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Abstract

An individual bank can put the whole banking system at risk if its losses in response to shocks push losses for the system as a whole above a critical threshold. We determine the contribution of banks to this systemic risk using a generalisation of the Shapley value; a concept originating in co-operative game theory. An important feature of this approach is that the order in which banks fail in response to a shock depends on the composition of the banks' asset portfolios and capital buffers. We show how these factors affect banks' contributions to systemic risk, and the extent to which these contributions depend on the level of the critical threshold.

Key words: Shapley value, systemic risk, bank regulation.

JEL classification: C71, G01, G21, G28.

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Summary

Policymakers have in the period since the crisis been discussing how to regulate banks in ways that reflect the potentially different contributions banks make to systemic risk in the financial system in the event of their failure. One aspect of how an individual bank's failure could contribute to systemic risk could be defined in terms of whether its failure is considered to be pivotal in tipping the banking system from a state of stability to a state of instability. Based on this idea, we develop an approach that can be used to calculate the marginal contributions of individual bank failures to systemic risk.

The approach is based on a measure originally introduced by the mathematician and economist Lloyd Shapley. The so-called Shapley value is a way of allocating the output produced by a group among its members in a way that reflects fairly their individual contributions. In this paper we apply the Shapley value to the situation where the group is a set of banks that fail due to shocks to the values of their assets and the good they produce is in fact something bad – in this approach the bad is the failure of a set of banks tipping the system from a state of stability to one of instability.

The framework requires two key inputs: the values of banks' exposures to different asset classes; and the levels of banks' capital available to absorb losses on their asset holdings. The banking system can be hit by a range of shocks, which are defined in terms of the extent to which they reduce the value of the different asset classes. The shocks are assumed to occur with equal probability. For each possible shock, banks can be lined up in the order that they would fail as a result of that shock. Banks with asset portfolios weighted more towards the assets affected more by the shock, and/or have lower levels of capital, tend to be higher up the order of failure. The pivotal bank is the one that, when it is added to the banks that fail before it, causes the value of the failed banks' assets to move above a critical threshold value – this is defined as a systemic event. The pivotal bank receives a score of one (and other banks receive a score of zero). By taking an average of a bank's score over the range of possible shocks we calculate a measure of a bank's contribution to systemic risk. We illustrate, using simple examples, how banks' contributions depend on their asset portfolio compositions and their capital levels as well as on the calibration of the critical threshold that defines a systemic event.

We outline several ways in which the framework could be extended to consider: different definitions of a systemic event; adjustments to the values of banks' asset exposures to reflect the riskiness of those exposures; and the possibility of interbank contagion. We conclude by identifying some possible key next steps and further extensions of the approach. A key next step will be to apply the approach to bank data so that it could be used as a risk assessment tool. Since our approach applies to circumstances in which the system is in a state of instability, it would be natural to use our approach as part of a reverse stress-testing exercise.

1 Introduction

Systemic risk in a banking system can arise when banks' losses tend to move together in response to shocks because of their common exposures to particular asset classes. A bank might be defined as contributing to systemic risk if its losses in response to shocks push losses for the system as a whole above a critical threshold. [(1)] The failure of such a bank is thus pivotal for the transition of the system from stability to instability – a situation in which a bank failure has system-wide consequences beyond the scope of the particular bank (e.g. asset fire sales, bank runs, and funding freezes). The practical problem is how to assess the contributions of bank failures to systemic risk. In some cases the failure of a single bank may be enough to trigger a transition to instability in the system, while in other cases it may take multiple failures. Given that banks individually and/or collectively have the ability to create a crisis the question is, how do we assign individual responsibility for the associated risk?

The problem has much in common with the question of how to allocate the total 'value' created by members of a group. In this case, however, the value generated by the group is something bad (a systemic event) rather than something good. An intuitive and computationally pragmatic way to allocate gains or losses produced by a group is the Shapley value (Shapley (1953)). In this context, where we are interested in the losses that can be generated by various combinations of bank failures, the Shapley value averages over the marginal contributions of each individual failure to the (negative) value generated by each possible subgroup of failed banks.

A special case of the Shapley value applies to situations where there are only two possible group outcomes. Shapley and Shubik (1954) developed their 'power index' as a measure of the power of a party in a coalitional bargaining game. In their context, the value of a coalition equals one if the members of the coalition control a majority of the votes, otherwise the value of the coalition is zero. A player's marginal contribution to a coalition is positive if and only if it is pivotal in turning an existing coalition from a losing one into a winning one. This paper contends that the power index is well suited to capturing contributions of bank failures to systemic risk. In this paper, a systemic event occurs if the combined share of all failed banks' assets exceeds a prescribed threshold value.

The power index delivers, for each bank, the frequency with which its failure is pivotal in causing a systemic event. But crucially, applying the power index on a standard (symmetric) basis would assume that all orderings of bank failures are equally likely. This is not true in practice. Rather, orderings depend on the compositions of banks' assets and their capital holdings. Some should even be ruled out as they are not economically possible (absent the possibility of an idiosyncratic shock to a bank like fraud). For example, for two banks with identical portfolios but different capital ratios, it should not be possible for the better capitalised bank to fail while the less well capitalised bank does not. Shapley (1977) provides us with a procedure for incorporating additional information about the probability distribution over orderings of bank failures into the computation of the power index. The resulting asymmetric indices that we calculate in this paper can be interpreted as the likelihood that the failure of each

[(1)] See Lehar (2005, Section 5.1).

particular bank will be pivotal in causing a systemic event, taking into account differences in bank balance sheets.

Asymmetric power indices for a system of banks can be calculated by first defining banks' positions in an l-dimensional 'leverage' space, where l denotes the number of distinct asset classes. We choose to define positions in terms of leverage – asset exposures relative to capital – because the ordering of bank failures in response to a particular type of shock ought to depend not only on the sizes of banks' exposures to that shock, but also on their ability to withstand it. In this paper, we consider a two-dimensional leverage space, although the model could be extended to more than two assets classes. Each bank is therefore represented as a point in a two-dimensional Euclidean space with coordinates defined by the ratio of holdings of each asset class to capital.

