BANK OF ENGLAND

Working Paper No. 417

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Martin M Andreasen

March 2011

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Abstract

This paper studies how non-Gaussian shocks affect risk premia in DSGE models approximated to second and third order. Based on an extension of the work by Schmitt-Grohe and Uribe to third order, we derive propositions for how rare disasters, stochastic volatility, and GARCH affect any risk premia in a wide class of DSGE models. To quantify these effects, we then set up a standard New Keynesian DSGE model where total factor productivity includes rare disasters, stochastic volatility, and GARCH. We find that rare disasters increase the mean level of the ten-year nominal term premium, whereas a key effect of stochastic volatility and GARCH is an increase in the variability of this premium.

Key words: Epstein-Zin-Weil preferences, GARCH, rare disasters, risk premia, stochastic volatility.

JEL classification: C68, E30, E43, E44, G12.

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(1) Bank of England. Email: martin.andreasen@bankofengland.co.uk

The views expressed in this paper are those of the author, and not necessarily those of the Bank of England. The author gratefully acknowledges access to computer facilities provided by the Danish Center for Scientific Computing (DCSC) and support from the Center for Research in Econometric Analysis of Time Series, CREATES, funded by the Danish National Research Foundation. For helpful comments and suggestions, we thank Eric Swanson, Glenn Rudebusch, Juan Rubio-Ramirez, and Marcelo Ferman. This paper was finalised on 26 January 2011.

The Bank of England's working paper series is externally refereed.

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Publications Group, Bank of England, Threadneedle Street, London, EC2R 8AH
Telephone +44 (0)20 7601 4030 Fax +44 (0)20 7601 3298 email mapublications@bankofengland.co.uk

ISSN 1749-9135 (on-line)

Contents

Summary

The current financial crisis and the recession that followed have highlighted the close link between the macroeconomy and asset prices. Unfortunately, standard economic tools are not well suited to examine this relationship. Economists often use dynamic stochastic general equilibrium (DSGE) models when studying the economy. These models use economic theory to describe how all agents in the economy interact through time. The term ‘stochastic’ refers to the crucial feature that there is uncertainty in the economy (ie the economy is constantly being hit by ‘shocks’, also known as innovations), and this affects agents’ behaviour.

The relationships implied by DSGE models determine all quantities and prices in the economy, and finding a set of rules which ensure that all markets clear is called solving the model. The exact solutions to most DSGE models are unfortunately unknown and economists therefore have to resort to approximations. This is normally done using linearisation, assuming that relationships are close to linearity near the equilibrium. This often delivers a fairly accurate approximation. But this method does not capture effects of uncertainty in the model; ie agents are effectively assumed to behave as if there were no uncertainty. This is an unfortunate assumption to impose, in particular in an asset pricing context, because it constrains all risk premia to be zero.

Luckily, there are many alternative solution methods to linearisation. The one considered in this paper is to approximate the solution by second and third-order expansions around the model’s deterministic steady state (ie the point at which the economy would arrive in the long run if there were no uncertainty). These expansions introduce the curvature that is needed to capture the consequences of risk. We then analyse how three types of ‘non-Gaussian’ shocks affect risk premia in a wide class of DSGE models. Gaussian shocks are well behaved; ie they follow a normal distribution which is unchanged over time. In practice, this assumption frequently does not hold. The first type of shock we consider captures rare disasters, which refer to the possibility that the economy may be hit by a very large negative shock on rare occasions, for instance four times during a century (roughly the frequency of major recessions). We then show that rare disasters do not affect risk premia in a second-order approximation but do affect the level of risk premia at third order. The *variability* of risk premia is however not affected at either second or third order by the presence of rare disasters in the model. The second type of shock we analyse

are stochastic volatility shocks which refer to the possibility that the variability of the fundamental innovations may change at random time points. One can think of stochastic volatility shocks as disturbances to the confidence level of the economic agents. We show that stochastic volatility may affect the mean level but not the variability of risk premia at second order. For a third-order approximation, stochastic volatility may affect the mean level and the variability of risk premia. The final non-Gaussian shock distribution we analyse is structural disturbances with a type of time variation known as generalised autoregressive conditional heteroscedasticity (GARCH). We find that GARCH may affect the mean level but not the variability of risk premia at second order, whereas GARCH may affect both the level and the variability of risk premia in a third-order approximation.

To explore the quantitative effects of these non-Gaussian shocks, we then examine how rare disasters, stochastic volatility, and GARCH in productivity shocks affect the ten-year nominal term premium in an otherwise standard New Keynesian DSGE model solved to third order. We find that the chosen specification of rare disasters can have substantial effects on the level of the term premium and values of skewness and kurtosis (which measure aspects of asymmetry and the probability of extreme events occurring) for several macro variables. However, rare disasters hardly affect the standard deviation of most macro variables. We also find that stochastic volatility can generate sizable variation in the term premium without distorting the model's ability to match characteristics of a number of key macroeconomic series. The effects of GARCH are slightly different from those generated by stochastic volatility. In particular, GARCH increases both the mean level and the variability of the term premium.

This analysis is unavoidably technical but it is not arcane. It is essential if we wish to understand the consequences of extreme shocks to the economy in an uncertain world. Never has this been more important than in the past few years.

1 Introduction

Macroeconomic models often struggle to explain the dynamics of asset prices and their related risk premia. The early work by Mehra and Prescott (1985), Mark (1985), and Backus, Gregory and Zin (1989) illustrate this for the equity premium, the foreign exchange risk premium, and the term premium, respectively. Much work has subsequently tried to improve the standard consumption endowment model along these dimensions. The paper by Campbell and Cochrane (1999) extend the model with consumption habits and heteroscedastic shocks, whereas Bansal and Yaron (2004) emphasise the importance of Epstein-Zin-Weil preferences and stochastic volatility in consumption growth. Another interesting extension is due to Rietz (1988) and Barro (2006) who also rely on non-Gaussian shocks as they incorporate rare disasters into the basic framework. These improvements of the consumption endowment model are all successful at reproducing several key moments of asset prices.

A second strand of the literature tries to explain the same asset pricing moments in fully specified dynamic stochastic general equilibrium (DSGE) models. The objective in these models is to provide further economic insight into the dynamics of asset prices through general equilibrium effects. Important contributions are Jermann (1998), Boldrin, Christiano and Fisher (2001), Chari, Kehoe and McGrattan (2002) and more recently Wu (2006), Uhlig (2007), De Paoli, Scott and Weeken (2007), Hordahl, Tristani and Vestin (2008), Rudebusch and Swanson (2009), and Bekaert, Cho and Moreno (2010). It is interesting to note that all these studies only consider Gaussian shocks when specifying structural disturbances. This is contrary to the aforementioned literature based on endowment economies which rely heavily on non-Gaussian shocks. Moreover, the exclusive focus on Gaussian shocks when studying asset prices in DSGE models also goes counter to recent findings suggesting that structural innovations may be non-Gaussian (see Fernández-Villaverde and Rubio-Ramírez (2007), Justiniano and Primiceri (2008), Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez and Uribe (2010), among others).

The contribution of this paper is to close this gap in the literature and study how non-Gaussian shocks in DSGE models affect risk premia. This is done for the general class of DSGE models considered in Schmitt-Grohé and Uribe (2004) when these models are solved by second and third-order perturbation approximations around the deterministic steady state.

The third-order terms are of great economic interest because they allow for time-varying risk premia as observed in the data (see for instance Campbell and Shiller (1991)). Schmitt-Grohé and Uribe (2004) derive the solution to all second-order terms and we extend their results to third order. These analytical formulas are useful because they provide key insights into how non-Gaussian shocks affect the approximated solution. Similar insights are hard to establish when approximations to DSGE models are computed in a high-level programming language like Mathematica (see Swanson, Anderson and Levin (2005), Arouba, Fernández-Villaverde and Rubio-Ramírez (2005), Rudebusch and Swanson (2009), among others).

Based on the general formulas for second and third-order terms in DSGE models, we then analyse how non-Gaussian shocks in general affect risk premia in these models. Given the aforementioned extensions of the standard endowment model, we choose to explore effects of rare disasters, stochastic volatility, and conditional heteroscedasticity modelled by GARCH. Our key findings are as follows. First, the presence of rare disasters does not affect risk premia in a second-order approximation and only change the level of risk premia at third order. Second, modelling time-varying uncertainty by stochastic volatility and GARCH do not generate time-varying risk premia in a second-order approximation. This is because all second moments remain constant at the approximation point even with these extensions. Third, when DSGE models are solved up to third order, stochastic volatility and GARCH may affect the level of risk premia, and these processes may generate additional variation in risk premia. We emphasise that these properties hold for all DSGE models belonging to the considered class and for any risk premia - ie whether it relates to equities, bonds, exchange rates, etc.

To explore the quantitative effects of non-Gaussian shocks, we then examine how rare disasters, stochastic volatility, and GARCH in productivity shocks affect the ten-year nominal term premium in an otherwise standard New Keynesian DSGE model solved to third order. We find that the chosen specification of rare disasters can have substantial effects on the level of the term premium and values of skewness and kurtosis for several macro variables. However, rare disasters hardly affect the standard deviation of most macro variables. We also find that stochastic volatility can generate sizable variation in the term premium without distorting the model's ability to match a key number of macroeconomic moments. A decomposition shows that stochastic volatility makes this premium more volatile by increasing the variation in the quantity of risk. Hence, our DSGE model has the same key feature as many finance models; a higher uncertainty level raises the term premium through an increase in the quantity of risk (see Cox, Ingersoll and Ross (1985), Dai and Singleton (2002), among others). We also find that the considered specification of stochastic volatility does not change the mean level of the term premium. The effects of GARCH are slightly different from those generated by stochastic volatility. In particular, GARCH increases both the mean level and the variability of the term premium. A decomposition shows that these effects arise from a higher and more volatile market price of risk. It is further shown that GARCH affects the variability of consumption growth but not the variation in inflation and interest rates.

From our analysis we therefore conclude that non-Gaussian shocks can have substantial effects on risk premia in DSGE models. In particular, non-Gaussian shocks may generate more realistic and less puzzling dynamics for risk premia without compromising the ability of the models to match other moments.

The rest of this paper is organised as follows. Section 2 introduces the set-up considered in Schmitt-Grohé and Uribe (2004), and Section 3 derives analytical expressions for all third-order terms in this class of DSGE models. The general expression for risk premia is studied in Section 4 where we also analyse the effects of innovations with non-symmetric distributions (ie rare disasters), stochastic volatility, and GARCH for risk premia. A standard New Keynesian DSGE model is considered in Section 5 to explore the quantitative effects of these shock specifications for the ten-year nominal term premium. Concluding comments are provided in Section 6. Unless stated otherwise, all proofs are deferred to a technical appendix available on the Bank of England's website or on request.

2 The general model

Following Schmitt-Grohé and Uribe (2004), we consider models with equilibrium conditions of the form

$$E_t\left[\mathbf{f}\left(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t\right)\right] = \mathbf{0}, \qquad \textbf{(1)}$$

where E_t is the conditional expectation given information available at period t. The state vector $\mathbf{x}_t$ has dimension $n_x \times 1$, and the vector $\mathbf{y}_t$ with dimension $n_y \times 1$ contains all the control variables. Here, $n_x + n_y \equiv n$. The function $\mathbf{f}$ takes elements from $\mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ into $\mathbb{R}^n$, and we assume that this mapping is at least three times differentiable in the deterministic steady state. This point is defined by $\sigma = 0$ and values of $(\mathbf{x}_{ss}, \mathbf{y}_{ss})$ which imply $\mathbf{f}(\mathbf{y}_{ss}, \mathbf{y}_{ss}, \mathbf{x}_{ss}, \mathbf{x}_{ss}) = \mathbf{0}$.

The vector $\mathbf{x}_t$ is partitioned as $\begin{bmatrix} \mathbf{x}'_{1,t} & \mathbf{x}'_{2,t} \end{bmatrix}'$ where $\mathbf{x}_{1,t}$ contains endogenous state variables and $\mathbf{x}_{2,t}$ contains exogenous state variables. The dimensions of these vectors are $n_{x_1} \times 1$ and $n_{x_2} \times 1$, respectively, where $n_{x_1} + n_{x_2} = n_x$. It is further assumed that

$$\mathbf{x}_{2,t+1} = \Gamma\left(\mathbf{x}_{2,t}\right) + \sigma\widetilde{\boldsymbol{\eta}}\boldsymbol{\epsilon}_{t+1}, \tag{2}$$

where $\boldsymbol{\epsilon}_{t+1}$ has dimension $n_\epsilon \times 1$ and is independent and identical distributed with mean zero and covariance matrix $\mathbf{I}$. That is, $\boldsymbol{\epsilon}_{t+1} \sim \mathcal{IID}(\mathbf{0}, \mathbf{I})$. The function Γ takes elements from $\mathbb{R}^{n_{x_2}}$ into $\mathbb{R}^{n_{x_2}}$ and is required to be at least three times differentiable in the deterministic steady state. Moreover, all eigenvalues of $\partial\Gamma/\partial\mathbf{x}_{2,t}$ evaluated in the deterministic steady state must have modulus less than one. Finally, $\sigma \geq 0$ and $\widetilde{\boldsymbol{\eta}}$ is a known matrix with dimension $n_{x_2} \times n_\epsilon$

As observed by Schmitt-Grohé and Uribe (2004), the solution to this model is given by

$$\mathbf{y}_t = \mathbf{g}(\mathbf{x}_t, \sigma) \tag{3}$$

$$\mathbf{x}_{t+1} = \mathbf{h}(\mathbf{x}_t, \sigma) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \tag{4}$$

$$\boldsymbol{\eta} \equiv \begin{bmatrix} \mathbf{0}_{n_{x_1} \times n_\epsilon} \\ \widetilde{\boldsymbol{\eta}} \end{bmatrix} \tag{5}$$

The function $\mathbf{g}$ maps elements from $\mathbb{R}^{n_x} \times \mathbb{R}^+$ into $\mathbb{R}^{n_y}$ whereas $\mathbf{h}$ takes elements from $\mathbb{R}^{n_x} \times \mathbb{R}^+$ into $\mathbb{R}^{n_x}$. Both functions are unknown and assumed to be at least three times continuous differentiable at the deterministic steady state.

