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Time-varying volatility, precautionary saving and monetary policy

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Abstract

This paper analyses the conduct of monetary policy in an environment where households' desire to amass precautionary savings is influenced by fluctuations in the volatilities of disturbances that hit the economy. It uses a simple New Keynesian model with external habit formation that is augmented with demand and supply disturbances whose volatilities vary over time. If volatility fluctuations are ignored by policy, interest rates are set at a suboptimal level. The extent of 'policy bias' is relatively small but of greater importance the higher the degree of habit formation. The reason is that habit-forming preferences raise risk aversion, increasing the importance of the precautionary savings channel through which volatility fluctuations impact upon inflation and output.

Key words: Time-varying volatility, precautionary saving, monetary policy, DSGE models.

JEL classification: E21, E32, E58, G12.

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Summary

In order to design effective monetary policy, central banks require an understanding of the mechanism by which economic shocks are transmitted to key macro variables like inflation, consumption and output. Economists therefore conduct policy analyses using models in which key economic relationships are spelt out but are subject to 'stochastic shocks' that represent unpredictable external events that influence the economy. A key task for monetary policy is to understand the transmission mechanism of such shocks, thereby enabling effective policy responses to be formulated.

Perhaps oddly, most policy analyses are carried out in a way that sidesteps the impact of uncertainty on households. Such models can match many features in the data and have a number of advantages. Notably, they can be represented in the form of a linear system of equations, making numerical simulations of medium and large-scale models feasible. However, an important drawback is that they cannot properly capture swings in uncertainty (fluctuations in the volatilities of economic disturbances), to understand the impact of such swings on the economy, or to evaluate potential policy responses. Yet, as exemplified by the recent financial crisis, changing uncertainty can be an important driver of economic behaviour. By ignoring such effects, these models provide policymakers with an incomplete picture and may lead to biased policy recommendations. Previous research at the Bank of England and elsewhere has examined the impact of uncertainty. But beyond that, there is an issue of whether changing levels of volatility also affect behaviour materially. This paper builds on that work and investigates the issue in more detail, focusing on a single aspect of household behaviour that is influenced by changes in uncertainty – precautionary saving.

Precautionary saving is additional saving driven by the possibility that if households are unlucky, consumption will fall to a low level, at which point an extra pound of spending is highly valued. This introduces a powerful non-linearity into economic models which has to be addressed explicitly. Furthermore, it has direct relevance for monetary policy, because an increase implies a reduction in current consumption, the main component of aggregate demand and an important factor influencing the extent of inflationary pressure in the economy. Thus we look at the monetary policy implications of ignoring precautionary savings effects arising from variations in the volatilities of demand and supply disturbances hitting the economy – an investigation which, by definition, cannot be conducted within a constant volatility framework.

In order to capture these effects in the model solution, the model is solved numerically using a higher-order approximation method. Given that the mechanism is driven by uncertainty, crucial to financial markets, consumer preferences are specified in a way that has been shown to provide a better 'match' to asset pricing data. Specifically, it is assumed that utility follows an 'external habits' specification, such that consumers value the difference between consumption

and a slow-moving reference value. This specification of preferences introduces cyclical variation in risk appetite and raises household aversion to risk, two effects that appear to be important features of financial markets. Given that the model itself is stylised, the quantitative results reported are intended to illustrate rather than estimate the monetary policy implications of volatility fluctuations.

A key finding is that volatility fluctuations can have a small but relevant impact on precautionary saving behaviour, and therefore upon the appropriate conduct of monetary policy. The main contribution of the paper is to clarify the mechanism by which volatility fluctuations are transmitted through the precautionary savings channel and to illustrate – both analytically and quantitatively – the implications for monetary policy. If volatility fluctuations are not taken into account by policy, interest rates will be set incorrectly. As a result, a central bank that follows an interest rate rule that ignores volatility fluctuations will increase inflation and output instability, albeit to a small degree. Moreover, sensitivity analysis shows that the extent of 'policy bias' falls as the importance of habits in preferences is decreased. Consequently, models which are not calibrated to match higher-order risk effects may understate the importance of volatility fluctuations for the economy.

1 Introduction

Most monetary policy analysis is carried out ignoring the role of uncertainty. Models of this kind can match many stylised 'macro facts' and have several advantages from a practical perspective. Notably, such models can be represented as linear, or log-linearised, systems of dynamic equations, making feasible simulations of medium and large-scale macro models with a wide array of economic transmission mechanisms. However, in order to capture the effects of volatility we need to take higher-order approximations. This has been examined in previous work, including some at the Bank of England (eg, De Paoli and Zabzyck (2011)). But in addition, volatility itself may vary over time. This may be particularly pertinent at the moment, given the recent financial crisis and subsequent economic turbulence. Constant volatility models may therefore give rise to misleading policy recommendations, and the current paper extends the literature to examine the implications for monetary policy. Thus this paper investigates this issue in further detail using a model in which changes in uncertainty affect economic behaviour because agents have a precautionary savings motive. The analysis is innovative in that it investigates the impact of fluctuations in uncertainty on precautionary saving behaviour and traces out the implications for monetary policy.

The paper also contributes to recent literature investigating the impact of fluctuations in macroeconomic risk upon the behaviour of households. Demand and supply shock volatility fluctuations of the kind modelled in the current paper have been documented by Fernández-Villaverde *et al* (2010), Justiniano and Primiceri (2008) and Fernández-Villaverde and Rubio-Ramirez (2007), who estimate medium-scale dynamic stochastic general equilibrium (DSGE) models of the US economy in order to investigate the underlying sources of the 'Great Moderation'.[(1)] Furthermore, Bloom (2009) and Fernández-Villaverde *et al* (2009) demonstrate that fluctuations in the volatilities of economic disturbances can have a quantitatively relevant impact on household behaviour within simulated models of the economy. The quantitative implications of volatility fluctuations for the conduct of monetary policy are potentially important but have not yet been assessed directly.

This paper takes up this challenge. It can be viewed as part of a wider research agenda investigating the impact of non-linearities and risk on macroeconomic dynamics along various dimensions (eg Andreasen (2011); Rudebusch and Swanson (2008); Fernández-Villaverde and Rubio-Ramirez (2005)). Rather than considering these numerous dimensions simultaneously, the paper focuses on a single channel through which changes in risk are transmitted to key macro variables – precautionary savings. The motivation for focusing on this channel comes from the direct link between precautionary savings and equilibrium interest rates in economic models, and the empirical finding that precautionary behaviour is an important factor driving the

[(1)] The Great Moderation refers to the dramatic declines in the volatilities of key macroeconomic variables observed over the past two decades. An example from the substantial literature on this topic is Stock and Watson (2003).

accumulation of household wealth (Benito (2006); Ludvigson and Michaelides (2001); Carroll and Samwick (1998)).[2] Moreover, an increase in precautionary saving implies a reduction in current consumption, the main component of aggregate demand and an important factor influencing the amount of inflationary pressure in the economy.

Using a New Keynesian model with external habit formation, De Paoli and Zabczyk (2011) show that ignoring precautionary saving leads to 'bias' in the interest rate set by monetary policy – that is, the interest rate deviates from the 'natural rate' that is necessary to maintain price stability, defined as zero inflation. Intuitively, consumers' incentive to engage in precautionary saving depends upon how uncertain they are about the future and the extent to which they expect current economic conditions to persist – higher-order effects that policymakers must take into account when setting policy to ensure that price stability is maintained. The finding that ignoring precautionary saving is a source of monetary policy bias suggests that models that only partially capture precautionary savings effects may also be vulnerable. One such class of models is those with constant volatility. The aim of this paper is to determine whether volatility fluctuations have potential quantitative importance for monetary policy. This investigation is carried out using a simple New Keynesian model with a precautionary savings channel that is augmented with demand and supply disturbances whose volatilities vary over time.

Given the focus on high-order uncertainty effects, the model adopts a specification for consumer preferences that has been shown to provide a better match to asset pricing data – namely, the external habits specification where utility depends on consumption relative to a slow-moving aggregate reference value (Abel (1990)). Habit-forming preferences of this kind perform better in terms of matching asset 'risk-premia' because they raise household aversion to risk and lead to cyclical variation in risk appetite. Indeed, Campbell and Cochrane (1999) show that such preferences do a better job at replicating the 'equity premium', whilst Rudebusch and Swanson (2008) show that they also perform better than standard preferences at matching the 'term premium' on nominal bonds. External habits also play an important role in the present paper because, by raising risk aversion, they increase the importance of the precautionary savings channel through which volatility fluctuations have an impact on inflation and output.

The model is solved using a third-order approximation in order to capture time variations in the precautionary savings motive.[3] Moreover, the tractability of the model means that third-order approximate analytical solutions can be derived, which are used to clarify the transmission mechanism of fluctuations in volatility *via* the precautionary savings channel. These analytical

[2] For instance, using the Panel Study of Income Dynamics (PSID), Carroll and Samwick (1998) estimate that between 32% and 50% of wealth accumulation is attributable to precautionary behaviour.
[3] In particular, the model is solved using Dynare++, which approximates non-linear economic models using perturbation methods (see Julliard (2001)).

solutions provide intuition for the quantitative results from simulating the model, including the key finding of the paper that fluctuations in the volatilities of disturbances can have a quantitatively relevant impact on precautionary saving and the conduct of monetary policy. Indeed, if innovations to uncertainty are not taken into account when formulating policy, there is a 'policy bias': the interest rate will not respond to all shocks that influence inflation and output, with the result that price stability – defined here as zero inflation – cannot be maintained. For instance, consider the impact of a positive innovation to volatility. Such an innovation raises future uncertainty and therefore increases the incentive for risk-averse consumers to engage in precautionary saving. There is thus downward pressure on both aggregate demand and prices, with the result that a reduction in the interest rate is needed to ensure that price stability is maintained.[(4)]

It is shown that this interest rate 'policy bias' is quantitatively relevant using third-order impulse responses to volatility innovations, which isolate the marginal impact of a change in risk. Moreover, one policy implication of this result is that a central bank which follows an interest rate rule that ignores volatility fluctuations will increase inflation and output gap instability non-trivially compared to the constant volatility case. However, in the calibrated model, it is only volatility innovations to the supply shock (a productivity shock) that have a quantitatively relevant impact, with volatility innovations to the demand shock (a consumption preference shock) having only trivial effects. The reason is that the supply shock in the model is considerably more important for overall consumption volatility and is subject to larger volatility fluctuations. Therefore, whilst volatility fluctuations are potentially relevant for monetary policy, whether they are in practice is likely to depend crucially upon the type of economic disturbance that is considered. An important finding from sensitivity analysis is that the extent of 'policy bias' falls somewhat as the importance of habits in utility is decreased. Consequently, economic models that are not calibrated to match higher-order risk effects may understate the importance of volatility fluctuations for policy.

The paper proceeds as follows. Section 2 sets out the model. In Section 3, analytical results are presented that clarify the transmission mechanism of volatility fluctuations in the model. The model is calibrated in Section 4. In Section 5 quantitative simulation results are presented, and in Section 6 a sensitivity analysis is conducted. Finally, Section 7 concludes.

[(4)] In the case of a negative innovation to volatility, an increase in the interest rate is necessary to ensure that price stability is maintained.

2 Model

The model is inhabited by a continuum of consumer-producers, or 'yeoman farmers', each indexed by j on the unit interval. Each consumer-producer produces a single differentiated good and consumes a basket of all the goods that are produced in the economy (Woodford (2003); Rotemberg and Woodford (1998)). Preferences over consumption follow an external habits specification. Households therefore receive utility from the difference between consumption and a slow-moving aggregate reference value, as in Campbell and Cochrane (1999) and Abel (1990). In order to keep the model as simple as possible, the economy is closed and there is no capital accumulation. Important technical details relating to the model are provided in Sections A.1 and A.2 of Appendix A.

The utility of consumer-producer j is given by

$$U_t(j) = E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\frac{\xi_{d,s} (C_s(j) - hX_s)^{1-\rho}}{1-\rho} - \frac{\xi_{y,s}^{-\eta} y_s(j)^{1+\eta}}{1+\eta} \right) \quad \textbf{(1)}$$

where $C_t(j)$ denotes consumption of agent j in period t, X_t is the level of 'habits', $\xi_{d,t}$ is a consumption preference shock, and $0 < h < 1$ is the 'habit size' parameter that determines the importance of habits in utility from consumption. The second term in the large brackets captures agent j's disutility from producing $y_t(j)$ units of the differentiated output good when productivity is equal to $\xi_{y,t}$. Hence $\xi_{y,t}$ is a productivity (or supply) shock, and $\xi_{d,t}$ is a consumption preference (or demand) shock.

Habits are 'external' and are given by

$$\log(X_t) = (1-\phi)\log(C_{t-1}) + \phi \log(X_{t-1}) \quad \textbf{(2)}$$

where C_t is aggregate consumption and ϕ controls the degree of persistence in habits.