The ordering of bank failures is determined by random shocks to the value of assets. A particular shock is represented by a direction in the leverage space. That is, we do not consider the magnitude of shocks. Rather, all that matters is the type of shock; is it biased more towards one asset than the other? For each type of shock, there is an implied order in which banks would fail. This is based in part on the proximity of the asset position to the shock (banks with asset mixes that are similar to the shock fail first) and in part on the distance of the asset position from the origin (banks with high ratios of assets to capital fail first). These two aspects determine the overall ordering of bank failures. Given a probability distribution over shocks, we thus generate a probability distribution over different orderings of bank failures. This in turn is used to compute the power index for each bank. Taking banks' asset mixes as exogenous and fixed, we explore the impact on power indices of changes to banks' capital, asset portfolios and the critical threshold that defines whether there is a systemic event.

The power index of each bank equals its share of responsibility for the instability state as computed from an ex ante perspective, before the nature of the actual shock that causes the system to switch from a state of stability to instability is known. If one were to compute the expected costs of being in the instability state and attribute to each bank its individual responsibility for this cost, then our measure of power indices could serve as a cost-allocation rule. Many studies advocate the desirability of the Shapley value for this purpose (e.g. Littlechild and Thompson (1977)). A theoretical justification in terms of bargaining theory is provided by Roth and Verrecchia (1979), who state that cost allocation determined by the Shapley value 'is consistent with the objectives of fairness, equity, and neutrality suggested by accounting theory' (page 296).

Banks' power indices could also be used as a risk assessment tool. These indices identify which banks are most likely to be pivotal in moving the system into a state of financial instability and hence, they could be used as a reverse stress-testing tool to help identify scenarios of bank failures that could lead to systemic risk crystallising. [2]

(2) We thank Jamie McAndrews for drawing the analogy to reverse stress-testing.

The paper proceeds as follows. Section 2 reviews several existing approaches to calculating the contributions of banks' distress to systemic risk based on the Shapley value concept. Section 3 describes how the effect of bank failures on systemic risk is measured in this paper using the Shapley's asymmetric power index. We compute power indices for a simple three-bank system and then illustrate the role of balance sheet strength and diversification using extreme examples. We also illustrate the impact on power indices of introducing extra banks into the system and discuss how the power of individual banks changes as we vary the critical threshold. Section 4 presents some extensions of the approach. Section 5 concludes.

2 Literature

Since the recent financial crisis, a number of approaches have been developed for assessing the effects individual bank failures have on systemic risk. A number of these share a common basis in the Shapley value concept.(3) Tarashev, Borio and Tsatsaronis (2010) were the first to demonstrate how the Shapley value concept can be used to measure banks' contributions to systemic risk. They derive results for two measures of systemic risk that can be applied to any subset of banks in the system: value-at-risk (VaR) and expected shortfall (ES). The VaR approach defines the 'worth' of a collection of banks as the *x*th percentile of the subsystem loss distribution, while the ES approach looks at the expected value of these tail losses. Under their approach, a bank's Shapley value is equal to the average change in VaR or ES for a subsystem that results from adding this bank to a smaller subsystem. Tarashev *et al* demonstrate their approach using numerical examples. Gauthier, Lehar and Souissi (2012) compute a Shapley value measure in the same way using data from the Canadian banking system.

The effect that a bank failure has on systemic risk could depend on a bank's interconnections with other banks in the system. Drehmann and Tarashev (2011) extend the Tarashev *et al* approach to include interbank linkages. In particular, they account for exposures banks have with other banks in the system and the possibility that one bank failure can lead to another. They show how taking into account the potential for interbank contagion changes banks' Shapley values.

Staum (2010) applies a Shapley value approach to allocate the costs of financing a deposit insurance scheme among participating banks. Staum generalises the methods used in the other papers to derive so-called Aumann-Shapley values (Aumann and Shapley (1974)). By computing these Aumann-Shapley values, the authors allow for the possibility that a bank participates in a subsystem of the banking system to some degree between zero and one (zero implying no participation and one means full participation) rather than a bank either participating or not (as in Tarashev *et al* and in Drehmann and Tarashev). Liu and Staum (2010)

(3) Other papers develop approaches to assessing banks' contributions to systemic risk outside the Shapley value framework. Lehar (2005) uses stock market information to estimate the joint dynamics of banks' asset portfolios and computes the expected shortfall of each bank in case of default. Adrian and Brunnermeier (2010) and Acharya, Pedersen, Philippon and Richardson (2010) also follow a portfolio-based approach that emphasizes aspects of tail risk. The former looks at how imposing large losses on each individual bank increases the value-at-risk of the entire system while the latter computes each bank's share of the expected shortfall associated with a systemic event. Webber and Willison (2011) use a similar approach to determine the level and distribution of capital across banks in a system that delivers a level of systemic risk that a policymaker targets.

show how Aumann-Shapley values can be derived in a banking system with a network of inter-bank linkages.

The use of the Shapley value concept in this paper differs from how it is used in these other papers in two crucial respects. The first is that we focus on the notion of a pivotal bank, i.e., one that tips system losses over a prescribed threshold, rather than looking at tail risk. One way to justify the focus on the pivotal bank is to think in terms of each bank's contribution to the expected cost of a bailout. Since a bailout is triggered by the pivotal bank, only that bank adds to the expected indemnity.[4] The second is that we use bank capital and the composition of asset portfolios explicitly to restrict attention to bank failure scenarios that are economically congruent. Bank capital and portfolios play a role in some of the other papers, but they are not used to determine the order in which banks would fail or the subsystems they analyse in order to calculate Shapley values.

