BANK OF ENGLAND

Working Paper No. 454

Fixed interest rates over finite horizons

Andrew P Blake

May 2012

BANK OF ENGLAND

Working Paper No. 454
Fixed interest rates over finite horizons

Andrew P Blake[(1)]

Abstract

We consider finite horizon conditioning paths for nominal interest rates in New Keynesian monetary policy models. This is done two ways. First, we develop a simple way to use policy interventions in the form of interest rate shocks to achieve the conditioning path and show this yields a unique solution. We then modify this method to generate an infinity of solutions making the model better behaved but effectively indeterminate. Second, we use two-part rules where a specially designed targeting rule generates fixed interest rates endogenously over the initial period before reverting to a more conventional instrument rule. We show that the two approaches are equivalent. We discuss appropriate selection criteria over the resulting equilibria.

Key words: Fixed nominal interest rates, uniqueness, indeterminacy.

JEL classification: C63, E47, E61.

---

(1) Centre for Central Banking Studies, Bank of England. Email: andrew.blake@bankofengland.co.uk

The views expressed in this paper are those of the author, and not necessarily those of the Bank of England. The author wishes to thank Tim Taylor for conversations that contributed to an earlier draft of this paper, and Simon Price, Ole Rummel and two anonymous referees who have helped improve the current one. Errors remain my own. This paper was finalised on 24 January 2012.

The Bank of England's working paper series is externally refereed.

Information on the Bank's working paper series can be found at
www.bankofengland.co.uk/publications/Pages/workingpapers/default.aspx

Publications Group, Bank of England, Threadneedle Street, London, EC2R 8AH
Telephone +44 (0)20 7601 4030 Fax +44 (0)20 7601 3298 email mapublications@bankofengland.co.uk

ISSN 1749-9135 (on-line)

Contents

Summary

Two natural questions to ask about monetary policy are 'what would happen to inflation if interest rates were a bit higher than forecast?' and 'what are the implications of interest rates not changing for some period of time?'. Satisfactory quantitative answers to both of these questions are, perhaps surprisingly, hard to come by. With many widely used forecasting models, this is not a problem. For example, the commonly used vector autoregression (VAR) — a system of equations explaining a set of interrelated variables — would allow us to simply impose a path for one of the variables with no practical consequences. But for policy we need to have a proper economic understanding, and one way of acquiring that is via a 'structural' model, which a VAR is not. Moreover, modern economics recognises the importance of forward-looking behaviour and expectations. Models where the forward-looking behaviour of agents helps explain the dynamic evolution of all variables in a coherent, equilibrium way are known as rational expectations (RE) models. Using an RE model to answer the questions just posed requires a forecaster to solve a number of quite difficult conceptual problems.

Using a general equilibrium RE model it is difficult to formalise how a higher (or indeed a fixed) interest rate is achieved. This is because such models are usually solved incorporating a monetary policy rule. These rules are conditional, and react to variables policymakers care about. Often, they are versions of the well-known Taylor rule that feeds back from inflation and growth. Departing from these rules to induce interest rates that are different from those already implied is hard to manage, and even if the technical problems are overcome it can be that the results sometimes seem perverse. Essentially, we cannot just 'fix' interest rates, as we can with VARs. In a structural model we have to have a coherent explanation of why interest rates follow the path they do (rather than what is implied by the policy rule embedded in the model). And the problem is compounded by the fact that behaviour in the model depends on what agents expect to happen after the fixed-rate path ends.

But the questions we began with are good ones that need reasonable answers. This paper explores a number of potential resolutions to modelling partially fixed interest rates in a common framework. These include imposing a sequence of anticipated or unanticipated interest rate 'shocks' that deliver the desired path, using a shock for each period the path is fixed, which

seems a natural way to handle things. Unfortunately, when the strengths and weaknesses of different existing methods are compared they are all found wanting, either because they imply excessively volatile or counterintuitive forecasts. So a new approach is developed that restores more normal behaviour; but at the cost of introducing a new problem.

The new approach takes as a starting point that permanently fixed interest rates imply a well defined trade-off between inflation and output growth, but do not imply any particular level of inflation. This is a well-known problem but (as we show) does not automatically apply in finite horizon problems, the case relevant for policymakers who publish fixed interest rate forecasts. Although at first sight the approach may seem somewhat perverse, the paper shows how to make sure it *does* apply for such problems. It can again be done by setting shocks, but using one more than the number of periods the rate is fixed; or by using a rule that specifically targets the interest rate, again for one period longer than the fixed-rate period. This restores intuitively sensible paths; but at the cost of introducing an equilibrium selection problem. This arises because when we use more shocks than we 'need' to fix rates, there are an infinity of well-behaved solutions that the forecaster must choose between. Equivalently, there are an infinity of rules we could use. A degree of arbitrariness in the selected solution is then inevitable. This is not as bad as it seems, though, as some paths are more 'sensible' than others (eg, a path that is close to that implied by a Taylor rule). Nevertheless, the paper concludes that there is no easy solution to the finite horizon problem, and any answer to the questions we started with must inevitably be strongly caveated.

1 Introduction

This paper sets out to answer a simple question. How do we fix the interest rate in a rational expectations macromodel at a given value for a set number of periods? A seemingly innocuous question, with – it turns out – many answers, and correspondingly many implications for the paths of inflation and output. These are related to the observation that a New Keynesian model with fixed nominal interest rates is indeterminate – there are infinitely many solutions that satisfy the equations of the model that do not imply divergent paths, a result familiar since Sargent and Wallace (1975) and seemingly robust to model specification. One might reason the following. Surely it must be that there is some (possibly quite short) horizon for simulating a particular model with fixed rates at which it switches from being determinate to indeterminate? We ask the same question slightly differently: what does our understanding of the behaviour of indeterminate, permanently fixed interest rate models, tell us about fixing the interest rate over a finite horizon?

This is not the first analysis of the fixed interest rate problem. Since Waggoner and Zha (1999), it has been commonplace to implement conditional forecasting using exogenous shocks to achieve a target path.[1] Leeper and Zha (2003) suggested a 'modest interventions' procedure, that can be interpreted as imposing an interest rate path by using shocks that are unanticipated by agents in the model. The model is then solved recursively, and agents are continually surprised by the fixed interest rates. If the shocks required to maintain the fixed path are 'modest' then it can be argued that a simulation is relatively unaffected by agents being deceived (and not learning). Laséen and Svensson (2011) suggest instead using anticipated shocks, which has the benefit of treating agents in the model as fully rational, but turns out to have some perhaps undesirable side effects.

Nor is this the first analysis of fixed interest rates that questions the determinacy properties of such models. Galí (2011, 2009) studies monetary policy rules that are both determinate and imply an exogenous path for interest rates. This somewhat puzzling result that fixed nominal interest rates can be determinate is reinterpreted here and made consistent with more conventional analysis. In so doing we extend his analysis and develop targeting versions of his and other rules that fix the nominal interest rate over a finite horizon and can also be exactly

[1] It used to be equally commonplace to do such exercises on older-generation, pre-New Keynesian rational expectations models. In Appendix A we compare the shock-based approach with 'old fashioned' Type 1 and Type 2 fixes.

replicated by an appropriate choice of anticipated shocks.

Our analysis of fixed interest rate models combines, extends and simplifies all of the previous approaches. We suggest several methods for fixing the interest rate, all of which are variations on two basic approaches which we show to be equivalent. These are:

1. Effectively exogenising both the interest rate and the initial value of some other variable by adding in a sequence of rationally understood monetary policy shocks such that the *ex post* interest rate is set to its target value. This generalises the results in Laséen and Svensson (2011).
2. Employing a 'two-part rule' where the policymaker follows a targeting rule which implies a fixed rate for a set number of periods and then reverts to a conventional Taylor-type rule. This generalises the results in Galí (2011, 2009).

It turns out that these two approaches — or seemingly innocuous variations — do not necessarily give the same answer for what appears to be the same question. Understanding when they do give the same answer — and how we can *ensure* they give the same answer — is the key to understanding the finite horizon fixed interest rate problem. The contrasting approaches give an important insight into how policy models behave and how we should think of indeterminacy with 'partially exogenous' interest rates.

The paper is organised as follows. In the next section we show an analytic solution to the general finite horizon problem where we use shocks to achieve a particular target path different from that implied by the policy rule. This replicates the Leeper and Zha (2003) and Laséen and Svensson (2011) analysis in a common, simple framework and illustrates the implications using the standard New Keynesian model. In Section 3 we demonstrate how using one extra anticipated shock makes the model behave like an indeterminate one even though we retain determinacy, using the model from Section 2. In Section 4 we develop a new method based on targeting rules that replicates the analysis in Section 3 without shocks, which reconciles the approaches of Laséen and Svensson (2011) and Galí (2011). In Section 5 we illustrate all the methods developed using a model with more complicated dynamics derived from Lindé (2005). In a final section we suggest a strategy for deriving a satisfactory simulation design. Two brief appendices cover some technical issues.

2 Fixed rates for n periods using n shocks

We begin with a rather general representation of a model and a policy rule. Write a general model as

$$\begin{bmatrix} z_t \\ x_{t+1}^e \end{bmatrix} = A \begin{bmatrix} z_{t-1} \\ x_t \end{bmatrix} + G\varepsilon_t + B\nu_t \tag{1}$$

where z_t is a vector of predetermined variables, x_t is a vector of expectational or jumping variables, ε_t is a vector of mean zero unforecastable shocks, and ν_t is a scalar monetary policy shock in period t. We will allow ν_t to be forecast rationally, and is modelled as a *policy intervention*.

We will use specific models below, but all we need for the analysis to make sense at the general algebraic level is that we include as one of the variables in the z_t vector an equation for the nominal interest rate, usually:

$$i_t = \gamma\, i_{t-1} + (1-\gamma)\,(\theta\pi_t + \chi y_t) + \nu_t \tag{2}$$

the standard Taylor rule with smoothing (we ignore constants without any loss of generality). The other equations do not matter for the principle, but will matter for the dynamics. We will use the canonical New Keynesian model, to be outlined below, as one of our examples; in the meantime readers may find it helpful to keep such a model in mind.

Before turning to the solution of the model it is worth noting that to permanently keep interest rates fixed is usually thought to render a rational expectations model such as we have outlined *indeterminate*. Thus there would be many solutions to the model for which expectations were fulfilled and the model was not explosive. However, Galí (2011, 2009) suggests some policy rules that fix the interest rate at some arbitrary value without indeterminacy. Using one of these rules removes any need to consider policy interventions as the interest rate will already be at the desired rate, but this approach does bring along some other problems. We will return to such rules below, as one way to resolve any indeterminacy issues.

