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Appendices to Working Paper No. 471
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Stephen Burgess,[1] Emilio Fernandez-Corugedo,[2] Charlotta Groth,[3] Richard Harrison,[4] Francesca Monti,[5] Konstantinos Theodoridis[6] and Matt Waldron[7]

(1) Bank of England. Email: stephen.burgess@bankofengland.co.uk
(2) IMF. Email: EFernandez-Coruged@imf.org
(3) Zurich Insurance Group: Email: charlotta.groth@zurich.com
(4) Bank of England. Email: richard.harrison@bankofengland.co.uk
(5) Bank of England. Email: francesca.monti@bankofengland.co.uk
(6) Bank of England. Email: konstantinos.theodoridis@bankofengland.co.uk
(7) Bank of England. Email: matthew.waldron@bankofengland.co.uk (corresponding author)

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Publications Group, Bank of England, Threadneedle Street, London, EC2R 8AH
Telephone +44 (0)20 7601 4030 Fax +44 (0)20 7601 3298 email publications@bankofengland.co.uk

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A Derivation of COMPASS

This appendix presents the derivation of the COMPASS model equations in four parts. First, it describes the model and derives the first-order conditions for the optimisation problems of agents in the model. Second, it discusses the procedure through which the resulting model equations are de-trended and presents a set of detrended, stationary equations. Third, it derives the stationary steady state. Fourth, it details a complete set of model equations log-linearised around that steady state.

Before detailing the model, we first introduce some notation:

- The model includes population, productivity and sector-specific tenchnology trends. The trend for a variable J is denoted $\tilde{\chi}_t^J$ (eg the population trend is given by $\tilde{\chi}_t^H$) and its growth rate is denoted $\Gamma_t^J = \frac{\tilde{\chi}_t^J}{\tilde{\chi}_{t-1}^J}$.
- These trends imply that many of the endogenous variables are non-stationary. The levels of non-stationary variables are denoted in the upper case with the "~" symbol (eg consumption is $\tilde{C}_t$).
- Unless otherwise stated, all variables are expressed in per capita terms (eg $\tilde{C}_t$ is per capita consumption).
- De-trended variables are denoted using capital letters without the "~" symbol (eg detrended consumption is C_t) and the stationary steady states of those de-trended variables are denoted in the same way without the time subscript (eg the steady state of consumption is C).
- Log deviations of variables from steady state are denoted using the lower case and are defined as $j_t \equiv \log J_t - \log J$ (eg the logarithmic deviation of consumption from steady state is given by $c_t \equiv \log C_t - \log C$).
- Parameters are typically denoted using Greek letters with groups of parameters sharing the same letter, but with different subscripts. For example, ϕ is used to refer to the parameters governing the costs of price adjustments (eg ϕ_Z and ϕ_W for final output and wage price adjustment costs), while ϵ refers to elasticities, ξ to price indexation, ψ to real rigidities, θ to monetary policy response parameters, σ to standard deviations of shocks, and ρ to persistence in forcing processes.
- Forcing processes are denoted ε (eg the forcing process for labour supply shocks is ε^L) with the exception of the mark-up processes, which are denoted μ (eg the value-added mark-up forcing process is μ_V). All shocks are denoted η (eg the labour supply shock is η^L).

A.1 Model description and first-order conditions

A.1.1 Households

There is a continuum of households defined on the unit interval. A share, ω_o, of households are 'optimising' or 'unconstrained'. Those households have access to financial markets and are able to save and borrow. The remaining share, $1 - \omega_o$, are 'rule of thumb' or 'constrained'. Those households have no access to financial markets, so consume all their labour income in each period. We also assume that they supply any labour demanded given the wage rate set by the optimising households. All individual households (regardless of type) are denoted with subscript i. Individual optimising households are denoted with superscript o so that consumption of an optimising household is referred to as C^o_{it}, while individual 'rule of thumb' households are denoted with superscript rot so that consumption of these households is given by C^{rot}_{it}.

Following Burriel et al. (2010), the size of each household is $\tilde{\chi}^H_t$, which grows at the rate $\Gamma^H_t = \frac{\tilde{\chi}^H_t}{\tilde{\chi}^H_{t-1}}$. Given that households are defined as a continuum on the unit interval, this means that the total population is also given by $\tilde{\chi}^H_t$.

Members of optimising households consume, hold money, save, invest, work and pay taxes. Each household derives utility from the sum of the utilities of the individual household members. Since the members of the individual households are identical and the size of each household is equal to the total population $\tilde{\chi}^H_t$, we can express the utility function for the representative optimising household in per capita terms. In any arbitrary period, s, an optimising household, i, maximises a utility function of the following form:

$$U_{is} = \mathrm{E}_s \left[\sum_{t=s}^{\infty} \Theta_t \tilde{\chi}^H_t \tilde{U}_{it} \left(\tilde{C}^o_{it}, \tilde{C}^o_{t-1}, L^o_{it}, \frac{\tilde{\mathcal{M}}^o_{it}}{\tilde{P}^C_t}, \varepsilon^L_t \right) \right] \tag{A.1}$$

where $\mathrm{E}_t[\cdot]$ is the expectations operator, Θ_t is a discount factor (defined below), $\tilde{U}_{it}(\cdot)$ is the period utility function (defined below), $\tilde{C}^o_{t-1}$ is lagged *aggregate* per capita consumption of optimising households, L^o_{it} is the household's labour supply, $\tilde{\mathcal{M}}^o_{it}/\tilde{P}^C_t$ is real money holdings and $\left(\varepsilon^L_t\right)$ is a disturbance that raises the disutility of supplying labour. The disturbance evolves according to a forcing process, which satisfies:

$$\begin{aligned} \log \varepsilon^L_t &= (1-\rho_L) \log \varepsilon^L + \rho_L \log \varepsilon^L_{t-1} + \left(1-\rho_L^2\right)^{1/2} \sigma_L \eta^L_t \\ \eta^L_t &\sim N(0,1) \end{aligned} \tag{A.2}$$

where η^L_t is an iid shock, σ_L is the standard deviation of the forcing process[185] and ρ_L is the persistence of the forcing process.

We make similar assumptions to An and Schorfheide (2007) and Chen et al. (2012) and define the period utility function as:

$$\tilde{U}_{it} = \frac{\left(\frac{\tilde{C}^o_{it}}{\tilde{\chi}^Z_t} - \psi_C \frac{\tilde{C}^o_{t-1}}{\tilde{\chi}^Z_{t-1}} \right)^{1-\epsilon_C} - 1}{1-\epsilon_C} - \frac{\nu_L \varepsilon^L_t {L^o_{it}}^{1+\epsilon_L}}{1+\epsilon_L} + \frac{\nu_M \left(\frac{\tilde{\mathcal{M}}^o_{it}}{\tilde{\chi}^Z_t \tilde{P}^C_t} \right)^{1-\epsilon_C} - 1}{1-\epsilon_C} \tag{A.3}$$

where the marginal utility of consumption and real money balances is defined relative to the trend in overall productivity growth, $\tilde{\chi}^Z_t$ (defined below), ϵ_C is the inverse of the intertemporal elasticity of substitution, ψ_C is the parameter governing external habit formation, ϵ_L is the elasticity of labour supply, ν_L is the relative weight on the disutility of working and ν_M is the relative weight on real money balances. Utility is maximised with respect to the per capita budget constraint:

$$\begin{aligned} & \Psi_W \left(\frac{\tilde{W}^o_{it}}{\tilde{W}^o_{it-1}} \right) \tilde{W}^o_{it} L^o_{it} + \frac{\tilde{\mathcal{M}}^o_{it-1}}{\Gamma^H_t} + \tilde{R}^K_t \tilde{K}^o_{it-1} + \frac{R^A_{t-1} \tilde{A}^o_{it-1}}{\Gamma^H_t} + \tilde{D}^{FI,o}_{it} + \frac{\tilde{\Xi}_t}{\omega_o} \\ = \; & \tilde{P}^C_t \tilde{C}^o_{it} + \tilde{\mathcal{M}}^o_{it} + \tilde{A}^o_{it} + \tilde{P}^I_t \tilde{I}^o_{it} + \tilde{P}^{IO}_t \tilde{I}^{O,o}_{it} + \tilde{T}^o_{it} + \tilde{P}^C_t \tilde{\chi}^C_t \tilde{Trans}^o_i \end{aligned} \tag{A.4}$$

