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Simple banking: profitability and the yield curve

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Summary

An alternative model for bank net interest margins (NIMs), adapted from Wong (1997), is presented. In contrast to the model in the text, Wong's model incorporates interest rate risk and credit risk, together with a risk averse bank. Relative to Wong (1997), we extend the model to include liquid assets.

Appendix

A model of the NIM from Wong (1997)

Here we set out an alternative theoretical framework within which to understand the impact of interest rates on the net interest margin. In contrast to the model in the text, this framework is static, but incorporates a richer description of the bank's problem. We extend Wong's (1997) model of a commercial bank subject to interest rate risk and credit risk to include government bonds, and study the determinants of such a bank's net interest margin. The model is a simple static one in which risk-averse banks choose loan rates to maximise the expected utility value of profits. The bank has the following balance sheet, comprised of real economy loans (L), government bonds (G), interbank borrowing (B), capital (K), and deposits (D), which must satisfy:

$$L + G = D + B + K. \qquad \textbf{(A-1)}$$

The bank's balance sheet is subject to two forms of binding constraint.[1] The first is a constraint on leverage, which limits the stock of debt relative to capital, and takes the form

$$D = \alpha K. \qquad \textbf{(A-2)}$$

Second, the bank is subject to a liquidity constraint, which requires that the bank holds a fraction γ of its debt in liquid assets

$$G = \gamma (B + D). \qquad \textbf{(A-3)}$$

Banks face downward-sloping loan demand of $L(R_L)$, where R_L is the loan rate charged. $L(\cdot)$ has the properties that $L' < 0$, and $L'' \leq 0$, where primes denote partial derivatives. Loans are subject to credit risk. When a bank makes $L(\cdot)$ loans, it receives a fraction $(1 - \widetilde{\theta})$ of the return R_L, where $\widetilde{\theta}$ is a random variable with distribution function $F_\theta(\cdot)$ and support [0, 1], $F_\theta : [0, 1] \to [0, 1]$. (Throughout, a tilde indicates a random variable.) Government bonds return a sure return of R_G, which we refer to as the 'long rate'. Deposits are remunerated at $R_f < R_G$, where we refer to R_f as the risk-free rate. That $R_f < R_G$ implies the presence of a term premium

[1] For simplicity we assume the constraints bind eg due to regulatory requirements in our analysis. But it is possible to cast the constraints as weak inequalities that bind in equilibrium. As will become clear, if $E[R(\widetilde{\theta})] > 0$, then $E[\widetilde{R}] > R_f$, so the leverage constraint binds whenever there are profitable opportunities. $R_G > R_f$ guarantees this. If $(1 - \overline{\theta})R_L^*$ is sufficiently large relative to R_G, where R_L^* is the equilibrium loan rate and $E[\widetilde{\theta}] = \overline{\theta}$, then the liquidity constraint will bind too.

$\phi \equiv R_G - R_f$, which parametrises the slope of the yield curve. When $B > 0$, the bank is an interbank borrower. The interbank interest rate is subject to random shocks drawn from two sources. In particular, the interbank rate $\widetilde{R}$ is:

$$\widetilde{R} = R_f + R(\widetilde{\theta}) + \widetilde{\varepsilon}, \quad R'(\theta) > 0. \tag{A-4}$$

As equation **(A-4)** makes clear, interbank borrowing is subject to liquidity shocks $\widetilde{\varepsilon}$, with $E[\widetilde{\varepsilon}] = 0$, and is more expensive when credit risk is more severe, by $R'(\theta) > 0$. Liquidity shocks are drawn from distribution function $F_\varepsilon(.)$ with support $[\varepsilon_1, \varepsilon_2]$, so $F_\varepsilon(\cdot) : [\varepsilon_1, \varepsilon_2] \rightarrow [0, 1]$. The presence of both of these shocks in equation **(A-4)** captures the ideas that (a) random disturbances to market conditions can generate random variation in interbank liquidity and that (b) banks, being relatively well informed about each others' activities (relative to depositors), will charge a premium in lending to competitors that is increasing in credit risk. The leverage constraint and the liquidity constraints are used to pin down balance sheet quantities, leaving the mix between loans and interbank borrowing endogenous. Profits are

$$\widetilde{\pi} = (1 - \widetilde{\theta}) R_L L(R_L) + R_G G - \widetilde{R} B - R_f D.$$

