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Abstract

Recent monetary policy experience suggests a simple test for models of monetary non-neutrality. Suppose the central bank pegs the nominal interest rate below steady state for a reasonably short period of time. Familiar intuition suggests that this should be inflationary. We pursue this simple test in three variants of the familiar Dynamic New Keynesian (DNK) model. Some variants of the model produce counterintuitive inflation reversals where an interest rate peg leads to sharp deflations.

Key words: Fixed interest rates, New Keynesian model, zero lower bound.

JEL classification: E32.

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Summary

Monetary models often assume that firms adjust their prices only slowly in response to changes in the economy. The most common assumption is that only some fraction of firms update their price in any period, as in the work of Calvo. At the same time firms and households are forward looking. They base their decisions not only on today's economic conditions but also on the outlook for the future, including expectations of future interest rates. The recent experience of low interest rates for an extended period of time provides a natural benchmark for testing such monetary models.

This paper examines the effects of an unconditional lowering of the nominal interest rate for an extended period of time in a model with infrequent price adjustment. One would expect that keeping the nominal interest rate low stimulates demand and thus increases inflation and output. In contrast, we show that the commonly used model of Calvo pricing implies unusual behaviour of inflation and output in such an environment.

First, the anticipated unconditional lowering of the interest rate for an extended period of time can make initial inflation and output response from the model unusually large. Second, as interest rates are kept low for a sufficiently long period of time, inflation and output may actually fall in the model we consider. We show that this counterintuitive result is not simply due to using a linear model, but also obtains in the full nonlinear sticky price model.

This is not an econometric test of sticky price models. Nor is it a statement about other possible shocks hitting the system. It is instead a question of *prima facie* plausibility. Our results suggest that these models can produce implausible behaviour under transient interest rate pegs. Future research should therefore examine whether similar results obtain under alternative models of price-setting or expectations formation.

1 Introduction

This paper examines a very simple question: what is the behaviour of inflation if a central bank credibly announces a fixed interest rate for an extended period of time? To be precise, we examine the perfect foresight path of sticky price models in which the nominal interest rate is pegged for a finite period of time. It is of course well known that if the peg is perpetual, then there is local equilibrium indeterminacy and thus sunspot equilibria. But if the peg is finite, and the subsequent monetary policy is consistent with a unique equilibrium, then this subsequent behaviour provides the needed terminal condition that implies determinacy along the entire path.

Our analysis is motivated by the use of such monetary policies in the aftermath of the 2008 financial crisis. But we do not view this paper as an investigation of policy during the financial crisis. Nor do we see this as a data-matching exercise. Instead we see a transient interest rate peg as an experiment for testing the reasonableness of monetary models.

Our experiments consider a fully anticipated and unconditional lowering of the monetary policy rate for a finite number of periods. We find that the workhorse Calvo (1983) models of time-dependent pricing may deliver unreasonably large responses of inflation and output in response to such a policy. Furthermore, if there are endogenous state variables, these models suggest that the initial responses can become arbitrarily large as the duration of the fixed interest rate regime approaches some critical value and then switch sign and become arbitrarily negative as this critical value is exceeded slightly. For empirically realistic models such as the Smets and Wouters (2007) model, the critical duration for which these asymptotes occur is around eight quarters.

We believe these results imply that this class of sticky price models should be used with great caution when studying monetary policy at the zero lower bound or when interest rates are fixed for other reasons. Furthermore, future research should investigate whether similar inflation reversals also occur in other models of nominal rigidities such as state-dependent pricing models as in Dotsey, King and Wolman (1999).

Several other authors have examined issues of this type. This previous work includes Blake (2012), Laseen and Svensson (2011) and Galí (2009). Laseen and Svensson (2011) provide a convenient mechanism for solving monetary models in which nominal rates are set exogenously for a finite period of time. These authors have already pointed out inflation reversals, but do not make them the centre of the analysis and do not consider the nonlinear model. Blake (2012) suggests novel ways to generate reasonable forecasts in models with fixed interest rates over finite horizons. Relatedly, Braun and Korber (2011) and Christiano, Eichenbaum and Rebelo (2011) have examined the size of the fiscal multiplier when the nominal rate is pegged for a finite time period.

Section 2 examines the issue in the simplest New Keynesian model. Section 3 adds inertia in the Phillips curve and Taylor rule and provides conditions for the existence of asymptotes. Section 4 shows that our results also obtain in the Smets and Wouters (2007) model at their estimated parameter values. Section 5 studies the issue in the nonlinear model. Section 6 concludes. The appendix contains proofs, an analysis of a stochastic exit from a fixed interest rate regime, and a stacked time approach to the issue.

2 The DNK model with no endogenous states

The canonical DNK model is given by the following:

$$\pi_t = \kappa y_t + \beta \pi_{t+1} \tag{1}$$

$$y_t = y_{t+1} - \frac{1}{\sigma}\left(i_t - \pi_{t+1}\right) \tag{2}$$

where π_t, y_t, and i_t, denote inflation, the output gap, and the nominal rate, respectively, all measured as deviations from the steady state. We dispense with expectations notation as we are focusing on the perfect foresight path. To close the model, we assume the central bank announces an interest rate rule given by the following:

$$i_t = \begin{cases} i^* & t = 1,2,...,\mathrm{T} \\ \phi_\pi \pi_t + \phi_y y_t & t = \mathrm{T}+1,\mathrm{T}+2, ... \end{cases} \tag{3}$$

Under standard assumptions on ϕ_y and ϕ_π there is a unique equilibrium for $t > T$. Since there are no state variables nor exogenous shocks, the unique equilibrium for $t > T$ is given by $\pi_t = y_t = 0$. For the first T periods, the constant interest rate suggests that there is equilibrium indeterminacy. But since this policy regime is finite, the subsequent uniqueness for $t > T$ serves as a terminal condition and thus ensures uniqueness of equilibrium along the entire path. During the constant interest rate policy, the inflation dynamics are governed by

$$\pi_t = -\frac{\kappa}{\sigma} i^* + \left(\frac{\kappa}{\sigma} + 1 + \beta\right) \pi_{t+1} - \beta \pi_{t+2} \tag{4}$$

with the two terminal conditions:

$$\pi_T = -\frac{\kappa}{\sigma} i^* \tag{5}$$

$$\pi_{T-1} = -\left(\frac{\kappa}{\sigma} + 2 + \beta\right) \frac{\kappa}{\sigma} i^* \tag{6}$$