Consumer-producer j maximises utility subject to the following budget constraint:

$$B_t + C_t(j) = R_{t-1}B_{t-1} + \frac{p_t(j)y_t(j)}{P_t} \quad \textbf{(3)}$$

where B_t denotes holdings of risk-free real bonds with gross return R_t, and $p_t(j)\, y_t(j)/P_t$ is real revenue received by agent j from selling its differentiated output good at price $p_t(j)$ when the aggregate price level is P_t. As in Rotemberg and Woodford (1998), financial markets are complete. Consequently, consumption levels are equalised across all consumer-producers.

Aggregate consumption is a composite of all goods in the economy, and the aggregate price level is an index that is appropriately aggregated across consumption goods:

$$C_t = \left[\int_0^1 c_t(j)^{(\sigma-1)/\sigma} dj\right]^{\sigma/(1-\sigma)} \qquad P_t = \left[\int_0^1 p_t(j)^{1-\sigma} dj\right]^{1/(1-\sigma)} \tag{4}$$

where $\sigma > 0$ is the elasticity of substitution between differentiated goods, $c_t(j)$ is consumption (and output) of the differentiated good produced by agent j, and $p_t(j)$ is the price at which it is sold. By market-clearing, aggregate consumption and aggregate output are equal – ie $C_t = Y_t$.

Households minimise the cost of buying their consumption basket and choose bond holdings to maximise utility, giving rise to an optimal demand for each output good and an Euler equation for aggregate consumption:

$$y_t(j) = \left(\frac{p_t(j)}{P_t}\right)^{-\sigma} Y_t \tag{5}$$

$$1 = R_t E_t\left[\frac{\beta \xi_{d,t+1}(C_{t+1} - hX_{t+1})^{-\rho}}{\xi_{d,t}(C_t - hX_t)^{-\rho}}\right] = R_t E_t(M_{t+1}) \tag{6}$$

where M_{t+1} is the stochastic discount factor.

The coefficient of relative risk aversion is given by

$$\vartheta(C_t, X_t) \equiv -C_t \frac{U_{cc}(C_t, X_t)}{U_c(C_t, X_t)} = \frac{\rho}{S_t} \tag{7}$$

where $S_t \equiv 1 - h(X_t / C_t)$ is surplus consumption.

Since $\partial S_t / \partial C_t > 0$, risk aversion is countercyclical: lower consumption relative to habit increases risk aversion, and higher consumption relative to habit reduces risk aversion.(5) Moreover, the presence of external habits raises risk aversion relative to the no-habits case when $h = 0$, since the steady-state coefficient of relative risk aversion is given by $\rho/(1\text{-}h)$.(6)

Producers of differentiated goods choose their output prices optimally taking the aggregates P_t and Y_t as given and knowing their individual demand functions. Following Calvo (1983), producer prices are rigid and follow a partial adjustment rule: a fraction α of producers are not allowed to change the nominal price of their output in each period, whilst the remaining fraction $1 - \alpha$ choose their nominal output price to maximise expected utility, taking into account the probability that the price they set will remain in place in subsequent periods.

(5) S_t is used as a measure of cyclical stance, as in Campbell and Cochrane (1999).
(6) It is possible, given the specification of habits, for surplus consumption to be zero or negative in some periods, implying infinite or negative risk aversion. However, this is not an issue in the simulations in this paper because the highest calibrated value for h is 0.8. Campbell and Cochrane (1999) avoid this potential problem by specifying a non-linear process for habits such that surplus consumption is always positive. This specification was not adopted here because it is analytically intractable.

Given the budget constraint faced by each consumer-producer j and the demand for their output good, we have following optimality condition for a producer resetting price at time t:

$$E_t \sum_{s=t}^{\infty} (\alpha\beta)^{s-t} \left(\frac{\xi_{d,s}}{(C_s(j) - hX_s)^{\rho}} \left(\frac{p_t(j)}{P_s} \right) - \frac{\sigma}{(\sigma - 1)} \xi_{y,s}^{-\eta} y_{t,s}(j)^{\eta} \right) y_{t,s}(j) = 0 \qquad \textbf{(8)}$$

where $y_{t,s}(j)$ is the time-t estimate of demand in period s if the price $p_t(j)$ is not reset.

Equation **(8)** implies that the aggregate price index is given by

$$P_t^{1-\sigma} = \alpha P_{t-1}^{1-\sigma} + (1-\alpha) p_t^{1-\sigma} \qquad \textbf{(9)}$$

where p_t is the optimal price set by all producers changing price at time t (hence the absence of the index j). Full derivations of equations **(8)** and **(9)** are given in Section A.1 of Appendix A.

Time-varying volatility is introduced into the model by making the variances of productivity and preference shocks time-varying. Productivity and preference shocks are given by the following stochastic processes:

$$\varepsilon_{d,t+1} = \rho_d \varepsilon_{d,t} + \sigma_{d,t} u_{d,t+1} \qquad \varepsilon_{y,t+1} = \rho_y \varepsilon_{y,t} + \sigma_{y,t} u_{y,t+1} \qquad \textbf{(10)}$$

where $\varepsilon_{i,t} \equiv \log(\xi_{i,t})$ and $u_{i,t}$ is an i.i.d. draw from a normal distribution with mean zero and standard deviation $\sigma_{i,u}$, $i \in \{d, y\}$.

The variables $\sigma_{d,t}$ and $\sigma_{y,t}$ are stochastic processes that drive variations in the conditional variances of preference and productivity shocks. It is easy to show that the (time-t) conditional standard deviations of preference and productivity shocks are given by $\sigma_{d,t}\,\sigma_{d,u}$ and $\sigma_{y,t}\,\sigma_{y,u}$ respectively. In what follows, the results under time-varying volatility are compared to the constant volatility case where $\sigma_{d,t} = \sigma_{y,t} = 1$ for all t.

Shephard (2005) gives a review of different time-varying volatility specifications that have been used in the literature. A parsimonious volatility specification that can capture key features in the data is the log-linear AR(1) process used by Fernández-Villaverde *et al* (2010), Justiniano and Primiceri (2008) and Fernández-Villaverde and Rubio-Ramirez (2007). For analytical tractability, the present paper assumes that the volatility driving processes are AR(1) in levels and not logs, as in Benigno *et al* (2010). A linear specification of this form can capture persistence observed in the data and is sufficient to determine whether volatility fluctuations are of potential quantitative importance for monetary policy.

The volatility processes for preference and productivity shocks are thus given by

$$\sigma_{i,t} = (1-\rho_{\sigma i})\sigma_{i,mean} + \rho_{\sigma i}\sigma_{i,t-1} + v_{i,t} \qquad \textbf{(11)}$$

where $0 < \rho_{\sigma i} < 1$ are volatility persistence parameters and $v_{i,t}$ is an i.i.d. draw from a normal distribution with mean zero and standard deviation $\sigma_{i,v}$, $i \in \{d, y\}$.

Since the constant volatility case sets $\sigma_{i,t} = 1$ for all t, the following normalisation is applied:

$$\sigma_{i,mean} = 1 \qquad \textbf{(12)}$$

This normalisation means that the volatilities of preference and productivity disturbances will fluctuate stochastically around their steady-state (ie constant volatility) values.

Together equations **(8)** and **(9)** define a non-linear New Keynesian Phillips curve. In order to derive tractable analytical results whilst capturing time variations in volatility, a partially linear version of this equation is considered. In particular, the Phillips curve is given by

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(\kappa_0 (1-h)^{-1} c_t - \rho(1-h)^{-1} h x_t - \eta \varepsilon_{y,t} - \varepsilon_{d,t}) \qquad \textbf{(13)}$$

where $\kappa = (1-\alpha\beta)(1-\alpha)/\alpha(1+\sigma\eta)$, $\kappa_0 = (1-h)\eta + \rho$ and the lowercase variables c_t ($= y_t$) and x_t denote log deviations from steady state. Inflation is given by $\pi_t \equiv \log(P_t / P_{t\text{-}1})$.

As is shown in Section A.2 of Appendix A, equation **(13)** is the Phillips curve that would result from log-linearisation of equations **(8)** and **(9)**, except that the log-linearised preference and productivity shocks are replaced with the non-linear ones from equation **(10)** that capture time variations in volatility. A 'partial linearisation' of this kind has previously been employed by Justiniano and Primiceri (2008) and Benigno *et al* (2010) in order to retain a role for time variations in risk, though both apply the method to a whole model rather than a single equation.(7) As a robustness check, the implications of having a fully non-linear Phillips curve are assessed in the sensitivity analysis section of this paper.

The model is closed with a rule describing monetary policy. In particular, most of the analysis that follows assumes that the central bank implements a 'targeting rule' that ensures price stability, defined as setting $\pi_t = 0$ for all t. However, the central bank could also achieve a price stability goal of this kind through an 'instrument rule' for the nominal interest rate. The implications of variations in volatility for a central bank pursuing Taylor-type interest rate rules are investigated in the sensitivity analysis section of the paper.

3 Time-varying volatility and precautionary saving

Since market-clearing requires $C_t = Y_t$, there is no saving in equilibrium. Nevertheless, we are able to examine the impact of precautionary saving via its implications for the natural rate of interest, that is, the interest rate that ensures perfect price stability. In particular, we can capture the motive for precautionary saving, also known as the 'precautionary savings effect'.

(7) Justiniano and Primiceri (2008) show in the appendix to their paper that a 'partially non-linear approximation' is a valid solution method for a general DSGE model.

Under conditional log-normality of M_{t+1}, the interest rate is given by[8]

$$-r_t = \underbrace{E_t(m_{t+1})}_{\text{Intertemporal substitution effect}} + \frac{1}{2}\underbrace{\mathrm{var}_t(m_{t+1})}_{\text{Precautionary savings effect}} \qquad \textbf{(14)}$$

where lowercase letters indicate log deviations from steady state.

The first term on the right-hand side is the intertemporal substitution effect. This effect arises from an intertemporal consumption-smoothing motive and is therefore present in a log-linearised model, as, for example, in Amato and Laubach (2004). However, the second term on the right-hand side – the precautionary savings effect – is ignored when the Euler equation is log-linearised. This term reflects the desire of risk-averse agents to engage in precautionary saving behaviour when faced with uncertainty about future consumption. This additional desire to save reduces the equilibrium rate of interest. For instance, following an increase in the desire to save for precautionary reasons, the interest rate must bring about market-clearing by falling sufficiently to ensure that actual saving is 'choked off' to zero. As noted by De Paoli and Zabczyk (2011), monetary policy will need to take this high-order effect into account in order to achieve price stability.

Focus is restricted to the implications of time-varying shock volatilities for the precautionary savings motive because shock volatility fluctuations enter the precautionary savings effect directly through their impact upon stochastic discount factor variability. In order to analyse the implications of volatility fluctuations for precautionary behaviour, we need to understand the factors that drive the precautionary savings effect. The precautionary savings effect was therefore approximated analytically up to third order – the lowest that allows for variation over time. It is important to note that although some of the analytical results that follow make use of the AR(1) specifications of the volatility processes given above, the key analytical expressions – equations **(16)**, **(18)** and **(19)** – will hold for any bounded volatility processes $\sigma_{i,t},\ i \in \{d, y\}$.[9]

Using the definition of the stochastic discount factor, the precautionary savings effect can be decomposed as follows:

$$\mathrm{var}_t(m_{t+1}) = \mathrm{var}_t(\widetilde{m}_{t+1}) + (2\,\mathrm{cov}_t(\widetilde{m}_{t+1}, \Delta\varepsilon_{d,t+1}) + \mathrm{var}_t(\Delta\varepsilon_{d,t+1})) \qquad \textbf{(15)}$$

where $\widetilde{m}_{t+1} \equiv \log(\widetilde{M}_{t+1})$, $\widetilde{M}_{t+1} \equiv (\xi_{d,t} / \xi_{d,t+1})M_{t+1}$ and $\mathrm{var}_t(\Delta\varepsilon_{d,t+1}) = \sigma^2_{d,t}\sigma^2_{d,u}$.

The first term on the right-hand side depends on the variances of both preference and productivity shocks, whilst the second term in brackets depends only on preference shock

[8] This relationship holds up to a second-order approximation for any distribution of the stochastic discount factor.
[9] The key to this result is that the innovation to $\varepsilon_{i,t+1}$ depends on $\sigma_{i,t}$ rather than $\sigma_{i,t+1}$. This specification, which follows the one in Benigno *et al* (2010), makes the algebra considerably easier and does not change the conclusions of the paper in any substantive way.

volatility, because preference and productivity shocks are uncorrelated. More focus is given to the first term in what follows, since in numerical simulations this was the driving term in the precautionary savings effect. However, both terms are approximated analytically up to third order for completeness. Details are given in Sections A.3 and A.4 of Appendix A, and key points are discussed in the text.