[185] This is guaranteed by the use of the scaling factor, $\left(1-\rho_L^2\right)^{1/2}$. It is useful when taking the model to the data because it means that the variances of the forcing processes depend on one parameter rather than two.

where $\tilde{W}_{it}^{o}$ is the nominal wage rate, $\Psi_W(\cdot)$ are adjustment costs associated with changing nominal wages, $\tilde{K}_{t-1}^{o}$ is physical capital inherited from the previous period, $\tilde{A}_{it}^{o}$ are savings (deposited with a portfolio packager), which provide a return to the household of R_t^A, $\tilde{\Xi}_{it}$ are the profits made by monopolistic firms that are re-distributed lump-sum to households, $\widetilde{D}_{it}^{FI,o}$ are dividends paid by the portfolio packager (explained below), $\tilde{P}_t^I \tilde{I}_{it}^o$ & $\tilde{P}_t^I \tilde{I}_{it}^{O,o}$ represent nominal investment (described below) and $\tilde{T}_{it}^o$ & $\tilde{Trans}_i^o$ are lump-sum taxes and transfers paid to the government.[186]

Optimising households are monopolistic suppliers of their own differentiated labour. They set their nominal wage $\tilde{W}_{it}^{o}$ and supply any amount of labour demanded at that wage. They face a quadratic adjustment cost for changing wages, measured in terms of the level of their nominal wages:

$$\Psi_W\left(\frac{\tilde{W}_{it}^{o}}{\tilde{W}_{it-1}^{o}}\right) \equiv 1 - \frac{\phi_W\left(\zeta_{it}^{W,o}-1\right)^2}{2\left(\mu^W-1\right)} \tag{A.5}$$

where ϕ_W is the parameter that governs the cost of adjustment and where:

$$\zeta_{it}^{W,o} \equiv \frac{\tilde{W}_{it}^{o}}{\tilde{W}_{it-1}^{o}\left(\Pi^W\right)^{1-\xi^W}\left(\Pi_{t-1}^W\right)^{\xi^W}} \tag{A.6}$$

There is full and costless indexation of wages to either steady-state wage growth, Π^W, or to lagged aggregate nominal wage growth, Π_{t-1}^W, with the fraction of each determined by the parameter ξ_W.

A "labour packager" costlessly transforms heterogenous household labour into a homogenous labour bundle, L, which is hired by firms. The labour packager operates under perfect competition and combines labour from both optimising and rule-of-thumb households to create the labour bundle. Aggregate labour is given by:

$$L_t = \left[\int_0^1 \left(L_{it}^{\frac{1}{\mu_t^W}}\right) di\right]^{\mu_t^W} \tag{A.7}$$

where L_t is aggregate per capita hours and L_{it} is per capita hours worked by household i. The variable μ_t^W, defined so that $\frac{\mu_t^W}{\mu_t^W-1}$ is the elasticity of demand between different households' labour, satisfies:

$$\begin{aligned} \log \mu_t^W &= \log \mu^W + \sigma_{\mu^W}\eta_t^{\mu^W} \\ \eta_t^{\mu^W} &\sim N(0,1) \end{aligned} \tag{A.8}$$

The per capita demand for labour of type i is given by:

$$L_{it} = \left(\frac{\tilde{W}_{it}}{\tilde{W}_t}\right)^{-\frac{\mu_t^W}{\mu_t^W-1}} L_t \tag{A.9}$$

and the aggregate per capita wage index satisfies:

$$\tilde{W}_t = \left[\int_0^1 \tilde{W}_{it}^{\frac{1}{1-\mu_t^W}} di\right]^{1-\mu_t^W} \tag{A.10}$$

[186] The transfers are also distributed to rule-of-thumb households in a way that ensures that optimising and non-optimising households have the same level of consumption in steady state, following Galí et al. (2007).

Optimising households own the capital stock and rent it out to firms at the rental rate $\tilde{R}_t^K$. Capital accumulates according to the following identity:[187]

$$\Gamma_{t+1}^H \tilde{K}_{it}^o = \left(1 - \delta^K\right) \tilde{K}_{it-1}^o + \Psi_I\left(\tilde{\zeta}_{it}^{I,o}, \varepsilon_t^I\right) \tilde{I}_{it}^o \tag{A.11}$$

where δ_K is the depreciation rate of capital and $\Psi_I(\cdot)$ is a function that determines the cost of adjusting investment away from it's long-run growth rate, specified as:

$$\Psi_I\left(\tilde{\zeta}_{it}^{I,o}, \varepsilon_t^I\right) = \left(1 - \frac{\psi_I\left(\tilde{\zeta}_{it}^{I,o} - \Gamma^H\Gamma^Z\Gamma^I\right)^2}{2}\right)\varepsilon_t^I \tag{A.12}$$

where:

$$\tilde{\zeta}_{it}^{I,o} \equiv \frac{\Gamma_t^H \tilde{I}_{it}^o}{\tilde{I}_{it-1}^o} \tag{A.13}$$

where ψ_I is a parameter that governs the size of the adjustment costs, $\Gamma^H\Gamma^Z\Gamma^I$ denotes investment growth along the balanced growth path (discussed below), and where ε_t^I is a shock that affects the rate at which investment is converted into capital, satisfying:

$$\begin{aligned} \log \varepsilon_t^I &= (1 - \rho_I) \log \varepsilon^I + \rho_I \log \varepsilon_{t-1}^I + \left(1 - \rho_I^2\right)^{1/2} \sigma_I \eta_t^I \\ \eta_t^I &\sim N(0,1) \end{aligned} \tag{A.14}$$

Optimising households also finance 'other investment', $\tilde{I}_{it}^{O,o}$, at the price $\tilde{P}_t^{I^O}$. This variable is included in the model so that the aggregate resource constraint comes as close as possible to matching the national accounting identity for GDP into expenditure components. The mapping of the model to the data implies that this variable captures expenditure components of GDP not explicitly included in the model like housing investment and stockbuilding.[188] For simplicity, growth in other investment is assumed to exhibit 'error correction' to its long-run trend:

$$\frac{\tilde{I}_{it}^{O,o}}{\tilde{I}_{it-1}^{O,o}} = \left(\frac{\tilde{I}_{it-1}^{O,o}}{\tilde{\chi}_{t-1}^Z}\right)^{\rho_{I^O} - 1} \varepsilon_t^{I^O} \tag{A.15}$$

where the disturbance to other investment satisfies:

$$\begin{aligned} \log \varepsilon_t^{I^O} &= \left(1 - \rho_{I^O}^2\right)^{1/2} \sigma_{I^O} \eta_t^{I^O} \\ \eta_t^{I^O} &\sim N(0,1) \end{aligned} \tag{A.16}$$

Each optimising household, i, solves the following Lagrangian in any arbitrary period,

[187] The capital accumulation identity is expressed in terms of the per capita capital stock, so Γ^H enters this equation to account for population growth between periods t and $t+1$.

[188] The model explicitly includes consumption, business investment, total government spending, exports and imports. The components of GDP (measured at market prices) not explicitly modelled are: dwellings investment, so-called 'other investment' (which includes stamp duty and so is correlated with dwellings investment), stockbuilding and the alignment adjustment.

s:

$$\mathcal{L}_{is} = \mathrm{E}_s \sum_{t=s}^{\infty} \Theta_t \tilde{\chi}_t^H \left\{ \begin{array}{c} \tilde{U}_{it}\left(\tilde{C}_{it}^o, \tilde{C}_{t-1}^o, L_{it}^o, \frac{\tilde{\mathcal{M}}_{it}^o}{\tilde{P}_t^C}, \varepsilon_t^L\right) \\ +\tilde{\Lambda}_{it}^C \left[\begin{array}{c} \Psi_W\left(\frac{\tilde{W}_{it}^o}{\tilde{W}_{it-1}^o}\right)\tilde{W}_{it}^o\left(\frac{\tilde{W}_{it}^o}{\tilde{W}_t}\right)^{-\frac{\mu_t^W}{\mu_t^W-1}} L_t + \frac{\tilde{\mathcal{M}}_{it-1}^o}{\Gamma_t^H} \\ +\tilde{R}_t^K \tilde{K}_{it-1}^o + \frac{R_{t-1}^A \tilde{A}_{it-1}^o}{\Gamma_t^H} + \widetilde{D}_{it}^{FI,o} \\ +\frac{\tilde{\Xi}_t}{\omega_o} - \tilde{P}_t^C \tilde{C}_{it}^o - \tilde{\mathcal{M}}_{it}^o - \tilde{A}_{it}^o - \tilde{P}_t^I \tilde{I}_{it}^o \\ -\tilde{P}_t^{IO} \tilde{I}_{it}^{O,o} - \tilde{T}_{it}^o - \tilde{P}_t^C \tilde{\chi}_t^C \tilde{T}rans_i^o \end{array} \right] \\ -\tilde{\Lambda}_{it}^K \left[\Gamma_{t+1}^H \tilde{K}_{it}^o - \left(1-\delta^K\right)\tilde{K}_{it-1}^o - \Psi_I\left(\tilde{\zeta}_{it}^{I,o}, \varepsilon_t^I\right)\tilde{I}_{it}^o\right] \end{array} \right\} \tag{A.17}$$