3 Measuring contributions to systemic risk

The set of banks is denoted by $N = \{1, \dots, n\}$ and banks' assets are divided into two classes, domestic (H_i) and foreign (F_i). [5] Each bank is represented as a point in a two-dimensional Euclidean leverage space with coordinates defined by the ratio of each type of asset to capital (C_i). That is, the position of bank $i \in N$ is a point $a_i = (h_i, f_i) \in \mathbb{R}^2_+$, where $h_i = H_i/C_i$ and $f_i = F_i/C_i$. [6] Types of asset shocks are characterised by two-dimensional, unit-length, vectors $z \in \mathbb{R}^2_+$ that span the two-dimensional leverage space $\{h_i, f_i\}$, centred on the origin. [7], [8] To determine the order of failure for a given shock and a given vector of bank positions we follow Shapley (1977) and assume that bank i fails before bank j if: [9]

$$z \cdot a_i > z \cdot a_j \qquad \textbf{(1)}$$

The order of bank failures is therefore determined by the relationship between the banks' positions in the leverage space and the type of asset shock. In particular, equation **(1)** holds if and only if $\|a_i\| \cos\theta_i > \|a_j\| \cos\theta_j$, where $\|a_k\|$ denotes the distance of a_k from the origin and θ_k denotes the angle between a_k and z. So, holding the angle between its position and the shock fixed, a bank will fail earlier in the overall ordering if its capital is decreased (i.e. $\|a_k\|$ is increased). Likewise, holding the distance from the origin fixed, a bank fails earlier if the direction implied by their asset mix is closer to the direction of the shock (i.e. the angle θ_k is smaller). Note that equation **(1)** implies that the order of arrivals of bank failure will be the same for any shock z that lies on the same ray from the origin. Shocks can therefore be distinguished only by their direction, not their magnitude.

In this paper, each bank's power index is measured by the proportion of shocks for which its failure is pivotal in causing a systemic event. A bank failure is pivotal when its inclusion in a

(4) We thank Chen Zhou for suggesting this.
(5) We use these asset names for convenience – other ways of splitting banks' assets into classes could, of course, be used.
(6) In reality, a bank's total assets equals the sum of a bank's capital and its other liabilities; i.e. $H_i + F_i \geq C_i$ or $h_i + f_i \geq 1$.
(7) So, generally $z = [a, \sqrt{1-a}]$ such that $|z| = 1$.
(8) One could instead assume $z \in \mathbb{R}^2$; i.e. consider shocks that increase the value of a bank's holdings of an asset class as well as shocks that decrease the value.
(9) As indicated by Shapley (1977), other methods of ordering bank failures could be used.

group of already-failed banks causes a systemic event. We define a systemic event to be one in which the total assets of failed banks $\sum_i w_i$ exceeds a pre-defined fraction of banking system assets ξ, where the vector of weights $w = \{w_1, \dots, w_n\}$ reflects banks' shares of total system assets: [(10), (11)]

$$w_i = \frac{H_i + F_i}{\sum_{i \in N}(H_i + F_i)} \tag{2}$$

Each subgroup of banks $G \subseteq N$ has a value $v(G) = 1$ if $\sum_{i \in G} w_i > \xi$ and $v(G) = 0$ otherwise. This is our so-called 'characteristic function'. [(12)]

Let π denote an ordering of the banks (i.e. an order in which banks fail). For each bank $i = 1, \dots, n$ let $p_\pi^i = \{j: \pi(i) > \pi(j)\}$ denote the set of players preceding i in the order π. The marginal contribution of bank i in order π is $v(p_\pi^i \cup i) - v(p_\pi^i)$.

We assume that asset shocks occur randomly with respect to a uniform distribution over all possible shocks (i.e. over all vectors $z \in \mathbb{R}_+^2$). [(13)] Then, for each possible order of banks π, we can compute the probability $\theta(\pi)$ that the random shock z generates the order π.

The contribution of each bank's failure to systemic risk is given by:

$$\phi_i(v; H, F, C) = \sum_{\pi \in \Pi} \theta(\pi)[v(p_\pi^i \cup i) - v(p_\pi^i)] \tag{3}$$

The term in square brackets in equation **(3)** takes a value of one or zero. These risk measures have the property that $\sum_i \phi_i = 1$.

3.1 Three-bank example

This section illustrates how the power index for each bank is calculated, for a system of three banks. Suppose banks' asset holdings are as specified in Table 1 and that banks' positions in a two-dimensional leverage space are as shown in Chart 1 with coordinates (h_i, f_i). Types of asset shock are represented by directional arrows. It is straightforward to show that the ordering

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(10) Our premise is that total assets of failed banks beyond this amount trigger a systemic crisis that could induce a costly policymaker intervention. Hence, this critical threshold would have to be determined by the policymaker. Other weighting schemes could have been considered which reflect additional characteristics of a bank (other than size) that mean its failure has system-wide consequences, such as the extent of interconnectivity with other financial institutions, extent to which its services are not readily substitutable and the complexity and opacity of its capital market activities (see Staff of the International Monetary Fund and the Bank for International Settlements, and the Secretariat of the Financial Stability Board (2009)). See also Basel Committee on Banking Supervision (2011) for a set of such characteristics.

(11) w_i is a function of the values of a bank's assets *before* a shock hits.

(12) Other forms for the characteristic function to estimate Shapley values are possible. For instance, rather than a binary function (in accordance with the Shapley-Shubik index) one could adopt a function such that $v(G) \in [0,1]$ depending on the value of $\sum_{i \in G} w_i$. See Section 4.

(13) Other distributions over possible shocks could be used. In fact, since the shocks we are contemplating are assumed to be of sufficient magnitude to put the system into a crisis state, it may be the case that shocks in certain directions should be excluded; i.e. assigned zero probability in our computations. Suppose, for example, that banks do not hold large values of foreign assets, so that a shock to only foreign assets would not cause any banks to fail. Then we would only want to put positive probability on shocks in directions that included both foreign and domestic assets. This issue would become more important if we went to a higher dimensional asset space that included more specialised assets.

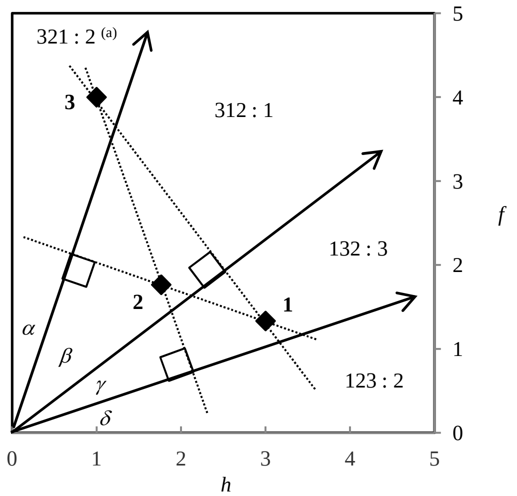
implied by equation **(1)** associated with any shock can be determined by dropping perpendiculars from the points to the shafts of the directional arrows. [14] Perpendiculars that lie higher up the shaft of an arrow denote earlier failure according to equation **(1)**. Moreover, as we rotate the arrows through the asset space, the order in which banks fail only changes when the arrow crosses a perpendicular associated with one of the sides of the triangle defined by the three asset positions.