Equation **(1)** can be solved using the formula (see Blanchard and Kahn (1980)):

$$
\begin{aligned}
x_t &= -M_{22}^{-1} M_{21} z_{t-1} - M_{22}^{-1} \Lambda_u^{-1} M_2 G \varepsilon_t - \sum_{i=0}^{\infty} M_{22}^{-1} \Lambda_u^{-i-1} M_2 B \nu_{t+i} \\
&= N z_{t-1} + P \varepsilon_t + \sum_{i=0}^{\infty} L_i \nu_{t+i} \qquad \textbf{(3)}
\end{aligned}
$$

when A is diagonalised such that $MA = \Lambda M$, where $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$. Λ_u is a diagonal matrix comprising of the unstable roots of A, $N = -M_{22}^{-1} M_{21}$, $P = -M_{22}^{-1} \Lambda_u^{-1} M_2$, L_i is defined as $-M_{22}^{-1} \Lambda_u^{-i-1} M_2 B$ for all $i \geq 0$ and we assume that the dimension of x_t matches the number of unstable roots.[2] The predetermined variables can be written in reduced form as:

$$
\begin{aligned}
z_t &= (A_{11} + A_{12} N) z_{t-1} + (G_1 + A_{12} P) \varepsilon_t + B_1 \nu_t + \sum_{i=0}^{\infty} A_{12} L_i \nu_{t+i} \\
&= \widetilde{A} z_{t-1} + \widetilde{G} \varepsilon_t + \sum_{i=0}^{\infty} \widetilde{B}_i \nu_{t+i} \qquad \textbf{(4)}
\end{aligned}
$$

where $\widetilde{A} = A_{11} + A_{12} N$, $\widetilde{G} = G_1 + A_{12} P$ with $\widetilde{B}_0 = B_1 + A_{12} L_0$ and $\widetilde{B}_i = A_{12} L_i$ for $i > 0$. Note that $L_i \to 0$ as $i \to \infty$ and hence so does $\widetilde{B}_i$.

Remember, one of the z_t variables is assumed to be the nominal interest rate. What if we wished to use a sequence of monetary policy shocks to set the interest rate to some value (or values) over a given horizon of, say, n periods? (We assume a zero value for target interest rates with no loss of generality.)

We can write **(4)** for a particular value of n, say 4, as:

$$
\begin{bmatrix} I & 0 & 0 & 0 \\ -\widetilde{A} & I & 0 & 0 \\ 0 & -\widetilde{A} & I & 0 \\ 0 & 0 & -\widetilde{A} & I \end{bmatrix}
\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}
=
\begin{bmatrix} \widetilde{B}_1 & \widetilde{B}_2 & \widetilde{B}_3 & \widetilde{B}_4 \\ 0 & \widetilde{B}_1 & \widetilde{B}_2 & \widetilde{B}_3 \\ 0 & 0 & \widetilde{B}_1 & \widetilde{B}_2 \\ 0 & 0 & 0 & \widetilde{B}_1 \end{bmatrix}
\begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{bmatrix}
+
\begin{bmatrix} \widetilde{A} \\ 0 \\ 0 \\ 0 \end{bmatrix} z_0
+
\begin{bmatrix} \widetilde{G} \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_0 \qquad \textbf{(5)}
$$

where we also assume that we have $m = 4$ policy interventions at our disposal. Inverting the left-hand side and multiplying all the terms on the right gives:

$$
z = \Xi \nu + \Phi z_0 + \Gamma \varepsilon_0 \qquad \textbf{(6)}
$$

[2] In Appendix B we describe the computational approach we use. In general eigenvalue decompositions should be avoided where possible for numerical reasons, and a Schur decomposition is used in practice with no consequences.

where:

$$\Xi = \begin{bmatrix} \widetilde{B}_1 & \widetilde{B}_2 & \widetilde{B}_3 & \widetilde{B}_4 \\ \widetilde{A}\widetilde{B}_1 & \widetilde{A}\widetilde{B}_2 + B_1 & \widetilde{A}\widetilde{B}_3 + \widetilde{B}_2 & \widetilde{A}\widetilde{B}_4 + \widetilde{B}_3 \\ \widetilde{A}^2\widetilde{B}_1 & \widetilde{A}^2\widetilde{B}_2 + \widetilde{A}B_1 & \widetilde{A}^2\widetilde{B}_3 + \widetilde{A}\widetilde{B}_2 + \widetilde{B}_1 & \widetilde{A}^2\widetilde{B}_4 + \widetilde{A}\widetilde{B}_3 + \widetilde{B}_2 \\ \widetilde{A}^3\widetilde{B}_1 & \widetilde{A}^3\widetilde{B}_2 + \widetilde{A}^2\widetilde{B}_1 & \widetilde{A}^3\widetilde{B}_3 + \widetilde{A}^2\widetilde{B}_2 + \widetilde{A}\widetilde{B}_1 & \widetilde{A}^3\widetilde{B}_4 + \widetilde{A}^2\widetilde{B}_3 + \widetilde{A}\widetilde{B}_2 + \widetilde{B}_1 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} \widetilde{A} \\ \widetilde{A}^2 \\ \widetilde{A}^3 \\ \widetilde{A}^4 \end{bmatrix}, \Gamma = \begin{bmatrix} \widetilde{G} \\ \widetilde{A}\widetilde{G} \\ \widetilde{A}^2\widetilde{G} \\ \widetilde{A}^3\widetilde{G} \end{bmatrix}, z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \text{ and } \nu = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{bmatrix}$$

If we further define a selector matrix $\imath$ such that $\imath z = \imath \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = i$, then using the selector matrix and denoting $\hat{X} = \imath X$ we can write:

$$i = \hat{\Xi}\nu + \widehat{\Phi}z_0 + \hat{\Gamma}\varepsilon_0. \tag{7}$$

The optimal value of ν for $m \leq n$ minimises $i'i$. This is:

$$\nu = -\left(\hat{\Xi}'\hat{\Xi}\right)^{-1}\hat{\Xi}'\left(\widehat{\Phi}z_0 + \hat{\Gamma}\varepsilon_0\right) \tag{8}$$

or more simply:

$$\nu = -\hat{\Xi}^{-1}\left(\widehat{\Phi}z_0 + \hat{\Gamma}\varepsilon_0\right) \tag{9}$$

to solve it exactly when $m = n$ and there are as many shocks as periods of 'zero' interest rates, as in **(5)**. This also indicates that we now have a condition for when the solution is unique: if we use n policy shocks to set interest rates on an arbitrary desired path (in this case zero) for n periods then this solution is unique as long as $\hat{\Xi}$ is non-singular. This is the exercise conducted in Laséen and Svensson (2011) for example.

Uniqueness is unequivocal. **(7)** defines the set of values that can solve the problem, and **(9)** the solution. If we look at the structure of $\hat{\Xi}$ we would probably expect it to be non-singular for very long fixed-rate horizons: many models will never become 'indeterminate'.

A further advantage to this set up is that it enables us to easily derive the Leeper and Zha (2003) modest interventions solution. This amounts to agents in the model assuming $B_i = 0$ for all $i > 1$ for every period of the simulation. The necessary shocks for this can be calculated using the above formulae with this imposed, ie instead of **(6)** use:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} \widetilde{B}_1 & 0 & 0 & 0 \\ \widetilde{A}\widetilde{B}_1 & B_1 & 0 & 0 \\ \widetilde{A}^2\widetilde{B}_1 & \widetilde{A}B_1 & \widetilde{B}_1 & 0 \\ \widetilde{A}^3\widetilde{B}_1 & \widetilde{A}^2\widetilde{B}_1 & \widetilde{A}\widetilde{B}_1 & \widetilde{B}_1 \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{bmatrix} + \begin{bmatrix} \widetilde{A} \\ \widetilde{A}^2 \\ \widetilde{A}^3 \\ \widetilde{A}^4 \end{bmatrix} z_0 + \begin{bmatrix} \widetilde{G} \\ \widetilde{A}\widetilde{G} \\ \widetilde{A}^2\widetilde{G} \\ \widetilde{A}^3\widetilde{G} \end{bmatrix} \varepsilon_0 \qquad \textbf{(10)}$$

Any model simulation must then be solved under the same assumptions, but actually this amounts to a standard simulation where all future shocks are simply ignored despite the fact that we already know what they are.

A further possibility is one considered (and indeed recommended) by Laséen and Svensson (2011), where the real rather than the nominal interest rate is fixed. Then instead of using a selector matrix to isolate the nominal rate we use one that selects the included equation for the real rate, r_t, which we denote $r = \iota_r z$, so then denoting $\hat{X}_r = \iota_r X$

$$\nu_r = -\hat{\Xi}_r^{-1} \left(\widehat{\Phi}_r z_0 + \hat{\Gamma}_r \varepsilon_0 \right) \qquad \textbf{(11)}$$

is the path of anticipated shocks required to fix real interest rates.[3]

All of these solutions are potentially unique: there may be some models for which the relevant $\hat{\Xi}$ is singular over some policy-relevant horizon, but not for the examples we investigate in this paper.[4] We now turn to what these approaches imply for a particular model, which allows us to replicate Leeper and Zha (2003) and Laséen and Svensson (2011) using our method.

2.1 Application 1

We apply these solutions to the simplest entirely forward-looking New Keynesian model (for details see Galí (2008)):

$$y_t = y_{t+1}^e - \sigma^{-1} r_t \qquad \textbf{(12)}$$

$$\pi_t = \beta \pi_{t+1}^e + \kappa y_t + g_t \qquad \textbf{(13)}$$

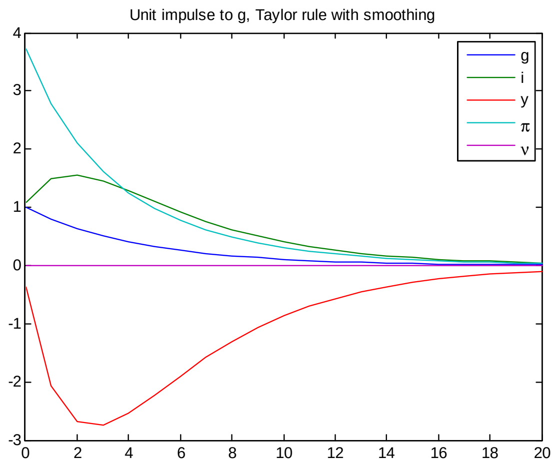
$$r_t = i_t - \pi_{t+1}^e \qquad \textbf{(14)}$$

$$g_t = \rho g_{t-1} + \varepsilon_t \qquad \textbf{(15)}$$

[3] This now becomes a Type 2 fix. See Appendix A.

[4] For example, the model used in the next section with fixed interest rates for 80 periods (for the calibration this implies 20 years) gives a unique answer.