where $\tilde{\Lambda}_{it}^C$ and $\tilde{\Lambda}_{it}^K$ are the Lagrange multipliers associated with the households' resource constraint (A.4) and capital accumulation equation (A.11) respectively. The term, Θ_t, is a discount factor defined in the following way:

$$\Theta_t = \Theta_{t-1}\mathbb{B}\left(\frac{\tilde{C}_{t-1}^o/\tilde{\chi}_{t-1}^Z}{C^o}\right) \tag{A.18}$$

where $\mathbb{B}$ is a function of the ratio of *aggregate* per capita consumption of optimising households, relative to its steady state level. We assume that the function is specified so that $\mathbb{B}(1) = \beta$ and has elasticity ϵ_β with respect to its argument when evaluated at steady state. Since Θ_t depends on *aggregate* per capita consumption of optimising households, each individual household treats it parametrically. The 'endogenous discount factor' is included in the model to ensure that the model returns to a unique steady state net foreign asset position following temporary shocks. There are a range of technical assumptions of this type that can be made to deliver this results: Schmitt-Grohe and Uribe (2003) argue that these approaches deliver similar quantitative properties if suitably parameterised.[189]

The first-order conditions for consumption, labour supply, money demand, savings, investment and capital are given by equations (A.19), (A.21), (A.26), (A.28), (A.29) & (A.32) respectively.
The first-order condition for consumption is:

$$\frac{\tilde{U}_{it}^{\tilde{C}_i^o}}{\tilde{P}_t^C} = \tilde{\Lambda}_{it}^C \tag{A.19}$$

where:

$$\tilde{U}_{it}^{\tilde{C}_i^o} = \left(\frac{\tilde{C}_{it}^o}{\tilde{\chi}_t^Z} - \psi_C \frac{\tilde{C}_{t-1}^o}{\tilde{\chi}_{t-1}^Z}\right)^{-\epsilon_C} \frac{1}{\tilde{\chi}_t^Z} \tag{A.20}$$

The first order condition for labour supply is:

$$\begin{aligned} \widetilde{MRS}_{it}^o \frac{\mu_t^W}{\mu_t^W - 1}\frac{L_{it}^o}{\tilde{W}_{it}^o} &= \frac{1}{\mu_t^W-1}\Psi_W\left(\frac{\tilde{W}_{it}^o}{\tilde{W}_{it-1}^o}\right)L_{it}^o - \Psi_W'\left(\frac{\tilde{W}_{it}^o}{\tilde{W}_{it-1}^o}\right)\tilde{W}_{it}^o L_{it}^o \\ &- \mathrm{E}_t\left[\frac{\Theta_{t+1}\tilde{\chi}_{t+1}^H \tilde{\Lambda}_{it+1}^C}{\Theta_t \tilde{\chi}_t^H \tilde{\Lambda}_{it}^C}\Psi_W'\left(\frac{\tilde{W}_{it+1}^o}{\tilde{W}_{it}^o}\right)\tilde{W}_{it+1}^o L_{it+1}^o\right] \end{aligned} \tag{A.21}$$

[189] We calibrate ϵ_β to be small, so that it does not play an important role in determining the quantitative properties of the model.

where the marginal rate of substitution, $\widetilde{MRS}^o_{it}$, is defined as:

$$\widetilde{MRS}^o_{it} = \frac{\tilde{U}^{\tilde{L}^o_i}_{it}}{\tilde{\Lambda}^C_{it}} \tag{A.22}$$

and where:

$$\tilde{U}^{L^o_i}_{it} = -\nu_L \varepsilon^L_t \left(L^o_{it}\right)^{\epsilon_L} \tag{A.23}$$

$$\Psi'_W\left(\frac{\tilde{W}^o_{it}}{\tilde{W}^o_{it-1}}\right) = -\frac{\phi_W\left(\zeta^{W,o}_{it}-1\right)\zeta^{W,o}_{it}}{\left(\mu^W-1\right)\tilde{W}^o_{it}} \tag{A.24}$$

$$\Psi'_W\left(\frac{\tilde{W}^o_{it+1}}{\tilde{W}^o_{it}}\right) = \frac{\phi_W\left(\zeta^{W,o}_{it+1}-1\right)\zeta^{W,o}_{it+1}}{\left(\mu^W-1\right)\tilde{W}^o_{it}} \tag{A.25}$$

The first order condition for money balances is:

$$\tilde{U}^{\tilde{\mathcal{M}}^o_i}_{it} = \frac{R^A_t-1}{R^A_t}\tilde{\Lambda}^C_{it} \tag{A.26}$$

where:

$$\tilde{U}^{\tilde{\mathcal{M}}^o_i}_{it} = \nu_M\left(\frac{\tilde{\mathcal{M}}^o_{it}}{\tilde{\chi}^Z_t\tilde{P}^C_t}\right)^{-\epsilon_C}\frac{1}{\tilde{\chi}^Z_t\tilde{P}^C_t} \tag{A.27}$$

The first order condition for deposits is:

$$\tilde{\Lambda}^C_{it} = \mathrm{E}_t\left[\frac{\Theta_{t+1}}{\Theta_t}\tilde{\Lambda}^C_{it+1}R^A_t\right] \tag{A.28}$$

The first order condition for investment is:

$$\begin{aligned}\tilde{P}^I_{it} = {} & \frac{\tilde{\Lambda}^K_{it}}{\tilde{\Lambda}^C_{it}}\left[\Psi_I\left(\tilde{\zeta}^{I,o}_{it},\varepsilon^I_t\right)+\Psi'_I\left(\tilde{\zeta}^{I,o}_{it},\varepsilon^I_t\right)\tilde{I}^o_{it}\right] \\ & + \frac{\Theta_{t+1}}{\Theta_t}\frac{\tilde{\chi}^H_{t+1}}{\tilde{\chi}^H_t}\frac{\tilde{\Lambda}^C_{it+1}}{\tilde{\Lambda}^C_{it}}\frac{\tilde{\Lambda}^K_{it+1}}{\tilde{\Lambda}^C_{it+1}}\Psi'_I\left(\tilde{\zeta}^{I,o}_{it+1},\varepsilon^I_{t+1}\right)\tilde{I}^o_{it+1}\end{aligned} \tag{A.29}$$

where:

$$\Psi'_I\left(\tilde{\zeta}^{I,o}_{it},\varepsilon^I_t\right) = -\psi_I\left(\zeta^{I,o}_{it}-\Gamma^H\Gamma^Z\Gamma^I\right)\frac{\zeta^{I,o}_{it}\varepsilon^I_t}{\tilde{I}^o_{it}} \tag{A.30}$$

$$\Psi'_I\left(\tilde{\zeta}^{I,o}_{it+1},\varepsilon^I_{t+1}\right) = \psi_I\left(\zeta^{I,o}_{it+1}-\Gamma^H\Gamma^Z\Gamma^I\right)\frac{\zeta^{I,o}_{it+1}\varepsilon^I_{t+1}}{\tilde{I}^o_{it}} \tag{A.31}$$

And the first order-condition for capital is the following:

$$\widetilde{TQ}_{it} = \mathrm{E}_t\left[\frac{\Theta_{t+1}}{\Theta_t}\frac{\tilde{\Lambda}^C_{it+1}}{\tilde{\Lambda}^C_{it}}\left\{\tilde{R}^K_{t+1}+\widetilde{TQ}_{it+1}\left(1-\delta^K\right)\right\}\right] \tag{A.32}$$

where Tobin's Q, $\widetilde{TQ}_{it}$, is defined as:

$$\widetilde{TQ}_{it} = \frac{\tilde{\Lambda}^K_{it}}{\tilde{\Lambda}^C_{it}} \tag{A.33}$$

Finally, rule-of-thumb or non-optimising households are assumed to consume their labour income plus a transfer from the government:

$$\tilde{P}_t^C \tilde{C}_{it}^{rot} = \tilde{W}_{it}^{rot} L_{it}^{rot} + \tilde{P}_t^C \tilde{\chi}_t^C \tilde{T}rans_i^{rot} \tag{A.34}$$

Note that it is assumed that rule-of-thumb households take the wage set by optimising households as given, and are prepared to supply any labour demanded under that wage. This means that (in a symmetric equilibrium) rule-of-thumb households supply the same amount of labour as optimising households.