Using the balance sheet constraint, write the amount of interbank borrowing as

$$B = \frac{L(R_L) - \left[1 + \alpha(1-\gamma)\right] K}{1-\gamma}, \tag{A-5}$$

which makes clear the dependence of B on the bank's chosen loan rate. Similarly, bond holdings satisfy

$$G = \frac{\gamma}{1-\gamma}\left[L(R_L) - K\right], \tag{A-6}$$

which, through the liquidity constraint, varies with interbank borrowing and hence also with the bank's optimal loan rate. Using these expressions, profits can be written[2]

$$\widetilde{\pi} = \left[(1-\widetilde{\theta}) R_L + \frac{\gamma R_G - \widetilde{R}}{1-\gamma}\right] L(R_L) + \left[\frac{\{1 + \alpha(1-\gamma)\}\,\widetilde{R} - \gamma R_G}{1-\gamma} - \alpha R_f\right] K. \tag{A-7}$$

The bank is risk averse, such that it seeks to maximise the expected utility of profits, where $U(\cdot)$ is the bank's utility function, which satisfies DARA in Ross' (1981) sense,[3] and $U'(\cdot) > 0, U''(\cdot) < 0$. Intuitively, this introduces an 'insurance' motive to the bank's choice of

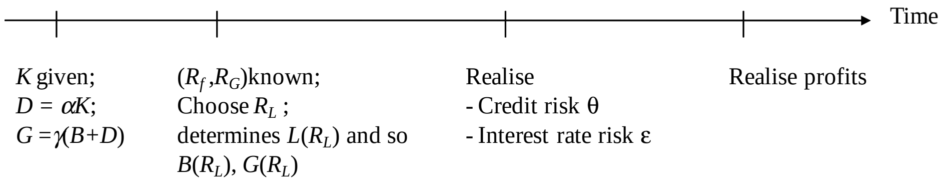
[2]Note that when $\gamma = 0$ (ie no liquidity constraint), then

$$\widetilde{\pi} = \left[(1-\widetilde{\theta}) R_L - \widetilde{R}\right] L(R_L) + \left[(1+\alpha)\,\widetilde{R} - \alpha R_f\right] K$$

which is closer to Wong's model. A further modification to the assumptions on interest rates, namely, letting R be known and R_f subject to interest rate and credit risk, yields Wong's model.

[3]This implies there exists some $\lambda > 0$ such that $-U'''(\pi)/U''(\pi) \geq \lambda \geq -U''(\pi)/U'(\pi)$, for all π.

Chart 1: Model timeline



loan rate to the extent that profits can be reduced by bad realisations of credit risk or interest rate risk *ex post*.

Chart 1 illustrates the timing of the bank's problem. Stated formally, the problem is to solve

$$\max_{R_L} E[U(\widetilde{\pi})] = \int_0^1 \int_{\varepsilon_1}^{\varepsilon_2} U[\pi(R_L;\widetilde{\theta},\widetilde{\varepsilon})]dF_\theta(\theta)dF_\varepsilon(\varepsilon), \qquad \textbf{(A-8)}$$

ie to choose its loan rate, knowing R_f and R_G, before the realisation of credit risk and interest rate risk.

The optimal interest margin

The bank's first-order condition is

$$E\left[U'\frac{\partial\widetilde{\pi}}{\partial R_L}\right] = 0, \qquad \textbf{(A-9)}$$

which defines the optimal loan rate R_L^*, conditional on exogenous variables, and hence the optimal spread $R_L^* - R_f$ over the risk-free rate.[4] Use that

$$\frac{\partial\widetilde{\pi}}{\partial R_L} = \widetilde{M}L'(R_L), \qquad \textbf{(A-10)}$$

where

$$\widetilde{M} \equiv (1-\widetilde{\theta})\left(1-\frac{1}{\eta_L}\right)R_L + \frac{\gamma}{1-\gamma}R_G - \frac{1}{1-\gamma}\widetilde{R},$$

in which $\eta_L \equiv -L'(R_L)R_L/L(R_L)$ is the interest elasticity of demand for loans, to write the optimality condition as

$$H(R_L^*;\varphi) \equiv E\left[U'\widetilde{M}\right]L'(R_L^*) = 0, \qquad \textbf{(A-11)}$$

where φ is a vector of exogenous variables, such that equation **(A-11)** summarises $R_L^*(\varphi)$ and hence determines equilibrium lending $L(R_L^*)$. This in turn pins down the optimal amount of

[4]The second-order condition is negative for $L'' \leq 0$.

interbank borrowing by equation **(A-5)** and the optimal holding of bonds according to equation **(A-6)**. Since more lending must be financed by more borrowing (higher B), more loans also require higher stocks of liquid assets (higher G). By use of the constraints, the choice of R_L^* pins down all balance sheet quantities.

