Naturally, these results are specific to this model and are inevitably subject to some uncertainty.

Appendix A: The difference between nominal and real interest rates

This section derives third-order approximated expressions for real and nominal interest rates using the same method as in Hordahl *et al* (2008). A third-order approximation the expression for a real bond prices is given by

$$P^{real}_{ss,n}\left(1+\hat{p}^{real}_{t,n}+\tfrac{1}{2}\left(\hat{p}^{real}_{t,n}\right)^2+\tfrac{1}{6}\left(\hat{p}^{real}_{t,n}\right)^3\right)=$$
$$E_t\left[D^{real}_{ss,n}\left(1+\hat{d}^{real}_{t,t+n}+\tfrac{1}{2}\left(\hat{d}^{real}_{t,t+n}\right)^2+\tfrac{1}{6}\left(\hat{d}^{real}_{t,t+n}\right)^3\right)\right]$$

$$\Updownarrow$$

$$\hat{p}^{real}_{t,n}+\tfrac{1}{2}\left(\hat{p}^{real}_{t,n}\right)^2+\tfrac{1}{6}\left(\hat{p}^{real}_{t,n}\right)^3=E_t\left[\hat{d}^{real}_{t,t+n}+\tfrac{1}{2}\left(\hat{d}^{real}_{t,t+n}\right)^2+\tfrac{1}{6}\left(\hat{d}^{real}_{t,t+n}\right)^3\right]$$

where $\hat{p}^{real}_{t,n}=\ln\left(P^{real}_{t,n}/P^{real}_{ss,n}\right)$ and $\hat{d}^{real}_{t,t+n}=\ln\left(D^{real}_{t,t+n}/D^{real}_{ss,ss+n}\right)$. We need third-order accurate expressions for $\left(\hat{p}^{real}_{t,n}\right)^2$ and $\left(\hat{p}^{real}_{t,n}\right)^3$ based on $P^{real}_{t,n}=E_t\left[D^{real}_{t,t+n}\right]$. These expressions are given by

$$\left(\hat{p}^{real}_{t,n}\right)^2=\left(E_t\left[\hat{d}^{real}_{t,n}\right]\right)^2+E_t\left[\hat{d}^{real}_{t,n}\right]E_t\left[\left(\hat{d}^{real}_{t,n}\right)^2\right]$$

and

$$\left(\hat{p}^{real}_{t,n}\right)^3=\left(E_t\left[\hat{d}^{real}_{t,n}\right]\right)^3$$

Hence,

$$\hat{p}^{real}_{t,n}=E_t\left[\hat{d}^{real}_{t,t+n}+\tfrac{1}{2}\left(\hat{d}^{real}_{t,t+n}\right)^2+\tfrac{1}{6}\left(\hat{d}^{real}_{t,t+n}\right)^3\right]-\tfrac{1}{2}\left(\hat{p}^{real}_{t,n}\right)^2-\tfrac{1}{6}\left(\hat{p}^{real}_{t,n}\right)^3$$

$$=E_t\left[\hat{d}^{real}_{t,t+n}+\tfrac{1}{2}\left(\hat{d}^{real}_{t,t+n}\right)^2+\tfrac{1}{6}\left(\hat{d}^{real}_{t,t+n}\right)^3\right]-\tfrac{1}{2}\left(\left(E_t\left[\hat{d}^{real}_{t,t+n}\right]\right)^2+E_t\left[\hat{d}^{real}_{t,t+n}\right]E_t\left[\left(\hat{d}^{real}_{t,t+n}\right)^2\right]\right)$$

$$-\tfrac{1}{6}\left(E_t\left[\hat{d}^{real}_{t,t+n}\right]\right)^3$$

$$= E_t\left[\hat{d}^{real}_{t,t+n}\right] + \tfrac{1}{2}Var_t\left(\hat{d}^{real}_{t,t+n}\right)$$

$$+\tfrac{1}{6}\left(E_t\left[\left(\hat{d}^{real}_{t,t+n}\right)^3\right] - \left(E_t\left[\hat{d}^{real}_{t,t+n}\right]\right)^3 - 3E_t\left[\hat{d}^{real}_{t,t+n}\right]E_t\left[\left(\hat{d}^{real}_{t,t+n}\right)^2\right]\right)$$