3 A third-order approximation

Substituting **(3)**-**(4)** into **(1)** gives

$$\mathbf{F}(\mathbf{x}, \sigma) \equiv E_t\left[\mathbf{f}\left(\mathbf{g}\left(\mathbf{h}(\mathbf{x}, \sigma) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}', \sigma\right), \mathbf{g}(\mathbf{x}, \sigma), \mathbf{h}(\mathbf{x}, \sigma) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}', \mathbf{x}\right)\right] = \mathbf{0}. \tag{6}$$

For simplicity, we omit the time subscript and use a prime to denote variables in period $t + 1$. A third-order approximation to $\mathbf{g}$ and $\mathbf{h}$ is stated in the appendix. The first and second-order derivatives of $\mathbf{g}$ and $\mathbf{h}$ at the deterministic steady state are computed in Schmitt-Grohé and Uribe (2004). We now derive the expression for all third-order terms using the fact that derivatives of

$\mathbf{F}(\mathbf{x},\sigma)$ are zero. Computing these third-order terms is greatly simplified by the fact that only linear systems need to be solved as noted by Judd and Guu (1997).

3.1 Computing the third-order terms

We start by computing $\mathbf{g_{xxx}}$ and $\mathbf{h_{xxx}}$ which is done from the third derivative of $\mathbf{F}(\mathbf{x},\sigma)$ with respect to $\mathbf{x}$ and evaluated in the deterministic steady state. This gives

$$\begin{aligned}
\left[\mathbf{F_{xxx}}(\mathbf{x}_{ss},0)\right]^i_{\alpha_1\alpha_2\alpha_3} &= \left[\mathbf{f_{y'}}\right]^i_{\beta_1}\left[\mathbf{g_{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{f_{y'}}\right]^i_{\beta_1}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h_{xxx}}\right]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3} \\
&\quad+\left[\mathbf{f_y}\right]^i_{\beta_1}\left[\mathbf{g_{xxx}}\right]^{\beta_1}_{\alpha_1\alpha_2\alpha_3}+\left[\mathbf{f_{x'}}\right]^i_{\gamma_1}\left[\mathbf{h_{xxx}}\right]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}+\left[b^1\right]^i_{\alpha_1\alpha_2\alpha_3} \\
&= 0
\end{aligned} \tag{7}$$

for $i=1,2,...,n$ and $\alpha_1,\alpha_2,\alpha_3,=1,2,...,n_x$. Here, we apply the tensor notation as in Schmitt-Grohé and Uribe (2004).[1] The expression for $\left[b^1\right]^i_{\alpha_1\alpha_2\alpha_3}$ is stated in the appendix and depends on $\mathbf{g_x}$, $\mathbf{h_x}$, $\mathbf{g_{xx}}$, $\mathbf{h_{xx}}$ along with first, second, and third-order derivatives of $\mathbf{f}$. It is important to note that $[b]^i_{\alpha_1\alpha_2\alpha_3}$ is known and different from zero. Hence, the linear system in **(7)** with $\left(n_y+n_x\right)\times n_x\times n_x\times n_x$ equations in as many unknowns is straightforward to solve and implies that $\mathbf{g_{xxx}}$ and $\mathbf{h_{xxx}}$ are non-zero matrices.

The terms $\mathbf{g}_{\sigma\sigma\mathbf{x}}$ and $\mathbf{h}_{\sigma\sigma\mathbf{x}}$ can be found by differentiating $\mathbf{F}(\mathbf{x},\sigma)$ twice with respect to σ and once with respect to $\mathbf{x}$. Evaluated at $(\mathbf{x}_{ss},0)$ we get

$$\begin{aligned}
\left[\mathbf{F}_{\sigma\sigma\mathbf{x}}(\mathbf{x}_{ss},0)\right]^i_{\alpha_3} &= \left[\mathbf{f_{y'}}\right]^i_{\beta_1}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}+\left[\mathbf{f_{y'}}\right]^i_{\beta_1}\left[\mathbf{g}_{\sigma\sigma\mathbf{x}}\right]^{\beta_1}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3} \\
&\quad+\left[\mathbf{f_y}\right]^i_{\beta_1}\left[\mathbf{g}_{\sigma\sigma\mathbf{x}}\right]^{\beta_1}_{\alpha_3}+\left[\mathbf{f_{x'}}\right]^i_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}+\left[b^2\right]^i_{\alpha_3} \\
&= 0
\end{aligned} \tag{8}$$

for $i=1,2,...,n$ and $\alpha_3=1,2,...,n_x$. From the expression of $\left[b^2\right]^i_{\alpha_3}$ in the appendix we see that its value is known and non-zero. The latter follows from the fact that second moments of ϵ_{t+1} are non-zero which also ensure that $\mathbf{g}_{\sigma\sigma}$ and $\mathbf{h}_{\sigma\sigma}$ are non-zero (see Schmitt-Grohé and Uribe (2004)). As a result, the linear system in **(8)** with $\left(n_y+n_x\right)\times n_x$ equations in the same number of unknowns allow us to solve for $\mathbf{g}_{\sigma\sigma\mathbf{x}}$ and $\mathbf{h}_{\sigma\sigma\mathbf{x}}$ which in general are non-zero matrices. Hence,

[1] For example, $\left[\mathbf{f_y}\right]^i_{\beta_1}$ is the (i,β_1) element in the $n\times n_y$ matrix $\mathbf{f_y}$ containing derivatives of $\mathbf{f}$ with respect to $\mathbf{y}$, and $\left[\mathbf{f_y}\right]^i_{\beta_1}\left[\mathbf{g_{xxx}}\right]^{\beta_1}_{\alpha_1\alpha_2\alpha_3}=\sum_{\beta_1=1}^{n_y}\frac{\partial f(i)}{\partial y(\beta_1)}\frac{\partial^3 g(\beta_1)}{\partial x(\alpha_1)\partial x(\alpha_2)\partial x(\alpha_3)}$.

a third-order approximation to **g** and **h** imply a correction for uncertainty in terms which are linear in the state vector **x.** As we will show in the next section, this is the uncertainty correction which generates time-variation in risk premia.

Proceeding in a similar manner, the values of $\mathbf{g}_{\sigma\mathbf{xx}}$ and $\mathbf{h}_{\sigma\mathbf{xx}}$ are given by

$$\begin{aligned}\left[\mathbf{F}_{\sigma\mathbf{xx}}\left(\mathbf{x}_{ss},0\right)\right]^{i}_{\alpha_2\alpha_3} &= \left[\mathbf{f}_{\mathbf{y}'}\right]^{i}_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\sigma\mathbf{xx}}\right]^{\gamma_1}_{\alpha_2\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'}\right]^{i}_{\beta_1}\left[\mathbf{g}_{\sigma\mathbf{xx}}\right]^{\beta_1}_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \\ &\quad +\left[\mathbf{f}_{\mathbf{y}}\right]^{i}_{\beta_1}\left[\mathbf{g}_{\sigma\mathbf{xx}}\right]^{\beta_1}_{\alpha_2\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'}\right]^{i}_{\gamma_1}\left[\mathbf{h}_{\sigma\mathbf{xx}}\right]^{\gamma_1}_{\alpha_2\alpha_3} \\ &= 0\end{aligned} \qquad \textbf{(9)}$$

for $i=1,2,...,n$ and $\alpha_2,\alpha_3=1,2,...,n_x$. This linear system of $\left(n_y+n_x\right)\times n_x\times n_x$ equations in as many unknowns is homogenous and it therefore follows that $\mathbf{g}_{\sigma\mathbf{xx}}=\mathbf{0}$ and $\mathbf{h}_{\sigma\mathbf{xx}}=\mathbf{0}$. Hence, a third-order approximation to **g** and **h** does not imply a correction for uncertainty in terms which are quadratic in the state vector **x.**

Finally, the derivatives $\mathbf{g}_{\sigma\sigma\sigma}$ and $\mathbf{h}_{\sigma\sigma\sigma}$ are determined by

$$\begin{aligned}\left[\mathbf{F}_{\sigma\sigma\sigma}\left(\mathbf{x}_{ss},0\right)\right]^{i} &= \left(\left[\mathbf{f}_{\mathbf{y}'}\right]^{i}_{\beta_1}+\left[\mathbf{f}_{\mathbf{y}}\right]^{i}_{\beta_1}\right)\left[\mathbf{g}_{\sigma\sigma\sigma}\right]^{\beta_1}+\left(\left[\mathbf{f}_{\mathbf{y}'}\right]^{i}_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}+\left[\mathbf{f}_{\mathbf{y}}\right]^{i}_{\beta_1}\right)\left[\mathbf{h}_{\sigma\sigma\sigma}\right]^{\gamma_1} \\ &\quad +\left[b^3\right]^{i} \\ &= 0\end{aligned} \qquad \textbf{(10)}$$

for $i=1,2,...,n$. From the expression of $\left[b^3\right]^{i}$ in the appendix, we first note that $\left[b^3\right]^{i}=0$ if all innovations have symmetric distributions, ie all third moments are zero. In this case, the linear system in **(10)** is homogenous and we therefore have $\mathbf{g}_{\sigma\sigma\sigma}=\mathbf{0}$ and $\mathbf{h}_{\sigma\sigma\sigma}=\mathbf{0}$. However, if some innovations have non-symmetric distributions, then $\left[b^3\right]^{i}$ may be different from zero and $\mathbf{g}_{\sigma\sigma\sigma}$ and $\mathbf{h}_{\sigma\sigma\sigma}$ may therefore also be different from zero. Innovations with this property are widely used in the literature which uses rare disasters to explain asset pricing puzzles (see Rietz (1988), Barro (2006), Gabaix (2008), Barro (2009), among others). Hence, when we allow for rare disasters in DSGE models, the constant terms in a third approximation may require a further correction for uncertainty than implied by the second-order terms $\mathbf{g}_{\sigma\sigma}$ and $\mathbf{h}_{\sigma\sigma}$.

This finding is contrary to the conjecture made in Schmitt-Grohé and Uribe (2004) about $\mathbf{g}_{\sigma\sigma\sigma}$ and $\mathbf{h}_{\sigma\sigma\sigma}$, as they always anticipate these terms to be zero. Schmitt-Grohé and Uribe (2004) form their conjecture based on numerical results in Aruoba, Fernandez-Villaverde and Rubio-Ramirez

(2006) for the neoclassical growth model with Gaussian distributed innovations where $\mathbf{g}_{\sigma\sigma\sigma}$ and $\mathbf{h}_{\sigma\sigma\sigma}$ are zero. It is well known that the Gaussian distribution is symmetric and our results are therefore in line with their findings. This example illustrates the usefulness of having closed-form expressions for the third-order terms as inferring their general values from numerical exercises may be difficult.

We summarise the key results from this section in the following theorem:

Theorem 1 For the class of models in **(1)-(2)** approximated around the deterministic steady state, it holds that $\mathbf{g}_{\sigma\mathbf{xx}} = \mathbf{0}$ and $\mathbf{h}_{\sigma\mathbf{xx}} = \mathbf{0}$. If all innovations have symmetric distributions, then $\mathbf{g}_{\sigma\sigma\sigma} = \mathbf{0}$ and $\mathbf{h}_{\sigma\sigma\sigma} = \mathbf{0}$.

3.2 Implementation of the derived formulas

The derived formulas are implemented in a set of Matlab functions which are publicly available on the Bank of England's website or on request. The structure of these codes is explained in the appendix.

We test our derived formulas and their implementation on three examples. First, our codes reproduce the third-order terms for the neoclassical growth model computed in the software program Mathematica by Aruoba *et al* (2006). Second, we also replicate the third-order terms in the model by Fernández-Villaverde *et al* (2010) which they compute in Mathematica.[2] The use of symbolic manipulations and solution algorithms for linear systems in Mathematica minimises the risk of errors, and we therefore consider the two sets of Mathematica codes as very reliable benchmarks.

Third, our codes also reproduce the third-order terms in the model by Lucas (1978) which Tsionas (2003) solves in closed form for innovations with arbitrary distributions. The correctness of our codes is verified in the case with Gaussian distributed innovations and innovations of the form $u_t \equiv 1 - \epsilon_t$ where ϵ_t is exponential distributed with a mean value of 1. The latter specification is interesting because the third moment is -2 and leads to a non-zero value of $\mathbf{g}_{\sigma\sigma\sigma}$.