Since we are interested in the transmission of exogenous interest rate changes onto margins, it is useful to consider pricing under risk-neutrality as a benchmark. In this case U is linear, so the first-order condition implies the risk-neutral (superscript n) loan rate R_L^n satisfies

$$(1-\overline{\theta})\left(1-\frac{1}{\eta_L^n}\right)R_L^n+\frac{\gamma}{1-\gamma}R_G-\frac{1}{1-\gamma}\overline{R}=0, \qquad \textbf{(A-12)}$$

where $\overline{R}=E[\widetilde{R}]$. Intuitively, the bank sets a loan rate such that the expected marginal profit of loans equals the expected interbank rate when it is risk neutral.[5] Under risk aversion, however, we have $M(\overline{\theta},\overline{\varepsilon})>0$, or

$$(1-\overline{\theta})\left(1-\frac{1}{\eta_L^*}\right)R_L^*+\frac{\gamma}{1-\gamma}R_G-\frac{1}{1-\gamma}\overline{R}>0,$$

where R_L^* is the optimal loan rate under risk aversion. Hence, from equation **(A-12)**, it must be that $R_L^*>R_L^n$; the bank charges a positive 'risk premium' for loans when it is risk-averse, in addition to a mark-up over costs. This means that for a risk-averse bank, exogenous changes in interest rates (or leverage and liquidity constraints) have both income effects and substitution effects. Substitution effects will arise from changes in the profitability of different activities. Income effects will operate through risk aversion, changing the bank's risk-bearing capacity and therefore its willingness to hold exposure to interest rate risk and credit risk.

Changes in the 'risk-free' rate, R_f

Using the first-order condition for the loan rate, we show that

Proposition 1 When equilibrium lending satisfies $L(R_L^*)>K$, it is the case that

$$\frac{dR_L^*}{dR_f}>0. \qquad \textbf{(A-13)}$$

Proof. See below.

[5]For example, when $\gamma=0$,

$$(1-\overline{\theta})\left(1-\frac{1}{\eta_L^n}\right)R_L^n=\overline{R}.$$

A rise in the risk-free rate leads the bank to raise its loan rate on two counts. First, the loan rate rises to preserve the bank's margin over debt costs (a substitution effect). Second, the bank contracts its lending, as its costs have risen and so its risk-bearing capacity has fallen (an income effect). It does this by raising its loan rate and shrinking its interbank borrowing, reducing its exposure to both interest rate risk and credit risk. The reduction in the bank's holding of risky debt means it can reduce its holding of liquid assets. Hence as R_f rises, the bank's bond holdings fall too. This contraction of the balance sheet means that, even though profits fall, the net interest margin can rise. In particular, the NIM on which we focus in the data is

$$\frac{\overline{\pi}}{L(R_L)+G(R_L)} = \left[(1-\overline{\theta})R_L + \frac{\gamma R_G - \overline{R}}{1-\gamma}\right]\frac{1}{1+\frac{\gamma}{1-\gamma}\left(1-\frac{K}{L(R_L)}\right)} + \left[\frac{\{1+\alpha(1-\gamma)\}\overline{R}-\gamma R_G}{1-\gamma} - \alpha R_f\right]\frac{K}{L(R_L)+G(R_L)} \qquad \textbf{(A-14)}$$

We can decompose the total effect of a rise in R_f on the NIM into three effects:

1. First, the loan rate increase mitigates the compression in the bank's spread, by $\frac{dR_L^*}{dR_f} > 0$. If $(1-\overline{\theta})\frac{dR_L^*}{dR_f} - \frac{1}{1-\gamma} > 0$, or
$$\frac{dR_L^*}{dR_f} > \frac{1}{1-\gamma}\frac{1}{1-\overline{\theta}} \qquad \textbf{(A-15)}$$
then the spread actually rises;
2. Second, the rise in R_f increases the value of the bank's interest-free 'capital shield'. The sign of this effect depends on the sign of $\left[\frac{1+\alpha(1-\gamma)}{1-\gamma} - \alpha\right]\frac{K}{L(R_L)+G(R_L)}$, which is always positive: the rise in the risk-free rate increases the marginal value of the interest-free liability, capital. The lower is the bank's leverage, the larger is this effect;
3. Finally, the rise in R_L^* contracts the bank's asset base, since $L'(R_L^*) + G'(R_L^*) < 0$. Since profits are deflated by a smaller asset base, the NIM tends to rise.

So for a rise in R_f to raise the NIM, condition **(A-15)** is sufficient. This in itself is more likely to be satisfied when $L'(R_L^*)$ is large, since then both the balance sheet contraction is large and the loan rate rises more strongly in response to increased funding costs.

It is straightforward to derive an analogous result for changes in R_f controlling for yield curve slope, ie holding $\phi \equiv R_G - R_f$ constant. Strictly interpreted, this is the relevant comparative static with respect to the empirical results we present below, though the intuition presented for Proposition 1 is analogous to the case in which the slope is held constant, so we do not discuss it further here.