It is convenient to invert this system, running time backwards from the end of the extended period. That is, let z_s denote the value of inflation s-periods before time T: $z_s \equiv \pi_{T+1-s}$. The difference equation **(5)** can now be written as:

$$z_s = \frac{\kappa}{\sigma} i^* + \left(\frac{\kappa}{\sigma} + 1 + \beta\right) z_{s-1} - \beta z_{s-2} \tag{7}$$

with the two *initial* conditions:

$$z_1 = \pi_T = -\frac{\kappa}{\sigma} i^* \tag{8}$$

$$z_2 = \pi_{T-1} = -\left(\frac{\kappa}{\sigma} + 2 + \beta\right) i^* \tag{9}$$

The two initial conditions **(8)**-**(9)** imply that there is a unique solution to the difference equation **(7)**. The solution has a simple form:

Proposition 1 The inflation rate during the extended period is given by

$$\pi_{T+1-s} \equiv z_s = i^* + m_1 e_1^s + m_2 e_2^s \quad \text{for } s = 0, 1, ..., T \tag{10}$$

for $s = 1, 2, .., T$, where $e_1 > 1$, $e_2 < 1$, and m_1 and m_2 come from the two restrictions **(8)**-**(9)**. Proof: see Appendix A.

Several remarks are in order. First, the solution to the difference equation is independent of T. That is, m_1 and m_2 are independent of T. This arises because there are only restrictions on inflation at the end of the extended period, and not at the beginning of the extended period. Second, the inflation path during the extended period is entirely independent of the Taylor rule that commences at the end of the extended period. But this subsequent policy is necessary as it provides a terminal condition to the indeterminate difference equation. These first two remarks will not be true if there are endogenous or exogenous state variables. Third, the inflation behaviour given by **(10)** is explosive since $e_1 > 1$. That is, if we run time backwards from the end of the extended period, the initial inflation rate explodes exponentially in T:

$$\pi_1 \equiv z_T = i^* + m_1 e_1^T + m_2 e_2^T \tag{11}$$

This unstable root is a manifestation of the familiar local indeterminacy induced by an interest rate peg. This explosive behaviour is already evident in **(5)**-**(6)**:

$$\pi_{T-1} = -\left(\frac{\kappa}{\sigma} + 2 + \beta\right) i^* = \left(\frac{\kappa}{\sigma} + 2 + \beta\right) \pi_T > \pi_T \tag{12}$$

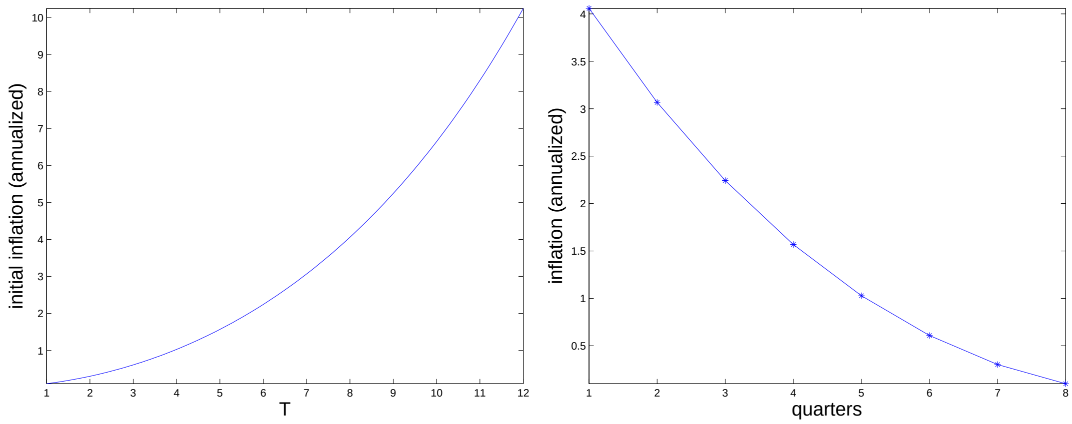
The unstable eigenvalue (e_1) during the extended period has a fairly simple form if we assume $\beta = 1$. It is given by:

$$e_1 = x + \sqrt{x^2 - 1} \quad \text{where: } x = 1 + \frac{\kappa}{2\sigma} \tag{13}$$

The key variable determining inflation during the extended period is $\frac{\kappa}{2\sigma}$. A smaller value for σ magnifies the inflation response, because it makes spending more interest rate sensitive. A larger value for κ increases the inflation response, because the more flexible prices are (the higher κ) the more the real rate interest rate falls. This stimulates demand and puts even further upward pressure on prices.

Chart 1 provides a quantitative example of these effects. The calibration is standard: $\kappa = 0.025$, $\sigma = 1$, $\beta = .99$. As noted, the policy rule subsequent to the extended period is irrelevant (as long as it produces determinacy). We assume an annualised nominal rate of 4% in the steady state, and consider a policy of pegging the nominal rate at zero for T periods.

Chart 1: Inflation dynamics in New Keynesian model with fixed interest rates, $\sigma = 1$



(a) Initial inflation as a function of T

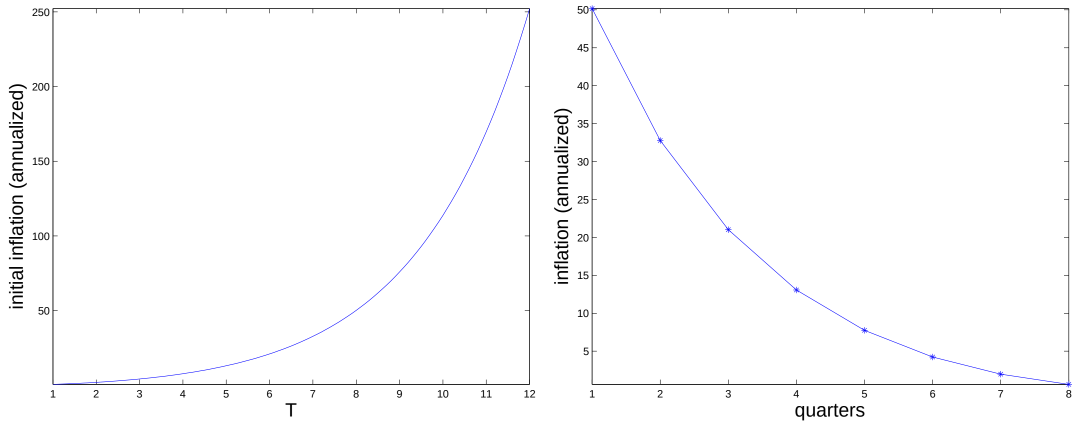
(b) Time path of inflation for $T = 8$

The reason that the initial inflation response is large is as follows. Firstly, this policy experiment is simply a large stimulus. In addition, the fact that interest rates are fixed shuts off any endogenous response of monetary policy from the ensuing inflationary pressure that would normally result in higher nominal rates when the Taylor principle is satisfied. But the combination of fixed rates and positive inflation *lowers* real rates and thus stimulates aggregate demand even further. This leads to even larger demand and higher inflation, which results in bigger falls in the real rate and so forth.