[2]For simplicity, a version of their model without stochastic volatility is considered. We are grateful to Juan Rubio-Ramirez for providing the third-order approximated solution to their model.

All these three tests support the derived formulas and their implementation.

4 Theoretical results for risk premia

We first present the general expression for all risk premia at third order in Section 4.1. The following three sections then apply our theoretical results from Section 3 to study how innovations with non-symmetric distributions, stochastic volatility, and GARCH processes affect risk premia in DSGE models approximated up to third order. Innovations with special fourth or fifth moments are not discussed, because only second and third moments of the innovations affect the approximated solution at third order as shown in the previous section.

4.1 The expression for risk premia at third order

We start by considering the general expression for all risk premia in DSGE models approximated up to third order around the deterministic steady state. The value of this risk premia is denoted P_t and could be equity risk premia, term premia, or exchange rate risk premia. The absence of uncertainty at the steady state means that all risk premia are zero in the approximation point. This further implies that derivatives solely with respect to the state vector, ie $P_{\mathbf{x}}$, $P_{\mathbf{xx}}$, and $P_{\mathbf{xxx}}$, are also zero as these terms do not capture effects of uncertainty.[3] On the other hand, derivatives with respect to the perturbation parameter σ account for the presence of uncertainty and may therefore be non-zero for risk premia. As a result, all risk premia in a third-order approximation around the deterministic steady state have the general form

$$\begin{aligned} P_t \;=\; & P_\sigma \sigma + [\mathbf{P}_{\sigma\mathbf{x}}]_{\alpha_2} \sigma [\mathbf{x}_t]^{\alpha_2} + \frac{1}{2} P_{\sigma\sigma}\sigma^2 \\ & + \frac{1}{6} P_{\sigma\sigma\sigma}\sigma^3 + \frac{3}{6}[\mathbf{P}_{\sigma\sigma\mathbf{x}}]_{\alpha_3}\sigma^2 [\mathbf{x}_t]^{\alpha_3} + \frac{3}{6}[\mathbf{P}_{\sigma\mathbf{xx}}]_{\alpha_2\alpha_3}\sigma [\mathbf{x}_t]^{\alpha_2}[\mathbf{x}_t]^{\alpha_3} \end{aligned} \tag{11}$$

for $\alpha_2, \alpha_3 = 1, 2, ..., n_x$.

4.2 Innovations with non-symmetric distributions

This section examines how risk premia are affected by innovations with non-symmetric distributions, for instance due to rare disasters as in Barro (2006). The second-order

[3] This property of a perturbation approximation around the steady state may even be taken to be a requirement when defining risk premia.

approximated solution is independent of third-order moments as shown in Schmitt-Grohé and Uribe (2004). We therefore have trivially that innovations with non-symmetric distributions do not affect risk premia at second order.

For a third-order approximation, the results in Schmitt-Grohé and Uribe (2004) imply $P_\sigma = 0$ and $\mathbf{P}_{\sigma\mathbf{x}} = \mathbf{0}$ and our Theorem 1 gives $\mathbf{P}_{\sigma\mathbf{xx}} = \mathbf{0}$. Hence, the expression for risk premia is

$$P_t = \frac{1}{2}P_{\sigma\sigma}\sigma^2 + \frac{1}{6}P_{\sigma\sigma\sigma}\sigma^3 + \frac{3}{6}\left[\mathbf{P}_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}\sigma^2\left[\mathbf{x}_t\right]^{\alpha_3}. \tag{12}$$

Theorem 1 also implies that $P_{\sigma\sigma\sigma} = 0$ when all third moments of the innovations are zero, otherwise $P_{\sigma\sigma\sigma} \neq 0$. Accordingly, the mean value of risk premia is affected by the constant $\frac{1}{6}P_{\sigma\sigma\sigma}\sigma^3$ when DSGE models have innovations with non-symmetric distributions. Our formulas in Section 3 also show that such distributions do not affect the value of $\mathbf{P}_{\sigma\sigma\mathbf{x}}$ because this term only depends on second moments of the innovations. Hence, for a given variance of $\mathbf{x}_t$, innovations with non-symmetric distributions do not affect the variability of risk premia which is determined by the term $\frac{3}{6}\left[\mathbf{P}_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}\sigma^2\left[\mathbf{x}_t\right]^{\alpha_3}$. Accordingly, accounting for rare disasters as in Barro (2006) affects the mean level of risk premia but does not generate additional variability in risk premia when DSGE models are solved up to third order.[4]

We summarise these results in the following propositions:

Proposition 1 A second-order approximation does not capture effects from innovations with non-symmetric distributions. Thus, innovations with non-symmetric distributions do not affect risk premia at second order.

Proposition 2 In a third-order approximation, innovations with non-symmetric distributions affect the level of risk premia but they do not, for a given variance of $\mathbf{x}_t$, affect the variability of risk premia.

[4]Rare disasters may in a fourth-order approximation generate additional variation in risk premia because the term $\mathbf{P}_{\sigma\sigma\sigma\mathbf{x}}$ may be non-zero.

4.3 *Stochastic volatility*

Recent work by Fernández-Villaverde and Rubio-Ramírez (2007) and Justiniano and Primiceri (2008) have documented the importance of stochastic volatility in the post-war US economy. This section briefly presents their specification of stochastic volatility before deriving the implication of such processes for risk premia in DSGE models.

Justiniano and Primiceri (2008) use a specification where an exogenous process a_t evolves according to

$$\ln\left(\frac{a_{t+1}}{a_{ss}}\right) = \rho_a \ln\left(\frac{a_t}{a_{ss}}\right) + \sigma_{a,t+1}\epsilon_{a,t+1}, \quad \textbf{(13)}$$

and $\epsilon_{a,t} \sim \mathcal{NID}(0,1)$. In the present discussion, we let a_t denote the level of productivity and a_{ss} is the steady-state level of a_t. The difference from the standard log-normal process is that the conditional volatility $\sigma_{a,t}$ in **(13)** is time-varying and changes according to

$$\ln\left(\frac{\sigma_{a,t+1}}{\sigma_{a,ss}}\right) = \rho_\sigma \ln\left(\frac{\sigma_{a,t}}{\sigma_{a,ss}}\right) + \epsilon_{\sigma,t+1}, \quad \textbf{(14)}$$

where $\epsilon_{\sigma,t} \sim \mathcal{NID}\left(0, Var\left(\epsilon_{\sigma,t}\right)\right)$. The innovations $\epsilon_{a,t+1}$ and $\epsilon_{\sigma,t+1}$ are assumed to be mutually independent at all leads and lags.

A potential problem with stochastic volatility in our framework in **(1)**-**(2)** is that $\epsilon_{a,t+1}$ do not enter linearly in **(13)** because $\epsilon_{a,t+1}$ is scaled by the state variable $\sigma_{a,t+1}$. However, it is straightforward to find an equivalent representation of **(13)** where $\epsilon_{a,t+1}$ enters linearly and therefore fits into our framework.[5] The representation we consider is given by

$$\ln\left(\frac{a_t}{a_{ss}}\right) = \sigma_{a,t} \ln \upsilon_t \quad \textbf{(15)}$$

$$\ln \upsilon_{t+1} = \rho_a \frac{\sigma_{a,t}}{\sigma_{a,t+1}} \ln \upsilon_t + \epsilon_{a,t+1} \quad \textbf{(16)}$$

where $\upsilon_t = 1$ in the steady state. To see that **(15)** and **(16)** are equivalent to **(13)**, first lead **(15)** by one period and insert **(16)**. This gives

$$\ln\left(\frac{a_{t+1}}{a_{ss}}\right) = \sigma_{a,t+1}\left(\rho_a \frac{\sigma_{a,t}}{\sigma_{a,t+1}} \ln \upsilon_t + \epsilon_{a,t+1}\right).$$

[5] The implementation of perturbation in the packages *Dynare*, *Dynare++*, and *Perturbation AIM* (see Kamenik (2005) and Swanson *et al* (2005), respectively) do not restrict shocks to enter linearly. Hence, no transformation is required in these packages when including stochastic volatility into DSGE models.

Form **(15)**, $\ln\left(\frac{a_t}{a_{ss}}\right)/\sigma_{a,t} = \ln \upsilon_t$ and **(17)** therefore simplifies to **(13)** as claimed. This alternative representation implies that the non-linearity in **(13)** is moved from the innovations to the local persistency coefficient in the auxiliary process $\ln \upsilon_t$, and the processes **(15)** and **(16)** therefore fit into our framework.[6]

Returning to the implication for risk premia, the results in Schmitt-Grohé and Uribe (2004) imply $P_\sigma = 0$ and $\mathbf{P}_{\sigma\mathbf{x}} = \mathbf{0}$, and we therefore have in a second-order approximation that

$$P_t = \frac{1}{2} P_{\sigma\sigma}\sigma^2. \tag{17}$$

The value of $P_{\sigma\sigma}$ is a linear combination of the second moments to the structural innovations (see Schmitt-Grohé and Uribe (2004)). Stochastic volatility adds additional state variables and innovations to the model, and this may therefore affect the value of $P_{\sigma\sigma}$. That is, stochastic volatility may affect the level of risk premia. We also note that stochastic volatility does not generate time-variation in risk premia at second order because $\mathbf{P}_{\sigma\mathbf{x}} = \mathbf{0}$. This result may at first appear surprising because risk premia is a combination of second moments and one might therefore expect that changing the conditional second moments by stochastic volatility would give rise to time-varying risk premia. The reason that this intuitive explanation does not carry through is because all second moments are constant at our approximation point, even with stochastic volatility, and the level correction $P_{\sigma\sigma}$ therefore remains constant. In other words, a second-order approximation does not capture the general effect that stochastic volatility results in time-varying risk premia.

For a third-order approximation, Theorem 1 in Section 3 implies the following expression for risk premia

$$P_t = \frac{1}{2} P_{\sigma\sigma}\sigma^2 + \frac{1}{6} P_{\sigma\sigma\sigma}\sigma^3 + \frac{3}{6}\left[\mathbf{P}_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3} \sigma^2 \left[\mathbf{x}_t\right]^{\alpha_3}, \tag{18}$$

when some innovations have non-zero third moments, otherwise $P_{\sigma\sigma\sigma} = 0$. We note that the term $P_{\sigma\sigma\sigma}$ may change if stochastic volatility induces additional innovations with non-symmetric distributions. As mentioned above, stochastic volatility adds the conditional volatilities to the state vector and this gives more terms in $\left[\mathbf{P}_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3} \sigma^2 \left[\mathbf{x}_t\right]^{\alpha_3}$. Hence, also the variability of risk

[6] It is straightforward to show that the alternative specifications of stochastic volatility considered in Andreasen (2010) can be re-expressed in a similar manner. Note also that transformations of shock processes are widely used when DSGE models are solved numerically. For instance, the well-known log-normal process $a_{t+1} = a_{ss}^{1-\rho_a} a_t^{\rho_a} e^{\sigma_a \epsilon_{a,t+1}}$ is in most cases represented through a log-transformation to simplify the approximation.

premia may be affected by the presence of stochastic volatility in DSGE models when a third-order approximation is used.

We summarise the results from this section in the next two propositions:

Proposition 3 In a second-order approximation, stochastic volatility may affect the level of risk premia but it does not generate time-variation in risk premia.

Proposition 4 In a third-order approximation, stochastic volatility may affect the level of risk premia and it may generate additional time-variation in risk premia.

4.4 GARCH processes

Following the work of Bollerslev (1986), GARCH processes have emerged as a popular way to specify conditional heteroscedasticity. This section describes how GARCH processes can be introduced in DSGE models before deriving their implications for risk premia.

We consider the implementation of the widely used GARCH(1,1) model with Gaussian distributed innovations.[7] One way to include this specification into DSGE models is to let a_t evolve according to

$$\ln\left(\frac{a_{t+1}}{a_{ss}}\right) = \rho_a \ln\left(\frac{a_t}{a_{ss}}\right) + \sigma_{a,t+1}\epsilon_{a,t+1} \tag{19}$$

$$\sigma^2_{a,t+1} = \sigma^2_{a,ss}\left(1-\rho_1\right) + \rho_1\sigma^2_{a,t} + \rho_2\sigma^2_{a,t}\epsilon^2_{a,t} \tag{20}$$

where $\epsilon_{a,t} \sim \mathcal{NID}\left(0, 1\right)$.[8] As in the standard log-normal specification of a_t without GARCH, we maintain the log-transformation of a_t in **(19)** to ensure positivity of the technology level. Contrary to stochastic volatility, the conditional volatility in a_{t+1} is in a GARCH process determined based on information in period t. The law of motion for σ^2_t in **(20)** follows the specification adopted in Bollerslev (1986). Here, $\sigma_{a,ss}, \rho_1, \rho_2 \geq 0$ are required to ensure non-negativity of $\sigma^2_{a,t}$, and wide-sense stationarity of $\sigma^2_{a,t}$ requires $\rho_1 + \rho_2 < 1$.

[7]Extensions to the general GARCH(p,q) model with potentially non-Gaussian innovations are straightforward.