Thus, in sum, we would expect a positive effect of risk-free rates on equilibrium loan rates, and if loan demand is sufficiently elastic, on the NIM.

Changes in the yield curve slope, R_G

We turn next to changes in long rates. An increase in R_G, for a given risk-free rate, implies a rise in the slope of the yield curve. As above, we can show:

Proposition 2 When equilibrium lending satisfies $L(R_L^*) > K$, it is the case that

$$\frac{dR_L^*}{dR_G} < 0. \tag{A-16}$$

Proof. See below.

When the long rate rises, the bank's profitability rises, increasing its risk-bearing capacity. Hence, in contrast to the case of a rise in R_f, a rise in R_G means the bank is willing to take a larger exposure to risk. It does so by lowering its loan rate and expanding its loan book. It also holds more bonds, and funds them through an increased reliance on interbank borrowing. Note that even though the loan rate falls, the bank will still charge a positive spread over R_G for its loans as long as, *inter alia*, expected credit risk is sufficiently severe.

The effect of an increase in R_G is to raise expected profits. As for the rise in risk-free rates however, the impact of a change in slope on the net interest margin (equation **(A-14)**) can be understood through three effects:

1. First, the spread of loans over debt falls as banks expand their loan books by moving down the loan demand curve, which works against the direct positive profit effect of the rise in R_G;

2. Second, the rise in R_G reduces the value of interest-free capital by $-\frac{\gamma}{1-\gamma}\frac{K}{L(R_L)+G(R_L)}$; since the bank can now transform debt into higher-yielding bonds, the interest rate 'shield' provided by capital is less valuable at the margin;
3. Third, the expansion of the balance sheet means profits are deflated by a larger total stock of assets. This tends to mitigate the increase in the NIM due to higher profits.

The sum of the three effects is ambiguous. The smaller is the reduction in the loan rate, the smaller are the first and third effects. So empirically, when the loan rate responds only minimally to changes in R_G, such that the pass-through from the long end of the yield curve to end borrowers is small, a steeper yield curve unambiguously raises the NIM, since the direct effect of R_G on NIM is simply

$$\frac{\gamma}{1-\gamma}\frac{L(R_L^*)-K}{L(R_L^*)+G(R_L^*)} > 0.$$

The higher the equilibrium leverage ratio of the bank, the larger is this effect. Hence the NIM can rise when long rates rise for highly levered banks that do not adjust their loan rates by very much in response to changes in R_G. This contrasts with the case of changes in short risk-free rates, which must bring forth much larger pass-through to borrowers if the return on assets is to rise with R_f. We can relate the response of the loan rate to both interest rates by $dR_L^*/R_f = -(1/\gamma)dR_L^*/dR_G$. When $\gamma < 1$, it is clear that the optimal loan rate responds by more to changes in the risk-free rate R_f than to changes in the long rate R_G. The effect is larger when γ is small. Intuitively, when this is the case, a large proportion of the bank's assets is subject to credit risk. By risk aversion, this makes the bank more sensitive to changes in funding costs. By contrast, when γ is large and income is insulated from credit risk by large bond holdings, the loan rate need not respond so elastically.[6]

Second-order condition for R_L^*

The second-order condition is

$$\frac{\partial^2 E[U(\widetilde{\pi})]}{\partial R_L^2} = E[U''\widetilde{M}^2]\left(L'\right)^2 + E[U'\frac{\partial \widetilde{M}}{\partial R_L}]L' + E[U'\widetilde{M}]L''$$

where

$$\frac{\partial \widetilde{M}}{\partial R_L} = (1-\widetilde{\theta})\frac{1}{\eta_L}\left[\eta_L - 1 + \frac{\eta_L'}{\eta_L}R_L\right]$$

[6] Note that, by $E[U''\widetilde{M}] \geq 0$, an increase in the curvature of the utility function around $E[U'(\widetilde{\pi})]$ causes the values of dR_L^*/dR_f and dR_L^*/dR_G to diverge, in support of this 'risk-bearing capacity' interpretation.

where $\eta_L = -L' R_L / L$, so

$$\begin{aligned} \eta_L' &= -\frac{L'' R_L + L'}{L} + \frac{(L')^2 R_L}{L^2} \\ \frac{\eta_L'}{\eta_L} &= -\left(-L'' R_L - L' + \frac{(L')^2 R_L}{L} \right) \frac{1}{L' R_L} \end{aligned}$$

so

$$\frac{\partial \widetilde{M}}{\partial R_L} = (1-\widetilde{\theta}) \frac{1}{\eta_L} \frac{1}{L'} \left[2\eta_L L' + L'' R_L \right]$$

Then the SOC is

$$\frac{\partial^2 E[U(\widetilde{\pi})]}{\partial R_L^2} = E[U'' \widetilde{M}^2] \left(L'\right)^2 + E[U'(1-\widetilde{\theta})] \frac{2\eta_L L' + L'' R_L}{\eta_L} + E[U' \widetilde{M}] L''$$

which is negative for $L'' \leq 0$.