Now let $\hat{d}^{real}_{t,t+n} = \hat{\lambda}_{t+n} - \hat{\lambda}_t$ and let $\Delta^n\hat{\lambda}_{t+n} \equiv \hat{\lambda}_{t+n} - \hat{\lambda}_t$. Then

$$r^{real}_{t,n} - r^{real}_{ss,n} = -\tfrac{1}{n}\{E_t\left[\Delta^n\hat{\lambda}_{t+n}\right] + \tfrac{1}{2}Var_t\left(\Delta^n\hat{\lambda}_{t+n}\right) + \tfrac{1}{6}E_t\left[\left(\Delta^n\hat{\lambda}_{t+n}\right)^3\right]$$

$$-\tfrac{1}{6}\left(E_t\left[\Delta^n\hat{\lambda}_{t+n}\right]\right)^3 - 3E_t\left[\Delta^n\hat{\lambda}_{t+n}\right]E_t\left[\left(\Delta^n\hat{\lambda}_{t+n}\right)^2\right]\}$$

Similarly, for a nominal bond we have (following some simple algebra) that

$$r_{t,n} - r_{ss,n} = -\tfrac{1}{n}\{E_t\left[\Delta^n\hat{\lambda}_{t+n} - \hat{\Pi}_{t+n}\right] + \tfrac{1}{2}Var_t\left(\Delta^n\hat{\lambda}_{t+n}\right) + \tfrac{1}{2}Var_t\left(\hat{\Pi}_{t+n}\right) - Cov_t\left(\Delta^n\hat{\lambda}_{t+n}, \hat{\Pi}_{t+n}\right)$$

$$+\tfrac{1}{6}\left(E_t\left[\left(\Delta^n\hat{\lambda}_{t+n}\right)^3\right] - \left(E_t\left[\Delta^n\hat{\lambda}_{t+n}\right]\right)^3 - 3E_t\left[\Delta^n\hat{\lambda}_{t+n}\right]E_t\left[\left(\Delta^n\hat{\lambda}_{t+n}\right)^2\right]\right)$$

$$+\tfrac{1}{6}\left(\left(E_t\left[\hat{\Pi}_{t+n}\right]\right)^3 - E_t\left[\left(\hat{\Pi}_{t+n}\right)^3\right] + 3E_t\left[\hat{\Pi}_{t+n}\right]E_t\left[\left(\hat{\Pi}_{t+n}\right)^2\right]\right)$$

$$+\tfrac{1}{2}Cov_t\left(\left(\hat{\Pi}_{t+n}\right)^2, \Delta^n\hat{\lambda}_{t+n}\right) - \tfrac{1}{2}Cov_t\left(\left(\Delta^n\hat{\lambda}_{t+n}\right)^2, \hat{\Pi}_{t+n}\right)$$

$$+\tfrac{1}{2}\left(3E_t\left[\Delta^n\hat{\lambda}_{t+n}\hat{\Pi}_{t+n}\right] - Cov_t\left(\Delta^n\hat{\lambda}_{t+n}, \hat{\Pi}_{t+n}\right)\right)\left(E_t\left[\Delta^n\hat{\lambda}_{t+n}\right] - E_t\left[\hat{\Pi}_{t+n}\right]\right)$$

$$\}$$

The difference between the nominal and real interest rates is then given as stated in the text.