[8]The constant term in equation **(20)** is deliberately scaled by $(1-\rho_1)$ because this ensures that $\sigma_{a,t} = \sigma_{a,ss}$ in the deterministic steady state where $\epsilon^2_{a,t} = 0$.

Using the same type of argument as in Section 4.3, an alternative representation of **(19)** is given by

$$\ln\left(\frac{a_t}{a_{ss}}\right) = \sigma_{a,t} \ln \upsilon_t \quad \textbf{(21)}$$

$$\ln \upsilon_{t+1} = \rho_a \frac{\sigma_{a,t}}{\sigma_{a,t+1}} \ln \upsilon_t + \epsilon_{a,t+1} \quad \textbf{(22)}$$

The process for $\sigma^2_{a,t}$ can therefore be expressed as

$$\sigma^2_{a,t+1} = \sigma^2_{a,ss}(1-\rho_1) + \rho_1 \sigma^2_t + \rho_2 \sigma^2_{a,t}\left(\ln \upsilon_t - \rho_a \frac{\sigma_{a,t-1}}{\sigma_{a,t}} \ln \upsilon_{t-1}\right)^2. \quad \textbf{(23)}$$

This equivalent representation shows that the GARCH process fits into our framework as the innovation $\epsilon_{a,t+1}$ only enters linearly in **(22)**. We also note that the GARCH(1,1) process for a_t induces $\sigma_{a,t}$, $\sigma_{a,t-1}$, and υ_{t-1} as additional state variables.

Based on this alternative representation of a process with GARCH, we clearly have for risk premia that $P_\sigma = 0$ and $\mathbf{P}_{\sigma\mathbf{x}} = \mathbf{0}$ due to the results in Schmitt-Grohé and Uribe (2004). That is, for a second-order approximation risk premia is given by

$$P_t = \frac{1}{2} P_{\sigma\sigma}\sigma^2. \quad \textbf{(24)}$$

The GARCH process adds additional state variables to the model and GARCH may therefore affect the value of $P_{\sigma\sigma}$ and hence the level of risk premia. More importantly, however, GARCH does not generate time-varying risk premia at second order because $\mathbf{P}_{\sigma\mathbf{x}} = \mathbf{0}$. This may seem counterintuitive because risk premia is a combination of second moments and GARCH generates time-variation in these moments. However, all second moments are constant at our approximation point, even with GARCH, and the level correction $P_{\sigma\sigma}$ therefore remains constant. Hence, a second-order approximation does not capture the general effect that GARCH results in time-varying risk premia.

For a third-order approximation, Theorem 1 in Section 3 implies the following expression for risk premia

$$P_t = \frac{1}{2} P_{\sigma\sigma}\sigma^2 + \frac{1}{6} P_{\sigma\sigma\sigma}\sigma^3 + \frac{3}{6}\left[\mathbf{P}_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3} \sigma^2 \left[\mathbf{x}_t\right]^{\alpha_3} \quad \textbf{(25)}$$

when some innovations have non-zero third moments, otherwise $P_{\sigma\sigma\sigma} = 0$. The additional state variables caused by the presence of GARCH induce more terms in $\left[\mathbf{P}_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3} \left[\mathbf{x}_t\right]^{\alpha_3}$, and GARCH may therefore affect the variability of risk premia.

The results from this section are summarised in the following two propositions:

Proposition 5 In a second-order approximation, GARCH may affect the level of risk premia but it does not generate time-variation in risk premia.

Proposition 6 In a third-order approximation, GARCH may affect the level of risk premia and it may generate additional time-variation in risk premia.

5 Quantitative effects for nominal term premia

This section examines the quantitative effects of innovations with non-symmetric distributions (ie rare disasters), stochastic volatility, and GARCH for the nominal term premia in a standard New Keynesian DSGE model solved to third order. We describe the model in Section 5.1 and discuss its calibration in Section 5.2. The following three sections examine the effects for the ten-year nominal term premium of introducing rare disasters, stochastic volatility, and GARCH into productivity shocks. We end the section by discussing which non-Gaussian shocks are most promising for matching term premia in the considered DSGE model.

5.1 A New Keynesian DSGE model

We consider a standard New Keynesian model extended with Epstein-Zin-Weil preferences as introduced by Epstein and Zin (1989) and Weil (1990). These preferences are included because Rudebusch and Swanson (2009) show that they help an otherwise standard DSGE model generate a more realistic nominal term premium.

Households: Following Rudebusch and Swanson (2009), the value function V_t for the representative household is given by

$$V_t \equiv \begin{cases} u_t + \beta \left(E_t \left[V_{t+1}^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}} & \text{for } u_t \geq 0 \\ u_t - \beta \left(E_t \left[(-V_{t+1})^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}} & \text{for } u_t \leq 0 \end{cases} . \tag{26}$$

with $\beta \in]0, 1[$ and $\alpha \in \mathbb{R}\backslash\{1\}$. As in Binsbergen, Fernandez-Villaverde, Koijen and Rubio-Ramirez (2010), the periodic utility function is assumed to be

$$u(c_t, n_t) \equiv \frac{\left(c_t^{\nu}(1-n_t)^{1-\nu}\right)^{1-\gamma}}{1-\gamma}, \qquad \textbf{(27)}$$

where $\gamma \in \mathbb{R}\backslash\{1\}$ and $\nu \in [0, 1]$. Here, c_t and n_t denotes consumption and labour supply, respectively. The specifications in **(26)** and **(27)** imply that the intertemporal elasticity of substitution is given by $1/(1-\nu(1-\gamma))$. A measure of the relative risk aversion which accounts for the leisure decision is $\gamma + \alpha(1-\gamma)$ according to Swanson (2010). Hence, the value of α controls the degree of relative risk aversion whereas ν and γ determine the size of the intertemporal elasticity of substitution.

The household's real budget constraint is given by

$$E_t M_{t,t+1} x_{t+1} + c_t = \frac{x_t}{\pi_t} + w_t n_t + d_t, \qquad \textbf{(28)}$$

where $M_{t,t+1}$ is the nominal stochastic discount factor, x_t are nominal state-contingent claims, π_t is inflation, w_t is the real wage, and d_t is a real lump-sum transfer.

The firms: Final output is produced by a perfectly competitive representative firm which uses a continuum of intermediate goods $y_t(i)$ and the production function

$$y_t = \left(\int_0^1 y_t(i)^{\frac{\eta-1}{\eta}} di\right)^{\frac{\eta}{\eta-1}} \qquad \textbf{(29)}$$

with $\eta > 1$. This implies

$$y_t(i) = \left(\frac{p_t(i)}{p_t}\right)^{-\eta} y_t, \qquad \textbf{(30)}$$

where the aggregate price level is given by $p_t = \left[\int_0^1 p_t(i)^{1-\eta} di\right]^{\frac{1}{1-\eta}}$.

All intermediate firms produce a slightly differentiated good using

$$y_t(i) = z_t a_t \bar{k}^{\theta} n_t(i)^{1-\theta} \qquad \textbf{(31)}$$

where $\bar{k}$ and $n_t(i)$ denote physical capital and labour services of the ith firm, respectively. The variable a_t represents exogenous stationary technology shocks specified below. We use z_t to capture a deterministic trend in technology, meaning that $\mu_{z,t} \equiv z_t/z_{t-1}$ and $\mu_{z,t} = \mu_{z,ss}$ for all t. Intermediate firms maximise the net present value of future profit when setting the optimal level of $n_t(i)$ and $p_t(i)$. Following Rotemberg (1982), we assume quadratic price adjustment costs controlled by $\xi \geq 0$, and the i'th firm therefore solves

$$\max_{n_t(i),p_t(i)} E_t \sum_{j=0}^{\infty} M_{t,t+j} \left[\frac{p_{t+j}(i)}{p_{t+j}} y_{t+j}(i) - w_{t+j} n_{t+j}(i) - \frac{\xi}{2} \left(\frac{p_{t+j}(i)}{p_{t+j-1}(i)} \frac{1}{\pi_{ss}} - 1 \right)^2 y_{t+j} \right]$$

subject to **(30)** and **(31)**.

The central bank: The behaviour of the central bank is given by a standard Taylor rule

$$r_t = r_{ss}(1-\rho_r) + \rho_r r_{t-1} + \phi_\pi \ln\left(\frac{\pi_t}{\pi_{ss}}\right) + \phi_y \ln\left(\frac{y_t}{z_t y_{ss}}\right) + \epsilon_{R,t} \quad \textbf{(32)}$$

where r_t is the continuously compounded nominal interest rate and $\epsilon_{R,t} \sim \mathcal{NID}\left(0, Var\left(\epsilon_{R,t}\right)\right)$. Similar to Justiniano and Primiceri (2008) and Rudebusch and Swanson (2009), the output gap is measured in terms of output in deviation from its balanced growth path.

Aggregation: Simple aggregation implies $y_t = z_t a_t \bar{k}^{\theta} n_t^{1-\theta}$ because all intermediate firms are identical. The presence of a government sector is specified in a standard way by letting $g_t z_t$ units of output being used for public consumption in every period. The value of g_t is exogenously given by

$$\ln\left(\frac{g_{t+1}}{g_{ss}}\right) = \rho_g \ln\left(\frac{g_t}{g_{ss}}\right) + \epsilon_{g,t+1}, \quad \textbf{(33)}$$

where $\epsilon_{g,t} \sim \mathcal{NID}\left(0, Var\left(\epsilon_{g,t}\right)\right)$. Following Rudebusch and Swanson (2009) we also assume that $\delta\bar{k}z_t$ units of output are used in every period to maintain the fixed capital stock. As a result, the aggregate resource constraint is

$$y_t = c_t + g_t z_t + \delta\bar{k}z_t. \quad \textbf{(34)}$$

Technology shock: The model is closed by specifying an exogenous process for technology, a_t. Our benchmark is the log-normal process where

$$\ln\left(\frac{a_{t+1}}{a_{ss}}\right) = \rho_a \ln\left(\frac{a_t}{a_{ss}}\right) + \epsilon_{a,t+1}, \quad \textbf{(35)}$$

and $\epsilon_{a,t} \sim \mathcal{NID}\left(0, Var\left(\epsilon_{a,t}\right)\right)$. The three alternative specifications of a_t we consider are as follows. The first deals with non-symmetric innovations (ie rate disasters) where a_t evolves as in **(35)** but $\epsilon_{a,t+1}$ is given by

$$\widetilde{\epsilon}_{a,t+1} = \begin{cases} \mathcal{N}(0,1) & \text{with probability } 1-p \\ \varphi & \text{with probability } p \end{cases}, \quad \textbf{(36)}$$

where

$$\epsilon_{a,t+1} = \frac{\widetilde{\epsilon}_{a,t+1} - E\left(\widetilde{\epsilon}_{a,t+1}\right)}{\sqrt{Var\left(\widetilde{\epsilon}_{a,t+1}\right)}}. \quad \textbf{(37)}$$

Hence, negative values of φ induce innovations with negative tails, and *vice versa* for positive values of φ. Our second specification considers the case where technology shocks display stochastic volatility as described in Section 4.3. The final specification analyses the case where technology evolves as in Section 4.4 and a GARCH(1,1) process controls the conditional volatility in technology.

5.2 Calibration and benchmark results

The model is calibrated to match first and second moments for consumption growth, inflation, the three-month nominal interest rate, the ten-year nominal interest rate, the ten-year nominal term premium, and the ten-year excess holding period return.[9] Our calibration is fairly standard and summarised in Table A. Some of the coefficients deserve a few comments. First, we let $\nu = 0.35$ and $\gamma = 2.5$ which give an intertemporal elasticity of substitution of 0.66. The coefficient related to the Epstein-Zin-Weil preferences α is set to -110 as this implies a mean value of 108 basis points for the ten-year nominal term premium.[10] The empirical moment is in the neighbourhood of 106 basis points according to Rudebusch and Swanson (2009). Given these values for ν, γ, and α, the relative risk aversion is 168 when using the measure stated in the previous section.

Second, β is assigned a relatively high value of 0.9995 to get a sufficiently low mean value for the three-month nominal interest rate. Our calibration implies a mean value of 5.82% in annual terms which is close to the empirical mean of 5.59%. Although the value of β may appear relatively high, the effective discount factor in the Euler-consumption equation is $\beta\mu_{z,ss}^{\nu(1-\gamma)-1} = 0.9919$ due to the deterministic trend in technology and hence fairly standard.

[9]We use data from the Federal Reserve Bank of St. Louis. The annualised growth rate in consumption is calculated from real consumption expenditures (PCECC96) and expressed in per capita based on the total population in the United States. The annual inflation rate is for consumer prices. The three-month nominal interest rate is measured by the rate in the secondary market (TB3MS), and the ten-year nominal rate is taken from Gürkaynak, Sack and Wright (2007). As in Rudebusch and Swanson (2009), observations for the ten-year interest rate from 1961 Q2 to 1971 Q3 are calculated by extrapolation of the estimated curves in Gürkaynak *et al* (2007). The first and second moments for the ten-year nominal term premium and the excess holding period return for the ten-year bond are from Rudebusch and Swanson (2009). Finally, all moments related to interest rates are in annualised terms.

[10]The 10-year nominal term premium in our model is computed by the difference between the 10-year interest rate and the yield to maturity on the corresponding risk neutral bond where payments are discounted by r_t instead of the stochastic discount factor.