Risk-neutral versus risk-averse

Under risk aversion, following Wong (1997), we have

$$E_\varepsilon[U'(\widetilde{\pi}^*)|\theta] > (<) E_\varepsilon[U'(\widetilde{\pi}^*)|\overline{\theta}] \quad \text{for} \quad \theta > (<) \overline{\theta}, \qquad \textbf{(A-17)}$$

where E_ε denotes the expectation over ε (ie conditional on $\widetilde{\theta} = \theta$). Since $\partial M / \partial \theta < 0$, multiplying both sides of equation **(A-17)** by

$$M(\theta, \varepsilon) - M(\overline{\theta}, \overline{\varepsilon}) > (<) 0 \quad \text{for } \theta < (>) \overline{\theta} \text{ and } \varepsilon \leq (\geq) \overline{\varepsilon}$$

and, since when $\theta < \overline{\theta}$

$$\begin{aligned} E_\varepsilon[U'(\widetilde{\pi}^*)|\theta] \left[M(\theta, \varepsilon) - M(\overline{\theta}, \overline{\varepsilon}) \right] &< E_\varepsilon[U'(\widetilde{\pi}^*)|\overline{\theta}] \left[M(\theta, \varepsilon) - M(\overline{\theta}, \overline{\varepsilon}) \right] \\ \Rightarrow E\left\{ U'(\widetilde{\pi}^*) \left[M(\widetilde{\theta}, \widetilde{\varepsilon}) - M(\overline{\theta}, \overline{\varepsilon}) \right] \right\} &< E[U'(\widetilde{\pi}^*)] \left[M(\overline{\theta}, \overline{\varepsilon}) - M(\overline{\theta}, \overline{\varepsilon}) \right] = 0, \end{aligned}$$

taking expectations over θ yields

$$E\left\{ U'(\widetilde{\pi}^*) \left[M(\widetilde{\theta}, \widetilde{\varepsilon}) - M(\overline{\theta}, \overline{\varepsilon}) \right] \right\} < E[U'(\widetilde{\pi}^*)] \left[M(\overline{\theta}, \overline{\varepsilon}) - M(\overline{\theta}, \overline{\varepsilon}) \right] = 0.$$

Using this together with equation **(A-11)** implies $M(\overline{\theta}, \overline{\varepsilon}) > 0$, or

$$(1-\overline{\theta}) \left(1 - \frac{1}{\eta_L^n} \right) R_L^* + \frac{\gamma}{1-\gamma} R_G - \frac{1}{1-\gamma} \overline{R} > 0.$$

Comparative statics

The sign of the impact of changes in any of the exogenous variables in $\boldsymbol{\varphi}$ on the optimal loan rate can be captured by $\partial H/\partial\varphi$, where φ is an element in $\boldsymbol{\varphi}$, since

$$\frac{dR_L^*}{d\varphi} = -\frac{1}{\Delta}\frac{\partial H(R_L;\boldsymbol{\varphi})}{\partial\varphi},$$

where $\Delta \equiv \frac{\partial H(R_L;\boldsymbol{\varphi})}{\partial R_L} = \frac{\partial^2 E[U(\widetilde{\pi})]}{\partial R_L^2} < 0$ by the second-order condition. So

$$\operatorname{sign}\left(\frac{dR_L^*}{d\varphi}\right) = \operatorname{sign}\left(\frac{\partial H(R_L;\boldsymbol{\varphi})}{\partial\varphi}\right). \qquad \textbf{(A-18)}$$

The sign of $E[U''\widetilde{M}]$

U exhibits DARA in Ross' (1981) sense iff there exists $\gamma > 0$ such that

$$-\frac{U'''(\pi)}{U''(\pi)} \geq \gamma \geq -\frac{U''(\pi)}{U'(\pi)}, \quad \forall\pi$$