The magnitude of the inflation response depends crucially on the ratio $\frac{\kappa}{\sigma}$ and our baseline calibration assumes conservative values for this. Thus, the response of inflation shown in Chart 1 may not seem unreasonable. Woodford (2003) argues that in a model without capital one should use a lower value for σ as that makes spending more interest rate sensitive, partly compensating for the absence of investment which is more interest rate sensitive.

For the dynamics of inflation, only the ratio κ/σ matters. So a graph similar to Chart 2 would obtain for changes in κ. But the output dynamics do depend on whether σ or κ are varied. Chart 2 shows that the initial inflation response can become extremely large when $\sigma = 0.16$ as in the Woodford calibration. This suggests that for reasonable calibrations of the simple DNK model, a reasonably short interest rate peg may deliver an unreasonable level of inflation.

Chart 2: Inflation dynamics in New Keynesian model with fixed interest rates, $\sigma = 0.16$



(a) Initial inflation as a function of T (b) Time path of inflation for $T = 8$

3 Adding state variables to the model

Many have argued that the simple DNK model is a poor description of macro behaviour because of its inability to generate inflation persistence, see Fuhrer and Moore (1995). One manifestation of this lack of inertia is the explosive behaviour of the initial inflation rate in Chart 2. If inflation in the initial period is linked to the inflation rate in the previous period (the period before the announcement of the extended period), then the initial inflation behaviour will likely be dampened. Hence, in this section we explore the effect of adding inflation inertia to the model as in Christiano, Eichenbaum and Evans (2005) and Smets and Wouters (2007). The model is the same as before, but the Phillips curve is now given by:

$$\pi_t = \frac{\kappa}{1+\beta\lambda} y_t + \frac{\beta}{1+\beta\lambda}\pi_{t+1} + \frac{\lambda}{1+\beta\lambda}\pi_{t-1} \tag{14}$$

where $0 \leq \lambda \leq 1$ is the degree of indexation. Similarly, the policy rule subsequent to the extended period is characterised by inertia $0 \leq \rho < 1$, so that for $t > T$ we have

$$i_t = \rho i_{t-1} + (1-\rho)\left(\phi_\pi \pi_t + \phi_y y_t\right) \tag{15}$$

Again, under standard assumptions on the policy rule, there is a unique equilibrium for $t > T$. We denote these decision rules for $t > T$ by:

$$\pi_t = b_1 \pi_{t-1} + b_2 i_{t-1} \tag{16}$$

$$i_t = d_1 \pi_{t-1} + d_2 i_{t-1} \tag{17}$$

$$y_t = f_1 \pi_{t-1} + f_2 i_{t-1} \tag{18}$$

Our analysis proceeds in two steps. We first work through a very simple example where we have a simple diagrammatic exposition of some key results. The general case is examined in a second step. Our simplest case considers pegging the nominal interest rate for two periods below steady state. We show that it is possible for such a policy to deliver perverse outcomes where output and inflation actually fall. To make the exposition as simple as possible, we consider a special case with $\sigma = \lambda = \beta = 1$. We drop the assumption of interest rate inertia and set $\rho = 0$ as this is not crucial to our results and merely complicates the algebra. The system of equations for a two period peg with $\pi_0 = 0$ can be written as

$$\pi_1 = \frac{\kappa}{2} y_1 + \frac{1}{2} \pi_2 \tag{19}$$

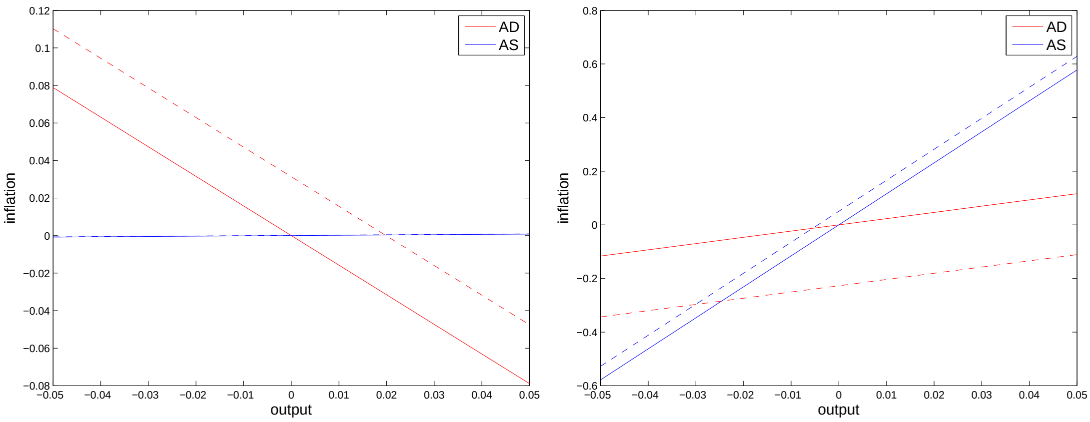
$$y_1 = y_2 - i^* + \pi_2 \tag{20}$$

$$\pi_2 = \frac{\kappa}{2} y_2 + \frac{1}{2} b_1 \pi_2 + \frac{1}{2} \pi_1 \tag{21}$$

$$y_2 = f_1 \pi_2 - i^* + b_1 \pi_2 \tag{22}$$

A graphical exposition of the equilibrium in this model can be obtained by plotting the aggregate demand and the aggregate supply equation for different values of price stickiness. These graphs are derived by eliminating y_2 and π_2 from the system **(19)**-**(22)** and then plotting π_1 as a function of y_1 for the Phillips curve and for the Bond Euler equation.

Chart 3: AD and AS curves for different degrees of price stickiness



(a) Very sticky prices

(b) Very flexible prices

When prices are sufficiently sticky as in panel (a) of Chart 3, the AD and AS have their usual slopes and lowering the nominal rate for two periods has the usual effect of shifting out the aggregate demand curve moving the equilibrium upwards on the supply curve. But when prices are sufficiently flexible, the demand curve is upward sloping as well. In this case an outward shift of the demand curve implies a fall in inflation and output. In addition, a lower policy rate shifts the supply curve inwards somewhat, contributing to the economic contraction.