Third, we set the coefficient for the quadratic price adjustment costs ξ to 260 which for a linearisation of the model corresponds to a Calvo coefficient of 0.75.[11]

Table A: Calibration of the New Keynesian model

γ	2.5	ρ_r	0.85
ν	0.35	n_{ss}	0.38
β	0.9995	$\frac{g_{ss}}{y_{ss}}$	0.17
α	-110	π_{ss}	1.008
θ	0.36	$\mu_{z,ss}$	1.005
η	6	ρ_a	0.98
ξ	260	ρ_g	0.90
δ	0.025	$std\left(\epsilon_{a,t}\right)$	0.0075
ϕ_π	1.5	$std\left(\epsilon_{g,t}\right)$	0.004
ϕ_y	0.3	$std\left(\epsilon_{r,t}\right)$	0.003

The first two columns in Table B show empirical moments and simulated moments for our benchmark model, respectively. We see that the model is fairly successful in matching the standard deviations for consumption growth, inflation, and the two interest rates. The model is also able to produce a sizable slope for the nominal term structure of 97 basis points, although this value is somewhat lower than the empirical value of 140 basis points. On the other hand, the model gives a mean excess holding period return for the 10-year nominal bond $xhr_{t,40}$ of 169 basis points which is slightly below the empirical moment of 176 basis points. The model struggles when it comes to explaining the variation in the 10-year term premium $std\left(P_{t,40}\right)$ which is only 2 basis points in the model compared to 54 basis points in the data. The benchmark model in Rudebusch and Swanson (2009) displays a similar shortcoming.

[11] It is straightforward to show for a linearised model that the general relationship between the Rotemberg parameter ξ and a Calvo parameter α_p, giving an average duration of prices of $\frac{1}{1-\alpha_p}$ periods, is $\xi = \frac{(1-\theta+\eta\theta)(\eta-1)\alpha_p}{(1-\alpha_p)(1-\theta)\left(1-\alpha_p\beta\mu_{Z,ss}^{\nu(1-\gamma)}\right)}$. Note that the presence of a deterministic trend and decreasing returns to scale in the production function (see Galí (2008)) modify the relation between ξ and α_p as stated in Keen and Wang (2007).

Table B: Technology shocks with non-symmetric innovations

Moments in the model are computed from a simulated time series of length 2,000,000 for a third-order approximation to the model.

	1961-2007	Benchmark	Non-symmetric Case I	Non-symmetric Case II
Key moments (in pct.)	(1)	(2)	(3)	(4)
$std\,(\Delta c_t)$	2.69	3.99	4.02	3.99
$std\,(\pi_t)$	2.49	2.23	2.19	2.22
$std\,(r_t)$	2.71	2.89	2.88	2.88
$std\,(r_{t,40})$	2.41	1.99	1.98	1.99
$mean\,(r_{t,40} - r_t)$	1.40	0.97	1.72	1.04
$std\,(r_{t,40} - r_t)$	1.39	1.19	1.18	1.18
$mean\,(P_{t,40})$	1.06	1.08	1.81	1.14
$std\,(P_{t,40})$	0.54	0.02	0.02	0.02
$mean\,(xhr_{t,40})$	1.76	1.69	2.95	1.81
$std\,(xhr_{t,40})$	23.43	15.03	14.99	15.03
Skewness				
Δc_t	-0.69	0.00	-7.94	-0.68
π_t	1.22	-0.04	1.59	0.07
r_t	1.05	-0.01	1.34	0.09
$r_{t,40}$	0.97	0.00	1.62	0.11
Kurtosis				
Δc_t	5.75	3.00	99.07	6.43
π_t	4.24	3.02	6.26	3.11
r_t	4.58	3.01	5.57	3.09
$r_{t,40}$	3.60	3.02	6.42	3.12
Properties				
$std\,(a_t)$	-	0.0377	0.0377	0.0376
$skew\,(a_t)$	-	0.00	-1.64	-0.12
$kurt\,(a_t)$	-	3.01	6.52	3.13
φ	-	0	-38.67	-7.00
$E\left[\epsilon_{a,t}^3\right]$	-	0	-12.19	-1.03

5.3 *Rare disasters in technology shock*

We use the work by Barro (2006) on rare disasters to calibrate the shape of the non-symmetric distribution in **(36)**. Barro (2006) estimates disasters to happen with a probability of 1.7% every year which in our quarterly model corresponds to $p = 0.0043$. The value of φ is initially set as in Barro (2006) to generate a reduction of 0.29 in technology, which in Barro's model gives an equivalent reduction in output.

The third column in Table B shows that this calibration of rare disasters hardly changes the standard deviations for consumption growth, inflation, and the two interest rates. A notable effect appears for the mean slope of the term structure which increases from 97 basis points in the benchmark model to 172 basis points. This increase is generated by lowering the mean level of the three-month rate (from 5.82% to 5.30%) and by increasing the mean level of the ten-year rate (from 6.79% to 7.02%). The economic intuition behind these results is as follows. A positive probability of a large reduction in consumption during a disaster leads the household to increase its level of precautionary saving. The household therefore requires a lower compensation for postponing consumption, and this explains the lower mean level for the three-month interest rate. Investing in the ten-year nominal bond becomes even more risky with the presence of disasters because such an event generates high inflation which erodes the real value of this bond when its pay-off is most needed to maintain a smooth consumption profile. This lowers the equilibrium price level for the ten-year bond which is equivalent to a higher interest rate.

The presence of rare disasters is further seen to increase the term premium from 108 to 181 basis points. The variation in this premium is, however, not affected in a third-order approximation to our model. As for the higher-order moments, we see that rare disasters increase skewness and kurtosis for consumption growth, inflation, and the two interest rates far beyond the values implied by a Gaussian distribution.

Another calibration of rare disasters is to account for the fact that a disaster lasts for several periods in our model and not just for one period as initially assumed when setting φ. In our second calibration, we consider the case where a disaster lasts for five years. Accounting for the trend in output of 1.005 per quarter, a disaster then reduces output by roughly 40% when compared to the trend level of output over a five-year period. We model this scenario by reducing the value of φ from -38.67 to -7.00, which effectively means lowering the size of rare disasters compared to the first calibration. Column four in Table B shows, as expected, that effects of rare disasters in this calibration are less pronounced. The values of all standard deviations are hardly affected compared to the benchmark model, and the mean slope and mean term premium only increase by 7 and 6 basis points, respectively. This second calibration broadly matches skewness and kurtosis for consumption growth whereas these moments for inflation and the two interest rates are too low compared to their empirical values.

Overall, we conclude that the presence of rare disasters in our DSGE model can have substantial effects on the level of the term premium along with skewness and kurtosis for consumption growth, inflation, and interest rates. However, rare disasters hardly affect the standard deviations of these macro variables.

5.4 Stochastic volatility in technology shock

We use the estimation results in Justiniano and Primiceri (2008) to calibrate the process for the stochastic volatility in technology. They find $\sqrt{Var\left(\epsilon_{\sigma,t}\right)}$ to be around 0.01 and assume $\rho_a = 1$. We initially choose the same value for $\sqrt{Var\left(\epsilon_{\sigma,t}\right)}$ but let $\rho_a = 0.99$ to get a highly persistent but stationary process for $\sigma_{a,t}$.

Comparing the second and third columns in Table C, we see that many of the moments are not affected by the presence of stochastic volatility. An important exception is the standard deviation in the term premium, which increases from 2 to 14 basis points. The mean values for term premium is, however, not affected by stochastic volatility.

To further explore effects of stochastic volatility, we next increase $\sqrt{Var\left(\epsilon_{\sigma,t}\right)}$ to 0.0265.[12] This leads to a slight increase in the variance of consumption growth, inflation, and the two interest rates when compared to the benchmark model. The mean level for term premia, the slope of the term structure, and the excess holding period return are still unchanged by stochastic volatility. A key difference, however, is a sizable increase in the standard deviation of the term premium from 2 basis points in the benchmark model to 35 basis points. The corresponding sample moment in the data is around 56 basis points, so stochastic volatility clearly brings the model closer to the data along this dimension. Hence, the inability of the benchmark model to generate sufficient variability in the term premium may be due to the omission of time-varying uncertainty in productivity shocks. We also note from Table C that this explanation is consistent with the fourth moments in consumption growth, inflation, and the two interest rates, because stochastic volatility increases kurtosis for these variables and also brings these moments closer to the data.

[12] We chose this value in order to make this calibration comparability to the second specification with GARCH in Section 5.5 as both models imply an unconditional standard deviation in $\sigma_{a,t}$ of 0.19.

Note in addition that stochastic volatility also induces a small increase in the variability of the excess holding period return.

The final column in Table C considers the benchmark model with the same unconditional variance in technology as in the model with the second calibration of stochastic volatility. We observe that the increased variability in the term premium along with higher values of kurtosis are not present in this version of the benchmark model. This means that these properties of the model are generated by stochastic volatility and not by the slightly higher variance in technology as induced by our calibration of stochastic volatility.

Table C: Technology shocks with stochastic volatility

Moments in the model are computed from a simulated time series of length 2,000,000 for a third-order approximation to the model. The calibration for the volatility process is $\rho_{\sigma_a} = 0.99$. All the remaining parameter values are as in Table A.

	1961-2007	Benchmark	Stoch. vol. Case I	Stoch. vol. Case II	No stoch. vol. High std.
Key moments (in pct.)	(1)	(2)	(3)	(4)	(5)
$std\,(\Delta c_t)$	2.69	3.99	4.00	4.13	4.19
$std\,(\pi_t)$	2.49	2.23	2.25	2.38	2.38
$std\,(r_t)$	2.71	2.89	2.92	3.06	3.06
$std\,(r_{t,40})$	2.41	1.99	2.02	2.14	2.12
$mean\,(r_{t,40} - r_t)$	1.40	0.97	0.97	0.97	1.10
$std\,(r_{t,40} - r_t)$	1.39	1.19	1.20	1.25	1.22
$mean\,(P_{t,40})$	1.06	1.08	1.08	1.08	1.23
$std\,(P_{t,40})$	0.54	0.02	0.14	0.35	0.03
$mean\,(xhr_{t,40})$	1.76	1.69	1.69	1.69	1.92
$std\,(xhr_{t,40})$	23.43	15.03	15.12	15.77	16.02
Skewness					
Δc_t	-0.69	0.00	0.00	0.00	0.00
π_t	1.22	-0.04	-0.03	-0.02	-0.05
r_t	1.05	-0.01	-0.01	-0.08	-0.01
$r_{t,40}$	0.97	0.00	0.02	0.09	0.00
Kurtosis					
Δc_t	5.75	3.00	3.04	3.26	3.00
π_t	4.24	3.02	3.10	3.71	3.02
r_t	4.58	3.01	3.08	3.59	3.01
$r_{t,40}$	3.60	3.02	3.10	3.71	3.02
Properties					
$std\,(a_t)$	-	0.0377	0.0382	0.0402	0.0403
$std\,(\sigma_{a,t})$	-	0	0.071	0.189	0
$skew\,(a_t)$	-	0.00	-0.01	-0.02	-0.01
$kurt\,(a_t)$	-	3.01	3.10	3.71	3.01
$std\,(\epsilon_{\sigma_a,t})$	-	0.00	0.01	0.0265	0

To explain the reactions in the economy when a volatility shock in technology hits the economy, it is useful to apply the terminology from finance where risk premia are decomposed into a market price of risk and a quantity of risk. Following Cochrane (2001), we define the market price of risk MPR_t in period t by $MPR_t = Var_t\left(M_{t,t+1}\right)/E_t\left(M_{t,t+1}\right)$. This means that the term premium P_t mechanically can be decomposed as

$$P_t = MPR_t \times \frac{P_t}{MPR_t},$$

where $\frac{P_t}{MPR_t}$ then is the quantity of risk. Table D shows mean values and standard deviations for $Var_t\left(M_{t,t+1}\right)$, the market price of risk, and the quantity of risk. We see that stochastic volatility does not change the mean level of the market price of risk, whereas the mean level for the quantity of risk increases slightly (from 0.157 to 0.166). Larger effects appear for the standard deviations in the market price of risk, which increases from 0.0009 in the benchmark model to 0.0065 with the second calibration of stochastic volatility. That corresponds to a 7-fold increase. The largest and most notable effect, however, is the increase in the standard deviation for the quantity of risk which increases from 0.0049 in the benchmark model to 0.1857 with our second calibration of stochastic volatility. This constitutes a 38-fold increase. Hence, the large standard deviation for the term premium in our model with stochastic volatility is primarily driven by a more volatile quantity of risk. This finding is similar to the implications from classical finance models such as Cox *et al* (1985) and Dai and Singleton (2002), where an increase in the uncertainty level increases the term premium through a higher quantity of risk.