The orderings generated by rotating an arrow 90 degrees through the asset space are shown in Chart 1. Consider a shock in the direction of the f-axis. Asset positions with high values of f relative to h imply the ordering 3,2,1. As that arrow rotates clockwise, it becomes perpendicular to the line through points 1 and 2 and the order of failure switches to 3,1,2. Further along the rotation it becomes perpendicular to the line through points 1 and 3 and the order switches to 1,3,2. Finally, for asset positions with low values of f relative to h the ordering is 1,2,3.

Table 1: A three-bank example

	Bank		
	1	2	3
Domestic assets (H_i)	90	30	20
Foreign assets (F_i)	40	30	80
Capital (C_i)	30	17	20
h_i	3	1.765	1
f_i	1.333	1.765	4
Weight (w_i)	0.448	0.207	0.345
Strength ($(H_i + F_i)/C_i$)	4.333	3.529	5
Diversification ($\min\{H_i, F_i\}/(H_i + F_i)$)	0.308	0.5	0.2

Chart 1: Illustration of a three-bank example



[14] We mentioned earlier that equation **(1)** is equivalent to $\|a_i\| \cos\theta_i > \|a_j\| \cos\theta_j$ and each term $\|a_k\| \cos\theta_k$ represents the distance of the projection of point a_k along the directional arrow.

To determine each bank's power index, we compute the fraction of shocks for which each bank failure is pivotal, in the sense that it causes the value of the characteristic function $v(G)$ to change from 0 to 1. With a chosen risk threshold of $\xi = 0.5$, in this example, the pivotal bank failure will always be the second bank to fail, as indicated by the number after the colon in Chart 1. The power indices for banks 1, 2 and 3 are given by $\beta/90$, $(\alpha+\delta)/90$ and $\gamma/90$, respectively. Using the positions stated in Table 1, the angle values are $\alpha = 19.3$, $\beta = 33.9$, $\gamma = 18.1$, and $\delta = 18.7$. The power indices for banks 1, 2 and 3 are therefore given by $\phi(v; H, F, C) = (0.38, 0.42, 0.20)$. In contrast to a standard (symmetric) power index, not all orderings are equally likely and some orderings are ruled out completely.

Bank 2 has the highest power index (largest ϕ_i) because its failure is pivotal in terms of causing a systemic event more often than either of the failure of the other two banks. Further inspection of Chart 1 indicates why this is the case. Bank 2 is centrally located relative to banks 1 and 3, reflecting its more diversified asset base. Asset shocks that are concentrated in the domestic sector bring down bank 1 first, because bank 1 has a large share of domestic assets on its balance sheet. Conversely, asset shocks that are concentrated overseas bring down bank 3 first because bank 3 has a large share of foreign assets on its balance sheet. In both cases, however, bank 2 is positioned to fail second and hence its power index is equal to the 42 per cent share of the asset space covered by these shocks. But notice that bank 2 is not the first to fail when asset shocks involve broadly even mixtures of domestic and foreign assets: both asset mix and balance sheet strength matter in determining the order of failures. Bank 2 has a diversified mix of domestic and foreign assets, but it is also well capitalised (it has a ratio of total assets to capital of 60/17=3.529) relative to its counterparts (4.333 and 5 for banks 1 and 3 respectively). Bank 2 is therefore relatively less likely to fail across possible vectors of asset shocks. It is only in extreme cases, where asset shocks fall almost entirely on one asset class or the other that one of the other banks is less likely to fail.

This example illustrates the importance of diversification and balance sheet strength, as measured by exposure to each asset class relative to capital, in determining banks' power indices. But on the face of it, the directions of these effects may seem counter-intuitive. The failure of banks which hold diversified asset portfolios are, ceteris paribus, more likely to be pivotal because they fail *after* less diversified banks. On the other hand being very strong (i.e. having low leverage), or even being very weak (i.e. having high leverage), tend to reduce a bank's power index. In the former case it is because the bank is likely to fail last (after the threshold defining a systemic event has already been reached) while in the latter case it is because the bank is likely to fail first (before the threshold for a systemic event is reached).

These implications of bank 2's diversified portfolio for its power index do, however, depend on the value of the threshold ξ. If $\xi \leq 0.207$, for any shock it is the first bank that fails which is pivotal. Table 1 and Chart 1 show that for $\xi \leq 0.207$ the power indices for banks 1, 2 and 3 would be $(\gamma+\delta)/90$, 0, and $(\alpha+\beta)/90$, respectively. In these cases, bank 2's diversified portfolio (and its capital level) means it has the lowest power index among the three banks. See Section 3.2.4 for a further discussion of how banks' power indices depend on the value of ξ.

The following subsection illustrates these issues further by considering some extreme examples. We also demonstrate how the procedure for computing power indices can be extended to more than three banks.

3.2 Strength, diversification and composition of the banking system

The role of balance sheet strength and diversification in determining power indices is illustrated in this section using two benchmark scenarios. The cases presented are overly simplistic because they involve only three banks, which make the determination of the pivotal bank trivial. Nevertheless, these examples illustrate well how balance sheet strength and diversification interact to determine power in more complex cases.

In one scenario, all three banks have the same total value of assets and capital holdings (and hence the same strength), but different asset mixes. In the other, all three banks have the same asset mix, but have different amounts of capital. In each case, banks' power indices are determined by their relative positions in the leverage space. Specifically, the failure of a bank positioned in the middle is always pivotal in creating a systemic event in these cases and, hence, has a power index equal to 1. Departures from these scenarios put us back into the world of Table 1 and Chart 1, where differences in strength and diversification jointly determine power indices.