A third-order approximation to the first term yields

$$\mathrm{var}_t(\widetilde{m}_{t+1}) = \frac{\rho^2}{\kappa_0^2}(\sigma_{d,t}^2\sigma_{d,u}^2 + \eta^2\sigma_{y,t}^2\sigma_{y,u}^2)(1-\kappa_y\varepsilon_{y,t} - \kappa_d\varepsilon_{d,t} + \kappa_x x_t) \qquad \textbf{(16)}$$

where

$$\kappa_0 = \rho + \eta(1-h), \qquad \kappa_x = \frac{2h(\rho+\eta)(\phi\kappa_0 + \rho h(1-\phi))}{\kappa_0^2}$$

$$\kappa_y = \frac{2h\eta((1-h)(\rho+\eta)(\phi-1)+\kappa_0\rho_y)}{\kappa_0^2}, \ \kappa_d = \frac{2h((1-h)(\rho+\eta)(\phi-1)+\kappa_0\rho_d)}{\kappa_0^2}.$$

The first bracket in equation **(16)** shows the impact of variations in preference and productivity shock volatilities on the precautionary savings motive. Positive innovations to preference volatility $v_{d,t}$ and productivity volatility $v_{y,t}$ increase the incentive to engage in precautionary saving – and do so persistently – by increasing $\sigma_{d,t}$ and $\sigma_{y,t}$ respectively; see equation **(11)**. Intuitively, volatility innovations increase uncertainty about the magnitude of shocks hitting the economy, raising uncertainty about future consumption. Given that consumers are risk-averse, this increase in uncertainty translates into a stronger incentive for 'self-insurance' via precautionary saving. It is also notable that, other things being equal, the impact of a productivity volatility innovation is larger (smaller) than a preference volatility innovation if η, the Frisch inverse elasticity of labour supply, is greater than (smaller than) one. The reason is that in order to achieve the price stability objective, the (natural) rate of interest must ensure that no inflationary or deflationary pressures build up via Phillips curve, which by equation **(13)** requires that consumption respond to productivity shocks with elasticity of $\eta(\kappa_0/(1\text{-}h))^{-1}$ and to preference shocks with an elasticity of $(\kappa_0/(1\text{-}h))^{-1}$.

The second bracket in equation **(16)** captures the impact of cyclical risk aversion on the precautionary savings motive, as discussed by De Paoli and Zabczyk (2011). In particular, the precautionary savings motive varies cyclically with the state of the economy, except if there is no habit formation (ie $h = 0$), since in this case utility follows a constant relative risk aversion (CRRA) specification.[(10)] It is also important to note that habit formation influences the average level of the precautionary savings effect as well as its dynamics. This point can be seen by taking the unconditional expectation of the precautionary savings effect and substituting for the coefficient κ_0 to arrive at

[(10)] Hence, in the CRRA case the precautionary savings effect is time-varying solely as a result of volatility fluctuations.

$$E[\mathrm{var}_t(\widetilde{m}_{t+1})] = \frac{\rho^2}{((1-h)\eta+\rho)^2}\left[\left(1+\frac{\sigma_{d,v}^2}{1-\rho_{\sigma d}^2}\right)\sigma_{d,u}^2 + \eta^2\left(1+\frac{\sigma_{y,v}^2}{1-\rho_{\sigma y}^2}\right)\sigma_{y,u}^2\right] \quad \textbf{(17)}$$

Hence the driving term in the precautionary savings effect is higher, on average, in an economy with a stronger degree of habit formation. Intuitively, an increase in habit formation raises risk aversion – see equation **(7)** – which in turn increases the incentive to engage in precautionary behaviour. By increasing this incentive, habit formation strengthens the transmission mechanism of volatility fluctuations. Consequently, an increase in habit formation will increase the bias imparted to interest rates if volatility fluctuations are not taken into account by monetary policy.

Finally, consider the proportional change compared to the constant volatility case which sets $\sigma_{d,t} = \sigma_{y,t} = 1$ for all t. Expressed in percentage terms, this change is given by

$$\%\Delta\,\mathrm{var}_t(\widetilde{m}_{t+1}) = 100\times\left(\chi_d(\sigma_{d,t}^2 - 1) + (1-\chi_d)(\sigma_{y,t}^2 - 1)\right) \quad \textbf{(18)}$$

where the weight $\chi_d \equiv \sigma_{d,u}^2/(\sigma_{d,u}^2 + \eta^2\sigma_{y,u}^2)$ indicates the relative importance of preference shocks in consumption volatility in the constant volatility case.

Equation **(18)** shows that the percentage change compared to the constant volatility case can be expressed as a weighted average of the deviations of the volatility driving processes from their steady-state values. The weights reflect the relative importance of each shock in consumption volatility in the *constant volatility case*. Intuitively, it is consumption volatility that matters because the log stochastic discount factor in equation **(16)** is defined to include only the marginal utility ratio; and it is the disturbance volatilities in the constant volatility case that are relevant because time variations in volatility are effectively small 'perturbations' (albeit persistent ones) around the steady-state variances. The importance of volatility variations in each of the shocks also depends crucially on the inverse elasticity of labour supply.

Now consider the second term in equation **(15)**, which depends only on the variance of preference shocks. This term can be approximated to third order as follows:

$$2\,\mathrm{cov}_t(\widetilde{m}_{t+1}, \Delta\varepsilon_{d,t+1}) + \mathrm{var}_t(\Delta\varepsilon_{d,t+1}) = \sigma_{d,t}^2\sigma_{d,u}^2(1 - 2\rho\kappa_0^{-1}(1+\theta_d\varepsilon_{d,t} + \theta_y\varepsilon_{y,t} + \theta_x x_t)) \quad \textbf{(19)}$$

where

$\theta_d = \frac{h}{\kappa_0^2}\left(h(\rho+\eta)(1-h)(1-\phi) - \rho_d\kappa_0\right)$, $\theta_y = \frac{h\eta}{\kappa_0^2}\left(h(\rho+\eta)(1-h)(1-\phi) - \rho_y\kappa_0\right)$, and

$\theta_x = \frac{h}{\kappa_0^2}(\rho+\eta)(\phi\kappa_0 + h(1-\phi)\rho)$.

The properties of this second term are somewhat different to the first term given in equation **(16)**, because preference shocks and the stochastic discount factor negatively covary. First, this term will increase with preference volatility innovations *only if* the whole term in brackets is positive. [11] Second, its average level is decreasing in the habit size parameter h, as can be seen formally by taking the unconditional expectations operator through equation **(19)**. As is shown in the numerical simulation exercises in Section 6, the negative covariance term means that changes in the 'size' of habits have different precautionary savings implications for preference and productivity volatility innovations. The next section discusses the calibration of the model and is followed by numerical simulations and sensitivity analysis. The analytical results discussed in this section are used to inform the quantitative results presented below.

4 Model calibration

4.1 Stochastic volatility calibrations

Direct estimates of stochastic volatility processes for shocks in DSGE models are relatively scarce. However, using quarterly US data, Fernández-Villaverde *et al* (2010), Justiniano and Primiceri (2008) and Fernández-Villaverde and Rubio-Ramirez (2007) estimate medium-scale New Keynesian DSGE models over the post-war period. These models include disturbances of a similar nature to the ones modelled here.[12] Justiniano and Primiceri (2008) set the persistence coefficients in their volatilities to one in order to reduce the number of free parameters they estimate, but Fernández-Villaverde *et al* (2010) and Fernández-Villaverde and Rubio-Ramirez (2007) estimate both persistence parameters and innovation variances. The calibration that follows is intended to be broadly consistent with these results; indeed, its aim is to capture similar short-run volatility dynamics to these papers. It should be noted that although the above papers use a log-linear specification for volatility, a linear specification with persistence as used here is sufficient to produce broadly similar dynamic behaviour.

First, consider the persistence parameters in the volatility processes. In this paper, a baseline persistence of $\rho_{\sigma d} = 0.75$ was chosen for preference volatility. This calibration is lower than the estimates of approximately 0.95 in Fernández-Villaverde *et al* (2010) and Fernández-Villaverde and Rubio-Ramirez (2007) in order to reflect uncertainty surrounding the correct value, and to enable sensitivity to higher and lower values to be tested. For productivity volatility persistence, there is only a single comparable estimate in the literature – an estimate of 0.13 in Fernández-

[11] This negative covariance explains, in part, why the first term on the right-hand side of equation **(15)** is the dominant one in the precautionary savings effect.
[12] More specifically, all three papers include discount factor shocks, which are similar to the consumption preference shocks modelled in this paper, whilst both Fernández-Villaverde *et al* (2010) and Justiniano and Primiceri (2008) include labour disutility shocks. Note that a labour disutility shock is effectively a shock to labour supply (or leisure) preferences, whilst the productivity shock in this paper is a shock to the disutility associated with producing a given amount of output.

Villaverde *et al* (2010), which is strongly at odds with the assumed value of one in Justiniano and Primiceri (2008). A mid-range value of $\rho_{\sigma y} = 0.50$ was therefore chosen for the baseline calibration. Sensitivity to higher and lower persistence calibrations is investigated in Section 6.2.

Second, the innovation variances in the volatility processes need to be calibrated. Conditional on the calibrated persistence parameters discussed above, the innovation variances were chosen to broadly match the dynamic behaviour of estimated standard deviations in the three papers discussed above. In particular, the graphical standard deviation results in these papers were used as a guide, with the aim of roughly matching, in magnitude, the observed *percentage deviations about trend*. This approach avoids the complicating issue that estimated volatilities vary across papers, whilst enabling the dynamic behaviour of volatilities to be roughly replicated.

Based on test simulations with the persistence parameters chosen as above, the variance of the innovation to productivity shock volatility was set at 0.004, and the variance of the innovation to preference shock volatility at 0.0001. The substantially lower innovation variance for preference volatility is consistent with the results in Fernández-Villaverde *et al* (2010) and Justiniano and Primiceri (2008).(13) Since there is considerable uncertainty surrounding the correct innovation variance calibrations, sensitivity is investigated in Section 6.2.

4.2 Model parameters

One period is defined as a quarter, so the discount factor was set equal to $\beta = 0.99$, yielding an annual steady-state real interest rate of 4%. The habit 'size' parameter h was set at 0.80, similar to the estimated values in Julliard *et al* (2006) and Banerjee and Batini (2003). The inverse elasticity of labour supply η was set equal to 6, which follows Canzoneri *et al* (2007) and ensures that consumption is not too 'smooth' relative to the data under the benchmark calibration of habits. The average length of price contracts α was set equal to three quarters. As in Campbell and Cochrane (1999), the risk aversion coefficient ρ was set equal to 2.37 and the degree of persistence in habits to 0.97. The elasticity of substitution between differentiated goods was set at 10 as in Benigno and Woodford (2005).

Following Smets and Wouters (2003), preference shock persistence was set equal to 0.90. Productivity shock persistence was set equal to 0.997, as in Smets and Wouters (2007). Following Smets and Wouters (2003), the ratio of productivity innovation volatility to preference innovation volatility was set equal to 3.5 at steady-state (ie when time variations in volatility are absent), with the levels of volatility chosen as in De Paoli and Zabczyk (2011). Table A presents the full model parameter calibration, plus the calibrations for the stochastic volatility processes.

(13) For example, the estimated variance ratio is almost 80 to 1 in Fernández-Villaverde *et al* (2010).

It is notable that the benchmark calibration of habit formation is similar to that in models which aim to match risk-premia size and dynamics (eg De Paoli and Zabczyk (2009); Campbell and Cochrane (1999)) and other higher-order uncertainty effects (eg De Paoli and Zabczyk (2011)). In particular, the process for habits is highly inertial, and habits play an important role in utility through the habit size parameter *h*, which raises the level of risk aversion and makes it strongly countercyclical.

Table A: Calibrated values in the quantitative analysis

Parameter	Value	Notes
β	0.99	Quarterly frequency
η	6	Following Canzoneri *et al* (2007)
ρ	2.37	Campbell and Cochrane (1999)
α	0.66	Average price contract length of 3 quarters
σ	10	Benigno and Woodford (2005)
h	0.80	Julliard *et al* (2006) and Banerjee and Batini (2003)
ϕ	0.97	Campbell and Cochrane (1999)
ρ_y	0.997	Smets and Wouters (2007)
ρ_d	0.90	Smets and Wouters (2003)
$\sigma^2_{y,u}$	0.0007	Smets and Wouters (2003)
$\sigma^2_{d,u}$	0.0002	Smets and Wouters (2003)
$\sigma^2_{d,mean}, \sigma^2_{y,mean}$	1	Normalisation
$\sigma^2_{y,v}$	0.004	See previous section
$\sigma^2_{d,v}$	0.0001	See previous section
$\rho_{\sigma y}$	0.50	See previous section
$\rho_{\sigma d}$	0.75	See previous section

5 Simulation results: impulse responses to volatility innovations

To obtain quantitative results, the model was simulated to third order using Dynare++. The impulse responses presented in this section show the impact of innovations to volatility on the natural rate of interest – that is, the interest rate that ensures price stability, defined as zero inflation. Since there are no innovations to volatility in constant volatility models, these impulse responses can be interpreted as showing the ‘policy bias’ that would arise in setting the interest rate if volatility innovations were ignored when formulating policy. Impulse responses to volatility innovations isolate the pure impact of a change in risk, because the standard deviations of the distributions from which shocks are drawn are ‘perturbed’ whilst the shocks themselves are held constant at their expected value of zero. The quantitative analysis below begins by assessing the response of the natural rate to a productivity volatility innovation.[14]

[14] As is typical, the analysis focuses on the impact of positive innovations. Negative innovations have an equal impact but the signs of the impulse responses are reversed.