Chart 1: Canonical New Keynesian model closed with Taylor rule; no policy shocks; unit cost-push shock

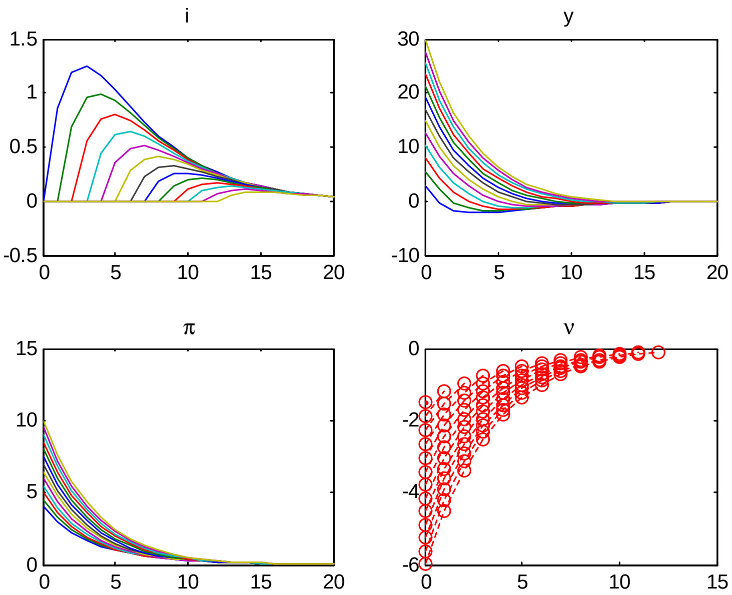


where **(12)** is the dynamic IS curve, **(13)** the Phillips curve, **(14)** the Fisher definition of the real interest rate and **(15)** defines a persistent cost-push shock. The parameters of the model can be micro-founded in the usual way, with $0 < \beta \leq 1$ and $\kappa, \sigma > 0$. In what follows, the shock ε_t is taken to be an unforecastable, mean-zero disturbance term. Model parameters used in the simulations are $\beta = 0.99$, $\sigma = 1$, $\kappa = 0.05$ and $\rho = 0.8$. We use the policy parameters of $\theta = 1.5$, $\chi = 0.5$ and $\gamma = 0.8$ which are representative of those often employed. All of the parameters of the model and policy rule have been varied considerably as a robustness check with no impact on the qualitative conclusions obtained.

This model is initially closed by appending **(2)**. We assume a unit shock for ε_0 and simulate the model with various values of ν_t for the first n periods. With no policy shocks the model responses are as shown in Chart 1.

The responses are familiar: output falls and inflation rises, with nominal interest rates rising by less than inflation in the short run due to the smoothing term. Interest rates then rise for the first few periods before falling back as inflation and output both converge back to zero. The output gap follows a similar but opposite sign path, reaching its lowest level in the third period after the shock. By the end of the five-year simulation horizon the responses have all but smoothly died away. Policy smoothing plays an important role here, as without it all variables jump to their

Chart 2: Canonical New Keynesian model; unit cost-push shock; n policy shocks



maximum deviations immediately and converge back to base at the same rate, ρ, as the shock.

Now what happens if we calculate an anticipated time series of shocks that places $i_t = 0$ over some initial interval? In Chart 2 we plot the result of applying the formula **(9)** to the model over an initial interval of up to n periods, for n ranging from one to thirteen. At no point is the matrix $\hat{\Xi}$ singular. The top left panel shows the response of interest rates, with the fixed period followed by a jumping-up of rates by successively smaller amounts as the horizon lengthens. The bottom right panel shows the shocks required to achieve this. The policy interventions start small and get progressively larger (the lower bound of the graph in the panel shows the thirteen-period intervention and a lone circle top left the single period one).

The results are more than a little alarming. Over even quite modest horizons of five to six periods, the jump in both inflation and output is positive and very large. The results are essentially nonsense: whilst the equations of the model all hold, the jumps in output and inflation are implausibly large by a factor of five to ten and of less-than-intuitive sign. Interest rates, by contrast, never get far from zero even after the Taylor rule on its own 'kicks in', as the other variables rapidly get back close to base.

Of course the moves in inflation and output do make sense: along the path the growth in output needs to be minus the inflation rate, as the IS curve with zero interest rates implies $\sigma \Delta y_t = -\pi_t$. If inflation increases in response to the shock (which might be expected with fixed nominal interest rates) then output must also jump up for this to hold so future growth rates can be negative.

To the casual observer the model might appear indeterminate, the responses seem so extreme; perhaps it is simply one of many solutions that has been selected, and the one selected is unsatisfactory. But we cannot select another as we have established uniqueness.[5] What appears to be a perfectly sensible future monetary policy rule goes along with extreme variation in the immediate simulation response.

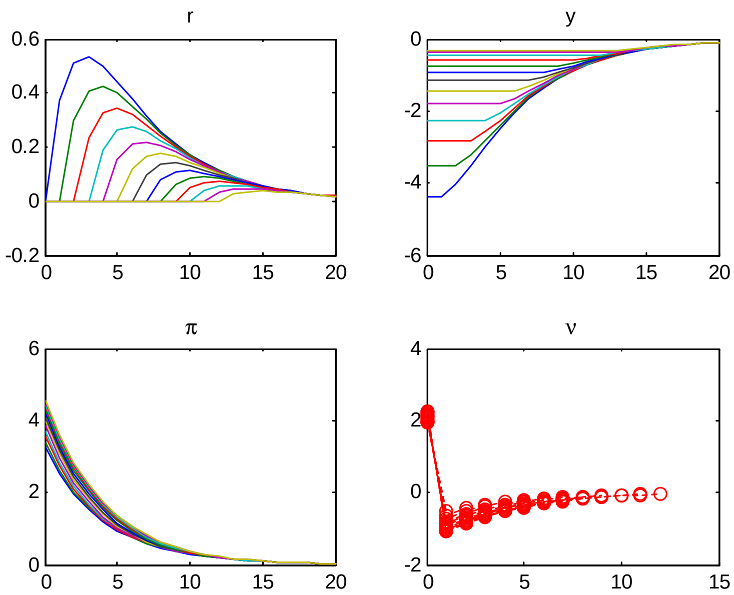
What should we make of this? It seems that uniqueness is not necessarily a virtue; good policy in the future goes along with disproportionate volatility in the short run. A policy rule that gives good, intuitive responses without fixing the interest rate in the initial periods can be completely the opposite with quite a short period of fixed rates.

If indeterminacy is not an issue, the simulation responses would seem to rule out any 'usable' solution to the finite horizon problem. Indeed, this is the conclusion reached by Laséen and Svensson (2011), who suggest instead that fixing the real interest rate path should be used for all but the shortest horizons. In their example models, fixed real interest rate simulations seem less prone to such problems. In Chart 3 we show the responses for our model with fixed real rates. These look much more sensible, but have two drawbacks. First, they now imply quite large *nominal* interest rate movements (the negative of the expected inflation rate), which was something the procedure was intended to avoid. Second, output dynamics are now governed by unit root behaviour because with zero real rates $y_t = y^e_{t+1}$ from **(12)**. Consequently, y_t is flat until real rates move, and for a long fixed real rate horizon stay quite close to zero. The path of the policy interventions, ν_t, is very similar for every horizon, and is just truncated when the fix stops.

Finally, turning to the Leeper-Zha method we obtain Chart 4. If interest rates are fixed for

[5] We do have an extra 'degree of freedom' as after the fixed-rate period the Taylor rule takes over. It could be that the responses are very sensitive to the rule. In fact, changing it changes the responses by a little, although very unconventional terminal rules can make this more marked.

Chart 3: Canonical New Keynesian model; fixed real interest rate; unit cost-push shock

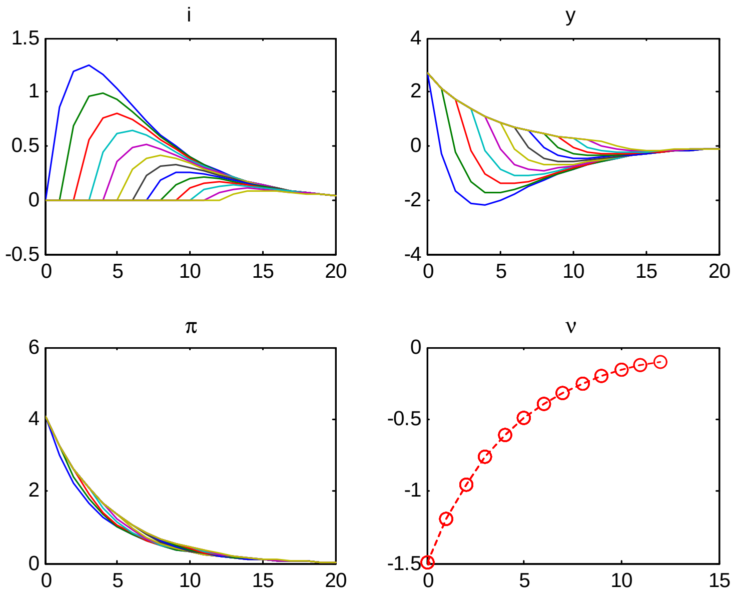


thirteen periods as before, the resulting simulation is just the envelope of the plotted responses. This seems very counterintuitive. The promised interest rate increase is sufficient to bring inflation down as effectively as the Taylor rule proper, but the actual (much lower) real interest rate stimulates activity. This leads to a rather incredible profile for output, supported by a strongly autocorrelated sequence of unanticipated shocks, which for every horizon now lie on top of each other in the graph. This reflects that by contrast to anticipated shocks, all shocks up to the fixed-rate horizon are unaffected by the horizon length. This simulation is difficult to support as one where modest interventions are being used; the responses are dominated by the simulation procedure itself.

These simulations are for one model with one calibration, and it may, of course, be that it generates extreme results when other models and calibrations do not. However, we have found this to be very typical for this particular model. We find that varying the calibration has little qualitative effect, and the results are consistent with the findings of Laséen and Svensson (2011) (and indeed Galí (2011)), although perhaps a little more pessimistic.

We require another approach. We know that models with *permanently* fixed nominal interest

Chart 4: Canonical New Keynesian model; fixed nominal interest rate using Leeper-Zha; unit cost-push shock



rates are indeterminate. Such models can be made to always have sensible simulation properties by use of an appropriate selection criterion, as in Lubik and Schorfheide (2004). Can we find a similar 'fixed interest rate' solution that yields *predictable* answers in the sense that they are similar to the usual responses, and essentially replicate the results that we would expect for indeterminate models?[6] It turns out that we can. We suggest two possible approaches to designing 'regimes' which essentially generate indeterminacy under partially fixed interest rates, either using additional policy interventions or using a time-varying policy rule. First, and in the next section, we give ourselves an 'extra' shock.

3 Fixed rates for n periods using $n + 1$ shocks

What, then, can we do to restore a greater degree of predictability to the simulation responses with anticipated shocks? We suggest two possible approaches, but they will generate the same result in the end. The first of these is to augment the problem with an explicit target for one of the jump variables, the second to use a time-varying policy rule.

[6] We have already seen that unit root behaviour seems tied down by the real interest rate simulations.

Before we describe these two different ways of recovering more normal behaviour, we should emphasise it is at the expense of returning to something akin to a 'conventionally' indeterminate model. This implies that there are infinitely many rational solutions to the model that cannot be distinguished between on stability grounds alone: the Taylor principle is no longer sufficient.