A.1.2 Portfolio packagers

There exists a continuum of perfectly competitive portfolio packagers, defined on the unit interval. These agents allocate household saving across the available assets: government and foreign bonds. An individual portfolio packager, indexed by j, receives deposits (from optimising households), $\tilde{A}_{jt}$ which they invest in domestic one-period government bonds, $\tilde{B}_{jt}$, and foreign one-period nominal bonds, $\tilde{B}_{jt}^F$, on households' behalf. Domestic and foreign bonds pay nominal one-period returns of R_t & R_t^F respectively. The balance sheet constraint of the portfolio packager is therefore:

$$\tilde{A}_{jt} = \tilde{B}_{jt} + \frac{\tilde{B}_{jt}^F}{\varepsilon_t^{B^F} \tilde{Q}_t} \tag{A.35}$$

which accounts for the fact that foreign bonds are denominated in foreign currency and so investments must be converted into foreign currency via the spot exchange rate market: $\tilde{Q}$ denotes the domestic price of foreign currency (the nominal spot exchange rate).[190] Investments in foreign bonds are also subject to an additional foreign exchange risk shock given by:

$$\begin{aligned} \log \varepsilon_t^{B^F} &= (1-\rho_{B^F}) \log \varepsilon^{B^F} + \rho_B \log \varepsilon_{t-1}^{B^F} + \left(1-\rho_{B^F}^2\right)^{1/2} \sigma_{B^F} \eta_t^{B^F} \\ \eta_t^{B^F} &\sim N(0,1) \end{aligned} \tag{A.36}$$

The portfolio packager pays a return R_t^A on assets invested with it, though this is subject to an exogenous ('risk premium') shock, so that the dividend that the packager aims to maximise is:

$$\tilde{D}_{jt}^{FI} \equiv \mathrm{E}_t \left[R_t \tilde{B}_{jt} + \frac{R_t^F \tilde{B}_{jt}^F}{\tilde{Q}_{t+1}} \right] - \frac{R_t^A}{\varepsilon_t^B} \tilde{A}_{jt}$$

subject to (A.35) and where the stochastic shock to returns is given by:

$$\begin{aligned} \log \varepsilon_t^{B} &= (1-\rho_{B}) \log \varepsilon^{B} + \rho_B \log \varepsilon_{t-1}^{B} + \left(1-\rho_{B}^2\right)^{1/2} \sigma_{B} \eta_t^{B} \\ \eta_t^{B} &\sim N(0,1) \end{aligned} \tag{A.37}$$

The first order conditions to the maximisation problem are:

$$\frac{R_t^A}{\varepsilon_t^B} = R_t \tag{A.38}$$

$$\frac{R_t^A}{\varepsilon_t^B \varepsilon_t^{B^F} \tilde{Q}_t} = \mathrm{E}_t \left[\frac{R_t^F}{\tilde{Q}_{t+1}} \right] \tag{A.39}$$

[190] This definition means that an increase in $\tilde{Q}$ represents an appreciation of the domestic currency.

These first order conditions can be combined to yield the following uncovered interest parity condition for the exchange rate:[191]

$$\mathrm{E}_t\left[\frac{\tilde{Q}_{t+1}}{\varepsilon_t^{B^F}\tilde{Q}_t}\right]=\frac{R_t^F}{R_t} \tag{A.40}$$

A.1.3 Firms

There are five types of firms in COMPASS: final output producers, value-added producers, importers, exporters and retailers. Final-output producers, value-added producers, importers and exporters are monopolistic competitive, so have some pricing power but face a cost for changing prices. Retailers are perfectively competitive firms with different technologies applying to each separate sector, giving rise to different trend growth rates for consumption, investment, government and trade. The rest of this sub-section describes each type of firm in turn.

Final output firms

There is a continuum of final output producers defined on the unit interval, indexed by n. The per capita final output production function for firm n is given by:

$$\tilde{Z}_{nt}=\tilde{V}_{nt}^{\alpha_V}\tilde{M}_{nt}^{1-\alpha_V} \tag{A.41}$$

where $\tilde{V}_{nt}$ denotes per capita value added and $\tilde{M}_{nt}$ per capita imported intermediates used by firm n. The prices of the bundle of import goods, $\tilde{P}_t^M$, and value added, $\tilde{P}_t^V$, are taken as given. Final output producers sell a differentiated good to retailers, with firm n facing the following demand function for their goods:

$$\tilde{Z}_{nt}=\left(\frac{\tilde{P}_{nt}^Z}{\tilde{P}_t^Z}\right)^{-\frac{\mu_t^Z}{\mu_t^Z-1}}\tilde{Z}_t \tag{A.42}$$

where $\tilde{P}_{nt}^Z$ is the price of final output set by firm n, $\tilde{P}_t^Z$ and $\tilde{Z}_t$ denote the aggregate price and quantity of final output, and where $\frac{\mu_t^Z}{\mu_t^Z-1}$ is retailers' elasticity of substitution between final goods, which follows the following process:

$$\begin{aligned}\log\mu_t^Z&=\sigma_{\mu^Z}\eta_t^{\mu^Z}\\ \eta_t^{\mu^Z}&\sim N(0,1)\end{aligned} \tag{A.43}$$

The aggregate per capita quantity of final output goods is given by the CES aggregator:

$$\tilde{Z}_t=\left[\int_0^1\left(\tilde{Z}_{nt}^{\frac{1}{\mu_t^Z}}\right)dn\right]^{\mu_t^Z} \tag{A.44}$$

The aggregate final output goods price index is then given by:

$$\tilde{P}_t^Z=\left[\int_0^1\left(\tilde{P}_{nt}^Z\right)^{\frac{1}{1-\mu_t^Z}}dn\right]^{1-\mu_t^Z} \tag{A.45}$$

[191]Note that this is not a function of the 'domestic' risk-premium shock ε_t^B.

Final output firms face a quadratic cost of adjusting prices. The cost for firm n is measured in terms of the final-output good and is proportional to the size of the sector as a whole. It is given by:[192]

$$\Phi_Z\left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_{nt-1}}, \tilde{P}^Z_t \tilde{Z}_t\right) = \frac{\phi_Z}{2\left(\mu^Z - 1\right)}\left(\zeta^Z_{nt} - 1\right)^2 \tilde{P}^Z_t \tilde{Z}_t \tag{A.46}$$

where:

$$\zeta^Z_{nt} \equiv \frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_{nt-1}\left(\Pi^Z\right)^{1-\xi_Z}\left(\Pi^Z_{t-1}\right)^{\xi_Z}} \tag{A.47}$$

where ϕ_Z is a parameter that determines the cost of price adjustment. There is full and costless indexation to either the steady state inflation rate, Π^Z, or to lagged final output price inflation, Π^Z_{t-1}, which ensures that price adjustment costs have no impact on the steady state.