Chart 1 shows the change in the natural rate of interest following a one standard deviation innovation to productivity volatility and decomposes this total effect into the intertemporal substitution effect and the precautionary savings effect.[15] As expected based on equation **(16)**, an innovation to productivity volatility increases the precautionary savings effect, causing a reduction in the natural rate of interest. On impact, the natural rate falls by around 0.09% on a quarterly basis – roughly one tenth of the steady-state real interest rate. From Chart 1 we can see that this fall is driven primarily by an increase in the precautionary savings effect of around 0.15% in response to a preference volatility innovation. Moreover, since the volatility process displays persistence, the precautionary savings effect remains above steady state for around eight quarters, and the natural rate below steady state for a similar period of time.

Interestingly, the intertemporal substitution effect is also increased by a productivity volatility innovation, though the impact is small by comparison to the increase in the precautionary savings effect. The intertemporal substitution effect responds to a volatility innovation because the combination of a 'partially linear' Phillips curve with a 'conditionally-linear' volatility process means that innovations to volatility can have an independent impact at second order, as discussed by Benigno *et al* (2010). In summary, there is a non-trivial bias in the interest rate if productivity volatility innovations are ignored by policy, due primarily to an increase in the precautionary savings motive.

Chart 1: Impulse responses following a one standard deviation productivity volatility innovation (quarterly, in percentage points)

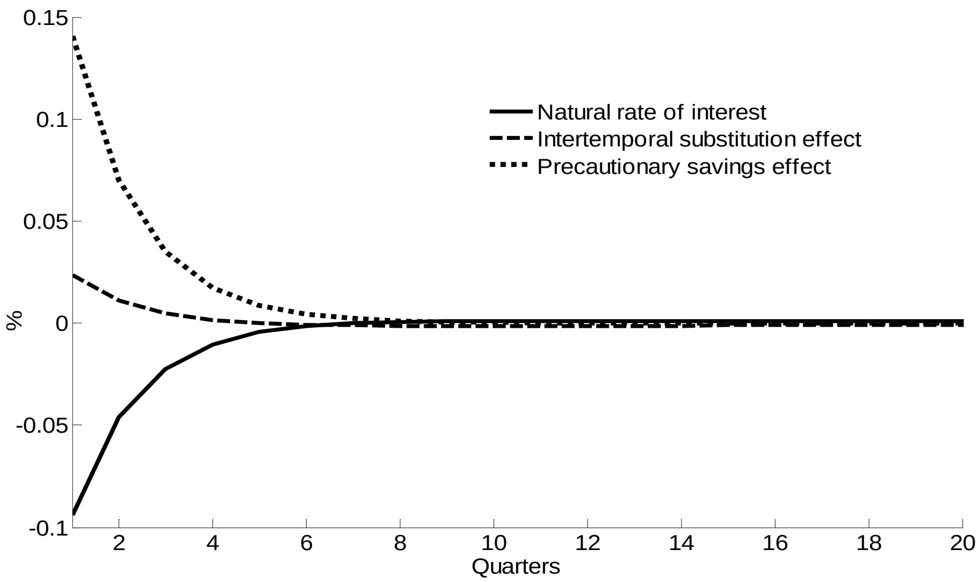


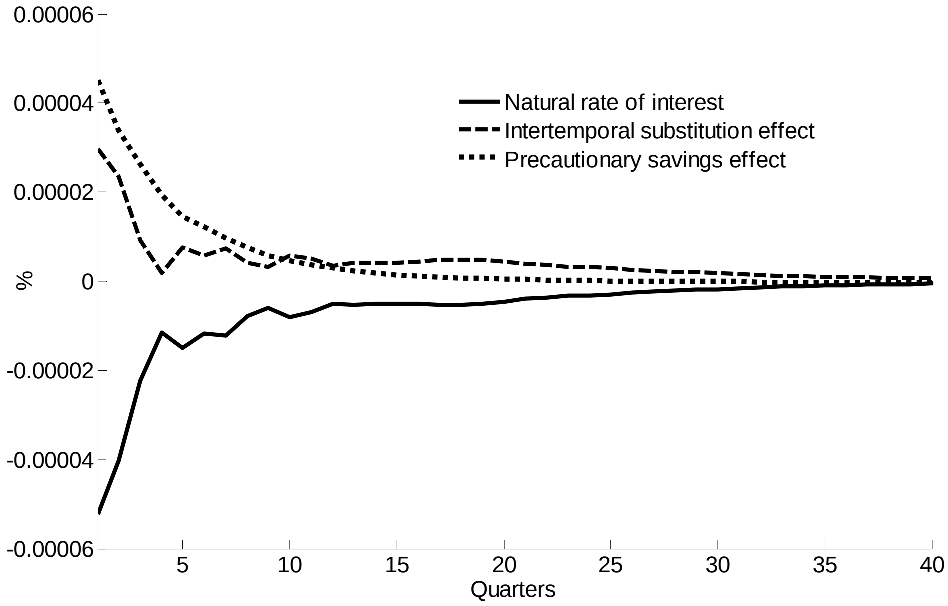
Chart 2 shows the impulse response of the natural rate of interest to a one standard deviation preference volatility innovation, with the response again decomposed into the intertemporal substitution effect and the precautionary savings effect. The precautionary savings effect also rises with a preference volatility innovation, but the impact is much smaller than for a

[15] Note from equation **(14)** that the precautionary savings effect enters the natural rate with a coefficient of -½.

productivity volatility innovation. Indeed, the impacts on the precautionary savings effect, the intertemporal substitution effect, and the natural rate of interest are all quantitatively trivial at less than one hundredth of 1 basis point.[16] There are three reasons for this stark difference in magnitude, all of which are related to the baseline calibration of the model.

First, in order to ensure price stability, consumption must respond to productivity shocks with an elasticity of $\eta(\kappa_0/(1\text{-}h))^{-1}$, but to preference shocks with an elasticity of $(\kappa_0/(1\text{-}h))^{-1}$. With the baseline calibration of $\eta = 6$, consumption uncertainty therefore depends much more strongly on productivity shock volatility than preference shock volatility – so that, intuitively, the precautionary motive responds more strongly to fluctuations in the former. Second, the steady-state level of productivity shock volatility is 3.5 times as high as the steady-state level of preference shock volatility, which also means that the contribution of productivity shocks to consumption volatility is somewhat higher. Third, the standard deviation of innovations to productivity volatility is 6.3 times higher under the baseline calibration (see Table A), so that preference volatility moves far less in response to a one standard deviation volatility innovation.

Chart 2: Impulse responses following a one standard deviation preference volatility innovation (quarterly, in percentage points)



That a lower innovation standard deviation leads to a smaller impact on the precautionary savings motive is intuitive and can be seen formally from equation **(16)**.[17] The working of the other two effects mentioned above can be seen formally from equation **(18)**, which shows how the impact of volatility fluctuations on the driving term in the precautionary savings effect depends on how important preference and productivity shocks are for steady-state consumption

[16] Again, the intertemporal substitution effect rises with a volatility innovation because innovations have an independent impact at second order (see Benigno *et al* (2010) for further detail).
[17] In particular, a one standard deviation preference volatility innovation $v_{d,t}$ will be somewhat smaller than a one standard deviation productivity volatility innovation $v_{y,t}$.

volatility. Indeed, under the baseline calibration, the relative weight on preference volatility fluctuations is only χ_d = 0.03, compared to a relative weight of $1-\chi_d$ = 0.97 on productivity volatility fluctuations. Hence preference shocks are responsible for only a small fraction of overall consumption volatility, with the result that fluctuations in preference volatility have much weaker implications for precautionary savings and monetary policy. Given that there is considerable uncertainty surrounding the correct calibrations of both η and the volatility innovation variances, the impact of different values is investigated in the next section.

To summarise, productivity volatility innovations have a non-trivial impact on the natural rate of interest through a substantial increase in the precautionary savings motive. Consequently, ignoring such innovations as in constant volatility models leads to a quantitatively relevant bias in the interest rate set by policy. By contrast, preference volatility innovations give rise to only quantitatively trivial policy bias. However, since the above conclusions depend crucially on the calibration of the model, the robustness of these results is tested in the next section.

6 Sensitivity analysis

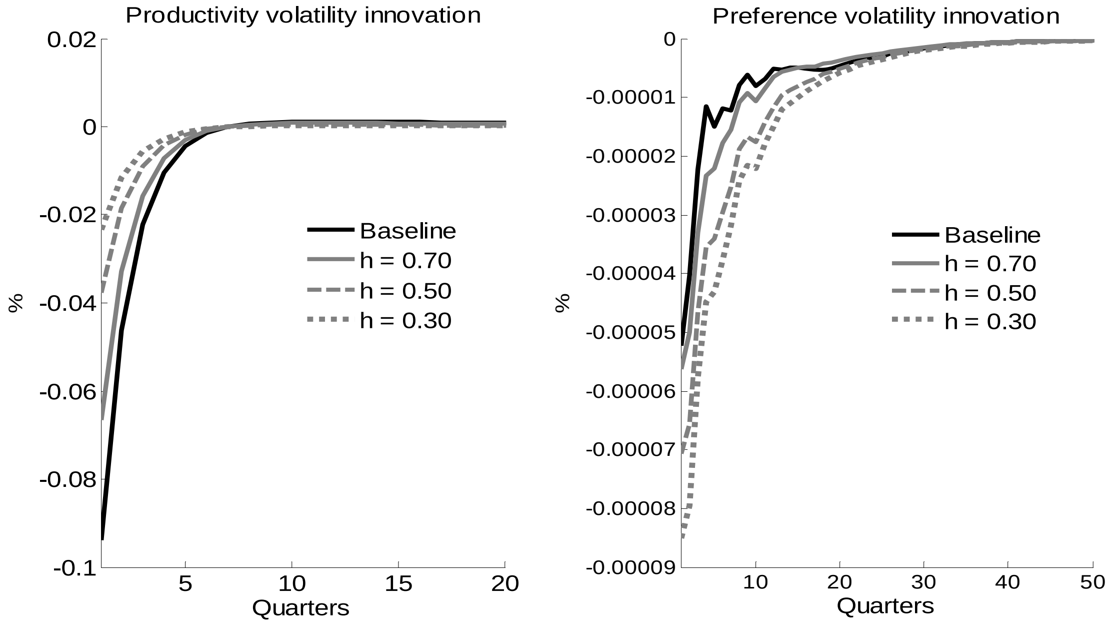
6.1 *Model parameter calibration*

This first section investigates sensitivity to two model parameters whose correct calibrations are somewhat uncertain and which the analytical results above suggest are important for the transmission of volatility shocks – namely, the habit 'size' parameter h and inverse elasticity of labour supply η.

Equation **(17)** shows that the magnitude of the driving term in the precautionary savings effect increases strongly with h, but from equation **(19)** the second term in the precautionary savings effect (which depends only on preference shock volatility) decreases with h due to the negative covariance between the stochastic discount factor and preference shocks. It is therefore instructive to investigate the impact of the habit size parameter h on the quantitative results of the previous section. The results of this exercise are shown in Chart 3.

In concordance with equation **(16)**, a reduction in the habit size parameter reduces the response of the natural rate that is required to ensure price stability in the face of a productivity volatility innovation. For instance, with h = 0.30, a productivity volatility innovation reduces the natural rate by only 0.02% on impact, compared to almost 0.09% under the baseline calibration that sets h = 0.80. Hence policy errors from ignoring productivity volatility innovations are higher the more important the role of external habit formation in consumer preferences. Due to the negative covariance between the stochastic discount factor and preference shocks, this result is reversed for preference volatility innovations: a fall in habit size raises the reduction in the natural rate on impact. Importantly, however, the quantitative impact of preference volatility innovations remains trivial even if the habit size is low.

Chart 3: Sensitivity of the natural rate impulse responses to the habit 'size' *h* (quarterly, in percentage points)



Overall, then, an increase in the habit size parameter increases the importance of volatility fluctuations through the precautionary savings channel. It is intuitive that habit formation makes higher-order precautionary saving effects of greater importance, since the curvature of the utility function with respect to consumption is given by $\rho/(1-h)$.[(18)] Hence under the baseline calibration of $h = 0.8$, the curvature of the utility function is more than doubled compared to the $h = 0.5$ case, whilst it is magnified by a factor of 3.5 compared to a calibration that sets $h = 0.3$. A potentially important policy implication from these results is that models which are not calibrated to match high-order risk effects may understate the importance of volatility fluctuations for monetary policy.