Appendix B: Proof related to inflation risk premia

This section shows that $TP_{t,n} = TP^{real}_{t,n} + TP^{infl}_{t,n}$ for affine Gaussian term structure models and DSGE models approximated up to second-order. Simple derivations for real interest rates imply

$$r^{real}_{t,n} = -\frac{1}{n}\left(E_t\left[\sum_{j=1}^{n} d^{real}_{t,t+j}\right] + \frac{1}{2}V_t\left[\sum_{j=1}^{n} d^{real}_{t,t+j}\right]\right).$$

For affine Gaussian models this expression holds provided $d^{real}_{t,t+j} \equiv \ln D^{real}_{t,t+j}$ and for DSGE models solved up to second-order if $d^{real}_{t,t+j} \equiv \ln\left(D^{real}_{t,t+n}/D^{real}_{ss,ss+n}\right)$. A similar expression holds for nominal interest rates, ie

$$r_{t,n} = -\frac{1}{n}\left(E_t\left[\sum_{j=1}^{n}\left(d^{real}_{t,t+j} - \pi_{t+j}\right)\right] + \frac{1}{2}V_t\left[\sum_{j=1}^{n}\left(d^{real}_{t,t+j} - \pi_{t+j}\right)\right]\right).$$

The standard decomposition of the real term structure implies

$$\begin{aligned} r^{real}_{t,n} - \frac{1}{n}\sum_{j=1}^{n} E_t\left[r^{real}_{t+j-1,1}\right] &= -\frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=1+j}^{n} Cov_t\left(d^{real}_{t,t+1+j}, d^{real}_{t,t+k}\right) \\ &\quad -\frac{1}{2n}\sum_{j=1}^{n} V_t\left[E_{t+j-1}\left[d^{real}_{t,t+j}\right]\right] \end{aligned}$$

and similarly for the nominal term structure, ie

$$\begin{aligned} r_{t,n} - \frac{1}{n}\sum_{j=1}^{n} E_t\left[r_{t+j-1,1}\right] &= -\frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=1+j}^{n} Cov_t\left(d^{real}_{t,t+1+j} - \pi_{t+1+j}, d^{real}_{t,t+k} - \pi_{t+k}\right) \\ &\quad -\frac{1}{2n}\sum_{j=1}^{n} V_t\left[E_{t+j-1}\left[d^{real}_{t,t+j} - \pi_{t+j}\right]\right] \end{aligned}$$

For these models, real and nominal term premia are given by

$$TP^{real}_{t,n} \equiv -\frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=1+j}^{n} Cov_t\left(d^{real}_{t,t+1+j}, d^{real}_{t,t+k}\right)$$

$$TP_{t,n} \equiv -\frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=1+j}^{n} Cov_t\left(d^{real}_{t,t+1+j} - \pi_{t+1+j}, d^{real}_{t,t+k} - \pi_{t+k}\right)$$

The break-even inflation rates are easily shown to be

$$\begin{aligned} r_{t,n} - r^{real}_{t,n} &= \frac{1}{n}\left(E_t\left[\sum_{j=1}^{n}\pi_{t+j}\right] - \frac{1}{2}\sum_{j=1}^{n} V_t\left[\pi_{t+j}\right]\right) \\ &\quad -\frac{1}{n}\left(\sum_{j=0}^{n-1}\sum_{k=1+j}^{n} Cov_t\left[\pi_{t+1+j}, \pi_{t+k}\right] - Cov_t\left[\sum_{j=1}^{n} d^{real}_{t,t+j}, \sum_{j=1}^{n}\pi_{t+j}\right]\right) \end{aligned}$$

Here, $-\frac{1}{2}\sum_{j=1}^{n} V_t\left[\pi_{t+j}\right]$ is the inflation convexity term, whereas inflation risk premia are given by

$$TP^{infl}_{t,n} \equiv -\frac{1}{n}\left(\sum_{j=0}^{n-1}\sum_{k=1+j}^{n} Cov_t\left[\pi_{t+1+j}, \pi_{t+k}\right] - Cov_t\left[\sum_{j=1}^{n} d^{real}_{t,t+j}, \sum_{j=1}^{n}\pi_{t+j}\right]\right).$$

Simple algebra then implies $TP_{t,n} = TP^{real}_{t,n} + TP^{infl}_{t,n}$ as claimed.