In models with a permanently fixed nominal interest rate this is a familiar property. In practice this means that we can select a solution by, say, arbitrarily either choosing an initial value for one of the jump variables or setting the trade-off between inflation and output in the first period. Lubik and Schorfheide (2004) do the latter, and choose the trade-off that most closely approximates a Taylor rule-based alternative. (As we show below, in our framework any such additional rule is simple to incorporate.) Once we have done this, then the solution is unique. But we must specify something other than asymptotic stability to determine the solution, the initial condition for (a combination of) some jump variable(s).

We did not *need* to concern ourselves with this before. As we have seen, if the interest rate is fixed for only n periods then the solution is unique given the non-singularity condition implied by **(9)** and the use of n policy shocks. However, we did not like what that implied. Instead we treat the model as if it were indeterminate, so we need to select some equilibrium. To do this we pick an initial value for a jump variable as well as set i_t to zero, and use a similar non-singularity condition that guarantees that we can satisfy these $n + 1$ conditions by using $n + 1$ expected shocks.[7]

We outline the procedure by setting up a representative problem. Say we wish to choose an initial value of the j^{th} jump variable, x_0^j. We can use the j^{th} row of **(3)** to write it as a function of variables known at time 0. Modify **(6)** to include x_0^j, written:

[7]This is now a multivariate version of a Type 2 fix.

$$
\begin{bmatrix} x_0^j \\ i_1 \\ i_2 \\ i_3 \end{bmatrix} = \hat{\imath}_j \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ \widetilde{B}_1 & \widetilde{B}_2 & \widetilde{B}_3 & \widetilde{B}_4 \\ \widetilde{A}\widetilde{B}_1 & \widetilde{A}\widetilde{B}_2 + B_1 & \widetilde{A}\widetilde{B}_3 + \widetilde{B}_2 & \widetilde{A}\widetilde{B}_4 + \widetilde{B}_3 \\ \widetilde{A}^2\widetilde{B}_1 & \widetilde{A}^2\widetilde{B}_2 + \widetilde{A}B_1 & \widetilde{A}^2\widetilde{B}_3 + \widetilde{A}\widetilde{B}_2 + \widetilde{B}_1 & \widetilde{A}^2\widetilde{B}_4 + \widetilde{A}\widetilde{B}_3 + \widetilde{B}_2 \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{bmatrix}
$$

$$
+\hat{\imath}_j \begin{bmatrix} N \\ \widetilde{A} \\ \widetilde{A}^2 \\ \widetilde{A}^3 \end{bmatrix} z_0 + \hat{\imath}_j \begin{bmatrix} P \\ \widetilde{G} \\ \widetilde{A}\widetilde{G} \\ \widetilde{A}^2\widetilde{G} \end{bmatrix} \varepsilon_0 \qquad \textbf{(16)}
$$

where $\hat{\imath}_j$ now selects both the equations for i_t, $t = 1, 2, 3$ and the j^{th} jump variable.[8] Written this way, where we have dropped i_4, we have as many targets, $\begin{bmatrix} x_0^j & i_1 & i_2 & i_3 \end{bmatrix}'$, as instruments $\begin{bmatrix} \nu_1 & \nu_2 & \nu_3 & \nu_4 \end{bmatrix}'$.[9] We now have a way of fixing the interest rate for n periods whilst at the same time selecting an initial value for one of the jump variables.

3.1 Application 1 revisited

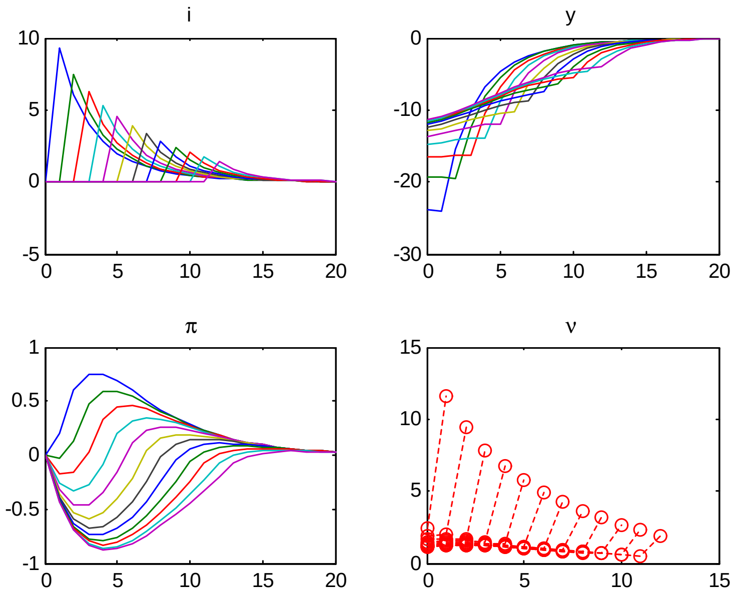
So what does this imply for model simulation responses? In Chart 5 we choose $\pi_0 = 0$, and now vary the number of shocks $(n + 1)$ from two to thirteen, fixing the interest rate for one to twelve periods respectively. In each case this should fix the interest rate for n periods and set $\pi_0 = 0$. Looking at the results (and despite achieving all our targets) first impressions are not good. Output collapses because inflation is forced negative and growth must therefore be positive; the results look as unusable as before. However, on the good side (and as we might expect), the solution now homes in on a long-horizon equilibrium, rather than exploding. This is the infinite horizon solution that would be obtained from the indeterminate model starting at $\pi_0 = 0$ (a result we establish later). The finite horizon solution is unique; our chosen initial condition and n periods of fixed interest rates are determined by simple algebra. The policy interventions now follow a path that is very similar for every period except for the shock in period $n + 1$, where there is a large jump up every time.

However, if we choose a different initial jump for π_0 — or for y_0 — we obtain a different response, again uniquely. In Chart 6 we choose the value of $y_0 = 0$ instead, and this looks much better with qualitatively acceptable simulation results. Again it homes in on the steady-state response, that which obtains for fixing the interest rate for all time and with the imposed jump in

[8] Targeting arbitrary values for the initial jump is a simple extension, as indeed is targeting linear combinations, which we will use later.

[9] Alternatively, we can keep the $n + 1$ targets and apply the least squares solution **(8)**. There is now a question of a trade-off between the interest rate versus the other target. A weighted least squares approach could be usefully adopted.

Chart 5: Canonical New Keynesian model; fixed nominal interest rates; $\pi_0 = 0$; unit cost-push shock



y_0. This is happening for all the early responses by about a horizon of ten periods of fixed interest rates. Rationally anticipating what follows in the paper, we should now be confident that there is no indeterminacy for the model solved this way. Even after 120 periods $\hat{\Xi}$ remains non-singular. Clearly for this model, the $n + 1$ periods needed for the $n + 1$ targets to 'home in' on the same solution at time $t = 0$ is around the maximum of the horizon we use for our simulations. Past this, the first six or eight periods are qualitatively the same. The policy shocks are qualitatively as before, with a very similar path up to period n with a marked jump up for the period $n + 1$ intervention.[10]

We noted above that we can choose a combination rather than a fixed value of a jump variable. In Chart 7 we choose the value of $\sigma y_0 = -\pi_0$. (Why will become clear in the next section.) As $\sigma = 1$ this implies that the jump in y_0 should be equal but opposite in sign to the jump in π_0. This is apparent in the graphs, and replicates the sign of the impact effect of the shock when the model was closed by a Taylor rule (compare Charts 1 and 7).

[10]If we had used only n policy shocks to fix the interest rate for n periods and pick an initial value for one of the jump variables then although this cannot be satisfied (if it were our uniqueness condition would be violated) this rapidly starts to look like the previous solution for most of the simulation period as we extend n. Simulations for this case are not shown but can be supplied on request.

Chart 6: Canonical New Keynesian model; fixed nominal interest rates; $y_0 = 0$; unit cost-push shock

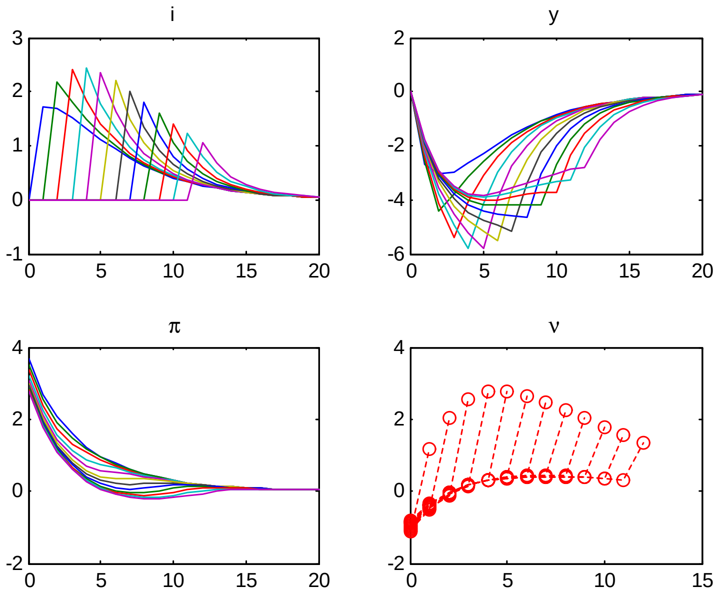
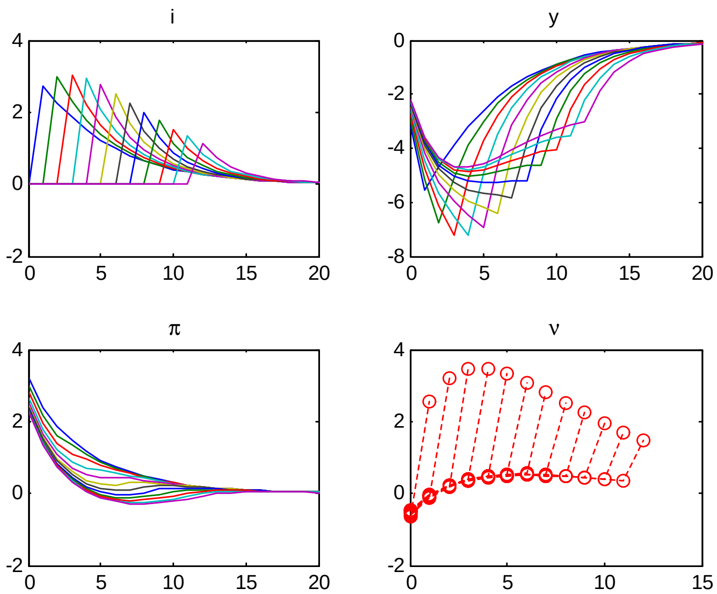


Chart 7: Canonical New Keynesian model; fixed nominal interest rates; $\sigma y_0 = -\pi_0$; unit cost-push shock



We close this section with the following observation. If we use $n + 1$ policy shocks we can generate infinitely many 'well-behaved' solutions for the endogenous variables and fix the nominal interest rate for n periods. We can thus replicate the same indeterminate behaviour that characterises permanently fixed interest rates by the addition of a single extra policy shock. This is as it should be. We have translated an n-instrument, n-target non-singular problem into an $(n + 1)$-dimensional non-singular problem, where we are free to choose the $(n + 1)^{\text{th}}$ dimension.