3.2.1 Equal strength

Suppose banks' asset holdings are as shown in Table 2. Bank asset positions fall on the same iso-strength line, as shown in Chart 2, because all three banks have the same asset to capital ratios.(15) In this case, the only thing that matters for the values of the power indices is diversification. Since bank 2 is more diversified than banks 1 and 3 (i.e., it is positioned in the middle of the two banks) it is always the second bank to fail. More specifically, for asset shocks to the left of the directional arrow the order of failure is 3,2,1 and for asset shocks to the right of the directional arrow the order of failure is 1,2,3. Assuming $\xi = 0.5$, bank 2 has a power index of 1, while banks 1 and 3 both have power indices of 0.

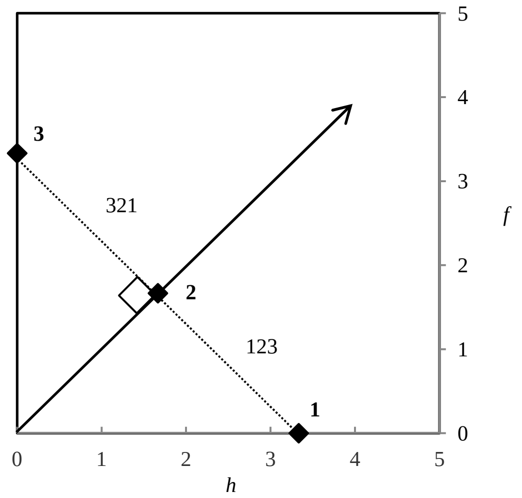
The idea that bank 2 can have the highest power index because (*given* the total assets of banks in the system and the value of ξ) it is diversified might appear to contradict standard finance theory, which explains how diversification reduces the risk in a portfolio. But the contrasting interpretations of diversification reflect the differences in perspectives. A bank could reduce the absolute risk of it failing by holding a diversified portfolio of assets. But still if we consider the propensity for the failure of that bank to trigger the systemic event (ignoring for these purposes the likelihood of it failing), it may be that its diversified portfolio implies it holds the position of being pivotal for the largest proportion of potential shocks to asset values. Thus, this approach to assessing systemic importance (as we define it) is consistent with standard finance theory.

(15) A bank's strength might depend on the structure of its funding as well as on its leverage. For instance, a bank that has more short-term wholesale liabilities might be less strong because it is more vulnerable to the risk of suffering a run by its creditors in response to shocks to the values of its assets. It would be possible to incorporate the effect of a bank's funding structure on its strength by dividing banks' asset positions by a function of the level of its capital and a measure of its funding vulnerability instead of just the level of its capital.

Table 2: Example of three banks of equal strength

	Bank		
	1	2	3
Domestic assets (H_i)	100	50	0
Foreign assets (F_i)	0	50	100
Capital (C_i)	30	30	30
h_i	3.333	1.667	0
f_i	0	1.667	3.333
Weight (w_i)	0.333	0.333	0.333
Strength ($(H_i + F_i)/C_i$)	3.333	3.333	3.333
Diversification ($\min\{H_i, F_i\}/(H_i + F_i)$)	0	0.5	0

Chart 2: Illustration of the example of three banks of equal stretch



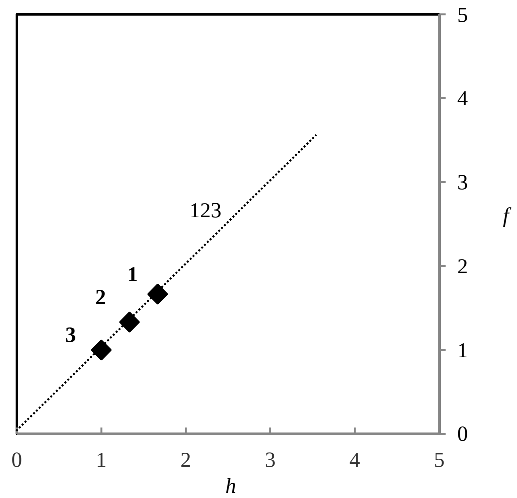
3.2.2 Equal diversification

Assuming that banks have the same level of diversification but different capital holdings we can easily see how balance sheet strength matters. Suppose the banks' asset exposures are as shown in Table 3. Banks' asset positions fall on the same iso-diversification line (the ray from the origin in Chart 3), reflecting the fact that all three banks have the same asset mix. Since all three banks have the same asset mix, the only thing that matters in determining power indices is balance sheet strength, which is reflected by each bank's position on the ray from the origin. Since bank 2 is weaker than bank 1, but stronger than bank 3 (i.e. it is again positioned in the middle of the two other banks), it is always the second bank to fail. The order of failure is 1,2,3 for all asset shocks. So once again, assuming $\xi = 0.5$, bank 2's power index is 1. Both banks 1 and 3 have power indices of 0.

Table 3: Example of three banks of equal diversification

	Bank		
	1	2	3
Domestic assets (H_i)	50	40	30
Foreign assets (F_i)	50	40	30
Capital (C_i)	30	30	30
h_i	1.667	1.333	1
f_i	1.667	1.333	1
Weight (w_i)	0.417	0.333	0.250
Strength ($(H_i + F_i)/C_i$)	3.333	2.667	2
Diversification ($\min\{H_i, F_i\}/(H_i + F_i)$)	0.5	0.5	0.5

Chart 3: Illustration of the example of three banks of equal diversification



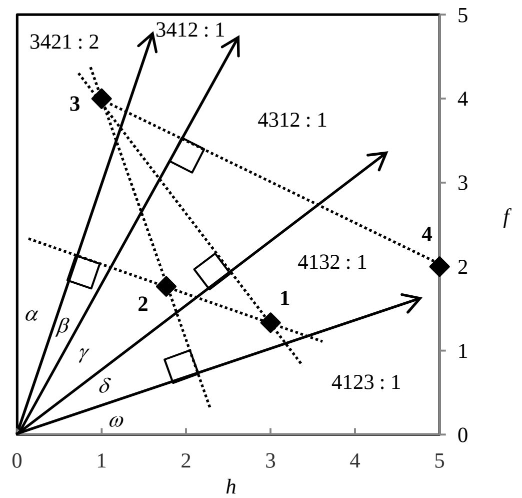
3.2.3 Incorporating additional banks in the system

Altering the population of banks alters the positions of institutions in the order in which institutions would fail in response to a particular shock and changes the specification of the pivotal bank for a given shock. This translates into different measures of contributions to systemic risk. Table 4 adds an extra bank (bank 4) to the population described in Table 1. In this particular case, the locations of the three critical vectors shown in Chart 1 remain unchanged and a new one appears between banks 2 and 3 (Chart 4). For the same threshold $\xi = 0.5$, banks' power indices change substantially: $\phi(v; H, F, C) = (0.79, 0.21, 0, 0)$. Bank 1 is now critical in pushing losses beyond the critical level of half of system assets in a vast majority of cases.