Appendix C: Using the POP method to compute conditional expectations in DSGE models

This section extends the POP method to compute conditional expectations of variables in DSGE models. Consider the case where the model reports a variable denoted r_t and we want to compute various conditional expectations for this variable, ie $r_t^{(1)} \equiv E_t\left[r_{t+1}\right]$, $r_t^{(2)} \equiv E_t\left[r_{t+2}\right]$, $r_t^{(3)} \equiv E_t\left[r_{t+3}\right]$, etc. The law of iterated expectations implies $r_t^{(2)} = E_t\left[E_{t+1}\left[r_{t+2}\right]\right] = E_t\left[r_{t+1}^{(1)}\right]$ etc. Accordingly, only a formula for $r_t^{(1)} \equiv E_t\left[r_{t+1}\right]$ is required because all the other expectations follows by iterating this formula. Below, $r_t^{(1)}$ is denoted by p for simplicity.

Consider the problem

$$T\left(p\left(\mathbf{x}_t, \sigma\right)\right) = E_t\left[T\left(r\left(\mathbf{x}_{t+1}, \sigma\right)\right)\right] \tag{C-1}$$

where σ is the perturbation parameter and $T(\cdot)$ is an invertible and differentiable transformation function. Observe that

$$F\left(\mathbf{x}_t, \sigma\right) \equiv E_t\left[-T\left(p\left(\mathbf{x}_t, \sigma\right)\right) + T\left(r\left(\mathbf{h}\left(\mathbf{x}_t, \sigma\right) + \sigma\boldsymbol{\eta}\epsilon_{t+1}, \sigma\right)\right)\right] = 0 \tag{C-2}$$

because

$$\mathbf{x}_{t+1} = \mathbf{h}\left(\mathbf{x}_t, \sigma\right) + \sigma\boldsymbol{\eta}\epsilon_{t+1} \tag{C-3}$$

Equation **(C-2)** must hold for all values of $(\mathbf{x}_t, \sigma)$, and this allow us to compute all derivatives of p with respect to $(\mathbf{x}_t, \sigma)$ around the deterministic steady state, ie $\mathbf{x}_t = \mathbf{x}_{ss}$ and $\sigma = 0$, given derivatives of $\mathbf{h}\left(\mathbf{x}_t, \sigma\right)$ and $r\left(\mathbf{x}_{t+1}, \sigma\right)$ around the same point.

For the indices we adopt the convention that the subscript indicates the order of differentiation. ie a 1 is for the first time we take derivatives and so on. Thus

$$\alpha_1, \alpha_2, \alpha_3 = 1, 2, ...n_x \qquad \gamma_1, \gamma_2, \gamma_3 = 1, 2, ...n_x \qquad \phi_1, \phi_2, \phi_3 = 1, 2, ..., n_\epsilon$$

where n_x is the number of state variables and n_ϵ is the number of elements in ϵ_{t+1}.

The first order terms:

For $\mathbf{x}_t$:

$$\left[F_{\mathbf{x}}\left(\mathbf{x}_{ss}, 0\right)\right]_{\alpha_1} = E_t\left[-T_p(p)\left[p_{\mathbf{x}}\right]_{\alpha_1} + T_r(r)\left[r_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\right] = 0$$

$\Updownarrow$

$$T_p(p)\left[p_{\mathbf{x}}\right]_{\alpha_1}=T_r(r)\left[r_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}$$

Using a log-transformation:

$$\mathbf{p}_{\mathbf{x}}(1;:)=\mathbf{r}_{\mathbf{x}}(1,:)\,\mathbf{h}_{\mathbf{x}}$$

For σ:

$$\left[F_\sigma\left(\mathbf{x}_{ss},0\right)\right]$$

$$=E_t\left[-T_p(p)\left[p_\sigma\right]+T_r(r)\left[r_{\mathbf{x}}\right]_{\gamma_1}\left(\left[\mathbf{h}_\sigma\right]^{\gamma_1}+\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)+T_r(r)\left[r_\sigma\right]\right]=0$$

$\Updownarrow$

$$\left[p_\sigma\right]=0$$

because $E_t\left(\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)=0$, $\left[\mathbf{h}_\sigma\right]^{\gamma_1}=0$, and $\left[r_\sigma\right]=0$.