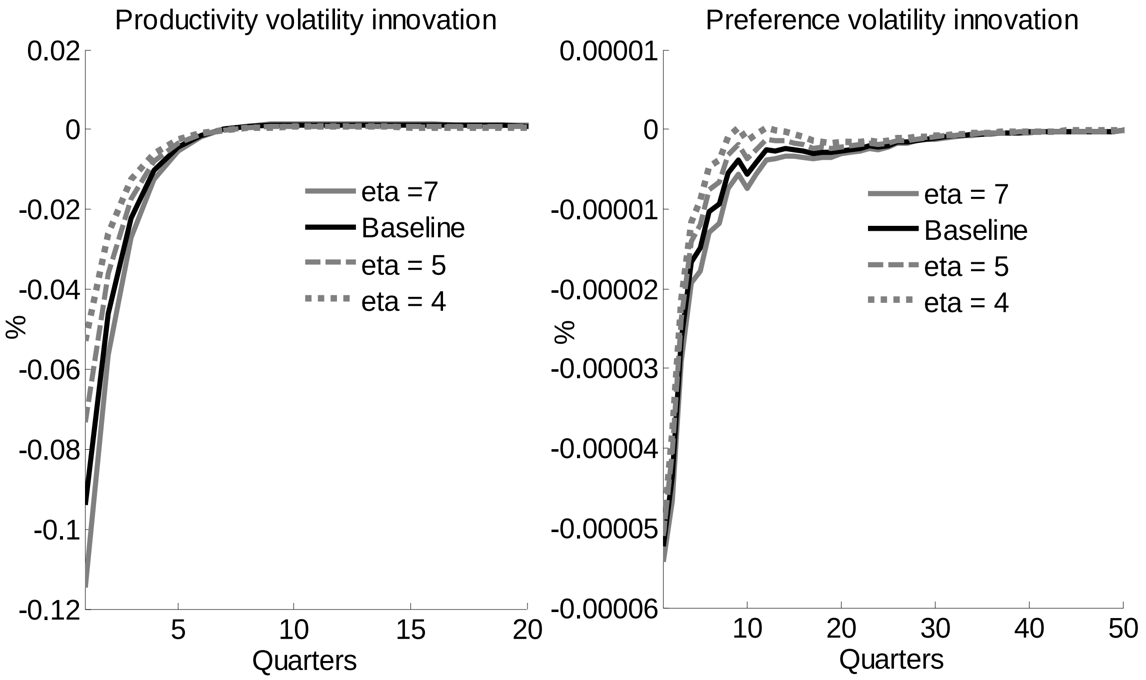
Chart 4 investigates robustness with respect to the calibration of the inverse elasticity of labour supply η. There is considerable uncertainty surrounding the correct calibration, with estimates in the literature ranging from 0.47 (Rotemberg and Woodford (1998)) to 7 (Canzoneri *et al* (2007)), compared to the baseline calibration of 6. In this section, sensitivity was tested for values from 7 down to 3.[(19)] As noted in Section 3, η is a key factor determining the relative importance of volatility innovations to productivity and preferences since, in order to ensure price stability, consumption must respond to preference shocks with an elasticity of $(\kappa_0/(1-h))^{-1}$ and to productivity shocks with an elasticity of $\eta(\kappa_0/(1-h))^{-1}$. Therefore, a reduction in η should

[(18)] Formally, this result can be derived by evaluating equation **(7)** at the deterministic steady state.
[(19)] Lower values were not considered because consumption tends to be too 'smooth' relative to the data if η is too small.

reduce the impact of productivity volatility innovations and increase the impact of preference volatility innovations.

The sensitivity results in Chart 4 confirm this intuition: as η is reduced, the impact of a productivity volatility innovation on the natural rate is reduced, whilst a preference volatility innovation has a larger initial impact – though still one which is quantitatively trivial. The productivity volatility innovation impacts, though smaller than in the baseline case, are still quantitatively relevant. For example, with $\eta = 4$, the natural rate still falls by 0.05% on impact, indicating a policy error of this magnitude if a one standard deviation a productivity innovation were ignored by policy.[20]

Chart 4: Sensitivity of the natural rate impulse responses to the inverse elasticity of labour supply η (quarterly, in percentage points)



6.2 *Stochastic volatility calibrations*

This section assesses sensitivity to the calibrations of the stochastic processes driving time variations in volatility. As noted in Section 4.1, there is considerable uncertainty surrounding the correct calibrations of these processes, primarily because there is relatively little literature estimating stochastic volatility in DSGE models.

The volatility persistence parameters were set at $\rho_{\sigma d} = 0.75$ and $\rho_{\sigma y} = 0.50$ in the baseline case. Sensitivity to alternative parameterisations is shown in Chart 5. As we would expect, an increase (decrease) in volatility persistence increases (decreases) persistence in the natural rate through its impact on the precautionary savings effect. For example, with preference volatility

[20] Of course, the implied policy error is somewhat larger if the impact of persistence is taken into account.

persistence of $\rho_{\sigma d}$ = 0.95, an innovation to preference volatility does not 'die out' until around 100 quarters, compared to around 40 under the baseline calibration, whilst with $\rho_{\sigma y}$ = 0.75 the impact of a productivity volatility innovation does not 'die out' until around 20 quarters, compared to 8 in the baseline case. The impulse response to preference volatility innovations remains quantitatively trivial, but for productivity volatility innovations the impact of changing persistence is relevant since greater persistence implies a somewhat larger *cumulative policy error* if innovations are ignored by policy. Lower persistence coefficients have the opposite effect: cumulative policy errors from ignoring volatility innovations are reduced.

Chart 5: Sensitivity of the natural rate impulse responses to volatility persistence (quarterly, in percentage points)

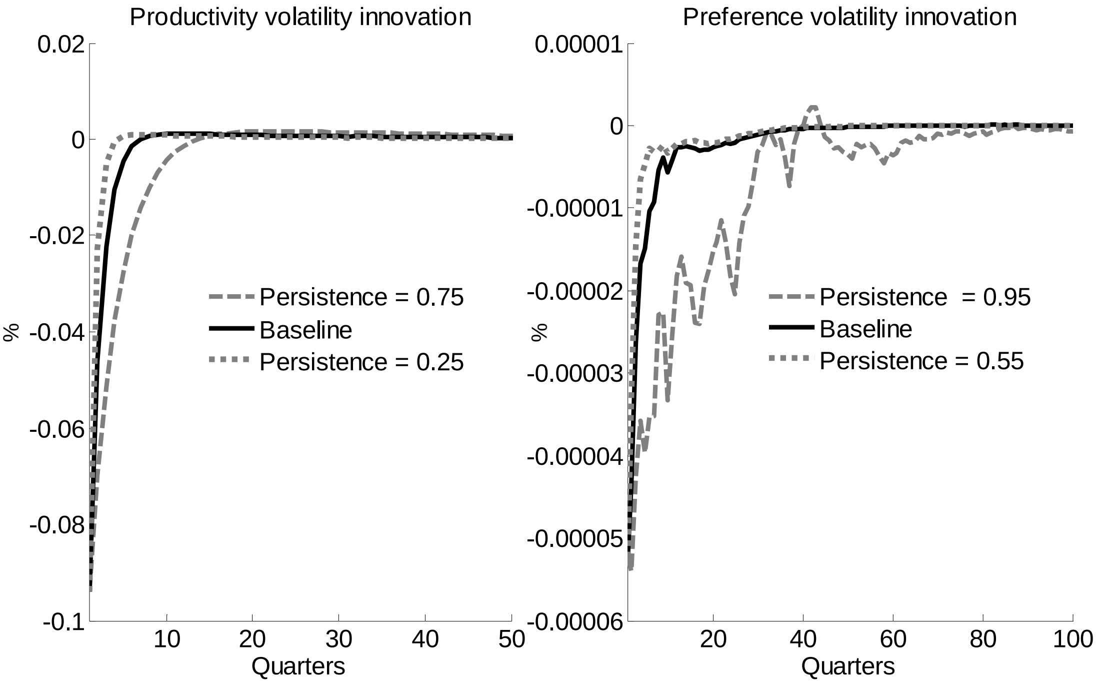
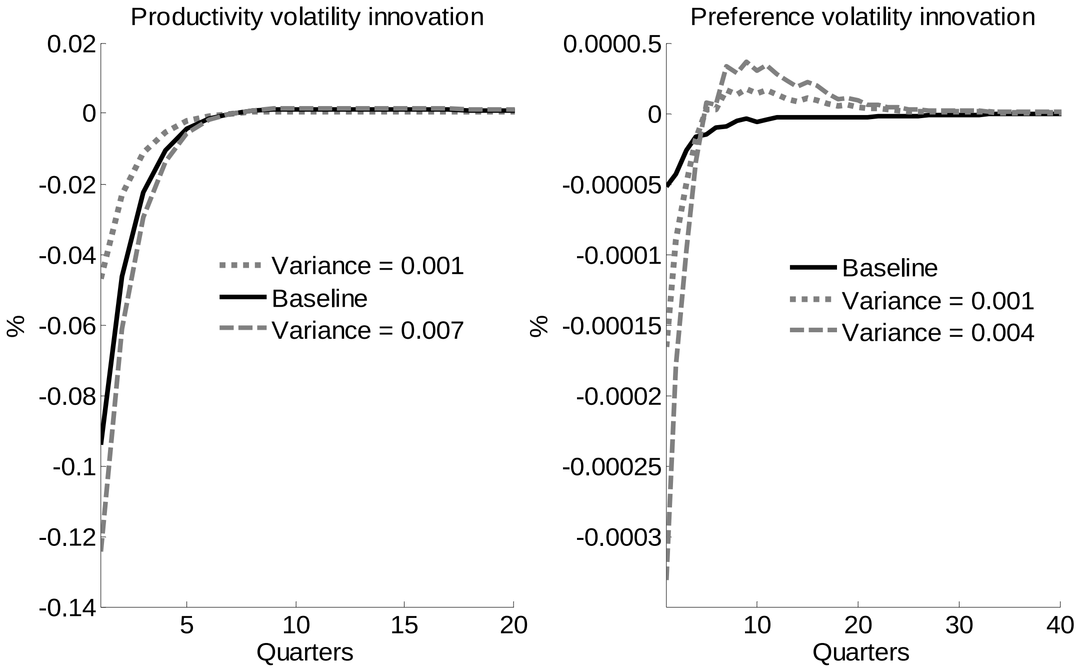


Chart 6 investigates sensitivity to alternative volatility innovation variances. Under the baseline calibration, the volatility innovation variance was set somewhat higher for productivity shocks than preference shocks, in line with findings from the time-varying volatility literature. The correct calibration of these variances is, nevertheless, rather uncertain. Consequently, sensitivity was tested for a wide range of alternative parameterisations. Given that preference volatility innovations have only a trivial impact in the baseline case when the innovation variance was set at 0.0001, sensitivity was tested to somewhat higher innovation variances of 0.001 and 0.004 – respectively, ten times and 40 times the baseline calibration. For productivity volatility, sensitivity was tested for a higher innovation variance of 0.007 and a lower innovation variance of 0.001, deviations of 0.003 either way from the baseline calibration.

As can be seen from Chart 6, the response of the natural rate is quite robust to changes in the productivity volatility innovation variance. With the higher innovation variance of 0.007 the natural rate falls by around 0.12% on impact, compared to 0.09% in the baseline case. The fall on impact is roughly halved with the lower innovation variance of 0.001 but is still quantitatively relevant at 0.04%. The response of the natural rate to preference volatility innovations appears more sensitive, but this is due primarily to the fact that the variances tested are so much higher than under the baseline calibration. If the preference volatility innovation variance is ten times the baseline at 0.001, the natural rate falls by three times as much on impact; and with the innovation variance raised to 40 times the baseline, the fall in the natural rate is more than six times as large, though still quantitatively trivial at only three hundredths of 1 basis point.

Chart 6: Sensitivity of the natural rate impulse responses to volatility innovation variances (quarterly, in percentage points)



To summarise, the results of this section show that the conclusion that productivity volatility innovations have a quantitatively relevant impact is robust to alternative calibrations of the stochastic process driving volatility fluctuations. On the other hand, preference volatility innovations have a quantitatively trivial impact even if volatility is highly persistent and innovations to volatility are of similar magnitude to productivity volatility innovations.