Table 4: A four-bank example

	Bank			
	1	2	3	4
Domestic assets (H_i)	90	30	20	50
Foreign assets (F_i)	40	30	80	20
Capital (C_i)	30	17	20	10
h_i	3	1.765	1	5
f_i	1.333	1.765	4	2
Weight (w_i)	0.361	0.167	0.278	0.194
Strength ($(H_i + F_i)/C_i$)	4.333	3.529	5	7
Diversification ($\min\{H_i, F_i\}/(H_i + F_i)$)	0.308	0.5	0.2	0.286

Chart 4: Illustration of a four-bank example



3.2.4 *Varying the critical threshold*

The systemic importance of individual banks depends crucially on the definition of the value of the critical threshold that defines a systemic event. Table 5 describes a system with 5 banks. Chart 5 illustrates the bank positions and shows the power indices of each of the 5 banks for different critical thresholds ranging from 0 (i.e. a single failure of any bank would imply a systemic event) to 1 (i.e. it is necessary for all banks to fail for there to be a systemic event).

Table 5: A five-bank example

	Bank				
	1	2	3	4	5
Domestic assets (H_i)	350	240	330	400	100
Foreign assets (F_i)	290	690	270	110	15
Capital (C_i)	45	90	80	40	8
h_i	7.778	2.667	4.125	10	12.5
f_i	6.444	7.667	3.375	2.75	1.875
Weight (w_i)	0.229	0.333	0.215	0.182	0.041
Strength ($(H_i + F_i)/C_i$)	14.222	10.333	7.5	12.75	14.375
Diversification ($\min\{H_i, F_i\}/(H_i + F_i)$)	0.453	0.258	0.45	0.216	0.13

Chart 5: Illustration of the effect of varying the value of the critical threshold

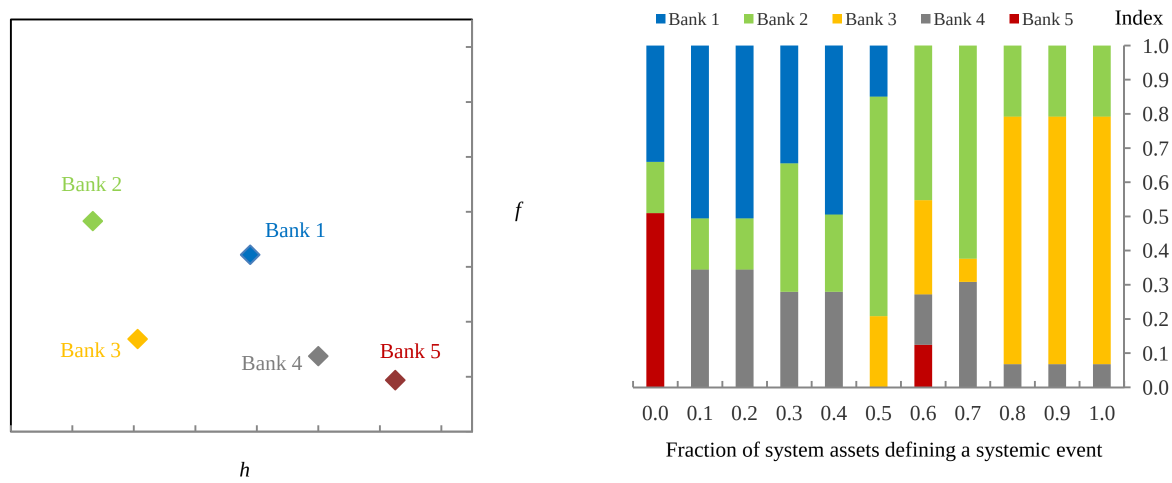


Chart 5 shows that banks' power indices vary considerably with the value of the critical threshold. Some worthwhile observations are immediately apparent.

In the case of very low thresholds, the banks with higher leverage are often the banks with the highest power indices. For instance, bank 1 tends to have the highest or second highest power index for values of the threshold 0.4 or less (see the right panel of Chart 5). But portfolio composition and relative size also matter for banks' power indices at low threshold values. For example, banks 1 and 5 have similar levels of leverage but for a critical threshold of zero, bank 5 has a higher power index reflecting the bias in its portfolio towards domestic assets compared to bank 1's more diversified portfolio. For a critical threshold greater than zero, however, bank 5's relatively minuscule size means its power index drops to zero while bank 1 still has a positive power index.

In the case of very high threshold values (0.8 and above) bank 3 has a highest power index. It is the least leveraged bank (as reflected by its close proximity to the origin in the left panel of Chart 5) and one of the most diversified. But a high threshold means that it tends to be the bank that fails last that causes the proportion of failed banks' assets to cross the critical threshold and bank 3's strength and diversification means that it is that last bank for many shocks. Only for shocks that are strongly biased towards domestic or foreign assets would other banks fail last.

This is why for high threshold values, bank 2 (biased towards foreign assets) and bank 4 (biased towards domestic assets) also have positive power indices, albeit much smaller than bank 3's (see right panel of Chart 5).

4 Possible extensions

4.1 Calculating generalised Shapley values

The framework can be extended in order to calculate asymmetric Shapley values, which include, as a special case, the asymmetric power indices. A reason for doing this is that one might want to incorporate the identity of other banks in a set of failed banks that triggers a systemic event into the risk measure, rather than only the bank in the set whose failure tips the system from stability to instability.