The second order terms:

For $(\mathbf{x}_t,\mathbf{x}_t)$:

$$\left[F_{\mathbf{xx}}\left(\mathbf{x}_{ss},0\right)\right]_{\alpha_1\alpha_2}=E_t[-T_{pp}(p)\left[p_{\mathbf{x}}\right]_{\alpha_1}\left[p_{\mathbf{x}}\right]_{\alpha_2}-T_p(p)\left[p_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}$$

$$+T_{rr}(r)\left[r_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\left[r_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}$$

$$+T_r(r)\left[r_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}$$

$$+T_r(r)\left[r_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}^{\gamma_1}]=0$$

$\Updownarrow$

$$T_p(p)\left[p_{\mathbf{xx}}\right]_{\alpha_1\alpha_2} = -T_{pp}(p)\left[p_{\mathbf{x}}\right]_{\alpha_1}\left[p_{\mathbf{x}}\right]_{\alpha_2}$$

$$+T_{rr}(r)\left[r_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\left[r_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}$$

$$+T_r(r)\left[r_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}$$

$$+T_r(r)\left[r_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}^{\gamma_1}$$

Using a log-transformation:

$$\mathbf{p}_{\mathbf{xx}} = \mathbf{h}_{\mathbf{x}}'\mathbf{r}_{\mathbf{xx}}\mathbf{h}_{\mathbf{x}} + \sum_{\gamma_1=1}^{n_x}\mathbf{r}_{\mathbf{x}}\left(1,\gamma_1\right)\mathbf{h}_{\mathbf{xx}}\left(\gamma_1,:,:\right)$$

For (σ,σ):

$$\left[F_{\sigma\sigma}\left(\mathbf{x}_{ss},0\right)\right] = E_t[-T_{pp}(p)\left[p_\sigma\right]\left[p_\sigma\right] - T_p(p)\left[p_{\sigma\sigma}\right]$$

$$+T_{rr}(r)\left(\left[r_{\mathbf{x}}\right]_{\gamma_2}\left(\left[\mathbf{h}_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[r_\sigma\right]\right)\left[r_{\mathbf{x}}\right]_{\gamma_1}\left(\left[\mathbf{h}_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$

$$+T_r(r)\left(\left[r_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left(\left[\mathbf{h}_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[r_{\mathbf{x}\sigma}\right]_{\gamma_1}\right)\left(\left[\mathbf{h}_\sigma\right]^{\gamma_1} + [\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right)$$

$$+T_r(r)\left[r_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}$$

$$+T_{rr}(r)\left(\left[r_{\mathbf{x}}\right]_{\gamma_2}\left(\left[\mathbf{h}_\sigma\right]^{\gamma_2} + [\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) + \left[r_\sigma\right]\right)\left[r_\sigma\right]$$

$$+T_r(r)\left[r_{\sigma\sigma}\right]] = 0$$

$$\Updownarrow$$

$$T_p(p)\left[p_{\sigma\sigma}\right] = T_{rr}(r)\left[r_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[r_{\mathbf{x}}\right]_{\gamma_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}[\mathbf{I}]_{\phi_2}^{\phi_1}$$

$$+T_r\left(r^{t+1}\right)\left[r_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}[\mathbf{I}]_{\phi_2}^{\phi_1}$$

$$+T_r(r)\left[r_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}$$

$$+T_r(r)[r_{\sigma\sigma}]$$

because $[\mathbf{h}_\sigma]^{\gamma_1}=0$ and $[r_{\mathbf{x}\sigma}]_{\gamma_1}=0$. Using a log-transformation:

$$p_{\sigma\sigma}=\mathbf{r_x}(1,:)\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{r_x}(1,:)'+trace(\boldsymbol{\eta}'\mathbf{r_{xx}}\boldsymbol{\eta})+\mathbf{r_x}(1,:)\mathbf{h}_{\sigma\sigma}+r_{\sigma\sigma}$$

The second-order term $p_{\mathbf{x}\sigma}$ is known to be zero (Schmitt-Grohé and Uribe (2004)).