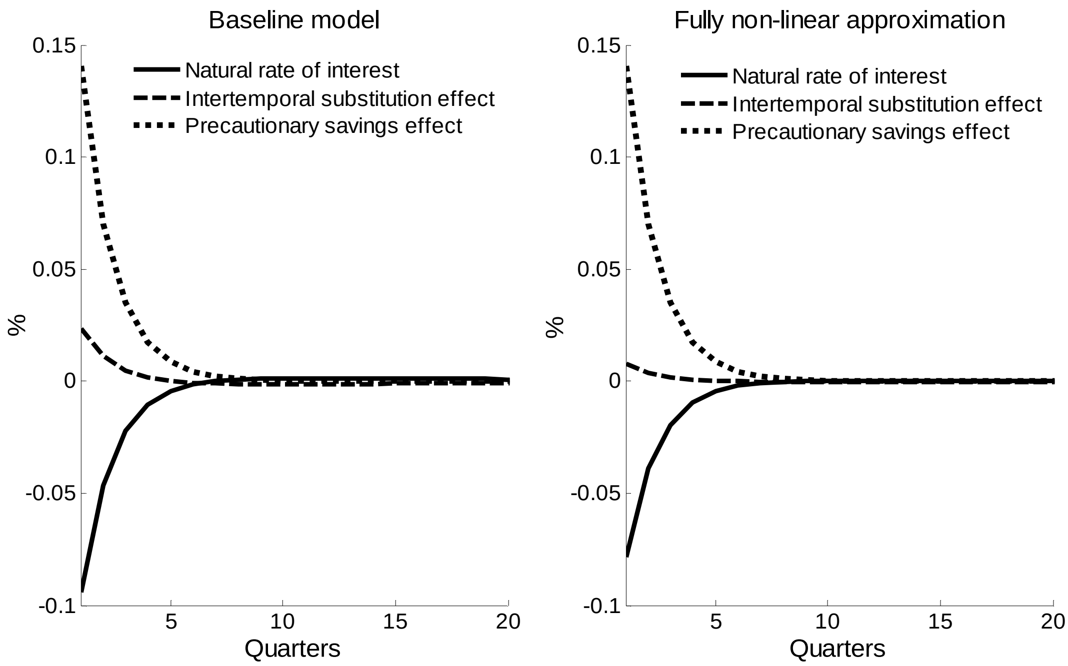
6.3 *Allowing for a fully non-linear approximation of the model*

In order to derive analytical results, the analysis above employed a ‘partially non-linear’ version of the New Keynesian Phillips curve. The results in this section show that the main conclusion from the analysis – namely, that ignoring productivity volatility innovations leads to quantitatively relevant policy errors – remains intact if a fully non-linear Phillips curve is specified. In order to derive these results, the model was simulated to third order in Dynare++

after imposing price stability (ie zero inflation in all periods) in equation **(8)**, the first-order condition for price-setting.

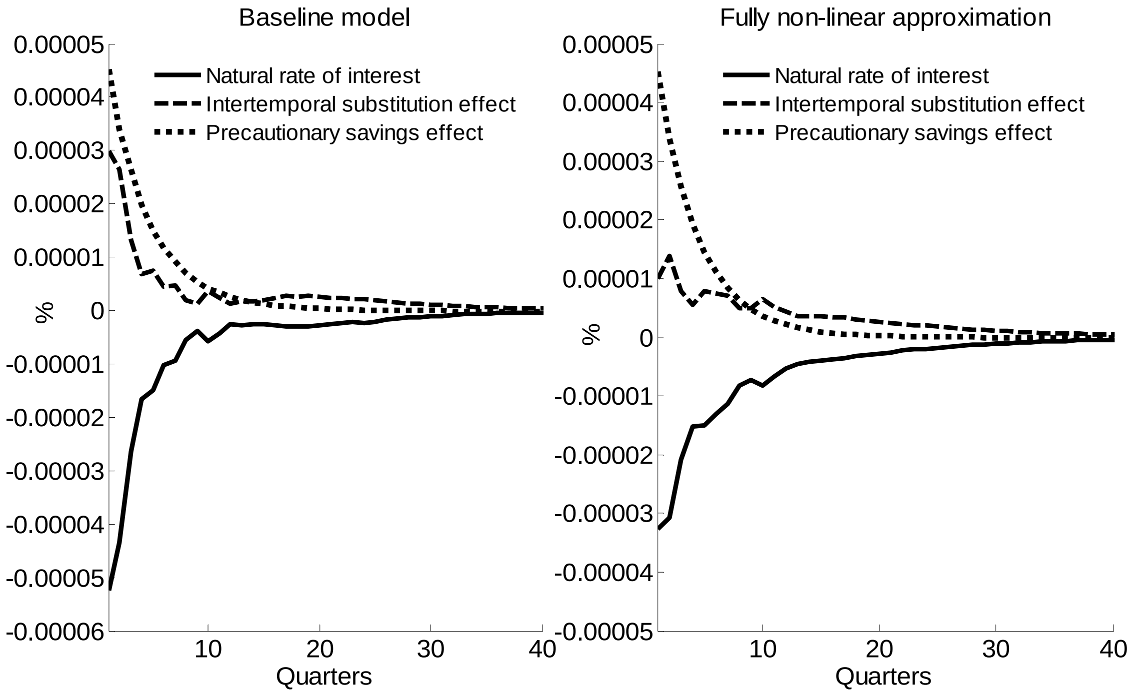
The impulses responses for the natural rate of interest, the intertemporal substitution effect and precautionary savings effect in a fully non-linear model are compared to those in the baseline model below. For productivity volatility innovations, the results are rather robust to this change in specification, as is shown in Chart 7. The response of the precautionary savings effect is essentially identical, but the intertemporal substitution effect responds somewhat less strongly than previously, rising by less than half of the amount it did on impact in the partially non-linear case. However, since the precautionary savings effect dominates in terms of magnitude, the impulse response of the natural rate remains similar at a fall of 0.08% on impact, compared to 0.09% in the partially non-linear case.

Chart 7: Impulses responses to a productivity volatility innovation when the Phillips curve is fully non-linear (quarterly, in percentage points)



In the case of a preference volatility innovation, Chart 8 shows that moving to a fully non-linear specification of the Phillips curve has a greater impact upon the impulse responses of the intertemporal substitution effect and the natural rate of interest. Indeed, although the impulse response of the precautionary savings effect is again essentially unchanged, the intertemporal substitution effect response is dampened somewhat, with the initial increase at around one third of its value in the partially non-linear case. Due to the substantially lower impact on the intertemporal substitution effect, the natural rate falls by only two thirds as much on impact. Therefore, allowing for a fully non-linear Phillips curve reduces even further the potential for policy bias if preference volatility innovations are ignored by policy – hence reinforcing the conclusion reached above in the partially non-linear case.

Chart 8: Impulses responses to a preference volatility innovation when the Phillips curve is fully non-linear (quarterly, in percentage points)



In summary, the key result that productivity volatility innovations have a quantitatively relevant impact on the natural rate holds also in a fully non-linear model – a result which offers support to the derived analytical expressions for the precautionary savings effect reported in Section 3.

6.4 Assessing the performance of a misspecified Taylor rule

It has been assumed so far that the central bank follows a 'targeting rule' that ensures price stability. With this in mind, the analysis thus far has sought to quantify the potential 'policy bias' from ignoring volatility fluctuations by examining impulse responses of the natural rate of interest to volatility innovations. The impulse response results clearly show that policy must respond directly to volatility innovations in order to ensure that price stability is maintained. However, it is important to note that an objective of zero inflation could also be achieved using an 'instrument rule' for the nominal interest rate.

The monetary policy implications arising from this alternative representation of policy are investigated in this section. In particular, consistent with the focus on the precautionary savings effect above, policy is represented by a Taylor rule that tracks the natural rate of interest. In order to assess the impact of ignoring volatility fluctuations through the precautionary savings channel, two different 'natural rates' are considered in which the targeted precautionary savings effect varies. The first captures the precautionary savings effect fully (up to third order), whilst the second captures the precautionary savings effect up to third order but (incorrectly) holds shock volatilities constant. Intuitively, comparing the results in these two cases gives a measure of the ‘marginal impact’ of volatility fluctuations on macroeconomic instability.

The central bank’s Taylor rule is thus given by:

$$r_{t,nom} = r_{t,nat} + (1+\theta_{\pi})\pi_t + \theta_{gap}(y_t - y_{t,nat}) \qquad \textbf{(20)}$$

where $r_{t,nom}$ is the nominal interest rate, $r_{t,nat}$ is the natural rate of interest, and $y_t - y_{t,nat}$ is the output gap, defined as the difference between actual output and its flex-price level.[21]

Table B reports unconditional standard deviations of inflation and the output gap from 2,000 simulations of the model in Dynare++ when the Taylor rule parameters were set at $\theta_{\pi} = 1.5$ and $\theta_{gap} = 0.2$.

Table B: Unconditional standard deviations (annualised, in percentage points)

Model	Inflation (%)	Output gap (%)
$r_{t,nat}$ is true natural rate	0	0
$r_{t,nat}$ ignoring volatility fluctuations through precautionary savings channel (Baseline, h = 0.8)	0.089	0.067
$r_{t,nat}$ ignoring volatility fluctuations through precautionary savings channel (h = 0.5)	0.016	0.054

The Taylor rule in equation **(20)** ensures perfect price stability if the natural rate targeted by policy is the true one that captures time variations in shock volatilities. This point is confirmed formally by the first row of results in Table B: both the inflation and output gap standard deviations are zero in this case. The second row of results in Table B reports the standard deviations of inflation and the output gap when the natural rate targeted by policy ignores time variations in the precautionary savings motive arising from fluctuations in shock volatilities, thereby isolating the ‘marginal impact’ of volatility fluctuations on macroeconomic instability. Ignoring the impact of volatility fluctuations on the precautionary saving motive increases the standard deviation of inflation to 0.09% and the standard deviation of the output gap to 0.07%. Therefore, ignoring volatility fluctuations when setting policy has a small but non-trivial impact on macroeconomic instability. Intuitively, if policy ignores such fluctuations, it fails to respond to inflationary or deflationary pressures building up as a result of variations in the precautionary savings motive. By the Phillips curve, these inflation variations then cause output to deviate from its flex-price level.

Notably, the results are also consistent with the impulse responses reported above. In particular, the increase in macroeconomic volatility is driven almost entirely by ignoring productivity

[21] Both actual and flex-price output are expressed in log deviations from steady state.

volatility fluctuations,[22] and reducing the habit size parameter to $h = 0.5$ lessens but by no means eliminates the impact of volatility fluctuations through the precautionary savings channel (see the third row of results in Table B). In summary, the main conclusions of the quantitative analysis are robust to the assumption that policy follows an 'instrument rule' rather than a 'targeting rule'. Indeed, the results in this section can be interpreted as showing that the non-trivial 'policy bias' highlighted in Section 5 translates into non-trivial implications for macroeconomic instability when compared to the constant volatility case.

7 Conclusions

The results in this paper show that variations in the volatilities of economic disturbances can have a quantitatively relevant impact on precautionary saving behaviour. Consequently, using constant volatility models that ignore such fluctuations may give rise to recommendations for monetary policy that are biased in a non-trivial way. The main contributions of this paper are to clarify the mechanism by which fluctuations in uncertainty are transmitted through the precautionary savings channel, and to demonstrate that such effects can be quantitatively important despite the fact that they enter only in a third-order approximation of the model. The analysis was conducted in a simple New Keynesian model with external habit formation introduced in previous work at the Bank, augmented to include demand and supply disturbances whose volatilities vary over time.

The main result that volatility fluctuations can be of quantitative relevance is robust along numerous dimensions, including alternative calibrations of the model; changes to the parameterisations of the stochastic volatility processes; the central bank rule for monetary policy; and the order to which the Phillips curve was approximated in the model solution. The external habits specification of utility plays an important role in the model because, by raising risk aversion, habit formation increases the importance of the precautionary savings motive, exacerbating 'policy errors' that arise from ignoring volatility fluctuations. Consequently, models which are not calibrated to match higher-order risk effects may understate the importance of volatility fluctuations for monetary policy.

Interestingly, in the calibrated model it is only volatility innovations to the supply shock (a productivity shock) that have a quantitatively relevant impact, with volatility innovations to the demand shock (a consumption preference shock) having only quantitatively trivial effects. The reason is that the supply shock in the model is considerably more important for overall consumption uncertainty and is also subject to larger volatility fluctuations – both of which magnify the impact of volatility fluctuations through the precautionary savings channel. The quantitative results in this paper therefore demonstrate that whilst volatility fluctuations are

[22] This was tested by setting the variance of preference volatility innovations to zero when simulating the model.

potentially relevant for monetary policy, whether they are in practice is likely to depend crucially on the type of economic disturbance that is considered.

The quantitative results reported in this paper are supported by third-order analytical solutions that provide intuition. Analytical results of this kind can be derived only in simple, stylised models of the economy. Such models are not realistic but are a useful tool for understanding the transmission mechanism of economic phenomena, and for gauging whether they are of potential importance for policy. As such, the quantitative results presented in this paper should not be taken as ‘estimates’ of the extent of policy bias due to time variations in volatility, but as indicative that changes in uncertainty – like those seen during the recent financial crisis – may have small but relevant implications for the conduct of monetary policy.