4 Fixed rates for n periods using a fixed-rate rule for $n + 1$ periods

Instead of working from a determinate solution and finding how to replicate an indeterminate one, we now turn to doing the same the other way round. We begin from effective indeterminacy and replicate the results we have just obtained by fixing the interest rate over a finite horizon. It has been shown by Galí (2011, 2009) that there exist policy rules that generate fixed interest rates over an infinite horizon, but there are many of them. We will take these rules (actually a generalisation of them) and use them for a finite period before switching back to more conventional policy.

We sketch how the Galí (2011, 2009) rules work for an infinite horizon. To do this we need to understand the structure of any particular model, so we illustrate how to do this by example rather than a general algebraic approach. Take the following somewhat unconventional rule:

$$i_t = \phi \left(r_{t-1} + \pi_t \right) \tag{17}$$

We can show easily that this yields a zero interest rate for all time when used to close our model. Equation **(17)** implies that the expected interest rate next period is $i^e_{t+1} = \phi \left(r_t + \pi^e_{t+1} \right)$. From equation **(14)** this implies that $i^e_{t+1} = \phi i_t$. If $|\phi| > 1$ this is explosive unless $i_t = 0, \forall t$. It is easy to show that ϕ is also an eigenvalue of the model[11] so all that is required to ensure determinacy and a zero interest rate is to choose an appropriate unstable value for ϕ. In order to understand what follows and why this rule is not a unique way of fixing nominal interest rates, we need to see what the rule implies for the behaviour of inflation and the output gap. What does **(17)** imply when the model is perturbed away from equilibrium? As then $r_{-1} = 0$ it must be that in the face of any shock $0 = \phi \left(r_{-1} + \pi_0 \right)$ implies $\pi_0 = 0$. This immediately means we should be able to

[11] In fact the implied equation $\left(r_t + \pi^e_{t+1} \right) = \phi \left(r_{t-1} + \pi_t \right)$ defines both an eigenvalue and an eigenvector of the model. These jointly define i_t as a canonical variable that must jump to zero for stability.

compare the results of this experiment directly with Chart 5 where we chose the initial value for inflation as zero.

But there are many more rules with similar fixed interest rate properties. Consider instead the policy rule:

$$i_t = \phi\,(\sigma\,\Delta y_t + \pi_t) \tag{18}$$

Leading this and taking expectations now implies $i^e_{t+1} = \phi\left(\sigma\,\Delta y^e_{t+1} + \pi^e_{t+1}\right)$ which together with **(12)** again implies $i^e_{t+1} = \phi i_t$. However now $0 = \phi\,(\sigma\,\Delta y_0 + \pi_0)$, so assuming $y_{-1} = 0$, the implies $\sigma\,y_0 = -\pi_0$, an initial period trade-off between output losses (gains) and inflation (disinflation). Given these proportions, we should expect to be be able to compare these results with those in Chart 7.

These two rules are investigated in Galí (2011); we now derive another one with similar properties. Take **(18)** and substitute out for π_t using **(13)** lagged one period, ie $\pi_t = \beta^{-1}(\pi_{t-1} - \kappa y_{t-1} - g_{t-1})$. This gives the rule:

$$i_t = \phi\left(\sigma\,\Delta y_t + \beta^{-1}(\pi_{t-1} - \kappa y_{t-1} - g_{t-1})\right). \tag{19}$$

This can easily be shown to again imply $i^e_{t+1} = \phi i_t$ (substitute the Phillips curve back in) but, following the same argument as before with all lagged values equal to zero, it must be that $y_0 = 0$, as we imposed in Chart 6.

Our analysis already extends Galí (2011) both by adding another candidate rule but more significantly by recognising that **(17)**, **(18)** and **(19)** have predictably different implications for the initial values of y_0 and π_0. In what follows we generalise his approach in three further directions. First, we further extend the number of possible fixed rate rules from three to infinity. Second, we develop a targeting rule approach as an alternative to the instrument rules above. Third, we show how to apply these rules to the finite horizon problem (and, indeed, why we need the targeting-rule approach).

Beginning with the last of these, an obvious approach would be to take one of the fixed rate rules just derived and combine it with a Taylor rule. Combining **(17)** and **(2)** would yield, for example, the following time-varying linear rule:

$$i_t = (1 - D_k)\,\phi\,(r_{t-1} + \pi_t) + D_k\,(\gamma\,i_{t-1} + (1-\gamma)(\theta\pi_t + \chi y_t)) \tag{20}$$

where D_k takes the value 0 before some given switch point k and 1 after. But this would not fix the interest rate at zero. As $i_{t+1} = \phi i_t$ in the initial periods, the interest rate must follow an unstable path until the Taylor rule takes over. Effectively the model follows the rule:

$$i_t = (1 - D_k)\,\phi i_{t-1} + D_k\,(\gamma\, i_{t-1} + (1-\gamma)(\theta\pi_t + \chi y_t))$$

from the initial condition $\pi_0 = 0$, assuming $r_{-1} = 0$. The predicted path for interest rates must be such that $i_t = \phi^{-1} i_{t+1}$ to join up with the path to equilibrium implied by the Taylor rule.[12]

Choosing a large value for ϕ is an approximate solution, but there is a better one. We can instead use a two-part rule which is part instrument rule and part targeting rule[13] which exactly implies fixed rates. To do this we first derive an appropriate targeting rule to replace the instrument rule. In the limit, it must be that:

$$\lim_{\phi\to\infty} i_t = \phi\,(r_{t-1} + \pi_t) \Rightarrow 0 = r_{t-1} + \pi_t \qquad \textbf{(21)}$$

so the instrument rule **(17)** is replaced by the targeting rule $0 = r_{t-1} + \pi_t$. If the model is closed by this targeting rule in every period would have an infinite eigenvalue, consistent with $\phi \to \infty$.[14]

Now we derive a two-part rule to replace **(20)**. This uses **(21)** for the first $n+1$ periods (we discuss why $n+1$ in a moment) followed by the Taylor rule for all subsequent periods. The model is therefore closed by a mixture of a targeting and an instrument rule such that:

$$0 = r_{t-1} + \pi_t \text{ for } t \le n+1 \qquad \textbf{(22)a}$$

$$i_t = \gamma\, i_{t-1} + (1-\gamma)(\theta\pi_t + \chi y_t) \text{ for } t > n+1. \qquad \textbf{(22)b}$$

Notice that **(22)**a implies for periods $s = 1, \ldots, n$ that the equations:

$$0 = r_s + \pi_{s+1} \qquad \textbf{(23)a}$$

$$i_s = r_s + \pi_{s+1} \qquad \textbf{(23)b}$$

ie the targeting rule and the Fisher identity, must simultaneously hold. This implies that $i_s = 0$ for $s = 0, \ldots, n-1$. For $s = 0$ all variables are predetermined except for π_0. This implies $\pi_0 = 0$ if the simulation begins from equilibrium with $r_{-1} = i_{-1} = 0$. So the addition of $n+1$ equations enforces the $n+1$ restrictions $i_s = 0$, $s = 0, \ldots, n-1$ and $\pi_0 = 0$.

[12] The mechanism by which the rules join up is reminiscent of the analysis (in a different context) of Wilson (1979).

[13] For a discussion of instrument versus targeting rules see Rudebusch and Svensson (1999).

[14] This also reflects the well-known property of targeting rules that they typically imply an infinite eigenvalue.

If instead we consider the pair of equations:

$$0 = \sigma (y_{s+1} - y_s) + \pi_{s+1} \tag{24)a}$$

$$i_s = \sigma (y_{s+1} - y_s) + \pi_{s+1} \tag{24)b}$$

where now a targeting rule and the IS curve should simultaneously hold. Then, as before, this implies that $i_s = 0$ for $s = 0, \ldots, n-1$. But now beginning from equilibrium (so that $y_{-1} = i_{-1} = 0$) this implies $\sigma y_0 = -\pi_0$. Now the addition of $n+1$ equations enforces the $n+1$ restrictions $i_s = 0$, $s = 0, \ldots, n-1$ and $\sigma y_0 + \pi_0 = 0$.

One more targeting rule replicates **(19)**, by substituting out for π_{s+1} in **(24)**a and **(24)**b:

$$0 = \sigma (y_{s+1} - y_s) + \beta^{-1}\pi_s - \beta^{-1}\kappa y_s \tag{25)a}$$

$$i_s = \sigma (y_{s+1} - y_s) + \beta^{-1}\pi_s - \beta^{-1}\kappa y_s. \tag{25)b}$$

As before, $i_s = 0$ for $s = 0, \ldots, n-1$ but it should be obvious from **(25)**a that now $y_0 = 0$ for $y_{-1} = \pi_{-1} = 0$.

Just as substituting out for a variable using some other equation of the model, it must be that adding in something that is always zero can have no effect either, so now augment **(24)**a and **(24)**b:

$$0 = \sigma (y_{s+1} - y_s) + \pi_{s+1} - (1 + \sigma + \sigma\delta) \left(\pi_{s+1} - \beta^{-1}\pi_s - \beta^{-1}\kappa y_s - g_s\right) \tag{26)a}$$

$$i_s = \sigma (y_{s+1} - y_s) + \pi_{s+1} - (1 + \sigma + \sigma\delta) \left(\pi_{s+1} - \beta^{-1}\pi_s - \beta^{-1}\kappa y_s - g_s\right) \tag{26)b}$$

where $(1 + \sigma + \sigma\delta) \left(\pi_{s+1} - \beta^{-1}\pi_s - \beta^{-1}\kappa y_s - g_s\right)$ is added in. The last bracketed term is the Phillips curve in implicit form, so is always zero. The term $(1 + \sigma + \sigma\delta)$ will allow us to generate any trade-off between initial inflation and output by choice of δ. Collecting terms this gives:

$$0 = \sigma \left(y_{s+1} - (1+\delta)\pi_{s+1}\right) + \beta^{-1}(1 + \sigma + \sigma\delta)\pi_s + \left(\beta^{-1}(1 + \sigma + \sigma\delta)\kappa - \sigma\right) y_s \tag{27)a}$$

$$i_s = \sigma \left(y_{s+1} - (1+\delta)\pi_{s+1}\right) + \beta^{-1}(1 + \sigma + \sigma\delta)\pi_s + \left(\beta^{-1}(1 + \sigma + \sigma\delta)\kappa - \sigma\right) y_s. \tag{27)b}$$

Written this way we now have a rule that can be used to generate an infinity of equilibria. For any finite[15] δ the implication of this rule is that (of course) $i_s = 0$ for $s = 0, \ldots, n-1$ and that:

$$y_0 = (1+\delta)\,\pi_0. \tag{28}$$

[15] Note that too large a value would not work, as this would effectively remove the first two terms in **(26)**a so that the targeting rule element is lost that implies fixed nominal interest rates.