Third order terms:

For $(\mathbf{x}_t,\mathbf{x}_t,\mathbf{x}_t)$:

$$T_p(p)\left[p_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3}=-T_{ppp}(p)\left[p_{\mathbf{x}}\right]_{\alpha_1}\left[p_{\mathbf{x}}\right]_{\alpha_2}\left[p_{\mathbf{x}}\right]_{\alpha_3}-T_{pp}(p)\left[p_{\mathbf{x}}\right]_{\alpha_1\alpha_3}\left[p_{\mathbf{x}}\right]_{\alpha_2}$$

$$-T_{pp}(p)\left[p_{\mathbf{x}}\right]_{\alpha_1}\left[p_{\mathbf{xx}}\right]_{\alpha_2\alpha_3}-T_{pp}(p)\left[p_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}\left[p_{\mathbf{x}}\right]_{\alpha_3}$$

$$+T_{rrr}(r)[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}[r_{\mathbf{x}}]_{\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}[r_{\mathbf{x}}]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}$$

$$+T_{rr}(r)[r_{\mathbf{xx}}]_{\gamma_1\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}[r_{\mathbf{x}}]_{\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}$$

$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h_{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}[r_{\mathbf{x}}]_{\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}$$

$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}[r_{\mathbf{xx}}]_{\gamma_2\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}$$

$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}[r_{\mathbf{x}}]_{\gamma_2}[\mathbf{h_{xx}}]^{\gamma_2}_{\alpha_2\alpha_3}$$

$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[r_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}$$

$$+T_r(r)[r_{\mathbf{xxx}}]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}$$

$$+T_r(r)[r_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\mathbf{h_{xx}}]^{\gamma_2}_{\alpha_2\alpha_3}[\mathbf{h_x}]^{\gamma_1}_{\alpha_1}$$

$$+T_r(r)[r_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\mathbf{h_x}]^{\gamma_2}_{\alpha_2}[\mathbf{h_{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}$$

$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h_{xx}}]^{\gamma_1}_{\alpha_1\alpha_2}$$

$$+T_r(r)[r_{\mathbf{xx}}]_{\gamma_1\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$$

$$+T_r(r)[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h_{xxx}}]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}$$

Using a log-transformation:

$$\mathbf{p_{xxx}}(\alpha_1,\alpha_2,\alpha_3)=\sum_{\gamma_3=1}^{n_x}\mathbf{h_x}(:,\alpha_1)'\,\mathbf{r_{xxx}}\left(:,:,\gamma_3\right)\mathbf{h_x}(:,\alpha_2)\,\mathbf{h_x}\left(\gamma_3,\alpha_3\right)$$

$$+\mathbf{h_x}(:,\alpha_1)'\,\mathbf{r_{xx}}\mathbf{h_{xx}}(:,\alpha_2,\alpha_3)$$

$$+\sum_{\gamma_1=1}^{n_x}\mathbf{r_{xx}}\left(\gamma_1,:\right)\mathbf{h_x}(:,\alpha_2)\,\mathbf{h_{xx}}\left(\gamma_1,\alpha_1,\alpha_3\right)$$

$$+\sum_{\gamma_1=1}^{n_x}\mathbf{r_{xx}}\left(\gamma_1,:\right)\mathbf{h_x}(:,\alpha_3)\,\mathbf{h_{xx}}\left(\gamma_1,\alpha_1,\alpha_2\right)$$

$$+\mathbf{r_x}(1,:)\,\mathbf{h_{xxx}}(:,\alpha_1,\alpha_2,\alpha_3)$$

For $\left(\sigma^2,\mathbf{x}_t\right)$:

$$T_p(p)\left[p_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}=-T_{pp}(p)\left[p_{\mathbf{x}}\right]_{\alpha_3}\left[p_{\sigma\sigma}\right]$$

$$+T_{rrr}(r)[r_{\mathbf{x}}]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[r_{\mathbf{x}}]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[r_{\mathbf{x}}]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$

$$+3T_{rr}(r)[r_{\mathbf{xx}}]_{\gamma_2\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[r_{\mathbf{x}}]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$

$$+T_r(r)[r_{\mathbf{xxx}}]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$