Table D: Market price of risk and quantity of risk

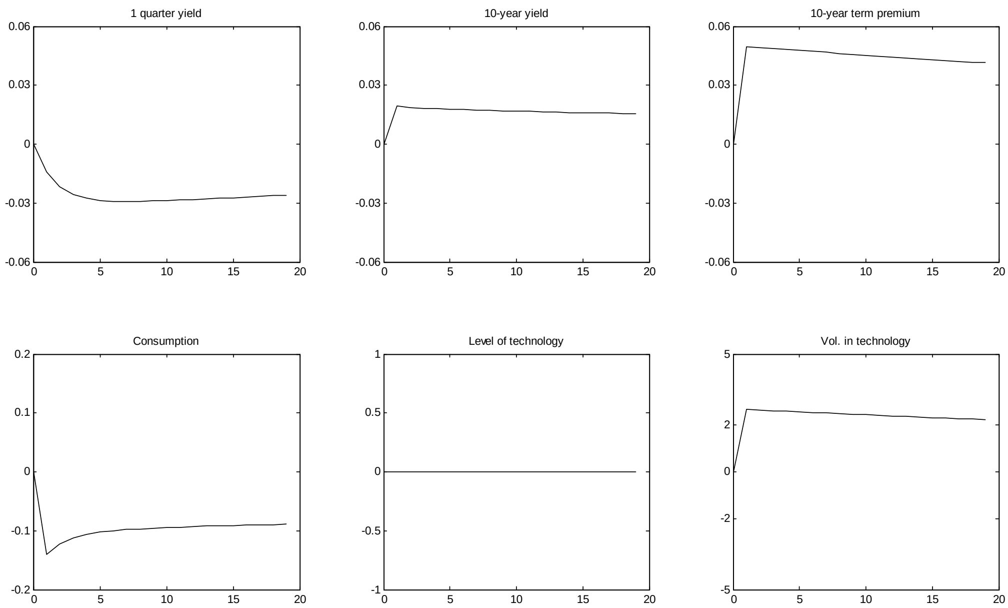
The moments are computed from a simulated time series of length 2,000,000 for a third-order approximation to the model. The term premium is not expressed in annual basis points for this decomposition.

	Benchmark	Stoch. vol. Case I	Stoch. vol. Case II	GARCH Case I	GARCH Case II
Mean					
$Var_t\left(M_{t,t+1}\right)$	0.017	0.017	0.017	0.021	0.042
Market price of risk	0.017	0.017	0.017	0.021	0.043
Quantity of risk	0.157	0.158	0.166	0.145	0.117
Standard deviation					
$Var_t\left(M_{t,t+1}\right)$	0.0008	0.0025	0.0064	0.0012	0.0061
Market price of risk	0.0009	0.0026	0.0065	0.0013	0.0062
Quantity of risk	0.0049	0.0060	0.1857	0.0044	0.0042

Impulse responses following a volatility shock in technology are reported in Chart 1. We first note that an increase in the uncertainty about the productivity level lowers consumption. This effect arises from the precautionary saving channel as the risk-averse household builds up a larger buffer stock during periods of higher uncertainty. As a result, the household requires a lower compensation for postponing consumption, and we therefore observe a fall in the three-month interest rate. The higher uncertainty level also generates an increase in the ten-year term premium, mainly due to an increase in the quantity of risk, and we therefore see a rise in the ten-year interest rate. Hence, a positive volatility shock in technology increases the slope of the term structure as the household requires a lower compensation for holding bonds with short maturities and a higher compensation for holding bonds with long maturities. Another interesting observation is that consumption falls and the term premium rises after a volatility shock in technology. This shock therefore helps the model generate countercyclical variation in the term premium as typically observed in the data.

Chart 1: Impulse responses to a volatility shock in technology

Impulse responses are for a one standard deviation shock to the conditional volatility in technology when the model is approximated up to third order. The effects for the two interest rates, the ten-year term premium, and consumption are expressed in annualised percentage deviation from the steady state. The level of technology and the volatility in technology are expressed in percentage deviation from the steady state. The calibration for the volatility process is $\rho_{\sigma_a} = 0.99$ and $\sqrt{Var\left(\epsilon_{\sigma,t}\right)} = 0.0265$. All the remaining parameter values are as stated in Table A.



To summarise, we find that stochastic volatility in technology shocks can generate sizable variation in the term premium without distorting the ability of the model to match a key number of macroeconomic moments. However, the considered specification of stochastic volatility does not affect the mean level of this premium.

5.5 *GARCH in technology shock*

The GARCH(1,1) process in technology is calibrated to mimic the specification of stochastic volatility for $\sigma_{a,t}$ in the previous section. To get a high degree of persistency in $\sigma_{a,t}$, we let the GARCH-coefficient ρ_1 be 0.95. Two values are considered for the ARCH-coefficient: $\rho_2 = 0.01$ and $\rho_2 = 0.04$.

Table D shows the effects of adding GARCH to our benchmark model. In our first calibration with $\rho_2 = 0.01$ (column three) we observe an increase in the standard deviations of consumption growth, inflation, and the two interest rates when compared to the benchmark model. The mean level for the slope of the term structure increases from 97 to 115 basis points, and we see a similar increase in the mean of the term premium (from 108 to 122 basis points). The standard deviation in this premium is still quite low (4 basis points), and all values of skewness and kurtosis are hardly affected given this calibration of GARCH.

In our second calibration where $\rho_2 = 0.04$, the standard deviations of consumption growth, inflation, and the two interest rates increases further as shown in column four of Table D. The mean slope of the term structure more than doubles from 97 in the benchmark model to 216 basis points. A similar increase is observed in the mean level for the term premium (from 97 to 199 basis points) and in the excess holding period return (169 to 328 basis points). We also see a sizable impact on the standard deviation of the term premium, which increases to 22 basis points. The corresponding empirical moment is around 54 basis points, so GARCH brings the benchmark model closer to the data along this dimension. GARCH also increases the standard deviation in the excess holding period return to 26.98 percent and is thus close to the empirical moment of 23.43 percent.

When interpreting these results it is important to note that GARCH increases the standard deviation of technology from 0.0378 in the benchmark model to 0.0437 in the second calibration. It is therefore interesting to see how much of the impact from GARCH is due to the larger variability in technology. Column five in Table D addresses this question by displaying moments for our model without GARCH but with a higher variance in technology. We see that the greater variability in technology can explain some of the increase in the standard deviation of consumption growth and all of the additional variation in inflation and the two interest rates.

However, the higher variability in technology only explains about 25% of the increase in the mean slope of the term structure and about 40% of the increase in the mean level of the term premium. More importantly, the variability in the term premium is hardly affected by the larger variance of technology, and this means that basically all of the additional variation in the term premium is due to GARCH. Broadly the same result holds for the variability in excess holding period return, where GARCH accounts for 80% of the increase in the standard deviation. We therefore conclude that the considered specification of GARCH has an independent impact on the level and the variability of the term premium in our model.

Table E: Technology shocks with GARCH

Moments in the model are computed from a simulated time series of length 2,000,000 for a third-order approximation to the model. The value of ρ_1 is 0.95. All the remaining parameter values are as in Table A.

	1961-2007	Benchmark	GARCH Case I	GARCH Case II	No GARCH High std.
Key moments (in pct.)	(1)	(2)	(3)	(4)	(5)
$std\,(\Delta c_t)$	2.69	3.98	4.35	6.53	4.49
$std\,(\pi_t)$	2.49	2.23	2.48	2.60	2.58
$std\,(r_t)$	2.71	2.89	3.19	3.26	3.30
$std\,(r_{t,40})$	2.41	1.99	2.22	2.30	2.31
$mean\,(r_{t,40} - r_t)$	1.40	0.97	1.15	2.16	1.30
$std\,(r_{t,40} - r_t)$	1.39	1.19	1.25	1.33	1.27
$mean\,(P_{t,40})$	1.06	1.08	1.22	1.99	1.45
$std\,(P_{t,40})$	0.54	0.02	0.04	0.22	0.03
$mean\,(xhr_{t,40})$	1.76	1.69	1.93	3.28	2.27
$std\,(xhr_{t,40})$	23.43	15.03	16.79	26.98	17.41
Skewness					
Δc_t	-0.69	0.00	-0.00	0.01	0.00
π_t	1.22	-0.04	-0.04	-0.04	-0.05
r_t	1.05	-0.01	-0.01	-0.01	-0.01
$r_{t,40}$	0.97	-0.00	0.00	0.00	-0.01
Kurtosis					
Δc_t	5.75	3.00	3.01	3.14	3.00
π_t	4.24	3.02	3.04	2.98	3.02
r_t	4.58	3.01	3.03	2.96	3.01
$r_{t,40}$	3.60	3.02	3.04	2.97	3.02
Properties					
$std\,(a_t)$	-	0.0377	0.0420	0.0437	0.0438
$std\,(\sigma_{a,t})$	-	0	0.0283	0.190	0
$skew\,(a_t)$	-	0.00	-0.01	0.00	-0.01
$kurt\,(a_t)$	-	3.01	3.04	2.97	3.01
ρ_2	-	0.00	0.01	0.04	0.00

Our results indicate that the effects of GARCH differ from those of stochastic volatility. Most importantly, GARCH affects the mean level of the premium (contrary to stochastic volatility), and GARCH has in general a smaller impact on the variability in the term premium when compared to stochastic volatility. The decomposition of the term premium in Table C illustrates another difference between the two methods to model conditional volatility in technology shocks. We see that GARCH increases the mean level for the market price of risk (from 0.017 to 0.043) and lowers the mean level for the quantity of risk (from 0.157 to 0.117) when compared to the benchmark model. Recall that stochastic volatility does not change the level for the market price of risk and marginally raises the mean quantity of risk. Secondly, GARCH increases the variability in the market price of risk and lowers the variation in the quantity of risk. The latter finding is different from the case with stochastic volatility.

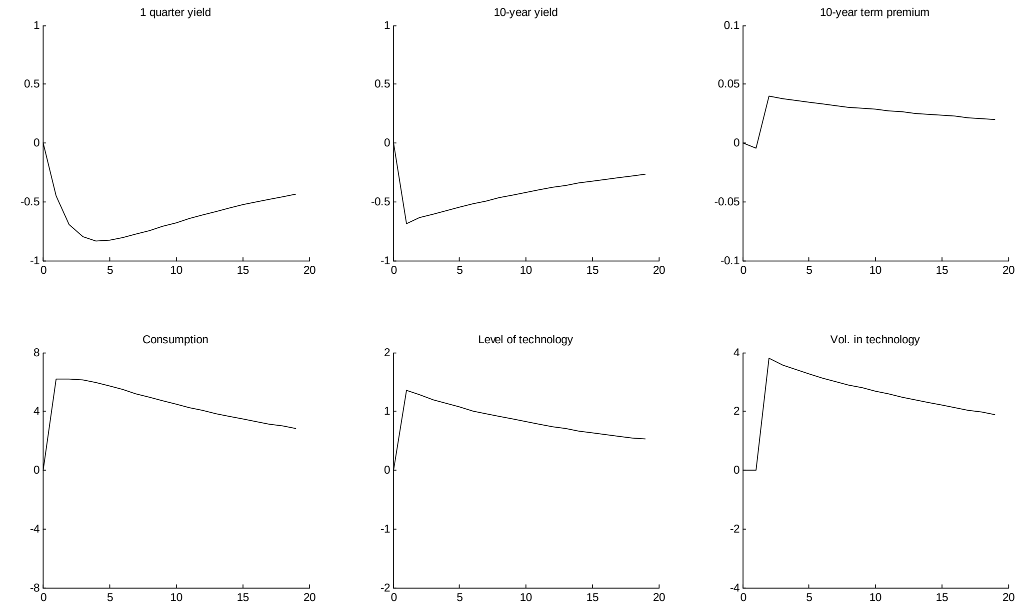
The explanation for the different effects of stochastic volatility and GARCH is as follows. Consider the version of our model with GARCH and a sizable negative shock to the technology level. This causes a large reduction in consumption and an increase in the conditional volatility of technology in the first period after the shock. The high value of the GARCH-coefficient and the absence of negative innovations to $\sigma_{a,t}$ imply that the conditional volatility in technology is high for many periods after the shock. The latter constitutes a risk to the household because the higher volatility increases the probability of another big negative technology shock and hence a further reduction in consumption. We see broadly the opposite effects following a positive technology shock, although the conditional volatility also increases in this case. However, the risk of these two events does not balance out because the risk-adverse household is mostly concerned with the first event. As a result, the household requires a higher compensation for entering into financial investments than without GARCH. The same mechanism is not present with stochastic volatility because the volatility level is not automatically high when consumption is low. This feature explains why the market price of risk is higher and more volatile with GARCH than with stochastic volatility.

Impulse response functions for a positive technology shock with GARCH are shown in Chart 2. On impact the conditional volatility in technology is unchanged, and we therefore have the well-known effects of a positive technology shock, ie lower nominal interest rates and an increase in consumption. Note also the small decrease in the term premium on impact. In the next period, the conditional volatility increases and this raises the term premium. Hence, we see a positive relationship between consumption and the term premium. Following a negative technology shock, which generates a reduction in consumption, we observe a similar increase in the conditional volatility and an increase in the term premium (not shown). Accordingly, GARCH does not generate additional countercyclical variation in the term premium because both negative and positive shocks to technology increase the conditional volatility.

To summarise, the presence of GARCH in technology shocks increases the mean level and the

Chart 2: Impulse responses to a technology shock with GARCH

Impulse responses are for a one standard deviation shock to technology when the model is approximated up to third order. The effects for the two interest rates, the ten-year term premium, and consumption are expressed in annualised percentage deviation from the steady state. The level of technology and the volatility in technology are expressed in percentage deviation from the steady state. The calibration for the GARCH process is $\rho_1 = 0.95$ and $\rho_2 = 0.04$. All the remaining parameter values are as stated in Table A.



standard deviation of the term premium. For a given variance in technology shocks, we also find that GARCH affects the variability of consumption growth but not the variability of inflation and interest rates.