In the illustrative example above, the perverse outcome rests on very flexible prices. But as we will show below, similar results can be obtained with more sticky prices as long as the duration of the fixed-rate regime is sufficiently long. We now proceed to examine the general case. During the extended period of time, lagged inflation is the only state variable. The difference equation governing dynamics is given by:

$$\pi_t = \frac{-\kappa i^*}{\sigma[1+\lambda(1+\beta)]} + \frac{\lambda}{[1+\lambda(1+\beta)]}\pi_{t-1} + \frac{\sigma[1+\lambda(1+\beta)]+\kappa}{\sigma[1+\lambda(1+\beta)]}\pi_{t+1} - \frac{\beta}{1+\lambda(1+\beta)}\pi_{t+2} \quad \textbf{(23)}$$

We will find it convenient to write this as:

$$\pi_t = c + \gamma_0\pi_{t-1} + \gamma_1\pi_{t+1} + \gamma_2\pi_{t+2} \quad \textbf{(24)}$$

The state variables imply that we have one initial and two terminal conditions:

$$\pi_0,$$

$$\pi_T = c + \gamma_0 \pi_{T-1} + \gamma_1(b_1\pi_T + b_2 i^*) + \gamma_2[b_1(b_1\pi_T + b_2 i^*) + b_2(d_1\pi_T + d_2 i^*)],$$

$$\pi_{T-1} = c + \gamma_0 \pi_{T-2} + \gamma_1 \pi_T + \gamma_2(b_1\pi_T + b_2 i^*)$$

As before, the system can be expressed in z-form:

$$z_s = c + \gamma_0 z_{s+1} + \gamma_1 z_{s-1} + \gamma_2 z_{s-2} \quad \textbf{(25)}$$

With two initial and one terminal condition in z-form:

$$z_{T+1} = \pi_0, \quad \textbf{(26)}$$

$$z_1 = c + \gamma_0 z_2 + \gamma_1(b_1 z_1 + b_2 i^*) + \gamma_2[b_1(b_1 z_1 + b_2 i^*) + b_2(d_1 z_1 + d_2 i^*)], \quad \textbf{(27)}$$

$$z_2 = c + \gamma_0 z_3 + \gamma_1 z_1 + \gamma_2(b_1 z_1 + b_2 i^*) \quad \textbf{(28)}$$

The solution to this system is given by the following.

Proposition 2 The system in **(25)** has one eigenvalue e_1 inside the unit circle and two eigenvalues (e_2 and e_3) outside the unit circle. The inflation rate during the extended period is then given by

1. If e_2 and e_3 are real:

$$\pi_{T+1-s} \equiv z_s = i^* + m_1 e_1^s + m_2 e_2^s + m_3 e_3^s \quad \textbf{(29)}$$

2. If e_2 and e_3 are complex, $(e_2, e_3) = a \pm bi$:

$$\pi_{T+1-s} \equiv z_s = i^* + m_1 e_1^s + m_2 r^s \cos(\theta s) + m_3 r^s \sin(\theta s) \quad \textbf{(30)}$$

where $r = \sqrt{a^2 + b^2} > 1$, $\theta = \tan^{-1}(\frac{b}{a})$, and m_1, m_2, and m_3, come from the three restrictions **(26)**-**(28)**. These constants do depend upon T because of the condition **(26)**.

Proof: See Appendix A.

Proposition 2 demonstrates that once again we have explosive inflation behaviour as we increase the length of the extended period. An additional implication of Proposition 2 is that we can get inflation reversals. Suppose that we are in a parameter region with complex roots. The initial inflation behaviour is then given by

$$\pi_1 \equiv z_T = i^* + m_1 e_1^T + m_2 r^T \cos(\theta T) + m_3 r^T \sin(\theta T) \qquad \textbf{(31)}$$

Since $r > 1$, the initial inflation rate is explosive in T. But the sign of this effect will switch between positive and negative. This switch is marked by an asymptote, where initial inflation goes from arbitrarily positive to arbitrarily negative as we increase T. Strictly speaking there are no asymptotes when T is restricted to take on integer values, but sign reversals will still occur. Chart 4 plots initial inflation against the duration of the fixed-rate regime and shows these reversals. The calibration is given by: $\kappa = 0.025$, $\sigma = 1$, $\beta = 0.99$, $\lambda = 1$, $\rho = 0.80$, $\phi_\pi = 1.5$, $\phi_y = 0.5$. The plot somewhat masks the asymptote that first arises between T = 6 and T = 7 (and periodically thereafter). But note that as we approach the asymptote, initial inflation goes from unboundedly positive to unboundedly negative.

Chart 4: Initial inflation as a function of the duration T

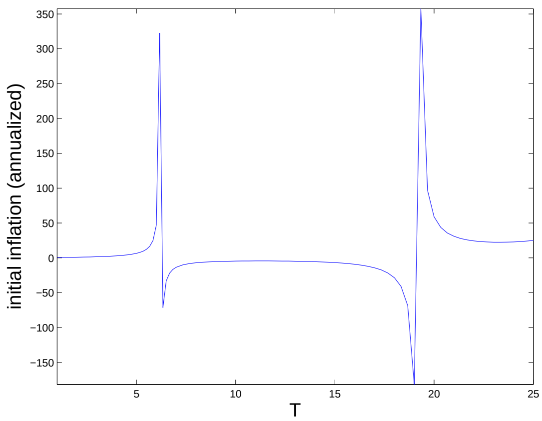
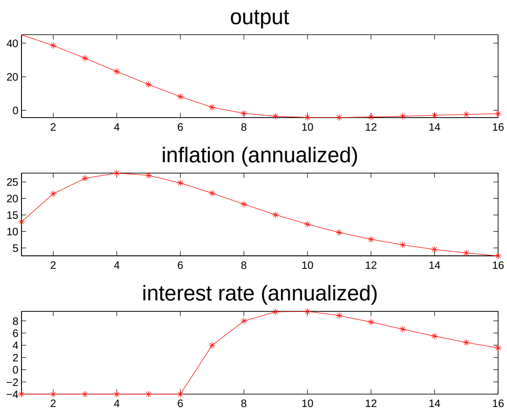
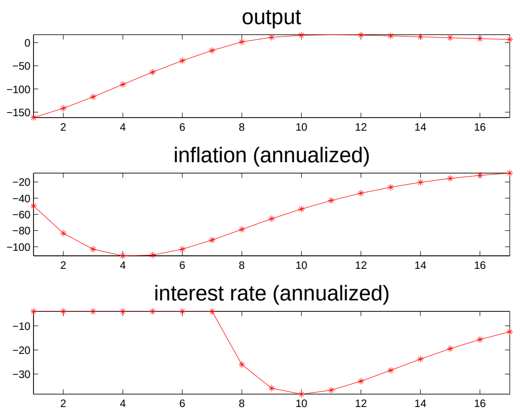


Chart 4 shows dynamics of the model for two different values of T on either side of the asymptote. The figures plot the inflation behaviour for $T = 6$, and $T = 7$ periods. Because these are trigonometric functions, we of course have further sign reversals as we increase T. Along with these peculiar reversals, we also have enormous inflation rates from very modest extended periods. This is in contrast to the model without state variables as it arises even with $\sigma = 1$. Hence, adding state variables has two peculiar implications: (i) the existence of enormous inflation rates for modest extended periods, and (ii) sign reversals so that a low nominal rate peg for six periods can have the opposite effect of a low nominal rate for seven periods.