Define

$$N(\theta) \equiv \frac{E_\varepsilon[U''(\widetilde{\pi})|\theta]}{E_\varepsilon[U'(\widetilde{\pi})|\theta]}$$

then

$$\frac{\partial N(\theta)}{\partial\theta} = \frac{1}{E_\varepsilon[U'(\widetilde{\pi})|\theta]}E_\varepsilon\left[U'''(\widetilde{\pi})\frac{\partial\widetilde{\pi}}{\partial\theta}|\theta\right] - \frac{E_\varepsilon[U''(\widetilde{\pi})|\theta]}{E_\varepsilon[U'(\widetilde{\pi})|\theta]^2}E_\varepsilon\left[U''(\widetilde{\pi})\frac{\partial\widetilde{\pi}}{\partial\theta}|\theta\right]$$

where

$$\begin{aligned}\frac{\partial\widetilde{\pi}}{\partial\theta} &= \left[-R_L - \frac{1}{1-\gamma}R'(\theta)\right]L + \frac{1+\alpha(1-\gamma)}{1-\gamma}R'(\theta)K \\ &= -R_L L + \left[\left[1+\alpha(1-\gamma)\right]K - L\right]\frac{1}{1-\gamma}R'(\theta) \\ &= -R_L L - BR'(\theta)\end{aligned}$$

which is negative. Then write

$$\frac{\partial N(\theta)}{\partial\theta} = \frac{1}{E_\varepsilon[U'(\widetilde{\pi})|\theta]}\frac{\partial\widetilde{\pi}}{\partial\theta}\left\{E_\varepsilon\left[U'''(\widetilde{\pi})|\theta\right] - \frac{E_\varepsilon[U''(\widetilde{\pi})|\theta]}{E_\varepsilon[U'(\widetilde{\pi})|\theta]}E_\varepsilon\left[U''(\widetilde{\pi})|\theta\right]\right\}.$$

Wong (1997) shows that the term in {} is positive by DARA in Ross' sense. Then, for $\widehat{\theta}$ defined by the level of credit risk (for given ε) at which the bank's margin is zero, $M(\widehat{\theta},\varepsilon) = 0$, we have

$$N(\theta) \geq N(\widehat{\theta}) \quad \text{if} \quad \theta \leq \widehat{\theta}$$

ie where the bank operates with a positive margin ($\theta \leq \widehat{\theta}$). Then multiply both sides by $E_\varepsilon[U'(\widetilde{\pi})|\theta]M(\theta,\varepsilon)$

$$N(\theta)E_\varepsilon[U'(\widetilde{\pi})|\theta]M(\theta,\varepsilon) \geq N(\widehat{\theta})E_\varepsilon[U'(\widetilde{\pi})|\theta]M(\theta,\varepsilon)$$

and taking expectations over θ

$$E\left[U''(\widetilde{\pi})\widetilde{M}\right] \geq N(\widehat{\theta})E[U'(\widetilde{\pi})\widetilde{M}] = 0$$

by the FOC for R_L^*. So

$$E\left[U''(\widetilde{\pi})\widetilde{M}\right] \geq 0$$

Proof of Proposition 1

Proof. Using equation **(A-18)**, it is sufficient to sign

$$\frac{\partial H}{\partial R_f} = E\left[U''\frac{\partial \widetilde{\pi}}{\partial R_f}\widetilde{M} + U'\frac{\partial \widetilde{M}}{\partial R_f}\right]L'(R_L^*)$$

where

$$\begin{aligned}
\frac{\partial \widetilde{\pi}}{\partial R_f} &= -\frac{1}{1-\gamma}L + \frac{1+\alpha(1-\gamma)}{1-\gamma}K - \alpha K = \frac{1}{1-\gamma}\left[K - L(R_L)\right] \\
\frac{\partial \widetilde{M}}{\partial R_f} &= -\frac{1}{1-\gamma}
\end{aligned}$$

so

$$\frac{\partial H}{\partial R_f} = \left\{E[U''\widetilde{M}]\left[K - L(R_L^*)\right] - E[U']\right\}\frac{1}{1-\gamma}L'(R_L^*) \qquad \textbf{(A-19)}$$

We show that $E[U''\widetilde{M}] \geq 0$ (when $U(.)$ satisfies DARA in Ross' (1981) sense – see above), such that $E[U''\widetilde{M}]\left[K - L\right] \leq 0$ when $L(R_L^*) > K$. When this is the case, it follows that

$$\frac{dR_L^*}{dR_f} > 0.$$

A rise in R_f, controlling for slope

Using $R_G = \phi + R_f$, where $\phi > 0$ captures the constant term premium, write profits as

$$\begin{aligned}
\widetilde{\pi} = &\left[(1-\widetilde{\theta})R_L + \frac{\gamma}{1-\gamma}\left(\phi + R_f\right) - \frac{1}{1-\gamma}\widetilde{R}\right]L(R_L) \\
&+ \left[\frac{1+\alpha(1-\gamma)}{1-\gamma}\widetilde{R} - \frac{\gamma}{1-\gamma}\left(\phi + R_f\right) - \alpha R_f\right]K
\end{aligned}$$