We can consider the implied values for y_0 and π_0 for any choice of δ.

For example (and always assuming lagged values are zero):

- $\delta = -1$: This implies $y_0 = 0$ as the equation straightforwardly reduces to **(25)**a.
- $\delta = -\frac{1+\sigma}{\sigma}$: This reduces **(27)**a to **(24)**a, implying $\sigma y_0 = -\pi_0$.
- $\delta = -2$: This implies $y_0 = -\pi_0$, so inflation and output are exactly traded off in the first period, whatever the value of σ.
- $\delta = 0$: This implies $y_0 = \pi_0$. This implies an *equal* jump in inflation and the output gap.
- $|\delta| \gg 0$: A large δ (either positive or negative) implies $\pi_0 = \frac{1}{1+\delta} y_0 \approx 0$; but notice that **(27)**a does not imply **(21)** for large $|\delta|$. It *does* still imply $i_s = 0$ for $s = 0, \ldots, n-1$ and $\pi_0 \approx 0$, almost the same as would happen by implementing the targeting rule **(23)**a, and is an equivalent limiting case.

Varying δ therefore generates the infinity of solutions consistent with indeterminacy. Increasing (decreasing) it will make y_0 larger (smaller) in absolute terms than π_0. Note that the sign of responses is not implied by **(28)**, merely the proportions.[16] Effectively δ indexes all of the possible equilibria.

4.1 Equivalence with the n and $n+1$ shock approaches

As we established above, adopting the targeting rule **(27)**a for $n+1$ periods fixes the nominal interest rate for n periods plus an additional initial condition determined by δ. We deliberately chose the initial conditions for π_0 and y_0 imposed by $n+1$ shocks in Section 3 to reflect the initial conditions implied by three specific rules: Chart 5 and rule **(22)**a, Chart 6 and rule **(25)**a and Chart 7 and rule **(24)**a.

[16]There are actually discontinuities in the responses of the model in Application 1. For the cost-push shock, as δ is increases y_0 rises until a point when y_0 and π_0 suddenly become very negative, but then rise again until $\pi_0 \approx 0$. Such discontinuities can be predicted from the implied identical behaviour as δ becomes either very positive or very negative.

We now have a full toolkit to replicate *any* simulation that we obtain with fixed interest rates over n periods either by choosing $n+1$ shocks or by choosing an appropriate two-part rule. Actually, we can replicate the n anticipated shock results in four different ways. These are:

1. Direct application of the optimal 'n shocks' formula **(9)**.
2. The trivial application of the optimal '$n+1$ shocks' formula **(9)** based on **(16)**, now choosing the shocks to replicate the jump generated by the n shock simulation, which of course will be the same n shocks and an extra zero.
3. Adding n equations of the form $i_s = 0$ (ie a Type 1 fix).
4. Adding $n+1$ equations of the form **(27)**a and choosing δ to replicate the jump generated by the n shock simulation.

Of course, for the second and fourth of these we need to know what the jump we wish to replicate is.[17] Looked at in this way the n shock result is simply a specific $n+1$ shock problem with a particular initial condition imposed. (We should not forget we know that this problem has a unique solution.) Equally, we can view the first and third methods (essentially the n anticipated shock case) as simply generating one of the infinity of solutions available.

We have now established that rather than calculate $n+1$ policy shocks we can simply use an appropriate time-varying rule and, or course, *vice versa*. We could, for example, use the fixed-rate rule to calculate the paths of the endogenous variables and then calculate the implicit shocks using the standard policy rule.

That we can fix interest rates in an infinity of different ways, each associated with a different path for output and inflation is just the indeterminacy problem, only in another guise. We are simply selecting a missing initial condition, just as we did explicitly when we allowed ourselves $n+1$ shocks. For the infinite horizon problem, the implication is that using any of **(17)**, **(18)** or **(19)** does not solve the indeterminacy problem despite the fact that each of them is a determinate rule. It simply relocates it to the choice of δ when the fixed-rate rule **(27)**a is used in every period. For finite horizon problems, some choice of δ maps directly to every x_0^j in **(16)**.

[17]For example, in Chart 2 by fixing nominal rates for six periods with six shocks generates $\pi_0 = 6.45$ and $y_0 = 14.86$. This would imply $\delta = \frac{14.86}{6.45} - 1 = 1.3$, equivalent to setting $0 = y_0 - 2.3\pi_0$.

4.2 Application 1 revisited with rules

For completeness we replicate our previous simulation responses with rules rather than anticipated shocks. In Chart 8 we plot four graphs by way of comparison. In every case the solution uses a stacked Newton approach, rather than the analytic Blanchard-Kahn one.[18] The simulation is as before, but we only depict the case where the rule changes at $n = 13$. In the top left panel, the standard Taylor-type response using **(2)** is plotted, and in the top right panel the response when the model is closed using the rule **(17)**. As we noted above, this is a fairly extreme experiment, as it implies that inflation must start at zero which exaggerates the downward push on output. When we use a mixture of the two rules when we revert to the Taylor rule after thirteen periods, we obtain the bottom left panel. Thirteen periods is clearly enough for the results to essentially mimic the 'zero rule' results, except for the blip when the Taylor rule takes over. In the bottom right panel, the model with the Taylor rule is solved using stacked Newton (as in the top left panel) but this time including anticipated shocks that we obtained analytically by solving the model using the approach outlined in Section 2. This is identical to the two-part solution depicted in the bottom left panel.

If we use a different policy rule to fix the interest rate we can get a completely different response. For example, it is a simple matter to replicate Chart 6 using the correct time-varying rule, ie **(25)**a and **(25)**b or Chart 7 using **(24)**a and **(24)**b. In Chart 9 we plot the responses for the three different rules that imply these different initial conditions for the jump variables. They replicate exactly the results we had before.

Thus we replicate the indeterminacy of the model associated with the infinite horizon fixed nominal interest rate problem at any finite horizon. We replace the unique but mostly unpredictable results associated with fixed nominal rates with an infinity of solutions, *some* of which are much more predictably behaved. The finite horizon solution becomes a problem of equilibrium selection.

[18]The models were all simulated for 120 periods, considerably further than the simulation horizon by which time all responses have died away, with the final 99 periods dropped from the graphs.

Chart 8: Canonical New Keynesian model; stacked Newton solutions; unit cost-push shock

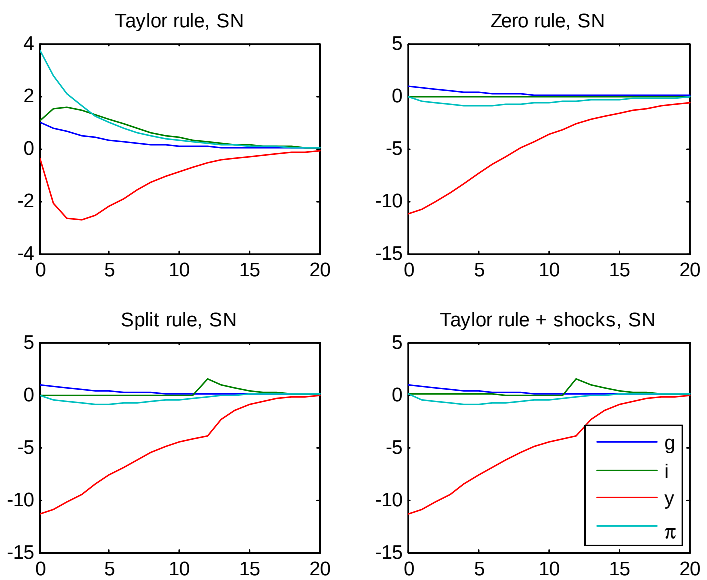


Chart 9: Canonical New Keynesian model, stacked Newton solutions of two-part rules

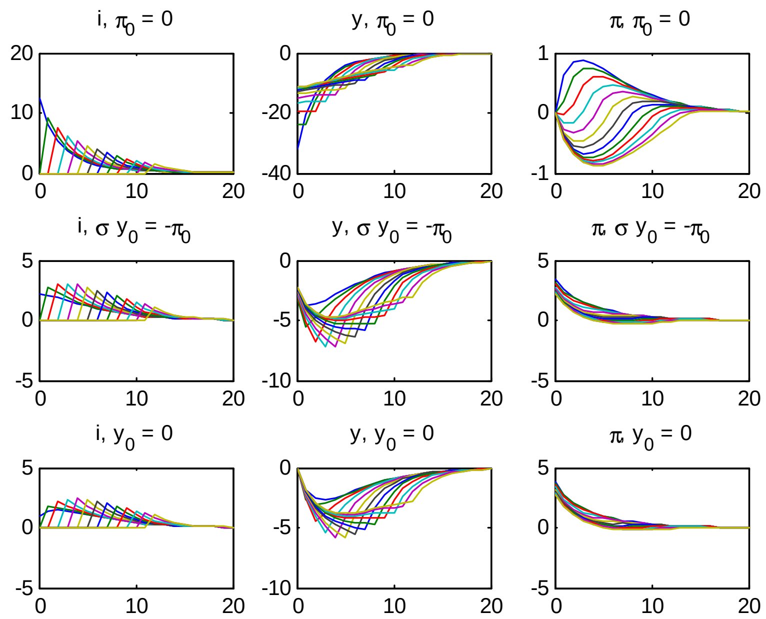
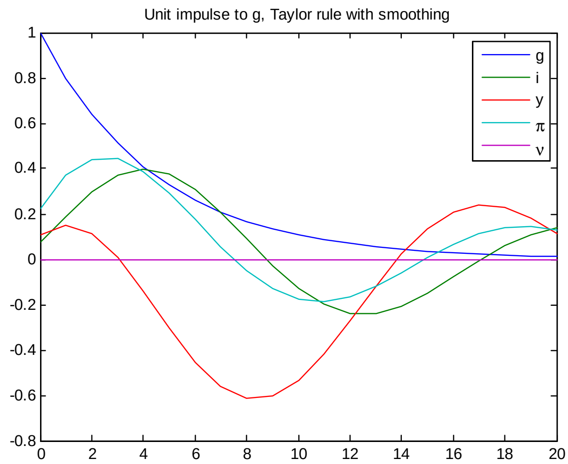


Chart 10: Lindé model; Taylor rule with smoothing in all periods; unit cost-push shock



5 Application 2

We now investigate the different approaches on a second model; it could be that the chosen method does not matter too much for models with predetermined state variables. As a check (and because it reveals some interesting further features of the solutions) we repeat our analysis with one of the models used by Laséen and Svensson (2011), which takes parameters from Lindé (2005). It is a modification of the canonical model above with a backwards/forwards specification for both the IS and Phillips curves. These are:

$$y_t = 0.425y^e_{t+1} + (1 - 0.425)y_{t-1} - 0.156r_t \tag{29}$$

$$\pi_t = 0.457\pi^e_{t+1} + (1 - 0.457)\pi_{t-1} + 0.048y_t + 0.05g_t \tag{30}$$

As before, the model comprises of five equations, which are now the IS curve **(29)**, the Phillips curve **(30)**, the autoregressive cost shock **(15)**, the Fisher identity **(14)** and a Taylor-type policy rule **(2)**.