Chart 5: Dynamics in New Keynesian model with inflation inertia



(a) Dynamics for $T = 6$



(b) Dynamics for $T = 7$

The nominal interest rate in panel (b) of Chart 5 violates the zero lower bound after the exit from the fixed-rate regime and is thus not an equilibrium. We show these dynamics nevertheless to emphasise the possibility of sudden reversals mentioned before. Taking the lower bound into account, we find numerically that the economy remains in the extended period until period $T = 12$. Keeping the interest rate low for a few extra quarters does not eliminate the perverse dynamics in the fixed-rate regime where output and inflation still collapse (inflation troughs at -13%, and output falls by -15%). This suggests that according to the model, it is simply not possible for the central bank to keep the rate pegged at zero for 7, 8, 9, 10, or 11, quarters and then revert to a Taylor rule.

A necessary condition for this perverse sign-switch behaviour is the existence of complex roots to the characteristic equation. If we assume $\beta = 1$, the condition for the existence of complex roots has a particularly simple form. There are complex roots if and only if $\Delta < 0$, with

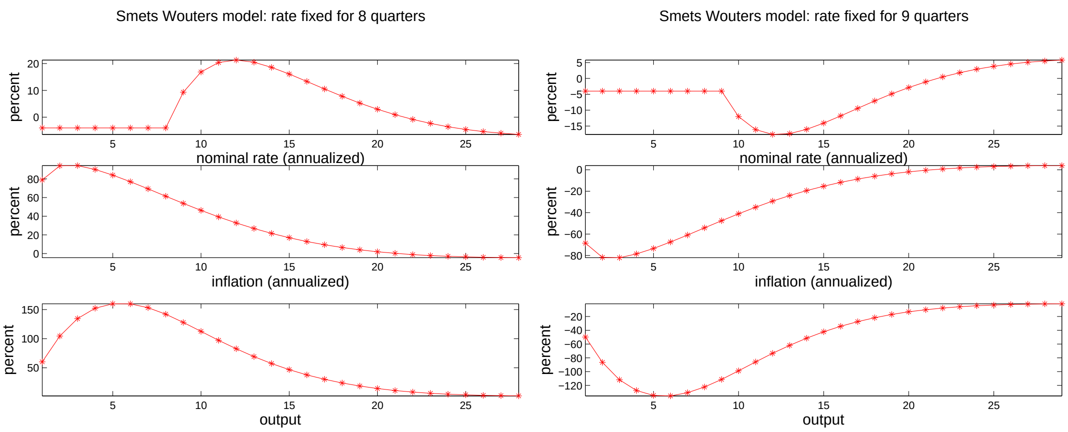
$$\Delta \equiv -4\frac{\kappa}{\sigma}\lambda^3 + \left[12\frac{\kappa}{\sigma} - 8\left(\frac{\kappa}{\sigma}\right)^2\right]\lambda^2 - 4\left[\left(\frac{\kappa}{\sigma}\right)^3 + 5\left(\frac{\kappa}{\sigma}\right)^2 + 3\left(\frac{\kappa}{\sigma}\right)\right]\lambda + \left(\frac{\kappa}{\sigma}\right)^2 + 4\left(\frac{\kappa}{\sigma}\right) \tag{32}$$

This relationship is a convex and decreasing mapping between $\frac{\kappa}{\sigma}$ and λ. For example, for $\kappa = 0.025$, there are complex roots if and only if $\lambda > 0.57$. For = $\frac{\kappa}{\sigma} = 0.25$, there are complex roots if $\lambda > 0.29$. The existence of complex roots implies that there is an asymptote at some value of T. Because the size of the complex part is increasing as Δ becomes more negative, the periodicity of the asymptotes is decreasing in $\frac{\kappa}{\sigma}$ and λ. Note from **(32)**, that interest rate inertia has no effect on the existence of complex roots (and thus asymptotes).

4 The Smets-Wouters Model

In this section we investigate whether the perverse effects outlined in Section 3 continue to arise in a more complete model economy. A classic reference here is the small-scale estimated DSGE model of Smets and Wouters (2007) that includes habit persistence in consumption, capital accumulation, nominal wage indexation, and nominal price indexation. We will use their benchmark model and estimated parameter values to investigate the existence of perverse behaviour in a more realistic model of the economy. The reader is referred to the paper by Smets and Wouters (2007) for the derivation of the model and the estimated parameter values.

Chart 6: Dynamics in Smets Wouters model with fixed interest rates



(a) Model dynamics for $T = 8$ (b) Model dynamics for $T = 9$

Remarkably, we again observe sign switches in the behaviour for inflation and output for reasonably short durations of the fixed-rate regime. Apparently, the additional frictions and features introduced in the empirically relevant model leave our main conclusion unchanged. We note again that dynamics in panel (b) of Chart 6 are not an equilibrium, because they violate the zero lower bound on the nominal interest rate. Imposing the lower bound, the interest rate would have to remain fixed until period $T = 13$. Nevertheless, in that equilibrium, annualised inflation falls by almost 20% on impact. So the qualitative features of a reversal of initial inflation dynamics still remain when taking the zero lower bound into account.

5 Does the linear approximation drive our results?

In the previous sections, we have treated the linear equilibrium conditions as the model economy. This section studies whether our key findings carry over to the nonlinear model that these equations are derived from.

We outline this nonlinear model only briefly, as it has been extensively described in the literature. The economy is populated by household with separable preferences in consumption and labour. The risk aversion parameter is σ and the inverse Frish elasticity of labour supply is given by θ. We assume that the Phillips curve is obtained from a Calvo price-setting problem. Firms are allowed to update their prices with probability $1-\alpha$. Non-optimising firms index their prices to past inflation with parameter λ. The price-setting problem is described in detail by Schmitt-Grohé and Uribe (2005). We refer the reader to their paper for a full exposition of the price-setting problem and a complete derivation of the equilibrium conditions.