Equation **(5)** shows a modified characteristic function.

$$\begin{aligned} v(G) &= 0 && \text{if } \textstyle\sum_{i \in G} w_i \leq \xi' \\ v(G) &= f\left(\sum_{i \in G} w_i\right) && \text{if } \textstyle\sum_{i \in G} w_i \in (\xi', \xi'') \\ v(G) &= 1 && \text{if } \textstyle\sum_{i \in G} w_i \geq \xi'' \end{aligned} \qquad \textbf{(5)}$$

where

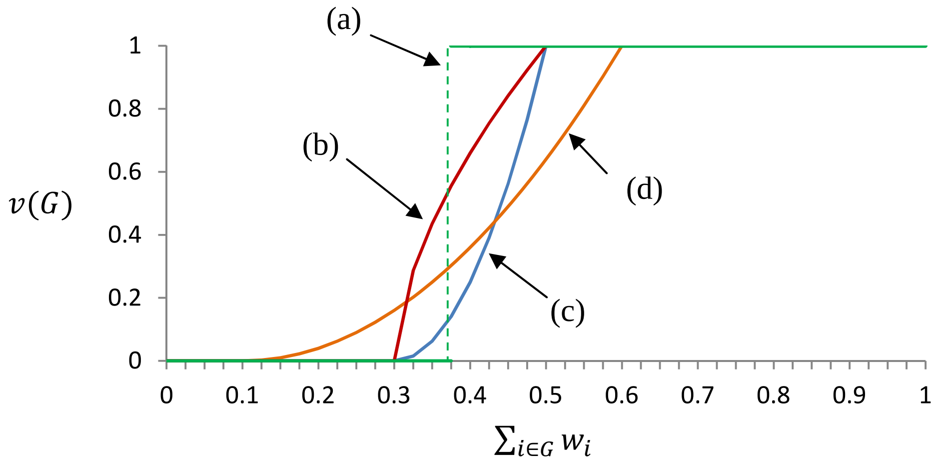
$f' > 0$

and, f is such that $v(G) \in (0,1)$ if $\sum_{i \in G} w_i \in (\xi', \xi'')$.

With this function, the value of the characteristic function always lies between zero and one. If the set of bank failures is small enough (i.e. the proportion of failed banks' assets is less than or equal to ξ'), the value is zero (i.e. there is no systemic event). If the set of failed banks is large enough (i.e. the proportion of failed banks' assets is greater than or equal to ξ'') , the value is one (i.e. there is a full systemic event). For intermediate levels (i.e. the proportion of failed banks' assets is between ξ' and ξ'') , the value is between zero and one (i.e. there is a partial systemic event). Of course, the characteristic function nests the binary function used to calculate asymmetric power indices; by setting $\xi' = \xi''$, the characteristic function can be the same as in Section 3.

Illustrations of possible characteristic functions are shown in Chart 6. Function (a) is a binary characteristic function like that explored in Section 3. Under both functions (b) and (c), there is a no systemic event if the share of total assets of bank failures is 0.3 or less and there is a full systemic event if the share is 0.5 or more. But under function (b), once the asset share of bank failures exceeds 0.3, it is the earlier incremental bank failures that have relatively larger effect on systemic risk (i.e. the function is concave). While under function (c) it is the later incremental bank failures that have relatively large effects (i.e. the function is convex). Thus, if one replaced function (b) with (c), this would tend to decrease the Shapley values of the earlier incremental bank failures and increase the Shapley values of the later incremental bank failures. The same is true under function (d) but under this characteristic function there is no systemic event if the share of total assets of bank failures is 0.1 or less and a full systemic event if this share is 0.6 or more; that is, over the region in which there is a partial systemic event, systemic risk builds more slowly than under function (c).

Chart 6: Different possible characteristic functions



(a) $\xi' = \xi'' = 0.375$

(b) $\xi' = 0.3$, $\xi'' = 0.5$, $f(\xi) = \left(\frac{\xi - \xi'}{\xi'' - \xi'}\right)^{0.6}$

(c) $\xi' = 0.3$, $\xi'' = 0.5$, $f(\xi) = \left(\frac{\xi - \xi'}{\xi'' - \xi'}\right)^{2}$

(d) $\xi' = 0.1$ $\xi'' = 0.6$, $f(\xi) = \left(\frac{\xi - \xi'}{\xi'' - \xi'}\right)^{2}$

4.2 The riskiness of asset exposures

Differences in the riskiness of banks' exposures to an asset class could also be captured by extending the framework. The reason for doing this could be that the relative riskiness of asset holdings affects the order in which banks would fail for a given shock. Banks that hold more risky exposures to the asset classes should tend to be towards the front of the order of bank failures.

One way to reflect differences in the riskiness of exposures is to apply risk weights to banks' holdings of asset classes; the values of risk weights would be increasing in the riskiness of holdings. For example, risk weights r_i^h and r_i^f could be applied to bank i's exposure to domestic and foreign assets, where $r_i^h, r_i^f \geq 0$, which mean bank i's position in the two-dimensional space becomes $a_i = (r_i^h h_i, r_i^f f_i) \in \mathbb{R}_+^2$.[16] To understand how risk weighting could increase (decrease) the power indices of banks that hold more (less) risky exposures to asset classes, consider the example of two banks 1 and 2, where $r_1^f = r_2^f$, $C_1 = C_2$, $F_1 = F_2$, and $H_1 > H_2$. If the bank's risk weights on their domestic asset holdings are the same or no risk

[16] Differences in the riskiness across asset classes could also be taken into account if risk weights are applied in this way. For example, if one asset class is less risky than another, risk weights for all banks will tend to be lower for the first asset than they are for the second asset.

weights on these exposures are applied, bank 1 would be before bank 2 in any order of bank failures that impacts both asset classes. But suppose bank 1 invests in less risky domestic assets than bank 2, which is reflected in the fact $r_1^h < r_2^h$. If bank 1's risk weight on domestic assets is sufficiently lower than bank 2's that $r_1^h H_1 < r_2^h H_2$, bank 2 is ahead of bank 1 in the order for any shock. In general, the introduction of risk weights will result in a relocation of all banks in the leverage space; the analysis of Section 3 can then be used to determine the impact of taking into account risk weights on banks' contributions to systemic risk.