The model subject to the same cost-push shock as before exhibits the responses in Chart 10.[19] These are, of course, somewhat similar to our earlier example, but now exhibit strong cyclicality. Whilst output is predominantly below base, it is not always (indeed not on impact). Similarly,

[19] Laséen and Svensson (2011) investigate the impact of changing the target interest rate rather than the response to an exogenous shock. This is simple to replicate using our analysis which can be easily extended to include a target nominal rate. We are able to compare our results to the 'Taylor rule in every period' alternative which we will also use in equilibrium selection later.

inflation is predominantly positive, but again goes below base. Both start quite close to zero, as the impact effects are considerably smaller than the maximum. Interest rates follow a similar path to inflation.

We report five fixed-rate policy experiments with this model. They are:

1. Fixed nominal interest rates as in Laséen-Svensson.
2. Fixed real interest rates following Laséen-Svensson's alternative suggestion.
3. Fixed nominal interest rates using Leeper-Zha.
4. Fixed nominal interest rates with the initial values of output and inflation constrained to be $\pi_0 = 0$.
5. Fixed nominal interest rates with the initial values of output and inflation constrained to be $\pi_0 = -\frac{0.425}{0.156} y_0$.

We use three-dimensional graphs to better explain the dynamics, as the cycles make two-dimensional graphs rather hard to read. We use the same simulation horizon and fixed interest rate intervals as before. The first experiment (Chart 11) illustrates nicely the problems that Laséen and Svensson found; changing the fixed-rate horizon from four to five periods, the interest rate jump in the first free quarter goes from positive to negative. This is manifested as a red 'cliff' in Chart 11. Then if the period of fixity is from seven to eleven periods, interest rates — and all the other variables — barely move, before suddenly becoming active again if the fixity period exceeds that, with another sign change. Output and inflation exhibit wild cycles (including over the period of fixed interest rates) or almost no movement at all with small variations in the number of periods of fixed interest rates.

If we instead fix the real interest rate (Chart 12) everything is much better behaved, but perhaps no more satisfactory. Real rates are flat, but so is output. This should not be much of a surprise; for the first n periods the IS curve is now:

$$y_t = 0.425 y^e_{t+1} + (1 - 0.425) y_{t-1}.$$

For long horizons, the jump in y_0 turns out to be zero because now we have to satisfy a second-order difference equation that gives a stationary value, unlike the first-order one we had

Chart 11: Lindé model; fixed nominal interest rates with *n* shocks; unit cost-push shock

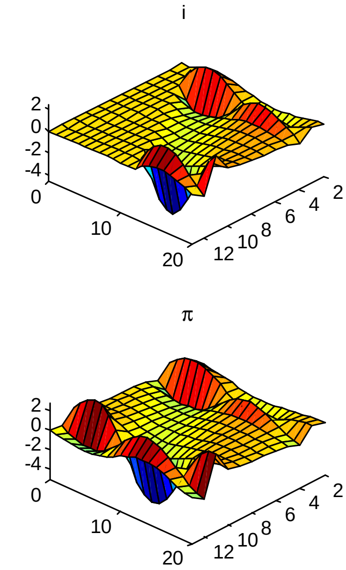


Chart 12: Lindé model; fixed real interest rates; unit cost-push shock

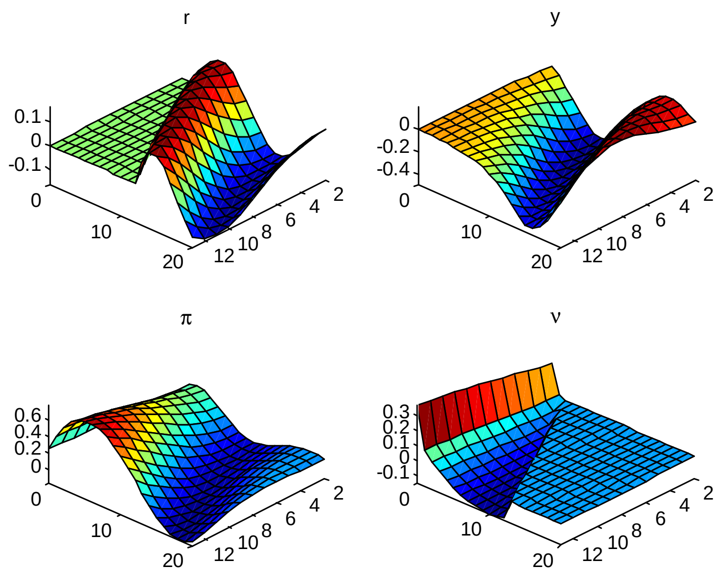
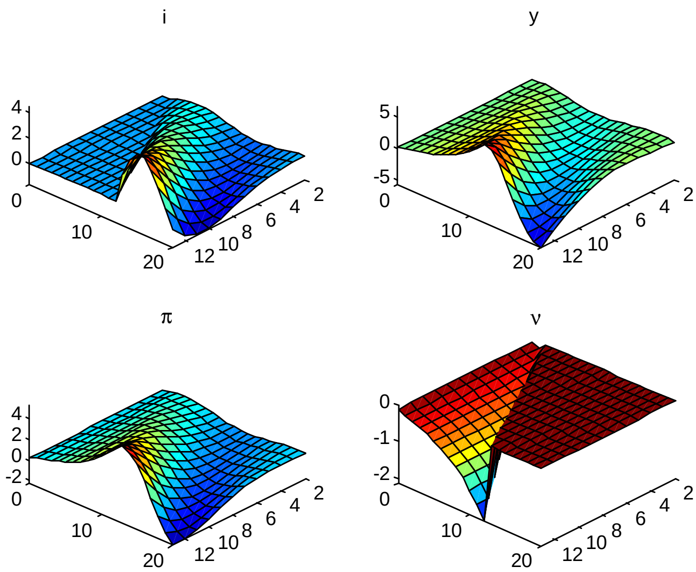


Chart 13: Lindé model; fixed nominal interest rates using Leeper-Zha; unit cost-push shock



before. Responses are smooth and relatively small, but perhaps no more credible, if we compare them with the Taylor rule.

Smoothness is also a property of the Leeper-Zha experiment (Chart 13). However, relatively little happens to either inflation or output until close to the end of the period of fixity, and both output and inflation are always positive whilst interest rates are fixed. The responses seem quite counterintuitive and again are unsatisfactory.

In Chart 13 we fix nominal rates and choose the initial value of inflation to be zero, ie we use rule **(22)**a–**(22)**b or the calculated policy shocks as illustrated. Now the responses, whatever the fixed interest rate horizon, are qualitatively similar to the Taylor rule ones in Chart 10. A major difference is that inflation immediately falls, whereas output dynamics are similar to the unconstrained case. Once interest rates are free they now typically jump up.

If we instead constrain inflation and output to be proportional to each other using the hybrid

Chart 14: Lindé model; fixed nominal interest rates; $\pi_0 = 0$; unit cost-push shock

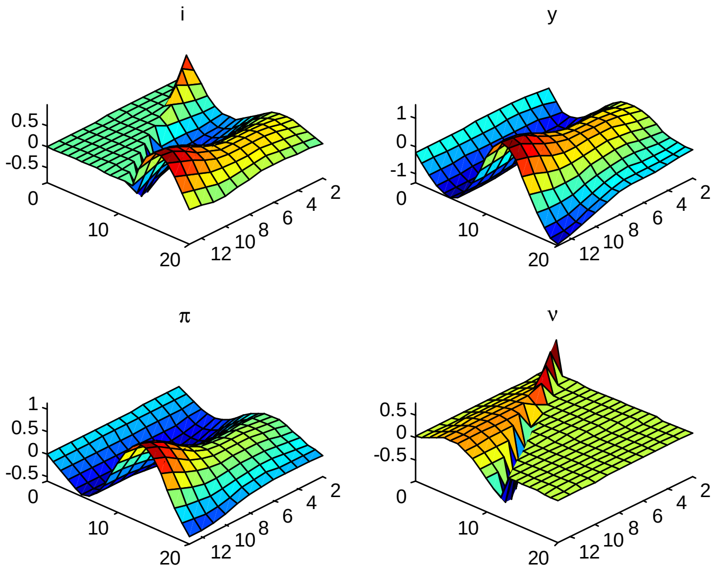
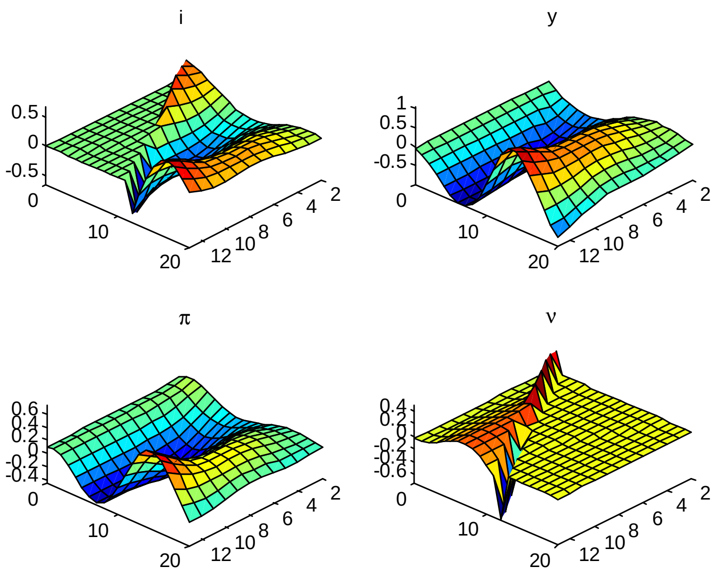


Chart 15: Lindé model; fixed nominal interest rates; $y_0 = -0.37\pi_0$; unit cost-push shock



targeting/instrument rule:

$$0 = 0.425y_t - y_{t-1} + (1 - 0.425)y_{t-2} + 0.156\pi_t \text{ for } t \leq n + 1$$

$$i_t = \gamma i_{t-1} + (1 - \gamma)(\theta\pi_t + \chi y_t) \text{ for } t > n + 1$$

obtained from **(29)** so that $y_t = -\frac{0.156}{0.425}\pi_t$ we get Chart 15. This is little different from fixing π_0 alone, and reflects the small initial jump in both variables seen in Chart 10.

What are we to make of these simulations? They broadly confirm the earlier picture; just fixed nominal or indeed fixed real rates are difficult to sustain both conceptually and practically. Simulations where we pick an initial condition are much more satisfactory from either a forecast or policy analysis point of view, but they need some additional judgement to be applied.