The price index implies

$$1 = \alpha \pi_t^{\eta-1} \pi_{t-1}^{\lambda(1-\eta)} + (1-\alpha)\tilde{p}_t^{1-\eta} \tag{33}$$

Here, η is the elasticity in the Dixit-Stiglitz aggregator of differentiated goods and $\tilde{p}_t$ is the relative price of optimising firms. Price dispersion d_t evolves as

$$d_t = (1-\alpha)\tilde{p}_t^{-\eta} + \alpha \left(\frac{\pi_t}{\pi_{t-1}^{\lambda}}\right)^{\eta} d_{t-1} \tag{34}$$

With subsidies that make the steady state efficient, the first-order condition for price-setting can be written as $x_{1,t} = x_{2,t}$. Here, the auxiliary variables $x_{1,t}$ and $x_{2,t}$ have a recursive representation given by

$$x_{1,t} = y_t mc_t \tilde{p}_t^{-\eta-1} + \alpha\beta \left(\frac{y_{t+1}}{y_t}\right)^{-\sigma} \left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta-1} \left(\frac{\pi_t^{\lambda}}{\pi_{t+1}}\right)^{-\eta} x_{1,t+1} \tag{35}$$

$$x_{2,t} = y_t \tilde{p}_t^{-\eta} + \alpha\beta \left(\frac{y_{t+1}}{y_t}\right)^{-\sigma} \left(\frac{\pi_t^{\lambda}}{\pi_{t+1}}\right)^{1-\eta} \left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta} x_{2,t+1} \tag{36}$$

We assume that the production function is linear in labour such that marginal cost is given by $mc_t = c_t^{\theta+\sigma} d_t^{\theta}$. The Bond Euler equation is

$$y_t^{-\sigma} = \beta y_{t+1}^{-\sigma} \frac{i_t}{\pi_{t+1}} \tag{37}$$

The interest rate rule is given by

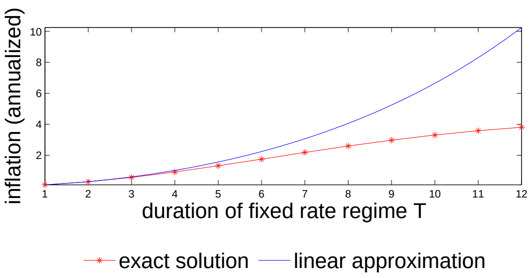
$$i_t = \begin{cases} i^* & t = 1,2,...,T \\ \frac{1}{\beta}\pi_t^{\phi_\pi(1-\rho)} y_t^{\phi_y(1-\rho)} (i_{t-1}\beta)^{1-\rho} & t = T+1,T+2, ... \end{cases} \quad \textbf{(38)}$$

When linearised around a steady state with $\pi = y = d = p = 1$, the Phillips curve collapses to that given in **(14)**.

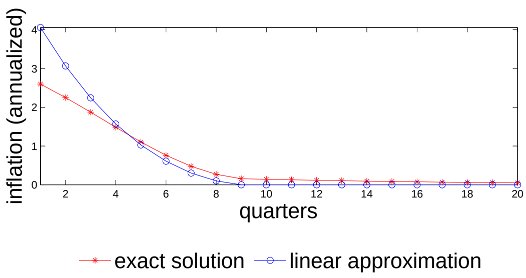
We compute the solution via stacking the set of nonlinear equilibrium conditions for $t = 1, 2, ..., K$. As a terminal condition, we impose that the model is in the deterministic steady state in period $K + 1$. Initial conditions assume $\pi_0 = d_0 = 1$. We set $K = T + 100$, the results do not depend on the choice of K as long as K is sufficiently large. This system is then solved via a nonlinear equation solver. Note that under the assumption of perfect foresight, the solution is exact up to machine precision without the need for any approximation. Our calibration sets $\eta = 11, \sigma = \theta = 1, \beta = 0.99, \phi_\pi = 1.5, \phi_y = 0.5$. We choose α such that the linearised model has a slope of the output gap version of the Phillips curve as in our previous analysis where $\kappa = 0.025$.

We first consider the nonlinear analog to the no-indexation model in Section 2, we thus set $\lambda = 0$ and $\rho = 0$. Panel (a) in Chart 7 plots initial inflation against the duration of the fixed-rate regime with $i^* = 1$ in the nonlinear model and equivalently with $i^* = -0.01$ in the linearised model.

Chart 7: Exact versus linear solution: no inflation inertia



(a) Initial inflation as a function of T

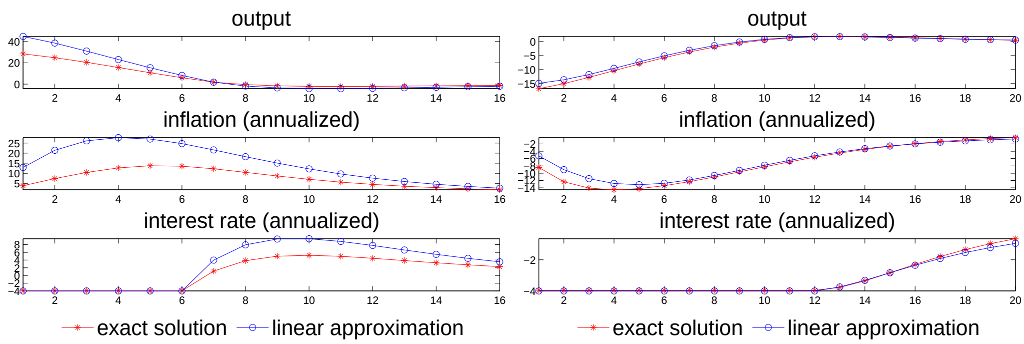


(b) Time path of inflation for $T = 8$

The initial inflation behaviour in the nonlinear model is tracked reasonably well by the solution from the linearised model for durations of up to about six quarters. For larger durations the approximation becomes increasingly bad and it grossly overstates the rise in initial inflation for large durations. Panel (b) in Chart 7 compares solution methods along a time path for inflation for a given $T = 8$. Since the nonlinear model has an endogenous state variable, namely price dispersion, the effect of the low interest rate on inflation is propagated somewhat into the period with an active Taylor rule.

Turning to the model with inflation inertia, we find the same qualitative behaviour for inflation in the nonlinear model as in the linearised model. In particular, initial inflation and output rise for certain values of the duration of the fixed-rate regime, but then there are sign reversals where inflation and output fall as the duration becomes larger. Chart 8 considers the model with indexation, setting $\lambda = 1$ and $\rho = 0.8$.

Chart 8: Exact versus linear solution: inflation inertia model



(a) Dynamics for $T = 6$ (b) Dynamics for $T = 12$

Panel (a) in Chart 8 assumes a duration of the fixed interest rate regime of six quarters. Inflation and output rise for both solutions techniques, but the magnitude is much bigger for the linear approximation. To illustrate that reversals are not an artifact of the linear approximation, we also show a fixed-rate regime of duration $T = 12$ quarters. We have chosen this value for T, because the nominal rate after the exit does not violate the lower bound at this value. Panel (b) in Chart 8 shows that sign reversals occur for both solution techniques. For $T = 12$, the linear approximation performs reasonably well quantitatively.