4.3 Interbank contagion

The framework set out above does not incorporate the risk of interbank contagion. For instance, the approach could not capture a scenario in which the failure of a relatively small bank has a significant effect on systemic stability because it is strongly interconnected with other banks. But it is possible to choose a weighting function that allows for contagion between banks to be taken into account in the calculation of banks' power indices. Interbank contagion arises through a network of interbank exposures, Ω. The function $\gamma(G,\Omega)$ maps the set of initial bank failures G (based on the order π) to a set of bank failures that are a result of contagion from the failed banks in G. There exists a unique function γ (Eisenberg and Noe (2001)).(17) The function γ has the following properties: $G \cap \gamma(G,\Omega) = \emptyset$ (i.e. the set of failed banks due to contagion does not include any of the initial failed banks); $\gamma(\emptyset,\Omega) = \emptyset$ (i.e. if there are no initial bank failures then there can be no contagious bank failures); $\gamma(N,\Omega) = \emptyset$ (i.e. if all banks in the system fail initially there can be no contagious failures); and $\gamma(G,\emptyset) = \emptyset$ (i.e. if there are no interbank links there can be no contagious failures). To incorporate interbank contagion replace the characteristic function v defined in Section 3 with the new characteristic function $\hat{v}$ defined such that each subgroup of banks G has a value $\hat{v}(G) = 1$ if $\sum_{i \in G} w_i + \sum_{j \in \gamma(G,\Omega)} w_j \geq \xi$ and $\hat{v}(G) = 0$ otherwise.

To understand how banks' power indices can change if the risk of interbank contagion is taken into account in this way consider again the four-bank system shown in Chart 4 in Section 3.2.3. An interbank network for this system is shown in Chart 7. Each of the arrows represents an interbank loan from the bank at the head of the arrow to the bank at the other end. That is, Chart 7 shows that bank 3 has borrowed from bank 2, bank 1 has borrowed from bank 3, and that bank 4 has borrowed from bank 1.

(17) In practice, one would calibrate the function γ by running multiple simulations of all possible bank failure scenarios for a given network. Eisenberg and Noe show that there is a unique contagious effect arising from those failures in each scenario.

Chart 7: An interbank network

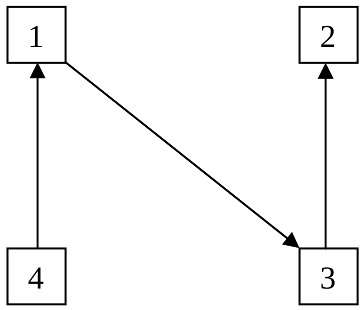


Table 6 shows all of the sets of initially failed banks that are possible for the network in Chart 7 (G) and maps these to banks that fail due to contagion through the interbank network ($\gamma(G, \Omega)$). Banks 1, 3 and 4 all pose risk to the banks they have borrowed from but bank 4 poses relatively more contagion risk. For instance, if bank 3 fails alone, it might default on its loan from 2 but we assume this is not sufficient to cause bank 2 to also fail (see top row of Table 6). But if bank 4 fails alone, it defaulting on its loan from bank 1 implies bank 1 fails, who in turn defaults on its loan from bank 3, which causes bank 3 to fail. Since bank 3 has incurred losses on its interbank lending, the amount that bank 2 recovers on its loan to bank 3 might be lower than if bank 3 had failed alone, which might be sufficient to mean bank 2 also fails. Thus, all four banks fail if bank 4 fails (see second row). It follows if bank 4 is among the banks that fail initially then all four banks fail either due to the shocks to assets or interbank contagion.

Table 6: Contagious bank failures

G	$\gamma(G, \Omega)$
{3}	∅
{4}	{1,2,3}
{3,4}	{1,2}
{1,4}	{2,3}
{2,3,4}	{1}
{1,3,4}	{2}
{1,2,4}	{3}
{1,2,3,4}	∅

If we use characteristic function $\hat{v}$ for the example in Chart 4 and given the interbank contagion process outlined in Table 6, bank 4 becomes the pivotal bank for all possible shocks and consequently has a power index of one. In the case of shocks for which it is the first to fail, the fact bank 4 triggers the failure of the other three banks means that the characteristic function takes the value of one. In the case of all other shocks, bank 3 is the first bank to fail but this is not enough to cause other banks to fail. The second bank to fail is then 4, which is enough to cause banks 1 and 2 to also fail. These power indices are in contrast with those if interbank contagion is not taken into account, where banks 1 and 2 had positive power indices but banks 3 and 4 had zero power indices (see Section 3.2.3).

Our example shows: (i) how the power indices of banks that borrow from other banks can increase if contagion risk is taken into account; and (ii) that power indices reflect how banks can trigger both the failure of banks they are directly connected to and the failure of banks they are indirectly connected to through the interbank network. With just four banks, it is difficult to illustrate fully the complexity of interbank contagion and the impact on power indices, and it is not surprising that by taking into account contagion a single bank is pivotal for all shocks. This will not be true in general and would not be expected in larger, more realistic networks.

5 Conclusion

This paper presents a measure of contributions of individual bank failures to systemic risk, based on the Shapley (1977) version of the Shapley-Shubik power index, which is calculated by identifying the prevalence with which each bank, in the event of its failure, would push the total assets of failed banks in the system beyond a critical threshold. The approach uses a very small set of variables from bank balance sheets to estimate banks' power indices.

The framework is an example of a more general methodology for analysing the identity of banks that are pivotal in the transition of a banking system from stability to instability. We describe how it is possible to generalise the framework to calculate generalised Shapley values, and take into account the riskiness of banks' asset holdings and the potential for interbank contagion. It is also possible to generalise the methodology in several other ways: banks' portfolios could be comprised of more than two asset classes; perturbations to bank balance sheets could be considered to derive confidence intervals around banks' power indices; alternative functions for determining the order in which banks fail could be used; and alternative characteristic functions based on weights that take into account factors other than banks' shares of total system assets are possible.

Finally, the approach is designed to be used as a tool for assessing the contributions of bank failures to systemic risk – an application to actual bank data is a goal for further work. The work in this paper suggests that the approach has the potential to be used in reverse stress testing.

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