The problem firm n solves in any arbitrary period s is:

$$\max_{\tilde{V}_{nt},\, \tilde{M}_{nt},\, \tilde{P}^Z_{nt}} \mathrm{E}_s\left[\sum_{t=s}^{\infty}\Theta_t \tilde{\chi}^H_t \frac{\tilde{\Lambda}^C_t}{\tilde{\Lambda}^C_s}\left(\tilde{P}^Z_{nt}\tilde{Z}_{nt} - \tilde{P}^V_t \tilde{V}_{nt} - \tilde{P}^M_t \varepsilon^M_t \tilde{M}_{nt} - \Phi_Z\left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_{nt-1}}, \tilde{P}^Z_t \tilde{Z}_t\right)\right)\right] \tag{A.48}$$

subject to the production function (A.41) and final output demand (A.42), where ε^M_t is a disturbance to the demand for imports that satisfies:

$$\log \varepsilon^M_t = (1-\rho_M)\log \varepsilon^M + \rho_M \log \varepsilon^M_{t-1} + \left(1-\rho^2_M\right)^{1/2}\sigma_M \eta^M_t \tag{A.49}$$

$$\eta^M_t \sim N(0,1) \tag{A.50}$$

The relevant Lagrangian for firm n in any arbitrary period t can be written as:

$$\begin{aligned}
\mathcal{L}_{nt} &= \tilde{P}^Z_{nt}\left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_t}\right)^{-\frac{\mu^Z_t}{\mu^Z_t-1}}\tilde{Z}_t - \tilde{P}^V_t \tilde{V}_{nt} - \tilde{P}^M_t \varepsilon^M_t \tilde{M}_{nt} \\
&- \widetilde{MC}^Z_{nt}\left[\left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_t}\right)^{-\frac{\mu^Z_t}{\mu^Z_t-1}}\tilde{Z}_t - \tilde{V}^{\alpha_V}_{nt}\tilde{M}^{1-\alpha_V}_{nt}\right] \\
&- \frac{\phi_Z}{2\left(\mu^Z-1\right)}\left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_{n,t-1}\left(\Pi^Z\right)^{1-\xi_Z}\left(\Pi^Z_{t-1}\right)^{\xi_Z}} - 1\right)^2 \tilde{P}^Z_t \tilde{Z}_t \\
&+ \mathrm{E}_t\left[\Gamma^H_{t+1}\frac{\Theta_{t+1}}{\Theta_t}\frac{\tilde{\Lambda}^C_{t+1}}{\tilde{\Lambda}^C_t}\left\{\begin{array}{c}
\tilde{P}^Z_{nt+1}\left(\frac{\tilde{P}^Z_{nt+1}}{\tilde{P}^Z_{t+1}}\right)^{-\frac{\mu^Z_{t+1}}{\mu^Z_{t+1}-1}}\tilde{Z}_{t+1} - \tilde{P}^V_{t+1}\tilde{V}_{nt+1} \\
-\tilde{P}^M_{t+1}\varepsilon^M_{t+1}\tilde{M}_{nt+1} \\
-\widetilde{MC}^Z_{nt+1}\left[\left(\frac{\tilde{P}^Z_{nt+1}}{\tilde{P}^Z_{t+1}}\right)^{-\frac{\mu^Z_{t+1}}{\mu^Z_{t+1}-1}}\tilde{Z}_{t+1} - \tilde{V}^{\alpha_V}_{nt+1}\tilde{M}^{1-\alpha_V}_{nt+1}\right] \\
-\frac{\phi_Z}{2(\mu^Z-1)}\left(\frac{\tilde{P}^Z_{nt+1}}{\tilde{P}^Z_{nt}(\Pi^Z)^{1-\xi_Z}\left(\Pi^Z_t\right)^{\xi_Z}} - 1\right)^2 \tilde{P}^Z_{t+1}\tilde{Z}_{t+1}
\end{array}\right\}\right] \\
&+ \ \ldots
\end{aligned} \tag{A.51}$$

[192]The log-linearised solution is identical if we assume that costs for firm n are proportional to nominal final output of firm n. We therefore choose the simpler setup.

where $\widetilde{MC}^Z_{nt}$ is the Lagrange multiplier on the constraint that output satisfies demand. This represents the shadow cost of one additional unit of output for the firm, which equals the nominal marginal cost.

The first-order condition for value-added demand is:

$$\alpha_V \widetilde{MC}^Z_{nt} \tilde{V}_{nt}^{\alpha_V - 1} \tilde{M}_{nt}^{1-\alpha_V} = \tilde{P}_t^V$$

which can be written as:

$$\widetilde{MC}^Z_{nt} = \frac{\tilde{P}_t^V \tilde{V}_{nt}}{\alpha_V \tilde{Z}_{nt}} \tag{A.52}$$

The first order-condition for import demand is:

$$0 = -\tilde{P}_t^M \varepsilon_t^M + (1-\alpha_V)\, \widetilde{MC}^Z_{nt} \tilde{V}_{nt}^{\alpha_V} \tilde{M}_{nt}^{-\alpha_V}$$

which can be written as:

$$\widetilde{MC}^Z_{nt} = \frac{\tilde{P}_t^M \tilde{M}_{nt}}{(1-\alpha_V)\, \tilde{Z}_{nt}} \varepsilon_t^M \tag{A.53}$$

And the first-order condition for the final output price is:

$$\begin{aligned}
0 &= \left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_t}\right)^{-\frac{\mu_t^Z}{\mu_t^Z-1}} \tilde{Z}_t - \frac{\tilde{P}^Z_{nt}\mu_t^Z}{\mu_t^Z - 1}\left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_t}\right)^{-\frac{\mu_t^Z}{\mu_t^Z-1}-1} \frac{\tilde{Z}_t}{\tilde{P}^Z_t} + \frac{\mu_t^Z}{\mu_t^Z-1}\widetilde{MC}^Z_{nt}\left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_t}\right)^{-\frac{\mu_t^Z}{\mu_t^Z-1}-1} \frac{\tilde{Z}_t}{\tilde{P}^Z_t} \\
&- \frac{\phi_Z}{(\mu^Z - 1)}\left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_{n,t-1}\left(\Pi^Z\right)^{1-\xi_Z}\left(\Pi^Z_{t-1}\right)^{\xi_Z}} - 1\right)\frac{\tilde{Z}_t \tilde{P}^Z_t}{\tilde{P}^Z_{n,t-1}\left(\Pi^Z\right)^{1-\xi_Z}\left(\Pi^Z_{t-1}\right)^{\xi_Z}} \\
&+ \mathrm{E}_t\left[\Gamma^H_{t+1}\frac{\Theta_{t+1}}{\Theta_t}\frac{\tilde{\Lambda}^C_{t+1}}{\tilde{\Lambda}^C_t}\frac{\phi_Z}{(\mu^Z-1)}\left(\frac{\tilde{P}^Z_{nt+1}}{\tilde{P}^Z_{nt}\left(\Pi^Z\right)^{1-\xi_Z}\left(\Pi^Z_t\right)^{\xi_Z}} - 1\right)\frac{\tilde{P}^Z_{t+1}\tilde{Z}_{t+1}\tilde{P}^Z_{n,t+1}}{\left(\tilde{P}^Z_{nt}\right)^2\left(\Pi^Z\right)^{1-\xi_Z}\left(\Pi^Z_t\right)^{\xi_Z}}\right]
\end{aligned}$$

$$\begin{aligned}
0 &= \left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_t}\right)^{-\frac{\mu_t^Z}{\mu_t^Z-1}} \tilde{Z}_t - \frac{\tilde{P}^Z_{nt}\mu_t^Z}{\mu_t^Z - 1}\left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_t}\right)^{-\frac{\mu_t^Z}{\mu_t^Z-1}-1} \frac{\tilde{Z}_t}{\tilde{P}^Z_t} + \frac{\mu_t^Z}{\mu_t^Z-1}\widetilde{MC}^Z_{nt}\left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_t}\right)^{-\frac{\mu_t^Z}{\mu_t^Z-1}-1} \frac{\tilde{Z}_t}{\tilde{P}^Z_t} \\
&- \frac{\phi_Z \tilde{Z}_t \tilde{P}^Z_t}{(\mu^Z-1)\tilde{P}^Z_{nt}}\left\{\left(\zeta^Z_{nt} - 1\right)\zeta^Z_{nt} - \mathrm{E}_t\left[\frac{\Gamma^H_{t+1}\Theta_{t+1}\tilde{\Lambda}^C_{t+1}}{\Theta_t\tilde{\Lambda}^C_t}\left(\zeta^Z_{nt+1} - 1\right)\zeta^Z_{nt+1}\frac{\tilde{P}^Z_{t+1}\tilde{Z}_{t+1}}{\tilde{P}^Z_t\tilde{Z}_t}\right]\right\}
\end{aligned} \tag{A.54}$$

Finally, note that setting the price adjustment cost parameter ϕ_Z to zero makes prices in the final output sector perfectly flexible, in which case the first-order condition for final output prices can be written as follows:

$$\frac{\mu_t^Z}{\mu_t^Z-1}\widetilde{MC}^Z_{nt}\left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_t}\right)^{-\frac{\mu_t^Z}{\mu_t^Z-1}-1}\frac{\tilde{Z}_t}{\tilde{P}^Z_t} = \frac{\mu_t^Z}{\mu_t^Z-1}\tilde{P}^Z_{nt}\left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_t}\right)^{-\frac{\mu_t^Z}{\mu_t^Z-1}-1}\frac{\tilde{Z}_t}{\tilde{P}^Z_t} - \left(\frac{\tilde{P}^Z_{nt}}{\tilde{P}^Z_t}\right)^{-\frac{\mu_t^Z}{\mu_t^Z-1}}\tilde{Z}_t \tag{A.55}$$