5.6 An overall assessment: non-Gaussian shocks and term premia

Based on these simulation exercises, we finally discuss which of the considered non-Gaussian shocks may be most useful for matching term premia in DSGE models. Throughout this discussion, our focus is devoted to models approximated up to third order.

Starting with rare disasters, the main effect of these non-Gaussian shocks is to change the mean level of term premia through $\frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$ which corrects for 'disaster risk'. Hence, the inclusion of this term provides additional degrees of freedom when matching term premia. For instance, our benchmark model requires a high degree of risk-aversion to generate a sizable mean term premium, but this reliance on risk aversion may be reduced by accounting for rare disasters. Using the first calibration in Table B, we find that lowering the absolute value of α from -110 to -70 in a model with rare disasters roughly gives the same mean term premium as in the benchmark model. This corresponds to reducing the relative risk aversion from 168 to 108.

A limitation of relying on rare disaster is that these non-Gaussian shocks hardly affect the variability in term premia. On the other hand, the main effect from stochastic volatility is its effect on the variability in term premia, whereas the mean level is unaffected by these non-Gaussian shocks. An obvious possibility would therefore be to exploit this dichotomy and account for rare disasters and stochastic volatility within the same DSGE model. We also note that both shocks have minor effects on the variability in key macro variables, and these shocks are therefore unlikely to distort the model's ability along this dimension. As illustrated in previous sections, restrictions on the magnitude of rare disasters and stochastic volatility could instead be derived from values of skewness and kurtosis for macro variables and asset prices.

Another promising possibility is to consider shocks with GARCH when matching moments for term premia. An attractive feature of these non-Gaussian shocks is the combined effect they have on the mean level and variance of term premia. Redoing the experiment from above based on the second calibration of GARCH, we find that a value of $\alpha = -55$ is sufficient to reproduce the same mean term premia as in the benchmark model. This corresponds to reducing the relative risk aversion from 168 to 85. A possible limitation of GARCH might be that these shocks affect the variability in some macro variables. As a result, it may be challenging to get sufficient variability in the term premia without compromising the model's ability to match the standard deviations of key macro variables.

Apart from simply matching first and second moments of term premia, it is also important to note that GARCH provides a different explanation to changes in uncertainty than the one implied by stochastic volatility. Viewed from the perspective of a GARCH specification, an increase in uncertainty originates from large innovations to the level of the process, and higher uncertainty is therefore a *response* to large shocks hitting the economy. This explanation and interpretation of uncertainty is therefore different from the one provided by stochastic volatility, as in Bloom

(2009), where changes in uncertainty are an independent source of variation to the economy. Given the different implications of GARCH and stochastic volatility on asset prices, it therefore seems obvious to study the dynamics of macro variables along with asset prices as the latter may help to distinguish between the two explanations of time-varying uncertainty.

6 Conclusion

This paper analyses the effects of non-Gaussian shocks for risk premia in DSGE models approximated to second and third order. Based on an extension of the results in Schmitt-Grohé and Uribe (2004) to third order, we derive propositions for how rare disasters, stochastic volatility, and GARCH affect any risk premia in a wide class of DSGE models. Our key findings are as follows. First, the presence of rare disasters does not affect risk premia when DSGE models are solved up to second order, and rare disasters only change the level of risk premia at third order. Second, modelling time-varying uncertainty by stochastic volatility and GARCH do not generate variation in risk premia when the model is solved up to second order. Third, for DSGE models approximated up to third order, stochastic volatility and GARCH may affect the level and the variability of risk premia.

The paper also examines the quantitative effects of non-Gaussian shocks in a standard New Keynesian DSGE model where productivity features rare disasters, stochastic volatility, and GARCH. Here, focus is devoted to the ten-year nominal term premium and a third-order approximation to the model. We find that the considered specification of rare disasters can have substantial effects on the level of the term premium and values of skewness and kurtosis for several macro variables. However, rare disasters hardly affect the standard deviation of these macro variables. We also find that stochastic volatility can generate sizable variation in the term premium without distorting the model's ability to match a key number of macroeconomic moments. The presence of GARCH in technology shocks is found to increase the mean level and the standard deviation of the term premium. For a given variance in technology shocks, we also find that GARCH affects the variance of consumption growth but not the variance of inflation and interest rates.

Although this paper settles with studying the quantitative effects of non-Gaussian shocks in technology, it would also be interesting to see how rare disasters, stochastic volatility, and

GARCH in other structural shocks affect the nominal term premium and other risk premia. Another possibility would be to explore the quantitative effects of non-Gaussian shocks in various extensions of the standard New Keynesian DSGE model. We leave these and other questions for future research.

Appendix A: The third-order approximation

The third-order approximation to **g** and **h** at the deterministic steady state is given by

$$
\begin{aligned}
\left[\mathbf{g}(\mathbf{x},\sigma)\right]^{\beta_1} &= \mathbf{g}(\mathbf{x}_{ss},0) + \left[\mathbf{g}_{\mathbf{x}}(\mathbf{x}_{ss},0)\right]^{\beta_1}_{\alpha_1}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_1} \\
&+\frac{1}{2}\left[\mathbf{g}_{\mathbf{xx}}(\mathbf{x}_{ss},0)\right]^{\beta_1}_{\alpha_1\alpha_2}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_1}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_2} \\
&+\frac{1}{2}\left[\mathbf{g}_{\sigma\sigma}(\mathbf{x}_{ss},0)\right]^{\beta_1}[\sigma][\sigma] \\
&+\frac{1}{6}\left[\mathbf{g}_{\mathbf{xxx}}(\mathbf{x}_{ss},0)\right]^{\beta_1}_{\alpha_1\alpha_2\alpha_3}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_1}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_2}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_3} \\
&+\frac{3}{6}\left[\mathbf{g}_{\sigma\sigma\mathbf{x}}(\mathbf{x}_{ss},0)\right]^{\beta_1}_{\alpha_3}[\sigma][\sigma]\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_3} \\
&+\frac{3}{6}\left[\mathbf{g}_{\sigma\mathbf{xx}}(\mathbf{x}_{ss},0)\right]^{\beta_1}_{\alpha_2\alpha_3}[\sigma]\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_2}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_3} \\
&+\frac{1}{6}\left[\mathbf{g}_{\sigma\sigma\sigma}(\mathbf{x}_{ss},0)\right]^{\beta_1}[\sigma][\sigma][\sigma]
\end{aligned}
\tag{A-1}
$$

and

$$
\begin{aligned}
\left[\mathbf{h}(\mathbf{x},\sigma)\right]^{\gamma_1} &= \mathbf{h}(\mathbf{x}_{ss},0) + \left[\mathbf{h}_{\mathbf{x}}(\mathbf{x}_{ss},0)\right]^{\gamma_1}_{\alpha_1}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_1} \\
&+\frac{1}{2}\left[\mathbf{h}_{\mathbf{xx}}(\mathbf{x}_{ss},0)\right]^{\gamma_1}_{\alpha_1\alpha_2}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_1}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_2} \\
&+\frac{1}{2}\left[\mathbf{h}_{\sigma\sigma}(\mathbf{x}_{ss},0)\right]^{\gamma_1}[\sigma][\sigma] \\
&+\frac{1}{6}\left[\mathbf{h}_{\mathbf{xxx}}(\mathbf{x}_{ss},0)\right]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_1}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_2}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_3} \\
&+\frac{3}{6}\left[\mathbf{h}_{\sigma\sigma\mathbf{x}}(\mathbf{x}_{ss},0)\right]^{\gamma_1}_{\alpha_3}[\sigma][\sigma]\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_3} \\
&+\frac{3}{6}\left[\mathbf{h}_{\sigma\mathbf{xx}}(\mathbf{x}_{ss},0)\right]^{\gamma_1}_{\alpha_2\alpha_3}[\sigma]\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_2}\left[(\mathbf{x}-\mathbf{x}_{ss})\right]^{\alpha_3} \\
&+\frac{1}{6}\left[\mathbf{h}_{\sigma\sigma\sigma}(\mathbf{x}_{ss},0)\right]^{\gamma_1}[\sigma][\sigma][\sigma]
\end{aligned}
\tag{A-2}
$$

for $\beta_1 = 1, 2, ..., n_y$, $\gamma_1 = 1, 2, ..., n_x$, and $\alpha_1, \alpha_2, \alpha_3 = 1, 2, ..., n_x$. The expressions in **(A-1)** and **(A-2)** have been simplified in two ways. First, Young's theorem implies that the order of differentiation with respect to **x** and σ is irrelevant when partial derivatives of **g** and **h** are continuous. Second, only non-zero first and second-order derivatives of **g** and **h** are included in **(A-1)** and **(A-2)**.

Appendix B: Constants for the third-order terms

B.1 The expression for $\left[b^1\right]^i_{\alpha_1\alpha_2\alpha_3}$

$$\left[b^1\right]^i_{\alpha_1\alpha_2\alpha_3} \equiv$$

$$+\left(\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}'}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}'}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}}\right]^i_{\beta_1\beta_2\alpha_3}\right)$$

$$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^i_{\beta_1\beta_2}(\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$

$$+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3})$$

$$+\left(\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}\mathbf{y}'}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}\mathbf{y}}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}\mathbf{x}'}\right]^i_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}\mathbf{x}}\right]^i_{\beta_1\beta_2\alpha_3}\right)$$

$$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}}\right]^i_{\beta_1\beta_2}\left(\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_2}_{\alpha_2\alpha_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\right)$$

$$+\left(\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{y}'}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{y}}\right]^i_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}'}\right]^i_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}}\right]^i_{\beta_1\gamma_2\alpha_3}\right)$$

$$\times\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}\right]^i_{\beta_1\gamma_2}\left(\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\right)$$

$$+\left(\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}\mathbf{y}'}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}\mathbf{y}}\right]^i_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}\mathbf{x}'}\right]^i_{\beta_1\alpha_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}\mathbf{x}}\right]^i_{\beta_1\alpha_2\alpha_3}\right)$$

$$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}}\right]^i_{\beta_1\alpha_2}\left(\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\right)$$

$$+\left(\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}}\right]^{i}_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}\right]^{i}_{\beta_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}}\right]^{i}_{\beta_1\alpha_3}\right)$$

$$\times\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{y}'}\right]^{i}_{\beta_1}\left(\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\right)$$

$$+\left(\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}}\right]^{i}_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}\right]^{i}_{\beta_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}}\right]^{i}_{\beta_1\alpha_3}\right)$$

$$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$$

$$+\left[\mathbf{f}_{\mathbf{y}'}\right]^{i}_{\beta_1}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$$

$$+\left(\left[\mathbf{f}_{\mathbf{y}\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}'\mathbf{y}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}'\mathbf{x}'}\right]^{i}_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}'\mathbf{x}}\right]^{i}_{\beta_1\beta_2\alpha_3}\right)$$

$$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}'}\right]^{i}_{\beta_1\beta_2}\left(\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}\right)$$

$$+\left(\left[\mathbf{f}_{\mathbf{y}\mathbf{y}\mathbf{y}'}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}\mathbf{y}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}\mathbf{x}'}\right]^{i}_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}\mathbf{x}}\right]^{i}_{\beta_1\beta_2\alpha_3}\right)$$

$$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}}\right]^{i}_{\beta_1\beta_2}\left(\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_2}_{\alpha_2\alpha_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}\right)$$

$$+\left(\left[\mathbf{f}_{\mathbf{y}\mathbf{x}'\mathbf{y}'}\right]^{i}_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}^{t}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{x}'\mathbf{y}}\right]^{i}_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{x}'\mathbf{x}'}\right]^{i}_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{x}'\mathbf{x}}\right]^{i}_{\beta_1\gamma_2\alpha_3}\right)$$

$$\times\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{y}\mathbf{x}'}\right]^{i}_{\beta_1\gamma_2}\left(\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}+\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}\right)$$

$$+\left(\left[\mathbf{f}_{\mathbf{y}\mathbf{x}\mathbf{y}'}\right]^{i}_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{x}\mathbf{y}}\right]^{i}_{\beta_1\alpha_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{x}\mathbf{x}'}\right]^{i}_{\beta_1\alpha_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{x}\mathbf{x}}\right]^{i}_{\beta_1\alpha_2\alpha_3}\right)\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{y}\mathbf{x}}\right]^{i}_{\beta_1\alpha_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\alpha_1\alpha_3}$$

$$+\left(\left[\mathbf{f}_{\mathbf{y}\mathbf{y}'}\right]^{i}_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{y}}\right]^{i}_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{x}'}\right]^{i}_{\beta_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}\mathbf{x}}\right]^{i}_{\beta_1\alpha_3}\right)\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\alpha_1\alpha_2}$$

$$+\left(\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}'\mathbf{y}'}\right]^{i}_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}'\mathbf{y}}\right]^{i}_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}'\mathbf{x}'}\right]^{i}_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}'\mathbf{x}}\right]^{i}_{\gamma_1\beta_2\alpha_3}\right)$$