Appendix A: Derivations

A.1 First-order condition for price-setters and the aggregate price index

Consumer-producer j will set the time-t price of their output good to solve the following problem:

$$\max_{p_t(j)} E_t \sum_{s=t}^{\infty} (\alpha\beta)^{s-t} \left(\frac{\xi_{d,s} (C_s(j) - hX_s)^{1-\rho}}{1-\rho} - \frac{\xi_{y,s}^{-\eta} y_{t,s}(j)^{1+\eta}}{1+\eta} \right) \qquad \textbf{(A-1)}$$

subject to

$$C_s(j) = R_{s-1} B_{s-1} - B_s + \frac{p_t(j) y_{t,s}(j)}{P_s} \qquad \textbf{(A-2)}$$

$$y_{t,s}(j) = \left(\frac{p_t(j)}{P_s} \right)^{-\sigma} Y_s \qquad \textbf{(A-3)}$$

The first-order condition for this maximisation problem is given by

$$E_t \sum_{s=t}^{\infty} (\alpha\beta)^{s-t} \left(\xi_{d,s} (C_s(j) - hX_s)^{-\rho} \frac{\partial C_s(j)}{\partial p_t(j)} - \xi_{y,s}^{-\eta} y_{t,s}(j)^{\eta} \frac{\partial y_{t,s}(j)}{\partial p_t(j)} \right) = 0 \qquad \textbf{(A-4)}$$

Using **(A-2)** and **(A-3)**, equation **(A-4)** can be written in the following form:

$$E_t \sum_{s=t}^{\infty} (\alpha\beta)^{s-t} \left(\frac{(\sigma-1) \xi_{d,s} p_t(j)^{-\sigma} P_s^{\sigma-1} Y_s}{(C_s(j) - hX_s)^{\rho}} - \sigma \xi_{y,s}^{-\eta} p_t(j)^{-1-\sigma(1+\eta)} P_s^{\sigma(1+\eta)} Y_s^{\eta+1} \right) = 0 \qquad \textbf{(A-5)}$$

Simplifying and collecting terms gives the first-order condition in the main text:

$$E_t \sum_{s=t}^{\infty} (\alpha\beta)^{s-t} \left(\frac{\xi_{d,s}}{(C_s(j) - hX_s)^{\rho}} \left(\frac{p_t(j)}{P_s} \right) - \frac{\sigma}{(\sigma-1)} \xi_{y,s}^{-\eta} y_{t,s}(j)^{\eta} \right) y_{t,s}(j) = 0 \qquad \textbf{(A-6)}$$

Since $p_t(j)$ can be taken outside the expectations operator, equation **(A-6)** can be solved for the optimal price as follows:

$$p_t(j) = p_t = \frac{\sigma}{\sigma-1} (PA_t / PB_t) \qquad \textbf{(A-7)}$$

where the absence of the index j indicates that the same price is set by all producers changing price at time t, and where two new variables have been defined as follows:

$$PA_t \equiv E_t \sum_{s=t}^{\infty} (\alpha\beta)^{s-t} \xi_{y,s}^{-\eta} y_{t,s}(j)^{\eta+1} \qquad PB_t \equiv E_t \sum_{s=t}^{\infty} (\alpha\beta)^{s-t} \frac{\xi_{d,s} y_{t,s}(j)}{P_s (C_s(j) - hX_s)^{\rho}} \qquad \textbf{(A-8)}$$

In order to express the aggregate price level in the recursive form reported in the text, note that by definition

$$P_t = \left[\int_0^1 p_t(j)^{1-\sigma} dj\right]^{1/(1-\sigma)} = \left[\sum_{z=0}^{\infty}(1-\alpha)\alpha^z (p_{t-z})^{1-\sigma}\right]^{1/(1-\sigma)} \quad \textbf{(A-9)}$$

where $(1-\alpha)\alpha^z$ is the proportion of producers who set price z periods ago and $p_{t-z} = p_{t-z}(j)$ is the price set by all producers who last changed price at time t-z.

Using **(A-9)** and its lagged value, the aggregate price level can be written as in the text:

$$P_t^{1-\sigma} = \alpha P_{t-1}^{1-\sigma} + (1-\alpha)p_t^{1-\sigma} \quad \textbf{(A-10)}$$

Note also that **(A-10)** can be written in terms of the optimal relative price as follows:

$$1 = \alpha\left(P_{t-1}/P_t\right)^{1-\sigma} + (1-\alpha)Q_t^{1-\sigma} \quad \textbf{(A-11)}$$

where $Q_t \equiv p_t(j)/P_t$.

A.2 Derivation of the 'partially non-linear' New Keynesian Phillips curve

Using the market-clearing condition, equation **(8)** of the main text can be written as follows:

$$E_t\sum_{s=t}^{\infty}(\alpha\beta)^{s-t}\left(\frac{\xi_{d,s}}{(C_s - hX_s)^{\rho}}\left(\frac{p_t(j)}{P_s}\right)^{1-\sigma} C_s - \frac{\sigma}{(\sigma-1)}\xi_{y,s}^{-\eta}\left(\frac{p_t(j)}{P_s}\right)^{-\sigma(1+\eta)} C_s^{\eta+1}\right) = 0 \quad \textbf{(A-12)}$$

Rearranging this expression gives the optimal relative price as

$$Q_t^{1+\sigma\eta}\left[E_t\sum_{s=t}^{\infty}(\alpha\beta)^{s-t}\xi_{d,s}(C_s - hX_s)^{-\rho}\frac{P_t^{1+\sigma\eta}}{P_s^{1-\sigma}}C_s\right] = \frac{\sigma}{\sigma-1}\left[E_t\sum_{s=t}^{\infty}(\alpha\beta)^{s-t}\xi_{y,s}^{-\eta}\frac{C_s^{\eta+1}}{P_s^{-\sigma(1+\eta)}}\right] \quad \textbf{(A-13)}$$

Log-linearising **(A-13)** around steady-state gives

$$q_t = \frac{(1-\alpha\beta)}{(1+\sigma\eta)}\sum_{s=t}^{\infty}(\alpha\beta)^{s-t}\begin{bmatrix}\dfrac{\eta(1-h)+\rho}{1-h}E_t c_s - \dfrac{h\rho}{1-h}E_t x_s - \eta E_t\hat{\varepsilon}_{y,s} \\ -E_t\hat{\varepsilon}_{d,s} + (1+\sigma\eta)(E_t p_s - p_t)\end{bmatrix} \quad \textbf{(A-14)}$$

where lowercase variables denote log deviations from steady state,(23) and the disturbances to productivity and preferences enter with 'hats' to show that they are log-linearised versions of the shocks in equation **(10)** of the text, ie $\hat{\varepsilon}_{d,t+1} = \rho_d\hat{\varepsilon}_{d,t} + u_{d,t+1}$ and $\hat{\varepsilon}_{y,t+1} = \rho_y\hat{\varepsilon}_{y,t} + u_{y,t+1}$.

Equation **(A-14)** can also be written as follows:

(23) Note that log deviations and percentage deviations are equivalent up to a first-order approximation.

$$q_t + p_t = (1-\alpha\beta)\sum_{s=t}^{\infty}(\alpha\beta)^{s-t}(E_t\chi_s + E_t p_s) \quad \textbf{(A-15)}$$

where $\chi_s \equiv \frac{\eta(1-h)+\rho}{(1-h)(1+\sigma\eta)}c_s - \frac{h\rho}{(1-h)(1+\sigma\eta)}x_s - \frac{\eta}{(1+\sigma\eta)}\hat{\varepsilon}_{y,s} - \frac{1}{(1+\sigma\eta)}\hat{\varepsilon}_{d,s}$.

Furthermore, equation **(A-15)** can be written in recursive form as

$$q_t + p_t = (1-\alpha\beta)(\chi_t + p_t) + \alpha\beta(E_t q_{t+1} + E_t p_{t+1}) \quad \textbf{(A-16)}$$

Subtracting p_t on both sides then gives

$$q_t = (1-\alpha\beta)\chi_t + \alpha\beta(E_t q_{t+1} + E_t \pi_{t+1}) \quad \textbf{(A-17)}$$

In order to substitute out for q_t, note that log-linearising equation **(A-11)** implies that

$$q_t = \frac{\alpha}{1-\alpha}\pi_t \quad \textbf{(A-18)}$$

Finally, substituting **(A-18)** into **(A-17)** and rearranging gives the log-linearised New Keynesian Phillips curve:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(\kappa_0(1-h)^{-1}c_t - \rho(1-h)^{-1}hx_t - \eta\hat{\varepsilon}_{y,t} - \hat{\varepsilon}_{d,t}) \quad \textbf{(A-19)}$$

where $\kappa = (1–\alpha\beta)(1–\alpha)/\alpha(1+\sigma\eta)$ and $\kappa_0 = (1–h)\eta + \rho$.

The log-linearised Phillips curve in equation **(A-19)** does not capture time variations in the volatilities of preference and productivity shocks. Therefore, the log-linearised disturbances in **(A-19)** were replaced with the non-linear ones from equation **(10)** of the text.

Hence the New Keynesian Phillips curve in the model is ‘partially non-linear’ and is given by

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(\kappa_0(1-h)^{-1}c_t - \rho(1-h)^{-1}hx_t - \eta\varepsilon_{y,t} - \varepsilon_{d,t}) \quad \textbf{(A-20)}$$

where $\varepsilon_{d,t+1} = \rho_d\varepsilon_{d,t} + \sigma_{d,t}u_{d,t+1}$ and $\varepsilon_{y,t+1} = \rho_y\varepsilon_{y,t} + \sigma_{y,t}u_{y,t+1}$.

A.3 Derivation of the $\mathrm{var}_t(\widetilde{m}_{t+1})$ term

Using the definition of surplus consumption in the main text,

$$\widetilde{m}_{t+1} = -\rho(c_{t+1} - c_t + s_{t+1} - s_t) \quad \textbf{(A-21)}$$

where s_t is the log deviation of surplus consumption from steady-state.

Hence we have

$$\mathrm{var}_t(\widetilde{m}_{t+1}) = \rho^2[\mathrm{var}_t(c_{t+1}) + 2\,\mathrm{cov}_t(c_{t+1}, s_{t+1}) + \mathrm{var}_t(s_{t+1})] \quad \textbf{(A-22)}$$

In order to get a third-order accurate expression, s_t was approximated to second order:

$$s_t \approx \psi_1\left(c_t + (1-h)^{-1}x_t c_t - \frac{1}{2}(1-h)^{-1}c_t^2 - x_t - \frac{1}{2}(1-h)^{-1}x_t^2\right) \quad \textbf{(A-23)}$$

where $\psi_1 \equiv h(1-h)^{-1}$.

First, the conditional variance of consumption is given by [24]

$$\mathrm{var}_t(c_{t+1}) = \psi_2^{-2}(\mathrm{var}_t(\varepsilon_{d,t+1}) + \eta^2\ \mathrm{var}_t(\varepsilon_{y,t+1})) = \psi_2^{-2}(\sigma_{d,t}^2\sigma_{d,u}^2 + \eta^2\sigma_{y,t}^2\sigma_{y,u}^2) \qquad \textbf{(A-24)}$$

where $\psi_2 \equiv \rho(1-h)^{-1} + \eta$.

The covariance between consumption and surplus consumption is given by:

$$\mathrm{cov}_t(c_{t+1}, s_{t+1}) = \psi_1\left((1 + (1-h)^{-1}x_{t+1})\,\mathrm{var}_t(c_{t+1}) - \frac{1}{2}(1-h)^{-1}\,\mathrm{cov}_t(c_{t+1}, c_{t+1}^2)\right) \qquad \textbf{(A-25)}$$

where x_{t+1} depends on variables known at time t.

The second term in equation **(A-25)** is as follows:

$$\mathrm{cov}_t(c_{t+1}, c_{t+1}^2) = \psi_2^{-3}\,\mathrm{cov}_t\begin{bmatrix}(\rho\psi_1 x_{t+1} + \varepsilon_{d,t+1} + \eta\varepsilon_{y,t+1}), \\ (\rho^2\psi_1^2 x_{t+1}^2 + \varepsilon_{d,t+1}^2 + \eta^2\varepsilon_{y,t+1}^2 + 2\rho\psi_1 x_{t+1}(\varepsilon_{d,t+1} + \eta\varepsilon_{y,t+1}) + 2\eta\varepsilon_{d,t+1}\varepsilon_{y,t+1})\end{bmatrix} \qquad \textbf{(A-26)}$$

Using the definition of a covariance, **(A-26)** is equal to

$$\mathrm{cov}_t(c_{t+1}, c_{t+1}^2) = \psi_2^{-3}E_t\begin{bmatrix}(\varepsilon_{d,t+1} - E_t\varepsilon_{d,t+1} + \eta\varepsilon_{y,t+1} - \eta E_t\varepsilon_{y,t+1})\times \\ \left((\varepsilon_{d,t+1}^2 - E_t\varepsilon_{d,t+1}^2) + \eta^2(\varepsilon_{y,t+1}^2 - E_t\varepsilon_{y,t+1}^2) + 2\rho\psi_1 x_{t+1}(\varepsilon_{d,t+1} - E_t\varepsilon_{y,t+1})\right. \\ \left. + 2\rho\psi_1 x_{t+1}\eta(\varepsilon_{y,t+1} - E_t\varepsilon_{y,t+1}) + 2\eta\varepsilon_{d,t+1}\varepsilon_{y,t+1} - 2\eta E_t(\varepsilon_{d,t+1}\varepsilon_{y,t+1})\right)\end{bmatrix} \qquad \textbf{(A-27)}$$