In a symmetric equilibrium, this reduces to:

$$\tilde{P}^Z_{nt} = \widetilde{MC}^Z_{nt}\mu_t^Z \tag{A.56}$$

Value-added firms

There is a continuum of value-added producing firms defined on the unit interval and indexed by j. The value added production function of firm j is given by:

$$\tilde{V}_{jt} = \varepsilon_t^{TFP} \tilde{K}_{jt}^{1-\alpha_L} \left(\tilde{\chi}_t^{LAP} L_{jt}\right)^{\alpha_L} \tag{A.57}$$

where ε_t^{TFP} is a temporary total factor productivity disturbance common to all value added firms, $\tilde{K}_{jt}$ denotes physical capital used by firm j, $\tilde{\chi}_t^{LAP}$ is a measure of labour-augmenting technology (referred to as LAP), so that $\tilde{\chi}_t^{LAP} L_{jt}$ is the effective labour input used by firm j in period t. The exogenous processes for the two technology terms are:

$$\log \varepsilon_t^{TFP} = (1-\rho_{TFP}) \log \varepsilon^{TFP} + \rho_{TFP} \log \varepsilon_{t-1}^{TFP} + \left(1-\rho_{TFP}^2\right)^{1/2} \sigma_{TFP} \eta_t^{TFP} \tag{A.58}$$

$$\eta_t^{TFP} \sim N(0,1)$$

$$\tilde{\chi}_t^{LAP} = \Gamma^{LAP} \tilde{\chi}_{t-1}^{LAP} \exp\left(\hat{\varepsilon}_t^{LAP}\right) \tag{A.59}$$

$$\hat{\varepsilon}_t^{LAP} = \rho_{LAP} \hat{\varepsilon}_{t-1}^{LAP} + \left(1-\rho_{LAP}^2\right)^{1/2} \sigma_{LAP} \eta_t^{LAP} \tag{A.60}$$

$$\eta_t^{LAP} \sim N(0,1)$$

Value-added firms sell a differentiated good to final output producers with firm j facing the demand schedule:

$$\tilde{V}_{jt} = \left(\frac{\tilde{P}_{jt}^V}{\tilde{P}_t^V}\right)^{-\frac{\mu_t^V}{\mu_t^V-1}} \tilde{V}_t \tag{A.61}$$

where $\tilde{P}_{jt}^V$ is the price set by firm j, $\tilde{V}_t$ and $\tilde{P}_t^V$ and are the aggregate per capita quantity and price of value added and μ_t^V fulfills:

$$\log \mu_t^V = \sigma_{\mu^V} \eta_t^{\mu^V} \tag{A.62}$$

$$\eta_t^{\mu^V} \sim N(0,1)$$

The aggregate quantity of per capita value-added goods is given by the CES aggregator:

$$\tilde{V}_t = \left[\int_0^1 \left(\tilde{V}_{jt}^{\frac{1}{\mu_t^V}}\right) dj\right]^{\mu_t^V} \tag{A.63}$$

and the aggregate value-added price index is given by:

$$\tilde{P}_t^V = \left[\int_0^1 \left(\tilde{P}_{jt}^V\right)^{\frac{1}{1-\mu_t^V}} dj\right]^{1-\mu_t^V} \tag{A.64}$$

Value-added producers face a quadratic cost of adjusting prices, similar to that for final-output producers, where ϕ_V is the parameter that determines the size of the adjustment cost in the value-added sector and where there is full indexation to either steady state value added inflation, Π^V, or to a lagged measure of value-added inflation. The adjustment cost function takes the following form:

$$\Phi_V\left(\frac{\tilde{P}_{jt}^V}{\tilde{P}_{jt-1}^V}, \tilde{P}_t^V \tilde{V}_t\right) = \frac{\phi_V}{2\left(\mu^Z-1\right)} \left(\zeta_{jt}^V - 1\right)^2 \tilde{P}_t^V \tilde{V}_t \tag{A.65}$$

where:

$$\zeta_{jt}^{V} \equiv \frac{\tilde{P}_{jt}^{V}}{\tilde{P}_{jt-1}^{V}\left(\Pi^{V}\right)^{1-\xi_V}\left(\Pi_{t-1}^{V}\right)^{\xi_V}} \tag{A.66}$$

Each firm j solves an analogous probem to that presented above for final output producers, which can be expressed using the following Lagrangian:

$$\begin{aligned}
\mathcal{L}_{jt} \;=\; & \tilde{P}_{jt}^{V}\left(\frac{\tilde{P}_{jt}^{V}}{\tilde{P}_{t}^{V}}\right)^{-\frac{\mu_t^V}{\mu_t^V-1}}\tilde{V}_t-\tilde{W}_tL_{jt}-\tilde{R}_t^K\tilde{K}_{jt} && \text{(A.67)}\\
- \; & \widetilde{MC}_{jt}^{V}\left(\left(\frac{\tilde{P}_{jt}^{V}}{\tilde{P}_{t}^{V}}\right)^{-\frac{\mu_t^V}{\mu_t^V-1}}\tilde{V}_t-\varepsilon_t^{TFP}\tilde{K}_{jt}^{1-\alpha_L}\left(\tilde{\chi}_t^{LAP}L_{jt}\right)^{\alpha_L}\right)\\
- \; & \frac{\phi_V}{2\left(\mu^V-1\right)}\left(\frac{\tilde{P}_{jt}^{V}}{\tilde{P}_{jt-1}^{V}\left(\Pi^V\right)^{(1-\xi_V)}\left(\Pi_{t-1}^{V}\right)^{\xi_V}}-1\right)^2\tilde{P}_t^V\tilde{V}_t\\
+ \; & \mathrm{E}_t\left[\Gamma_{t+1}^{H}\frac{\Theta_{t+1}}{\Theta_t}\frac{\tilde{\Lambda}_{t+1}^{C}}{\tilde{\Lambda}_{t}^{C}}\left\{\begin{array}{c}
\left(\frac{\tilde{P}_{jt+1}^{V}}{\tilde{P}_{t+1}^{V}}\right)^{-\frac{\mu_{t+1}^V}{\mu_{t+1}^V-1}}\tilde{V}_{t+1}-\tilde{W}_{t+1}L_{j,t+1}-\tilde{R}_{t+1}^K\tilde{K}_{j,t+1}\\
-\widetilde{MC}_{jt+1}^{V}\left(\begin{array}{c}\left(\frac{\tilde{P}_{jt+1}^{V}}{\tilde{P}_{t+1}^{V}}\right)^{-\frac{\mu_{t+1}^V}{\mu_{t+1}^V-1}}\tilde{V}_{t+1}-\\ \varepsilon_{t+1}^{TFP}\tilde{K}_{jt+1}^{1-\alpha_L}\left(\tilde{\chi}_{t+1}^{LAP}L_{jt+1}\right)^{\alpha_L}\end{array}\right)\\
-\frac{\phi_V}{2(\mu^V-1)}\left(\frac{\tilde{P}_{jt+1}^{V}}{\tilde{P}_{jt}^{V}(\Pi^V)^{(1-\xi_V)}\left(\Pi_t^V\right)^{\xi_V}}-1\right)^2\tilde{P}_{t+1}^{V}\tilde{V}_{t+1}
\end{array}\right\}\right]\\
+ \; & \ldots
\end{aligned}$$

where $\widetilde{MC}_{jt}^{V}$ is the shadow cost of one additional unit of output for the firm, equivalent to the nominal marginal cost.