$$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}'}\right]^{i}_{\gamma_1\beta_2}\left(\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\right)$$

$$+\left(\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}\mathbf{y}'}\right]^{i}_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}\mathbf{y}}\right]^{i}_{\gamma_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}\mathbf{x}'}\right]^{i}_{\gamma_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}\mathbf{x}}\right]^{i}_{\gamma_1\beta_2\alpha_3}\right)$$

$$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}}\right]^{i}_{\gamma_1\beta_2}\left(\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\alpha_2\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\right)$$

$$+\left(\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{y}'}\right]^{i}_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{y}}\right]^{i}_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{x}'}\right]^{i}_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{x}}\right]^{i}_{\gamma_1\gamma_2\alpha_3}\right)$$

$$\times\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}'}\right]^{i}_{\gamma_1\gamma_2}\left(\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}+\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_3}\right)$$

$$+\left(\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}\mathbf{y}'}\right]^{i}_{\gamma_1\alpha_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}\mathbf{y}}\right]^{i}_{\gamma_1\alpha_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}\mathbf{x}'}\right]^{i}_{\gamma_1\alpha_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}\mathbf{x}}\right]^{i}_{\gamma_1\alpha_2\alpha_3}\right)\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}$$

$$+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}}\right]^{i}_{\gamma_1\alpha_2}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$

$$+\left(\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}'}\right]^{i}_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}}\right]^{i}_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}'}\right]^{i}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}}\right]^{i}_{\gamma_1\alpha_3}\right)\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$$

$$+\left(\left[\mathbf{f}_{\mathbf{x}\mathbf{y}'\mathbf{y}'}\right]^{i}_{\alpha_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{y}'\mathbf{y}}\right]^{i}_{\alpha_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{y}'\mathbf{x}'}\right]^{i}_{\alpha_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{y}'\mathbf{x}}\right]^{i}_{\alpha_1\beta_2\alpha_3}\right)$$

$$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}$$

$$+\left[\mathbf{f}_{\mathbf{x}\mathbf{y}'}\right]^{i}_{\alpha_1\beta_2}\left(\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\right)$$

$$+\left(\left[\mathbf{f}_{\mathbf{x}\mathbf{y}\mathbf{y}'}\right]^{i}_{\alpha_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{y}\mathbf{y}_t}\right]^{i}_{\alpha_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{y}\mathbf{x}_{t+1}}\right]^{i}_{\alpha_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{y}\mathbf{x}_t}\right]^{i}_{\alpha_1\beta_2\alpha_3}\right)\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\alpha_2}$$

$$+\left[\mathbf{f}_{\mathbf{x}\mathbf{y}}\right]^{i}_{\alpha_1\beta_2}\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\alpha_2\alpha_3}$$

$$+\left(\left[\mathbf{f}_{\mathbf{x}\mathbf{x}'\mathbf{y}'}\right]^{i}_{\alpha_1\gamma_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{x}'\mathbf{y}}\right]^{i}_{\alpha_1\gamma_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{x}'\mathbf{x}'}\right]^{i}_{\alpha_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{x}'\mathbf{x}}\right]^{i}_{\alpha_1\gamma_2\alpha_3}\right)\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}$$

$$+\left[\mathbf{f}_{\mathbf{x}\mathbf{x}'}\right]^{i}_{\alpha_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}\mathbf{x}}\right]^{\gamma_2}_{\alpha_2\alpha_3}$$

$$+\left(\left[\mathbf{f}_{\mathbf{x}\mathbf{x}\mathbf{y}'}\right]^{i}_{\alpha_1\alpha_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{x}\mathbf{y}}\right]^{i}_{\alpha_1\alpha_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{x}\mathbf{x}'}\right]^{i}_{\alpha_1\alpha_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}\mathbf{x}\mathbf{x}}\right]^{i}_{\alpha_1\alpha_2\alpha_3}\right)$$

for $i = 1, 2, ..., n$ and $\alpha_1, \alpha_2, \alpha_3 = 1, 2, ..., n_x$.

B.2 The expression for $\left[b^2\right]^{i}_{\alpha_3}$

$$\left[b^2\right]^{i}_{\alpha_3} =$$

$$\left(\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}}\right]^{i}_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}'}\right]^{i}_{\beta_1\beta_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}}\right]^{i}_{\beta_1\beta_2\alpha_3}\right)$$

$$\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$

$$+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_2}\left(\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_2}_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}\right)$$

$$+2\left(\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{y}'}\right]^{i}_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{y}}\right]^{i}_{\beta_1\gamma_2\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}'}\right]^{i}_{\beta_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}}\right]^{i}_{\beta_1\gamma_2\alpha_3}\right)$$

$$\times\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$

$$+2\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}\right]^{i}_{\beta_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}$$

$$+\left(\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^{i}_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}}\right]^{i}_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}\right]^{i}_{\beta_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}}\right]^{i}_{\beta_1\alpha_3}\right)$$

$$\times\left(\left[\mathbf{g}_{\mathbf{x}\mathbf{x}}\right]^{\beta_1}_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}+\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}+\left[\mathbf{g}_{\boldsymbol{\sigma\sigma}}\right]^{\beta_1}\right)$$

$$+\left[\mathbf{f}_{\mathbf{y}'}\right]^i_{\beta_1}\left(\left[\mathbf{g}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}+\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}\right)$$

$$+\left(\left[\mathbf{f}_{\mathbf{yy}'}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}_\mathbf{x}\right]^{\beta_3}_{\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{yy}}\right]^i_{\beta_1\beta_3}\left[\mathbf{g}_\mathbf{x}\right]^{\beta_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{yx}'}\right]^i_{\beta_1\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{yx}}\right]^i_{\beta_1\alpha_3}\right)\left[\mathbf{g}_{\sigma\sigma}\right]^{\beta_1}$$

$$+\left(\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{y}'}\right]^i_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}_\mathbf{x}\right]^{\beta_3}_{\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{y}}\right]^i_{\gamma_1\gamma_2\beta_3}\left[\mathbf{g}_\mathbf{x}\right]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{x}'}]^i_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{x}}]^i_{\gamma_1\gamma_2\alpha_3}\right)$$

$$\times[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2}$$

$$+\left(\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}'}\right]^i_{\gamma_1\beta_3}\left[\mathbf{g}_\mathbf{x}\right]^{\beta_3}_{\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+\left[\mathbf{f}_{\mathbf{x}'\mathbf{y}}\right]^i_{\gamma_1\beta_3}\left[\mathbf{g}_\mathbf{x}\right]^{\beta_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}'}]^i_{\gamma_1\gamma_3}[\mathbf{h}_\mathbf{x}]^{\gamma_3}_{\alpha_3}+[\mathbf{f}_{\mathbf{x}'\mathbf{x}}]^i_{\gamma_1\alpha_3}\right)[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}$$

for $i=1,2,...,n$ and $\alpha_3=1,2,...,n_x$.

B.3 The expression for $\left[b^3\right]^i$

We introduce the notation that

$$\left[\mathbf{m}^3\left(\epsilon'\right)\right]^{\phi_1}_{\phi_2\phi_3}=\begin{cases}m^3\left(\epsilon'\left(\phi_1\right)\right) & \text{if } \phi_1=\phi_2=\phi_3\\ 0 & \text{otherwise}\end{cases}$$

where $m^3\left(\epsilon'\left(\phi_1\right)\right)$ denotes the third moment of $\epsilon_{t+1}\left(\phi_1\right)$ for $\phi_1=1,2,...,n_\epsilon$. Notice that $\mathbf{m}^3\left(\epsilon'\right)$ has dimensions $n_\epsilon\times n_\epsilon\times n_\epsilon$. Then:

$$\left[b^3\right]^i=\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{y}'}\right]^i_{\beta_1\beta_2\beta_3}\left[\mathbf{g}_\mathbf{x}\right]^{\beta_3}_{\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}_\mathbf{x}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_\mathbf{x}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\epsilon'\right)\right]^{\phi_1}_{\phi_2\phi_3}$$

$$+3\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'\mathbf{x}'}\right]^i_{\beta_1\beta_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}\left[\mathbf{g}_\mathbf{x}\right]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_\mathbf{x}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\epsilon'\right)\right]^{\phi_1}_{\phi_2\phi_3}$$

$$+3\left[\mathbf{f}_{\mathbf{y}'\mathbf{y}'}\right]^i_{\beta_1\beta_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_2}_{\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_\mathbf{x}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\epsilon'\right)\right]^{\phi_1}_{\phi_2\phi_3}$$

$$+3\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'\mathbf{x}'}\right]^i_{\beta_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_\mathbf{x}\right]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\epsilon'\right)\right]^{\phi_1}_{\phi_2\phi_3}$$

$$+3\left[\mathbf{f}_{\mathbf{y}'\mathbf{x}'}\right]^i_{\beta_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\epsilon'\right)\right]^{\phi_1}_{\phi_2\phi_3}$$

$$+\left[\mathbf{f}_{\mathbf{y}'}\right]^i_{\beta_1}\left[\mathbf{g}_{\mathbf{xxx}}\right]^{\beta_1}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\epsilon'\right)\right]^{\phi_1}_{\phi_2\phi_3}$$

$$+\left[\mathbf{f}_{\mathbf{x}'\mathbf{x}'\mathbf{x}'}\right]^{i}_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]^{\gamma_3}_{\phi_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3(\epsilon')\right]^{\phi_1}_{\phi_2\phi_3}$$

for $i = 1, 2, ..., n$.

B.4 Matlab implementation

Our implementation extends the one provided by Schmitt-Grohé and Uribe (2004) for DSGE models approximated up to second order. That is, the user only needs to provide the set of equilibrium conditions in the function $\mathbf{f}(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t)$ and values of $\mathbf{y}_t$ and $\mathbf{x}_t$ in the deterministic steady-state, ie $(\mathbf{y}_{ss}, \mathbf{x}_{ss})$. The function Anal_derivatives.m then computes all required analytical derivatives of $\mathbf{f}$ up to third order using the Symbolic Toolbox in Matlab. Given the steady-state values $(\mathbf{y}_{ss}, \mathbf{x}_{ss})$ and the structural coefficients, the function num_eval_3rd.m then computes the numerical values of these derivatives. The functions gx_hx.m, gxx_hxx.m, and gss_hss.m by Schmitt-Grohé and Uribe (2004) are used to compute the first and second-order derivatives of $\mathbf{g}$ and $\mathbf{h}$. All third-order derivatives of $\mathbf{g}$ and $\mathbf{h}$ are finally computed by the function g_h_3rd.m.

If the user in relation to estimation or sensitivity analysis requires to solve the model many times, then it is computationally faster to print the analytical derivatives of $\mathbf{f}$ into a function and evaluate them as real matrices. That is the user can settle with only differentiating the same model once. The script Display_matlab.m is useful in this context because it allows the user to print the analytical derivatives of $\mathbf{f}$ into a textfile.

The output from for the first and second-order terms are stored as in Schmitt-Grohé and Uribe (2004). For the third-order terms, the matrices $\mathbf{g}_{\mathbf{xxx}}$ and $\mathbf{h}_{\mathbf{xxx}}$ have dimensions $n_y \times n_x \times n_x \times n_x$ and $n_x \times n_x \times n_x \times n_x$, respectively. Here, $\mathbf{g}_{\mathbf{xxx}}(\beta_1, \alpha_1, \alpha_2, \alpha_3) = \left[\mathbf{g}_{\mathbf{xxx}}\right]^{\beta_1}_{\alpha_1\alpha_2\alpha_3}$ and $\mathbf{h}_{\mathbf{xxx}}(\gamma_1, \alpha_1, \alpha_2, \alpha_3) = \left[\mathbf{h}_{\mathbf{xxx}}\right]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}$ for $\beta_1 = 1, 2, ..., n_y$ and $\gamma_1, \alpha_1, \alpha_2, \alpha_3 = 1, 2, ..., n_x$, where the arguments for $\mathbf{g}_{\mathbf{xxx}}$ and $\mathbf{h}_{\mathbf{xxx}}$ index the elements in these matrices. Similarly, $\mathbf{g}_{\sigma\sigma\mathbf{x}}$ and $\mathbf{h}_{\sigma\sigma\mathbf{x}}$ have dimensions $n_y \times n_x$ and $n_x \times n_x$, respectively, and $\mathbf{g}_{\sigma\sigma\mathbf{x}}(\beta_1, \alpha_3) = \left[\mathbf{g}_{\sigma\sigma\mathbf{x}}\right]^{\beta_1}_{\alpha_3}$ and $\mathbf{h}_{\sigma\sigma\mathbf{x}}(\gamma_1, \alpha_3) = \left[\mathbf{h}_{\sigma\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}$. Finally, $\mathbf{g}_{\sigma\sigma\sigma}$ and $\mathbf{h}_{\sigma\sigma\sigma}$ have dimensions $n_y \times 1$ and $n_x \times 1$, and $\mathbf{g}_{\sigma\sigma\sigma}(\beta_1, 1) = \left[\mathbf{g}_{\sigma\sigma\sigma}\right]^{\beta_1}$ and $\mathbf{h}_{\sigma\sigma\sigma}(\gamma_1, 1) = \left[\mathbf{h}_{\sigma\sigma\sigma}\right]^{\gamma_1}$.

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