6 Conclusions and implications for simulation design

The main message of this paper is that fixing nominal interest rates for a few periods by dropping the monetary policy rule is often not a good idea. This is not because at some horizon models become indeterminate, but mostly because just fixing it has unforeseen consequences. Indeterminacy may 'kick in', but at very long fixed-rate horizons, and model responses may be effectively unusable a long way before this is reached. By contrast, we can make a virtue out of indeterminacy and instead make any model behave as if it were indeterminate at these short horizons. Properly done, this restores the intuition to the responses, but at the cost of introducing a selection problem.

One of the innovations in this paper is showing how to do this, and to essentially replicate the properties of a model with permanently fixed nominal interest rates. This can be achieved by either using more shocks than periods to fix the interest rate or, conveniently, by using a properly designed two-part rule. The drawback to this is that there is an infinity of such solutions — many seemingly well behaved — and some external mechanism needs to be used to choose between them.

The appropriate equilibrium choice for a given simulation experiment is likely to vary, but for the cost-push shock considered here we have a natural, relatively judgement-free, contender. Following a similar approach to Lubik and Schorfheide (2004), we could choose from the

Chart 16: Canonical New Keynesian model; $y_0 = -0.1\pi_0$; unit cost-push shock

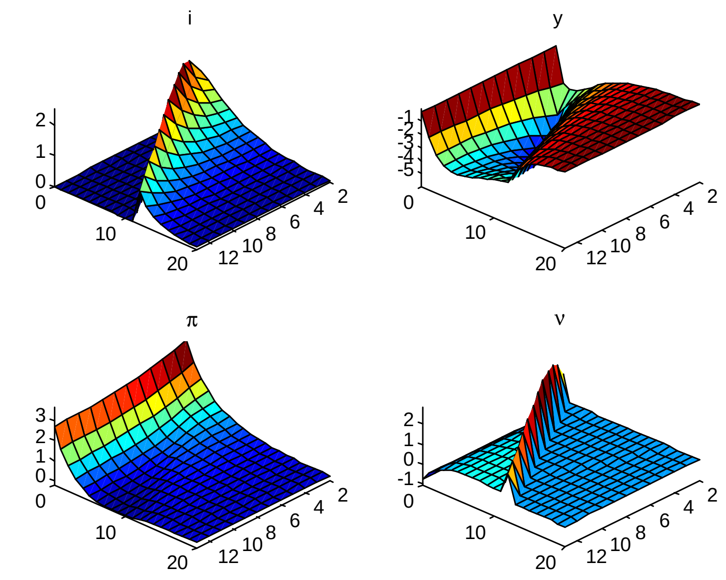
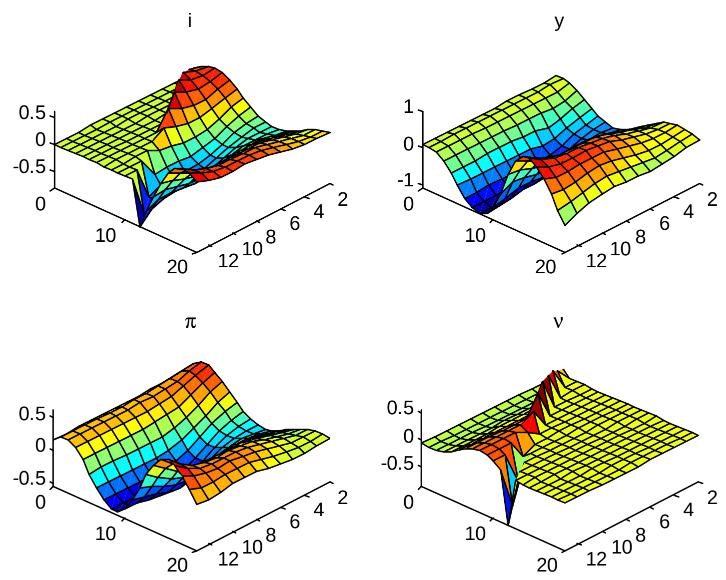


Chart 17: Lindé model; $y_0 = 0.5\pi_0$; unit cost-push shock



indeterminate solutions by in some way mimicking the behaviour of the Taylor rule.[20] For our canonical New Keynesian model, the initial trade-off between output and inflation that the Taylor rule implies is $y_0 \approx -0.1\pi_0$ (Chart 1). In Chart 16 we impose this for all of the fixed-rate horizons. After a very few periods where there is some (but not much) variation in simulation behaviour; the qualitative responses are the same. For our modified New Keynesian model, the trade-off between output and inflation shown in Chart 10 is $y_0 \approx 0.5\pi_0$. In Chart 17 we demonstrate an extraordinary degree of coherence across periods of fixity for this choice of first period output-inflation trade-off.

The experiment we have just conducted and the resolution suggested (replicating Taylor rule behaviour) will not always be the solution to the equilibrium selection problem. Often this will need to reflect the reason for doing the fixing in the first place. This could imply that we would wish to minimise the volatility of inflation or output (or some other variable), or impose a different trade-off between them. Throughout the paper we have argued that fixing interest rates over a finite horizon often results in model simulations that can be difficult to interpret and heavily dependent on the way the fixed interest rates were achieved. Applying existing methods all seem to have significant drawbacks, often with counterintuitive results. The novel approach in this paper transforms the model responses to ones which are better behaved but at the cost of introducing the new problem of equilibrium selection.

[20] Indeed Galí (2011) goes as far as to describe the Taylor rule responses as the ‘actual’ values, with the implied criticism that the fixed rate simulations are unsatisfactory *because* they are different.

Appendix A: Shocks, Type 1 and Type 2 fixes

In some conventional macroeconometic models there is a natural correspondence between certain variables and a particular residual or shock, through an equation that is typically normalised on that variable. A VAR, for example, is naturally set up this way, but even models with concurrent simultaneity may have an obvious association. In our models the natural correspondence is between the Taylor rule and the policy shocks. This, of course, is not necessary in general, but for such models where it does 'fixing' a variable at a given value is often achieved in one of two ways. These are sometimes described, for example in Wallis, Andrews, Bell, Fisher and Whitley (1985), as Type 1 and Type 2 fixes.

Consider for a moment static models without expectations. A Type 1 fix to achieve some desired value of an endogenous variable just sets the residual in a particular equation equal to minus the endogenous prediction in, say, the equation:

$$y_t = x_t\beta + \varepsilon_t \quad \textbf{(A1)}$$

so whatever the value of x_t, the residual is adjusted to achieve $y_t = y_t^*$. In this paper, we treat the desired value as zero, so $\varepsilon_t^* = -x_t\beta$. A Type 2 fix would instead manipulate x_t so that for our equation $x_t^* = \beta^{-1}y_t^*$. We then need to fix x_t at x_t^* by an appropriate Type 1 fix as before, so the 'shock' to a second equation is effectively used. In practice, it is not necessary to do any complicated calculation for a Type 1 fix, as the implied value of ε_t is most conveniently calculated by 'skipping' **(A1)** and setting $y_t = y_t^*$ directly, and then calculating ε_t^* from:

$$\varepsilon_t^* = y_t^* - x_t\beta \quad \textbf{(A2)}$$

using the endogenously calculated value of x_t. This is only feasible for a Type 1 fix, where we can identify the equation to skip.

For models with rational expectations, the assumed shocks in future periods matter, but for the problem of fixing the nominal interest rate for n periods with n shocks the same logic holds, and we can just skip the Taylor rule in periods 1 to n. The solution to equation **(16)** is probably best described as a 'multivariate Type 2 fix', which, of course, subsumes both Type 1 and Type 2 fixes.

Appendix B: Note on computation

All reported calculations are in Matlab, and were checked in *WinSolve* for errors. Two solution methods are used in Matlab to produce the results. The simple Blanchard-Kahn relationships **(3)** and **(4)** are implemented using a real Schur decomposition for numerical stability. The optimal interventions calculated both by solving the analytic expression **(9)** and setting up a direct minimisation problem solved using the Matlab `fminunc` routine as an additional robustness check. These all give the same answer.

The time-varying rule simulations are easiest to do using a *stacked Newton* approach (see Armstrong, Black, Laxton and Rose (1998)) which for small enough models can be coded in a few lines, and can take advantage of Matlab's sparse array handling. Two models with common state variables are stacked together at the appropriate point to generate the time-varying rule. These are easy to check against the other methods as it is apparent when they should produce the same result. An appropriate stacked Newton solution was also used to calculate the analytic coefficients obtained from the Blanchard-Kahn solution as yet another cross-check. All results for all directly comparable methods are identical with either three or four different methods used for each simulation experiment.

Matlab codes for the two models and all solution methods reported are available from the author.

References

Armstrong, J, Black, R, Laxton, D and Rose, D (1998), 'A robust method for simulating forward-looking models', *Journal of Economic Dynamics and Control*, Vol. 22, No. 4, pages 489–501.

Blanchard, O and Kahn, C (1980), 'The solution of linear difference models under rational expectations', *Econometrica*, Vol. 48, pages 1,305–11.

Galí, J (2008), *Monetary policy, inflation and the business cycle: an introduction to the New Keynesian framework*, Princeton, NJ: Princeton University Press.

Galí, J (2009), 'Constant interest rate projections without the curse of indeterminacy: a note', *International Journal of Economic Theory*, Vol. 5, No. 1, pages 61–68.

Galí, J (2011), 'Are central banks' projections meaningful?', *Journal of Monetary Economics*, Vol. 58, No. 6-8, pages 537–550.

Laséen, S and Svensson, L E (2011), 'Anticipated alternative policy-rate paths in policy simulations', *International Journal of Central Banking*, Vol. 7, No. 3, pages 1–35.

Leeper, E and Zha, T (2003), 'Modest policy interventions', *Journal of Monetary Economics*, Vol. 50, pages 1,673–700.

Lindé, J (2005), 'Estimating New-Keynesian Phillips curves: a full information maximum likelihood approach', *Journal of Monetary Economics*, Vol. 52, pages 1,135–49.

Lubik, T A and Schorfheide, F (2004), 'Testing for indeterminacy: an application to US monetary policy', *American Economic Review*, Vol. 94, No. 1, pages 190–217.

Rudebusch, G and Svensson, L E O (1999), 'Policy rules for inflation targeting', in Taylor, J B (ed), *Monetary Policy Rules*, Chicago, IL: University of Chicago Press, pages 203–46.

Sargent, T and Wallace, N (1975), ''Rational' expectations, the optimal monetary instrument, and the optimal money supply rule', *Journal of Political Economy*, Vol. 83, No. 2, pages 241–54.

Waggoner, D F and Zha, T (1999), 'Conditional forecasts in dynamic multivariate models', *The Review of Economics and Statistics*, Vol. 81, No. 4, pages 639–51.

Wallis, K F, Andrews, M J, Bell, D N F, Fisher, P G and Whitley, J D (1985), *Models of the UK economy: a second review by the ESRC Macroeconomic Modelling Bureau*, Oxford: Oxford University Press.

Wilson, C A (1979), 'Anticipated shocks and exchange rate dynamics', *Journal of Political Economy*, Vol. 87, No. 3, pages 639–47.