For a given slope ϕ, we have that

$$\left.\frac{\partial H}{\partial R_f}\right|_{\phi} = E\left[U''\left.\frac{\partial \widetilde{\pi}}{\partial R_f}\right|_{\phi}\widetilde{M} + U'\left.\frac{\partial \widetilde{M}}{\partial R_f}\right|_{\phi}\right]L'(R_L^*)$$

where

$$\begin{aligned}
\left.\frac{\partial\widetilde{\pi}}{\partial R_f}\right|_{\phi} &= -L(R_L) + \left[\frac{1+\alpha(1-\gamma)}{1-\gamma} - \frac{\gamma}{1-\gamma} - \alpha\right]K \\
&= -L(R_L) + K \\
\left.\frac{\partial\widetilde{M}}{\partial R_f}\right|_{\phi} &= -1
\end{aligned}$$

so

$$\left.\frac{\partial H}{\partial R_f}\right|_{\phi} = \left\{E\left[U''\widetilde{M}\right]\left[K - L(R_L^*)\right] - E[U']\right\} L'(R_L^*) \qquad \textbf{(A-20)}$$

Then, as above, when $L(R_L^*) > K$,

$$\left.\frac{dR_L^*}{dR_f}\right|_{\phi} > 0 \qquad \textbf{(A-21)}$$

Corollary 3 A rise in R_f for a given term premium $\phi = R_G - R_f$, ie for an upwards level shift in the yield curve, results in a higher loan rate when $L(R_L^*) > K$, such that

$$\left.\frac{dR_L^*}{dR_f}\right|_{\phi} > 0.$$

Proof of Proposition 2

Proof. We need to sign

$$\frac{\partial H}{\partial R_G} = E\left[U''\frac{\partial\widetilde{\pi}}{\partial R_G}\widetilde{M} + U'\frac{\partial\widetilde{M}}{\partial R_G}\right] L'(R_L^*)$$

Use

$$\frac{\partial\widetilde{\pi}}{\partial R_G} = \frac{\gamma}{1-\gamma}\left[L(R_L) - K\right]$$

which is positive for $L(R_L^*) > K$. And

$$\frac{\partial\widetilde{M}}{\partial R_G} = \frac{\gamma}{1-\gamma}$$

so

$$\frac{\partial H}{\partial R_G} = \left\{E\left[U''\widetilde{M}\right]\left[L(R_L^*) - K\right] + E[U']\right\}\frac{\gamma}{1-\gamma}L'(R_L^*) < 0 \qquad \textbf{(A-22)}$$

such that

$$\frac{dR_L^*}{dR_G} < 0$$

Even though the loan rate falls with long rates, the bank will still charge a positive spread over long rates when credit risk is sufficiently severe. To see this, use the risk-neutral margin to write

$$R_L^{n*} = \frac{\frac{1}{1-\gamma}\left(\overline{R} - \gamma R_G\right)}{(1-\overline{\theta})\left(1 - \frac{1}{\eta_L^n}\right)}$$

If the risk-neutral loan rate exceeds the long rate, then so must the risk-averse loan rate. For simplicity, suppose the interest elasticity of demand for loans is constant at η. Then the loan rate exceeds the long rate when

$$E[R(\widetilde{\theta})] > \left[(1-\gamma)(1-\overline{\theta})\left(1 - \frac{1}{\eta}\right) + \gamma\right] R_G - R_f$$

where we used that $\overline{R} = R_f + E[R(\widetilde{\theta})]$. Since the left-hand side of this expression is increasing in expected credit risk and the right-hand side is decreasing in expected credit risk, we conclude that for $\overline{\theta}$ sufficiently high, $R_L^{n*} > R_G$. If credit risk did not affect the bank's funding costs (such that $E[R(\widetilde{\theta})] = 0$), this condition would reduce to

$$R_f > \left[(1-\gamma)(1-\overline{\theta})\left(1 - \frac{1}{\eta}\right) + \gamma\right] R_G$$

which says that the term premium must not be too large or, again, that credit risk must be sufficiently big. If the term premium were large, the bank could effectively manage its interest rate risk by simply transforming its liabilities into risk-free bonds. The return on the bonds would be sufficient for the bank to 'insure' itself. When this is not the case, the bank holds loans on which it charges a positive spread over long rates.