We note that our nonlinear analysis leaves interesting questions untouched. The nonlinear model has parameters that drop out of the linearised equations, such as η. Similarly, the Rotemberg model of quadratic price adjustment is isomorphic to the Calvo model in its linearised form, but the two differ in their nonlinear aspects. Exploring these issues is beyond the scope of this paper. We conclude that while the linear approximation technique sometimes significantly distorts the quantitative results, our key qualitative result is unchanged once we allow for a full nonlinear model. There exists at least one reversal of the initial inflation and output response as the duration of the extended period is increased.

6 Conclusion

This paper documents the 'unreasonable'behaviour of inflation and output that can arise in sticky price models under a transient interest rate peg. We acknowledge that most of our analysis considers only local linear approximations to the nonlinear model. Furthermore, the variables move somewhat far away from their expansion points. But allowing for a nonlinear model and solution does not change our results qualitatively.

This is not an econometric test of sticky price models. Nor is it a statement about other possible shocks hitting the system. It is instead a question of *prima facie* plausibility. Our results suggest that these models can produce implausible behaviour under transient interest rate pegs. Therefore, these models should be used with great caution in this environment. Furthermore, competing models of price-setting and expectations formation should be studied in this environment.

A standard rejoinder to this conclusion is that the DNK model with time-dependent pricing would break down into state-dependent pricing along path with large movements in inflation and output. Consequently, it seems worthwhile to study the inflation behaviour in such pricing models under a transient interest rate peg.

Appendix A: Proofs

Proposition 1 The inflation rate during the extended period is given by

$$\pi_{T+1-s} \equiv z_s = i^* + m_1 e_1^s + m_2 e_2^s \quad \text{for } s = 0, 1, ..., T$$

for $s = 1, 2, ..., T$, where $e_1 > 1$, $e_2 < 1$, and m_1 and m_2 come from the two restrictions **(8)**-**(9)**.

Proof.

The difference equation is given by

$$z_s = -\frac{\kappa}{\sigma} i^* + \left(\frac{\kappa}{\sigma} + 1 + \beta\right) z_{s-1} - \beta z_{s-2} \equiv c + \gamma_1 z_{s-1} + \gamma_2 z_{s-2}$$

with the two initial conditions:

$$z_1 = c$$

$$z_2 = c + \gamma_1 c$$

The particular solution is given by: $m = c + \gamma_1 m + \gamma_2 m$. Solving this we have $m = i^*$. The characteristic equation of the homogeneous system is given by

$$q^2 - \gamma_1 q - \gamma_2 = 0$$

or

$$h(q) \equiv q^2 - \left(\frac{\kappa}{\sigma} + 1 + \beta\right) q + \beta$$

Since h is convex, $h(0) > 0$, and $h(1) < 0$, the difference equation has two real eigenvalues denoted by $e_1 > 1$ and $e_2 < 1$. The general solution is thus given by $z_s = m + m_1 e_1^s + m_2 e_2^s$.

Proposition 2 The system in **(25)** has one eigenvalue e_1 inside the unit circle and two eigenvalues (e_2 and e_3) outside the unit circle. The inflation rate during the extended period is then given by

1. If e_2 and e_3 are real:

$$\pi_{T+1-s} \equiv z_s = i^* + m_1 e_1^s + m_2 e_2^s + m_3 e_3^s$$

2. If e_2 and e_3 are complex, $(e_2, e_3) = a \pm bi$:

$$\pi_{T+1-s} \equiv z_s = i^* + m_1 e_1^s + m_2 r^s \cos(\theta s) + m_3 r^s \sin(\theta s)$$

where $r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1}(\frac{b}{a})$, and m_1, m_2, and m_3, come from the three restrictions **(26)**-**(28)**. These constants do depend upon T because of the condition **(28)**.

Proof. The difference equation **(25)** is given by

$$z_s = c + \gamma_0 z_{s+1} + \gamma_1 z_{s-1} + \gamma_2 z_{s-2}$$

The particular solution to this system is given by the m that satisfies

$$m = c + \gamma_0 m + \gamma_1 m + \gamma_2 m$$

or

$$m = \frac{c}{1 - \gamma_0 - \gamma_1 - \gamma_2} = i^*$$

The characteristic equation of the homogeneous system is

$$g(q) \equiv \frac{\lambda}{[1 + \lambda (1 + \beta)]} q^3 - q^2 + \left[\frac{\sigma (1 + \beta\lambda + \beta) + \kappa}{\sigma [1 + \lambda (1 + \beta)]} \right] q - \frac{\beta}{[1 + \lambda (1 + \beta)]}$$

We can express this more conveniently as

$$g(q) \equiv \lambda q^3 - [1 + \lambda (1 + \beta)] q^2 + \left[(1 + \beta\lambda + \beta) + \frac{\kappa}{\sigma} \right] q - \beta$$

The product of the three roots is $\frac{\beta}{\lambda} > \beta$. We also have $g(0) < 0$, $g(\beta) > 0$, so there is a real root in $(0, \beta)$. But since the product of the roots exceeds β, the other two roots must be outside the unit circle. The solution to the difference equation is then given by the sum of the particular solution and the homogeneous solution.

Appendix B: Stacked time approach

A complementary look at the issue of reversal can be obtained via a stacked time solution. During the fixed interest rate period, the dynamic system can be expressed in terms of leads, lags and contemporaneous values of the vector $Z_t \equiv [\pi_t, y_t]'$ as below

$$BZ_{t+1} + CZ_t + DZ_{t-1} = V$$

Here, B, C and D are coefficient matrices of appropriate dimension and V is a vector of constants. After the fixed interest rate period ($t > T$) and under perfect foresight, the solution is of the form $Z_t = AZ_{t-1}$. This decision rule is then used to substitute out expectations of future values in the ultimate period of the fixed interest rate regime.