$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}$$

$$+T_r(r)[r_{\mathbf{xx}}]_{\gamma_1\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}$$

$$+T_r(r)[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\sigma\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3}$$

$$+T_{rr}(r)[r_{\mathbf{x}}]_{\gamma_3}[\mathbf{h_x}]^{\gamma_3}_{\alpha_3}[r_{\sigma\sigma}]$$

$$+T_r(r)[r_{\sigma\sigma\mathbf{x}}][\mathbf{h_x}]^{\gamma_3}_{\alpha_3}$$

Using a log-transformation:

$$\mathbf{p}_{\sigma\sigma\mathbf{x}} = 2\mathbf{r_x}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{r_{xx}}\mathbf{h_x} + \sum_{\gamma_3=1}^{n_x} trace\left(\boldsymbol{\eta}'\mathbf{r_{xxx}}(:,:,\gamma_3)\boldsymbol{\eta}\right)\mathbf{h_x}\left(\gamma_3,:\right)$$

$$+\mathbf{h}'_{\sigma\sigma}\mathbf{r_{xx}}\mathbf{h_x} + \mathbf{r_x}\mathbf{h}_{\sigma\sigma\mathbf{x}} + \mathbf{r}_{\sigma\sigma\mathbf{x}}\mathbf{h_x}$$

For $\left(\sigma^3\right)$:

$$T_p(p)\left[p_{\sigma\sigma\sigma}\right] =$$

$$T_{rrr}(r)[r_{\mathbf{x}}]_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[r_{\mathbf{x}}]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[r_{\mathbf{x}}]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$

$$+3T_{rr}(r)[r_{\mathbf{xx}}]_{\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[r_{\mathbf{x}}]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$

$$+T_r(r)[r_{\mathbf{xxx}}]_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3}$$

$$+T_r(r)[r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\sigma\sigma\sigma}]^{\gamma_1}$$

$$+T_r(r)[r_{\sigma\sigma\sigma}]$$

Here we introduce the additional notation

$$\left[\mathbf{m}^3(\epsilon_{t+1})\right]^{\phi_1}_{\phi_2\phi_3} = \begin{cases} m^3(\epsilon_{t+1}(\phi_1,1)) & \text{if } \phi_1 = \phi_2 = \phi_3 \\ 0 & \text{else} \end{cases}$$

where $m^3(\epsilon_{t+1}(\phi_1,1))$ denotes the third moment of $\epsilon_{t+1}(\phi_1,1)$. Using a log-transformation we get

$$\left[p^t_{\sigma\sigma\sigma}\right] = \sum_{\phi_2=1}^{n_e}\sum_{\phi_3=1}^{n_e}\mathbf{r_x}\boldsymbol{\eta}\left(:,\phi_3\right)\mathbf{r_x}\boldsymbol{\eta}\left(:,\phi_2\right)\mathbf{r_x}\boldsymbol{\eta}\mathbf{m}^3\left(:,\phi_2,\phi_3\right)$$

$$+3\sum_{\phi_2=1}^{n_e}\sum_{\phi_3=1}^{n_e}\boldsymbol{\eta}\left(:,\phi_2\right)'\mathbf{r_{xx}}\boldsymbol{\eta}\left(:,\phi_3\right)\mathbf{r_x}\boldsymbol{\eta}\mathbf{m}^3\left(:,\phi_2,\phi_3\right)$$

$$+\sum_{\gamma_1=1}^{n_x}\sum_{\phi_2=1}^{n_e}\sum_{\phi_3=1}^{n_e}\boldsymbol{\eta}(:,\phi_2)'\,\mathbf{r}_{\mathbf{xxx}}\left(\gamma_1,:,:\right)\boldsymbol{\eta}\left(:,\phi_3\right)\boldsymbol{\eta}(\gamma,:)\,\mathbf{m}^3\left(:,\phi_2,\phi_3\right)$$

$$+\mathbf{r}_{\mathbf{x}}^{t+1}\mathbf{h}_{\sigma\sigma\sigma}^{t}$$

$$+r_{\sigma\sigma\sigma}^{t+1}$$

The third-order term $p_{\mathbf{xx}\sigma}$ is known to be zero (Andreasen (2010a)).

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