Why less profitable banks might be more sensitive to interest rates

We argue that the optimal loan rates of banks with lower average profitability are more sensitive to interest rates than highly profitable banks. To get at this, introduce a constant marginal cost of managing loans given by $c > 0$. For simplicity, suppose interbank borrowing and holdings of bonds are zero. Abstract from interest rate risk, but retain credit risk. Let capital be fixed and let the choice of the loan rate determine balance sheet size. Then the balance sheet is simply $L(R_L) = D + K$, and profits are

$$\pi(\widetilde{\theta}) = (1-\widetilde{\theta})R_L L(R_L) - R_f L(R_L) + R_f K - cL(R_L).$$

The first-order condition for the loan rate is as before, namely

$$H \equiv E\left[U'(\widetilde{\pi}^*)\widetilde{M}^*\right] L'(R_L^*) = 0,$$

where

$$\widetilde{M} = (1-\widetilde{\theta})\left(1-\frac{1}{\eta_L}\right)R_L - R_f - c,$$

is the margin which now accounts for loan costs. As before, the comparative static with respect to the risk-free rates is

$$\frac{dR_L^*}{dR_f} = -\frac{1}{\Delta}\frac{\partial H}{\partial R_f},$$

where $\Delta = \partial^2 E\left[U'(\widetilde{\pi}^*)\widetilde{M}^*\right]L'(R_L^*)/\partial R_f^2 < 0$. Like before, we have that

$$\frac{\partial H}{\partial R_f} = E\left[U''(\widetilde{\pi}^*)\widetilde{M}^*(-D) - U'(\widetilde{\pi}^*)\right]L'(R_L^*) > 0,$$

(since we can show that $E\left[U'''\widetilde{M}^*\right] > 0$), such that

$$\frac{dR_L^*}{dR_f} > 0.$$

Now hold constant Δ, and ask: what is the effect of a rise in c on the response of the optimal loan rate to the risk-free rate? This can be seen from

$$\frac{\partial^2 H}{\partial R_f \partial c} = \left\{E\left[U'''\widetilde{M}^*\right]L + E\left[U''(\widetilde{\pi}^*)\right]\left(1+\frac{L}{D}\right)\right\}DL'(R_L^*).$$

Since $E\left[U''(\widetilde{\pi}^*)\right] < 0$ and $L'(R_L^*) < 0$, a sufficient condition for $\frac{\partial^2 H}{\partial R_f \partial c} > 0$ is that $E\left[U'''\widetilde{M}^*\right] \leq 0$ too. In the case where $U''' = 0$, (ie U'' is linear), we are left with a pure 'risk aversion' effect working through U''. Here, the comparative static is

$$\frac{\partial^2 H}{\partial R_f \partial c} = E\left[U''(\widetilde{\pi}^*)\right]\left(1+\frac{L}{D}\right)DL'(R_L^*) > 0,$$

and higher cost banks' loan rates are more sensitive to risk-free rates, for given Δ.

We can incorporate the effects through prudence (the properties of U''') too, however. Decreasing absolute prudence implies

$$p(z) = -\frac{U'''(z)}{U''(z)} \quad \text{satisfies} \quad p'(z) < 0.$$

So by decreasing absolute prudence and $\pi'(\theta) < 0$, we have that[7]

$$-\frac{U'''(\theta)}{U''(\theta)} \leq -\frac{U'''(1)}{U''(1)} \quad \text{for} \quad \theta \leq 1$$

or equivalently,

$$\frac{U'''(\theta)}{U''(\theta)} \geq \frac{U'''(1)}{U''(1)} \quad \text{for} \quad \theta \leq 1$$

Manipulating gives

$$U'''(\theta) \leq \frac{U'''(1)}{U''(1)}U''(\theta) \quad \text{for} \quad \theta \leq 1$$

[7] If $p(z)$ is decreasing in z, and $\pi(\theta)$ is decreasing in θ, then $p(\pi(\theta))$ is increasing in θ.

since $U''(\theta) < 0$. Multiplying both sides by $M^* > 0$ and taking expectations over θ gives

$$E\left[U'''(\theta)\widetilde{M}^*\right] \leq \frac{U'''(1)}{U''(1)} E\left[U''(\theta)\widetilde{M}^*\right] \quad \text{for} \quad \theta \leq 1$$

Then by $E\left[U''(\theta)\widetilde{M}^*\right] \geq 0$ (see above) and $\frac{U'''(1)}{U''(1)} < 0$, we have that $E\left[U'''(\theta)\widetilde{M}^*\right] \leq 0$. This implies that

$$\frac{\partial^2 H}{\partial R_f \partial c} > 0,$$

or that a bank's loan rate sensitivity to interest rates rises as its costs rise.

What is the intuition? As costs rise, a bank's profitability falls, *ceteris paribus*. This reduces its risk-bearing capacity. This reduction in risk-bearing capacity becomes more severe as costs rise higher and higher (or profits fall lower and lower), since U'' becomes more and more steeply negative. Changes in short rates then have a larger impact on the bank's loan rate, as it must contract its lending by more in order to curb its exposure to risk as interest rates rise.

References

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