We can define a stacked vector $Z \equiv [Z_1', Z_2', ..., Z_{T-1}', Z_T']'$ and express the entire system of equilibrium conditions during the T periods in a stacked system of equations

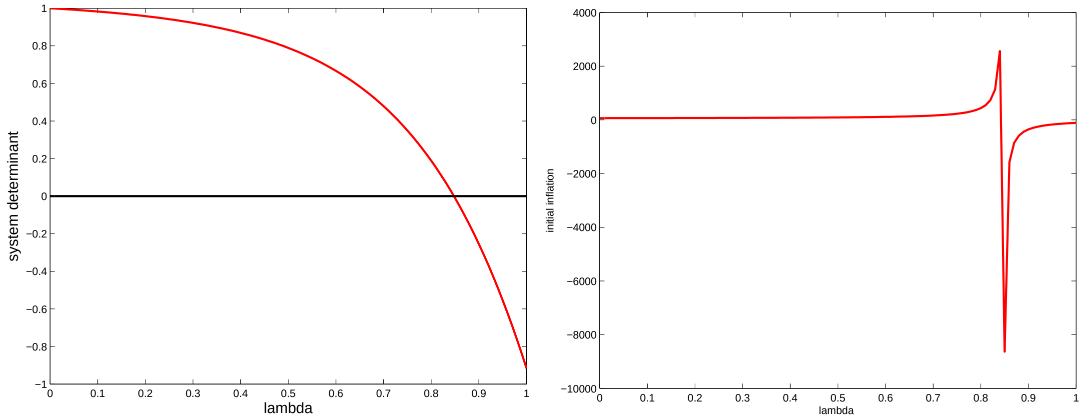
$$JZ = M$$

Here, the matrix J is given by

$$J = \begin{bmatrix} C & B & 0 & \cdots & 0 & 0 & 0 \\ D & C & B & \cdots & 0 & 0 & 0 \\ 0 & D & C & B & \cdots & 0 & 0 \\ \vdots & & & \cdots & & & \vdots \\ 0 & 0 & 0 & \cdots & D & C & B \\ 0 & 0 & 0 & \cdots & 0 & D & C + BA \end{bmatrix}$$

and the matrix M collects constants. The solution can then be obtained via simple matrix inversion, ie $Z = J^{-1}M$. Clearly, the system determinant, $\det(J)$, is crucial for the solution of the endogenous variables. Chart 9 below plots the determinant of this system and the initial inflation response when the interest rate is fixed for eight quarters as a function of the inertia in price-setting λ.

Chart 9: Sensitivity with respect to λ for given $T = 8$



(a) System determinant as a function of λ

(b) Initial inflation as a function of λ

Clearly, as λ exceeds a critical value in the neighbourhood of the value 0.85, the determinant approaches zero and switches sign. This implies that the dynamics of the model also asymptotes and switches sign at that value. These results are reminiscent of the bifurcation analysis in the New Keynesian model analysed by Barnett and Duzhak (2010).

Appendix C: Stochastic exit

This section considers a stochastic exit from the fixed-rate regime. This is motivated by the influential papers of Eggertsson and Woodford (2003) and Christiano *et al* (2011) that assume Markov process for the shocks that drive the economy to the zero lower bound. We assume that as long as no exit has occurred previously, there is a probability p that the interest rate remains pegged, and probability $(1-p)$ that an exit to an active interest rate regime occurs. Exit is an absorbing state.

In this model with no endogenous states, when the exit occurs $\pi_t = y_t = 0$. Hence, the equilibrium is given by two numbers: the levels of inflation and gap during the extended period. The model is given by equations **(1)** and **(2)** from Section 2.

The solution for inflation is given by:

$$\pi_t = -\frac{\frac{\kappa}{\sigma} i^*}{(1-p)(1-\beta p) - p\frac{\kappa}{\sigma}} \tag{C-1}$$

The mean duration of the peg is given by $T = 1/(1-p)$, or $p = (T-1)/T$, so that we can write **(C-1)** as

$$\pi_t = -\frac{\frac{\kappa}{\sigma} i^* T^2}{\beta + T(1-\beta) + T(T-1)\frac{\kappa}{\sigma}} \tag{C-2}$$

A simple case is to set $\beta = 1$ so that we have

$$\pi_t = \frac{-\frac{\kappa}{\sigma} i^* T^2}{1 - T(T-1)\frac{\kappa}{\sigma}} \tag{C-3}$$

The normal case is where the denominator is positive, or

$$T_{crit} = \left(1 + \sqrt{1 + 4\frac{\sigma}{\kappa}}\right)/2 \tag{C-4}$$

There is an asymptote and then sign switch at T_{crit}. If we use $\kappa = .025$, and $\sigma = 1$, we have that $T_{crit} = 6.84$. Thus, asymptotes and sign switches are possible even without endogenous states in the simple DNK model as long as the exit is stochastic.

But with a stochastic exit, there is also the possibility of equilibrium indeterminacy whenever the expected duration of the fixed-rate regime is too large. It is thus of interest to see whether the sign switch occurs in the determinacy region of the parameter space. Combining the Phillips curve with the IS curve, we have:

$$\beta p^2 \pi_{t+2} - \left[\frac{\kappa}{\sigma} + (1+\beta)\right] p\pi_{t+1} + \pi_t + \frac{\kappa}{\sigma} i^* = 0 \qquad \textbf{(C-5)}$$

To compute the determinacy region, we postulate a candidate equilibrium with unknown coefficients a,b, and c:

$$\pi_t = ca^t + b \qquad \textbf{(C-6)}$$

Put into **(C-5)**:

$$\beta p^2 (ca^{t+2} + b) - \left[\frac{\kappa}{\sigma} + (1+\beta)\right] p \left(ca^{t+1} + b\right) + ca^t + b + \frac{\kappa}{\sigma} i^* = 0 \qquad \textbf{(C-7)}$$

The value of b is the particular solution and is given by the solution to:

$$\beta p^2 b - \left[\frac{\kappa}{\sigma} + (1+\beta)\right] pb + b + \frac{\kappa}{\sigma} i^* = 0 \qquad \textbf{(C-8)}$$

But this is just the solution given by (3).

$$b = \frac{\frac{\kappa}{\sigma} i^*}{p\frac{\kappa}{\sigma} - (1-p)(1-\beta p)} \qquad \textbf{(C-9)}$$

The value of a is given by the root of the characteristic equation:

$$h(a) \equiv \beta p^2 a^2 - \left[\frac{\kappa}{\sigma} + (1+\beta)\right] pa + 1 \qquad \textbf{(C-10)}$$

This equilibrium is stationary if a is in the unit circle. Note that h is quadratic, convex, $h(0) > 1$, $h'(0) < 0$. The key issue then is the value of $h(1)$. The value of a is in the unit circle if $h(1) < 0$.

$$h(1) = \beta p^2 - \left[\frac{\kappa}{\sigma} + (1+\beta)\right] p + 1 < 0 \qquad \textbf{(C-11)}$$

But this is equivalent to the issue of whether there are reversals. That is, if $h(1) < 0$, then $a < 1$ and there are equilibrium of the form **(C-6)**. But c is free, there are an infinite number of equilibria. Hence, the boundary of the determinacy region is also the boundary of the region for which reversals occur. Thus for parameters in the determinacy region, the stochastic exit model agrees with the deterministic exit model that reversals are not possible without endogenous states.

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