Hence using $\varepsilon_{i,t} = \rho_i\varepsilon_{i,t-1} + \sigma_{i,t-1}u_{i,t}$, $i \in \{d, y\}$, we can write

$$\mathrm{cov}_t(c_{t+1}, c_{t+1}^2) = \psi_2^{-3}E_t\begin{bmatrix}(\sigma_{d,t}u_{d,t+1} + \eta\sigma_{y,t}u_{y,t+1})\times(2\rho_d\varepsilon_{d,t}\sigma_{d,t}u_{d,t+1} + \sigma_{d,t}^2(u_{d,t+1}^2 - \sigma_{d,u}^2) + \eta^2\times \\ \left(2\rho_y\varepsilon_{y,t}\sigma_{y,t}u_{y,t+1} + \sigma_{y,t}^2(u_{y,t+1}^2 - \sigma_{y,u}^2) + 2\rho\psi_1 x_{t+1}(\sigma_{d,t}u_{d,t+1} + \eta\sigma_{y,t}u_{y,t+1})\right. \\ \left. + 2\eta(\rho_d\varepsilon_{d,t}\rho_y\varepsilon_{y,t} + \rho_d\varepsilon_{d,t}\sigma_{y,t}u_{y,t+1} + \sigma_{d,t}u_{d,t+1}\rho_y\varepsilon_{y,t} + \sigma_{d,t}u_{d,t+1}\sigma_{y,t}u_{y,t+1})\right)\end{bmatrix} \qquad \textbf{(A-28)}$$

Since $E_t[u_{i,t+1}^3] = \mathrm{cov}_t(u_{i,t+1}, u_{i,t+1}^2) = 0$ and innovations are uncorrelated, expanding **(A-28)** gives

$$\mathrm{cov}_t(c_{t+1}, c_{t+1}^2) = 2\psi_2^{-3}(\rho\psi_1 x_{t+1} + \rho_d\varepsilon_{d,t} + \eta\rho_y\varepsilon_{y,t})[\sigma_{d,t}^2\sigma_{d,u}^2 + \eta^2\sigma_{y,t}^2\sigma_{y,u}^2] \qquad \textbf{(A-29)}$$

Using **(A-29)** and **(A-24)** in **(A-25)** gives the following result:

$$\begin{aligned}\mathrm{cov}_t(c_{t+1}, s_{t+1}) &= \psi_2^{-2}(\sigma_{d,t}^2\sigma_{d,u}^2 + \eta^2\sigma_{y,t}^2\sigma_{y,u}^2) \\ &\times\left[\psi_1\left(1 + (1-h)^{-1}x_{t+1}\right) - (1-h)^{-1}(\psi_1/\psi_2)(\rho\psi_1 x_{t+1} + \rho_d\varepsilon_{d,t} + \eta\rho_y\varepsilon_{y,t})\right]\end{aligned} \qquad \textbf{(A-30)}$$

Up to a third-order approximation, the third and final term in the expression for $\mathrm{var}_t(\widetilde{m}_{t+1})$ is given by

$$\mathrm{var}_t(s_{t+1}) = \psi_1^2\left((1 + 2x_{t+1}(1-h)^{-1})\,\mathrm{var}_t(c_{t+1}) - (1-h)^{-1}\,\mathrm{cov}_t(c_{t+1}, c_{t+1}^2)\right) \qquad \textbf{(A-31)}$$

[24] Note that $c_t = (1/\psi_2)(\rho(1-h)^{-1}hx_t + \eta\varepsilon_{y,t} + \varepsilon_{d,t})$ from imposing price stability in the Phillips curve.

Hence, using **(A-31)**, **(A-30)** and **(A-24)** in **(A-22)** and collecting terms, we arrive at

$$\mathrm{var}_t(\widetilde{m}_{t+1}) = \frac{\rho^2(\sigma_{d,t}^2\sigma_{d,u}^2 + \eta^2\sigma_{y,t}^2\sigma_{y,u}^2)}{(\eta(1-h)+\rho)^2}\left(1 + \frac{2h(\rho+\eta)x_{t+1}}{(\rho+\eta(1-h))} - \frac{2h(\rho_d\varepsilon_{d,t} + \eta\rho_y\varepsilon_{y,t})}{(\rho+\eta(1-h))}\right) \quad \textbf{(A-32)}$$

Finally, substituting for x_{t+1} and c_t and simplifying gives the expression reported in the text:

$$\mathrm{var}_t(\widetilde{m}_{t+1}) = \frac{\rho^2}{\kappa_0^2}(\sigma_{d,t}^2\sigma_{d,u}^2 + \eta^2\sigma_{y,t}^2\sigma_{y,u}^2)(1-\kappa_y\varepsilon_{y,t} - \kappa_d\varepsilon_{d,t} + \kappa_x x_t) \quad \textbf{(A-33)}$$

where

$$\kappa_0 = \rho + \eta(1-h), \qquad \kappa_x = \frac{2h(\rho+\eta)(\phi\kappa_0 + \rho h(1-\phi))}{\kappa_0^2},$$

$$\kappa_y = \frac{2h\eta((1-h)(\rho+\eta)(\phi-1) + \kappa_0\rho_y)}{\kappa_0^2}, \ \kappa_d = \frac{2h((1-h)(\rho+\eta)(\phi-1) + \kappa_0\rho_d)}{\kappa_0^2}.$$

A.4 Derivation of the $\mathrm{cov}_t(\widetilde{m}_{t+1}, \Delta\varepsilon_{d,t+1})$ term

Note first that $\mathrm{cov}_t(\widetilde{m}_{t+1}, \Delta\varepsilon_{d,t+1}) = \mathrm{cov}_t(\widetilde{m}_{t+1}, \varepsilon_{d,t+1})$.

Using $\widetilde{m}_{t+1} = -\rho(c_{t+1} - c_t + s_{t+1} - s_t)$, we have

$$\mathrm{cov}_t(\widetilde{m}_{t+1}, \varepsilon_{d,t+1}) = -\rho\,\mathrm{cov}_t(c_{t+1}, \varepsilon_{d,t+1}) - \rho\,\mathrm{cov}_t(s_{t+1}, \varepsilon_{d,t+1}) \quad \textbf{(A-34)}$$

First, using $c_t = \psi_2^{-1}(\rho(1-h)^{-1}hx_t + \eta\varepsilon_{y,t} + \varepsilon_{d,t})$ we have

$$\mathrm{cov}_t(c_{t+1}, \varepsilon_{d,t+1}) = \psi_2^{-1}\,\mathrm{var}_t(\varepsilon_{d,t+1}) = \psi_2^{-1}\sigma_{d,t}^2\sigma_{d,u}^2 \quad \textbf{(A-35)}$$

Second, using the second-order approximation for s_t, we have

$$\begin{aligned}\mathrm{cov}_t(s_{t+1}, \varepsilon_{d,t+1}) &= \psi_1(1+(1-h)^{-1}x_{t+1})\mathrm{cov}_t(c_{t+1}, \varepsilon_{d,t+1}) - \psi_1(1/2)(1-h)^{-1}\,\mathrm{cov}_t(\varepsilon_{d,t+1}, c_{t+1}^2) \\ &= (\psi_1/\psi_2)(1+(1-h)^{-1}x_{t+1})\sigma_{d,t}^2\sigma_{d,u}^2 - \psi_1(1/2)(1-h)^{-1}\,\mathrm{cov}_t(\varepsilon_{d,t+1}, c_{t+1}^2)\end{aligned} \quad \textbf{(A-36)}$$

The term $\mathrm{cov}_t(\varepsilon_{d,t+1}, c_{t+1}^2)$ is derived below.

First, note that

$$\mathrm{cov}_t(\varepsilon_{d,t+1}, c_{t+1}^2) = \psi_2^{-2}\,\mathrm{cov}_t\begin{bmatrix}\varepsilon_{d,t+1}, \\ \rho^2\psi_1^2x_{t+1}^2 + \varepsilon_{d,t+1}^2 + \eta^2\varepsilon_{y,t+1}^2 \\ +2\rho\psi_1x_{t+1}(\varepsilon_{d,t+1} + \eta\varepsilon_{y,t+1}) + 2\eta\varepsilon_{d,t+1}\varepsilon_{y,t+1}\end{bmatrix} \quad \textbf{(A-37)}$$

So, using the definition of a covariance and $\varepsilon_{i,t} = \rho_i\varepsilon_{i,t-1} + \sigma_{i,t-1}u_{i,t}$, $i \in \{d, y\}$, we can write

$$\mathrm{cov}_t(\varepsilon_{d,t+1}, c_{t+1}^2) = \psi_2^{-2}E_t\begin{bmatrix}(\sigma_{d,t}u_{d,t+1})\times \\ \left(\begin{array}{l}(2\rho_d\varepsilon_{d,t}\sigma_{d,t}u_{d,t+1} + \sigma_{d,t}^2(u_{d,t+1}^2 - \sigma_{d,u}^2)) + \eta^2(2\rho_y\varepsilon_{y,t}\sigma_{y,t}u_{y,t+1} + \sigma_{y,t}^2(u_{y,t+1}^2 - \sigma_{y,u}^2)) \\ +2\rho\psi_1x_{t+1}(\sigma_{d,t}u_{d,t+1} + \eta\sigma_{y,t}u_{y,t+1}) \\ +2\eta(\rho_d\varepsilon_{d,t}\rho_y\varepsilon_{y,t} + \rho_d\varepsilon_{d,t}\sigma_{y,t}u_{y,t+1} + \sigma_{d,t}u_{d,t+1}\rho_y\varepsilon_{y,t} + \sigma_{d,t}u_{d,t+1}\sigma_{y,t}u_{y,t+1})\end{array}\right)\end{bmatrix} \quad \textbf{(A-38)}$$

Since $E_t[u_{d,t+1}^3] = \text{cov}_t(u_{d,t+1}, u_{d,t+1}^2) = 0$ and innovations are uncorrelated, expanding **(A-38)** gives

$$\text{cov}_t(\varepsilon_{d,t+1}, c_{t+1}^2) = 2\psi_2^{-2}\sigma_{d,t}^2\sigma_{d,u}^2(\rho\psi_1 x_{t+1} + \rho_d\varepsilon_{d,t} + \eta\rho_y\varepsilon_{y,t}) \qquad \textbf{(A-39)}$$

Using **(A-39)** in **(A-36)** gives

$$\text{cov}_t(s_{t+1}, \varepsilon_{d,t+1}) = (\psi_1/\psi_2)\sigma_{d,t}^2\sigma_{d,u}^2\left(1 + (1-h)^{-1}(1-\rho(\psi_1/\psi_2))x_{t+1} - (1-h)^{-1}\psi_2^{-1}(\rho_d\varepsilon_{d,t} + \eta\rho_y\varepsilon_{y,t})\right) \quad \textbf{(A-40)}$$

Hence, using **(A-40)** and **(A-35)**, the $\text{cov}_t(\tilde{m}_{t+1}, \Delta\varepsilon_{d,t+1})$ term is as follows:

$$\text{cov}_t(\tilde{m}_{t+1}, \Delta\varepsilon_{d,t+1}) = -\frac{\rho}{\psi_2}\sigma_{d,t}^2\sigma_{d,u}^2\begin{pmatrix}1 + \psi_1 + \psi_1(1-h)^{-1}(1-\rho\psi_1/\psi_2)x_{t+1} \\ -(\psi_1/\psi_2)(1-h)^{-1}(\rho_d\varepsilon_{d,t} + \eta\rho_y\varepsilon_{y,t})\end{pmatrix} \qquad \textbf{(A-41)}$$

Multiplying the numerator and denominator of this equation by $(1 - h)$, we have:

$$\text{cov}_t(\tilde{m}_{t+1}, \Delta\varepsilon_{d,t+1}) = -\frac{\rho}{\kappa_0}\sigma_{d,t}^2\sigma_{d,u}^2\begin{pmatrix}1 + \psi_1(1-\rho h/\kappa_0)x_{t+1} \\ -h\kappa_0^{-1}(\rho_d\varepsilon_{d,t} + \eta\rho_y\varepsilon_{y,t})\end{pmatrix}$$

Finally, substituting for x_{t+1} and c_t and collecting terms, this equation can be written as follows:

$$\text{cov}_t(\tilde{m}_{t+1}, \Delta\varepsilon_{d,t+1}) = -\frac{\rho}{\kappa_0}\sigma_{d,t}^2\sigma_{d,u}^2\left(1 + \theta_d\varepsilon_{d,t} + \theta_y\varepsilon_{y,t} + \theta_x x_t\right) \qquad \textbf{(A-42)}$$

where

$$\theta_d = \frac{h}{\kappa_0^2}\left(h(\rho+\eta)(1-h)(1-\phi) - \rho_d\kappa_0\right),\ \theta_y = \frac{h\eta}{\kappa_0^2}\left(h(\rho+\eta)(1-h)(1-\phi) - \rho_y\kappa_0\right),$$

$$\theta_x = \frac{h}{\kappa_0^2}(\rho+\eta)(\phi\kappa_0 + h(1-\phi)\rho).$$

Hence, as reported in the text, the second term in the precautionary savings effect is given by

$$2\,\text{cov}_t(\tilde{m}_{t+1}, \Delta\varepsilon_{d,t+1}) + \text{var}_t(\Delta\varepsilon_{d,t+1}) = \sigma_{d,t}^2\sigma_{d,u}^2(1 - 2\rho\kappa_0^{-1}(1 + \theta_d\varepsilon_{d,t} + \theta_y\varepsilon_{y,t} + \theta_x x_t)) \quad \textbf{(A-43)}$$

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