The first-order conditions for labour, capital, and the price of value-added are given by equations (A.68), (A.69) & (A.70) respectively.
Labour demand:

$$\begin{aligned}
0 &= -\tilde{W}_t+\widetilde{MC}_{jt}^{V}\varepsilon_t^{TFP}\tilde{K}_{jt}^{1-\alpha_L}\left(\tilde{\chi}_t^{LAP}\right)^{\alpha_L}\alpha_L L_{jt}^{\alpha_L-1}\\
\widetilde{MC}_{jt}^{V} &= \frac{\tilde{W}_tL_{jt}}{\alpha_L\tilde{V}_{jt}} && \text{(A.68)}
\end{aligned}$$

Capital demand:

$$\begin{aligned}
0 &= -\tilde{R}_t^K+\widetilde{MC}_{jt}^{V}\left(1-\alpha_L\right)\varepsilon_t^{TFP}\tilde{K}_{jt}^{-\alpha_L}\left(\tilde{\chi}_t^{LAP}L_{jt}\right)^{\alpha_L}\\
\widetilde{MC}_{jt}^{V} &= \frac{\tilde{R}_t^K\tilde{K}_{jt}}{\left(1-\alpha_L\right)\tilde{V}_{jt}} && \text{(A.69)}
\end{aligned}$$

Value-added price:[193]

$$
\begin{aligned}
0 \;=\;& \left(\frac{\tilde{P}^V_{jt}}{\tilde{P}^V_t}\right)^{-\frac{\mu^V_t}{\mu^V_t-1}} \tilde{V}_t - \frac{\mu^V_t}{\mu^V_t-1}\tilde{P}^V_{jt}\left(\frac{\tilde{P}^V_{jt}}{\tilde{P}^V_t}\right)^{-\frac{\mu^V_t}{\mu^V_t-1}-1}\frac{\tilde{V}_t}{\tilde{P}^V_t} \\
&+ \frac{\mu^V_t}{\mu^V_t-1}\widetilde{MC}^V_{jt}\left(\frac{\tilde{P}^V_{jt}}{\tilde{P}^V_t}\right)^{-\frac{\mu^V_t}{\mu^V_t-1}-1}\frac{\tilde{V}_t}{\tilde{P}^V_t} \\
&- \frac{\phi_V}{(\mu^V-1)}\left(\frac{\tilde{P}^V_{jt}}{\tilde{P}^V_{jt-1}\left(\Pi^V\right)^{(1-\xi_V)}\left(\Pi^V_{t-1}\right)^{\xi_V}}-1\right)\frac{\tilde{P}^V_t\tilde{V}_t}{\tilde{P}^V_{jt-1}\left(\Pi^V\right)^{(1-\xi_V)}\left(\Pi^V_{t-1}\right)^{\xi_V}} \\
&+ \mathrm{E}_t\left\{\begin{array}{c}\Gamma^H_{t+1}\frac{\Theta_{t+1}}{\Theta_t}\frac{\tilde{\Lambda}^C_{t+1}}{\tilde{\Lambda}^C_t}\frac{\phi_V}{(\mu^V-1)}\left(\frac{\tilde{P}^V_{jt+1}}{\tilde{P}^V_{jt}\left(\Pi^V\right)^{(1-\xi_V)}\left(\Pi^V_t\right)^{\xi_V}}-1\right)^2 \\ \times\frac{\tilde{P}^V_{t+1}\tilde{V}_{t+1}\tilde{P}^V_{jt+1}}{\left(\tilde{P}^V_{jt}\right)^2\left(\Pi^V\right)^{(1-\xi_V)}\left(\Pi^V_t\right)^{\xi_V}}\end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& \left(\frac{\tilde{P}^V_{jt}}{\tilde{P}^V_t}\right)^{-\frac{\mu^V_t}{\mu^V_t-1}} \tilde{V}_t - \frac{\mu^V_t\tilde{P}^V_{jt}}{\mu^V_t-1}\left(\frac{\tilde{P}^V_{jt}}{\tilde{P}^V_t}\right)^{-\frac{\mu^V_t}{\mu^V_t-1}-1}\frac{\tilde{V}_t}{\tilde{P}^V_t} + \frac{\mu^V_t\widetilde{MC}^V_{jt}}{\mu^V_t-1}\left(\frac{\tilde{P}^V_{jt}}{\tilde{P}^V_t}\right)^{-\frac{\mu^V_t}{\mu^V_t-1}-1}\frac{\tilde{V}_t}{\tilde{P}^V_t} \\
&- \frac{\phi_V\tilde{P}^V_t\tilde{V}_t}{(\mu^V-1)\tilde{P}^V_{jt}}\left\{\left(\zeta^V_{jt}-1\right)\zeta^V_{jt} - \mathrm{E}_t\left[\frac{\Gamma^H_{t+1}\Theta_{t+1}\tilde{\Lambda}^C_{t+1}}{\Theta_t\tilde{\Lambda}^C_t}\left(\zeta^V_{jt+1}-1\right)\zeta^V_{jt+1}\frac{\tilde{P}^V_{t+1}\tilde{V}_{t+1}}{\tilde{P}^V_t\tilde{V}_t}\right]\right\} \qquad \text{(A.70)}
\end{aligned}
$$

Using equations (A.68) & (A.69) it is possible to show that the marginal cost for firm j is given by a weighted average of the factor prices:[194]

$$
\widetilde{MC}^V_{jt} = \frac{\tilde{W}^{\alpha_L}_t\left(\tilde{R}^K_t\right)^{1-\alpha_L}}{\left(\alpha_L\right)^{\alpha_L}\left(1-\alpha_L\right)^{1-\alpha_L}\varepsilon^{TFP}_t\left(\tilde{\chi}^{LAP}_t\right)^{\alpha_L}} \qquad \text{(A.71)}
$$

Importers

There is a continuum of importing firms defined on the unit interval indexed by f that buy a homogenous tradeable good on the world market at a price $\tilde{P}^{XF}_t$ (or $\tilde{P}^{XF}_t/\tilde{Q}_t$ in domestic currency). They differentiate the generic world export good and sell it on to final-output producers, facing demand given by:

$$
\tilde{M}_{ft} = \left(\frac{\tilde{P}^M_{ft}}{\tilde{P}^M_t}\right)^{-\frac{\mu^M_t}{\mu^M_t-1}}\tilde{M}_t \qquad \text{(A.72)}
$$

where $\tilde{M}_t$ is a per-capita measure of imports, $\tilde{P}^M_{ft}$ are import prices set by firm f (expressed in domestic currency) and μ^M_t fulfills:

$$
\begin{aligned}
\log\mu^M_t &= \sigma_{\mu^M}\eta^{\mu^M}_t \qquad \text{(A.73)} \\
\eta^{\mu^M}_t &\sim N(0,1)
\end{aligned}
$$

[193]It is straightforward to derive the flexible-price equivalent (where $\phi_V = 0$) in the same way as for final output pricing.

[194]And there is a similar expression for the marginal cost of final output prices.

Aggregate per capita imports are given by the CES aggregator:

$$\tilde{M}_t = \left[\int_0^1 \left(\tilde{M}_{ft}\right)^{\frac{1}{\mu_t^M}} df\right]^{\mu_t^M} \tag{A.74}$$

and the aggregate import price index is:

$$\tilde{P}_t^M = \left[\int_0^1 \left(\tilde{P}_{ft}^M\right)^{\frac{1}{1-\mu_t^M}} df\right]^{1-\mu_t^M} \tag{A.75}$$

As in the case of final output and value-added producers, importers face a quadratic cost of adjusting their prices measured in terms of the differentiated import and given by:

$$\Phi_M\left(\frac{\tilde{P}_{ft}^M}{\tilde{P}_{ft-1}^M}, \tilde{P}_t^M \tilde{M}_t\right) = \frac{\phi_M}{2\left(\mu^M - 1\right)}\left(\zeta_{ft}^M - 1\right)^2 \tilde{P}_t^M \tilde{M}_t \tag{A.76}$$

where:

$$\zeta_{ft}^M \equiv \frac{\tilde{P}_{ft}^M}{\tilde{P}_{f,t-1}^M \left(\Pi^M\right)^{(1-\xi_M)} \left(\Pi_{t-1}^M\right)^{\xi_M}} \tag{A.77}$$

where ϕ_M is a parameter that determines the size of the adjustment cost in the import sector and where there is full indexation of import prices to either steady state import price inflation or to lagged import price inflation.
Firm f maximises the discounted flow of profits defined by:

$$\begin{aligned}
\mathcal{D}_{ft} &= \tilde{P}_{ft}^M \left(\frac{\tilde{P}_{ft}^M}{\tilde{P}_t^M}\right)^{-\frac{\mu_t^M}{\mu_t^M-1}} \tilde{M}_t - \frac{\tilde{P}_t^{X^F}}{\tilde{Q}_t}\left(\frac{\tilde{P}_{ft}^M}{\tilde{P}_t^M}\right)^{-\frac{\mu_t^M}{\mu_t^M-1}} \tilde{M}_t \\
&- \frac{\phi_M}{2\left(\mu^M-1\right)}\left(\frac{\tilde{P}_{ft}^M}{\tilde{P}_{ft-1}^M\left(\Pi^M\right)^{1-\xi_M}\left(\Pi_{t-1}^M\right)^{\xi_M}} - 1\right)^2 \tilde{P}_t^M \tilde{M}_t \\
&+ \mathrm{E}_t\left[\Gamma_{t+1}^H \frac{\Theta_{t+1}}{\Theta_t}\frac{\tilde{\Lambda}_{t+1}^C}{\tilde{\Lambda}_t^C}\left\{\begin{array}{c}
\tilde{P}_{ft+1}^M\left(\frac{\tilde{P}_{ft+1}^M}{\tilde{P}_{t+1}^M}\right)^{-\frac{\mu_{t+1}^M}{\mu_{t+1}^M-1}}\tilde{M}_{t+1} \\
-\frac{\tilde{P}_{t+1}^{X^F}}{\tilde{Q}_{t+1}}\left(\frac{\tilde{P}_{ft+1}^M}{\tilde{P}_{t+1}^M}\right)^{-\frac{\mu_{t+1}^M}{\mu_{t+1}^M-1}}\tilde{M}_{t+1} \\
-\frac{\phi_M}{2(\mu^M-1)}\left(\frac{\tilde{P}_{ft+1}^M}{\tilde{P}_{ft}^M\left(\Pi^M\right)^{1-\xi_M}\left(\Pi_t^M\right)^{\xi_M}} - 1\right)^2 \tilde{P}_{t+1}^M\tilde{M}_{t+1}
\end{array}\right\}\right] \\
&+ \ldots
\end{aligned} \tag{A.78}$$

The first-order condition for import pricing is as follows:

$$
\begin{aligned}
0 &= \left(\frac{\tilde{P}_{ft}^{M}}{\tilde{P}_{t}^{M}}\right)^{-\frac{\mu_t^M}{\mu_t^M-1}} \tilde{M}_t - \frac{\mu_t^M}{\mu_t^M-1}\tilde{P}_{ft}^{M}\left(\frac{\tilde{P}_{ft}^{M}}{\tilde{P}_{t}^{M}}\right)^{-\frac{\mu_t^M}{\mu_t^M-1}-1}\frac{\tilde{M}_t}{\tilde{P}_t^M} \\
&+ \frac{\mu_t^M}{\mu_t^M-1}\frac{\tilde{P}_t^{X^F}}{\tilde{Q}_t}\left(\frac{\tilde{P}_{ft}^{M}}{\tilde{P}_{t}^{M}}\right)^{-\frac{\mu_t^M}{\mu_t^M-1}}\frac{\tilde{M}_t}{\tilde{P}_t^M} \\
&- \frac{\phi_M}{(\mu^M-1)}\left(\frac{\tilde{P}_{ft}^{M}}{\tilde{P}_{ft-1}^{M}\left(\Pi^M\right)^{1-\xi_M}\left(\Pi_{t-1}^M\right)^{\xi_M}}-1\right)\frac{\tilde{P}_t^M\tilde{M}_t}{\tilde{P}_{ft-1}^{M}\left(\Pi^M\right)^{1-\xi_M}\left(\Pi_{t-1}^M\right)^{\xi_M}} \\
&+ \mathrm{E}_t\left[\begin{array}{c}\frac{\Gamma_{t+1}^H\Theta_{t+1}\tilde{\Lambda}_{t+1}^C}{\Theta_t\tilde{\Lambda}_t^C}\frac{\phi_M}{(\mu^M-1)}\left(\frac{\tilde{P}_{ft+1}^M}{\tilde{P}_{ft}^M(\Pi^M)^{1-\xi_M}\left(\Pi_t^M\right)^{\xi_M}}-1\right)\\ \times\frac{\tilde{P}_{t+1}^M\tilde{M}_{t+1}\tilde{P}_{ft+1}^M}{\left(\tilde{P}_{ft}^M\right)^2(\Pi^M)^{1-\xi_M}\left(\Pi_t^M\right)^{\xi_M}}\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(\frac{\tilde{P}_{ft}^{M}}{\tilde{P}_{t}^{M}}\right)^{-\frac{\mu_t^M}{\mu_t^M-1}} \tilde{M}_t - \frac{\mu_t^M}{\mu_t^M-1}\tilde{P}_{ft}^{M}\left(\frac{\tilde{P}_{ft}^{M}}{\tilde{P}_{t}^{M}}\right)^{-\frac{\mu_t^M}{\mu_t^M-1}-1}\frac{\tilde{M}_t}{\tilde{P}_t^M} \qquad \text{(A.79)} \\
&+ \frac{\mu_t^M}{\mu_t^M-1}\frac{\tilde{P}_t^{X^F}}{\tilde{Q}_t}\left(\frac{\tilde{P}_{ft}^{M}}{\tilde{P}_{t}^{M}}\right)^{-\frac{\mu_t^M}{\mu_t^M-1}}\frac{\tilde{M}_t}{\tilde{P}_t^M} \\
&- \frac{\phi_M\tilde{P}_t^M\tilde{M}_t}{(\mu^M-1)\tilde{P}_{ft}^M}\left\{\left(\zeta_{ft}^M-1\right)\zeta_{ft}^M - \mathrm{E}_t\left[\frac{\Gamma_{t+1}^H\Theta_{t+1}\tilde{\Lambda}_{t+1}^C}{\Theta_t\tilde{\Lambda}_t^C}\left(\zeta_{ft+1}^M-1\right)\zeta_{ft+1}^M\frac{\tilde{P}_{t+1}^M\tilde{M}_{t+1}}{\tilde{P}_t^M\tilde{M}_t}\right]\right\}
\end{aligned}
$$

Exporters

There is a continuum of export firms defined on the unit interval and indexed by k that buy the export good from the export retailer (discussed below) at the price $\tilde{P}_t^X$ and differentiate it by branding it. They then sell the export goods on the world market at the foreign currency prices of $\tilde{P}_{kt}^{EXP}$, subject to the following demand schedule:

$$\tilde{X}_{kt} = \left(\frac{\tilde{P}_{kt}^{EXP}}{\tilde{P}_t^{EXP}}\right)^{-\frac{\mu_t^X}{\mu_t^X-1}}\tilde{X}_t \qquad \text{(A.80)}$$

where $\tilde{X}_{kt}$ is demand for export firm $k's$ goods, $\tilde{P}_t^{EXP}$ is the aggregate price of export goods denominated in foreign currency and μ_t^X satisfies:

$$\log\mu_t^X = \sigma_{\mu^X}\eta_t^{\mu^X} \qquad \text{(A.81)}$$
$$\eta_t^{\mu^X} \sim N(0,1)$$

Aggregate per capita demand for export goods, $\tilde{X}_t$, is given by the CES aggregator:

$$\tilde{X}_t = \left[\int_0^1\left(\tilde{X}_{kt}\right)^{\frac{1}{\mu_t^X}}dk\right]^{\mu_t^X} \qquad \text{(A.82)}$$

The aggregate export price index, expressed in foreign currency, satisfies:

$$\tilde{P}_t^{EXP} = \left[\int_0^1 \left(\tilde{P}_{kt}^{EXP} \right)^{\frac{1}{1-\mu_t^X}} dk \right]^{1-\mu_t^X} \tag{A.83}$$

Each exporter faces a quadratic cost of adjusting prices set on the world export market, similar to that for the other sectors, where ϕ_X is the parameter that determines the extent of the cost and where there is full indexation of export prices to either steady state export price inflation, Π^{EXP}, or lagged export price inflation (expressed in foreign currency):

$$\Phi_X \left(\frac{\tilde{P}_{kt}^{EXP}}{\tilde{P}_{kt-1}^{EXP}}, \tilde{P}_t^{EXP} \tilde{X}_t \right) = \frac{\phi_X}{2\left(\mu^X - 1\right)} \left(\zeta_{kt}^X - 1 \right)^2 \tilde{P}_t^{EXP} \tilde{X}_t \tag{A.84}$$

where:

$$\zeta_{kt}^X \equiv \frac{\tilde{P}_{kt}^{EXP}}{\tilde{P}_{k,t-1}^{EXP} \left(\Pi^{EXP}\right)^{(1-\xi_X)} \left(\Pi_{t-1}^{EXP}\right)^{\xi_X}} \tag{A.85}$$

Firm k maximises the discounted flow of profits defined by:

$$\begin{aligned}
\mathcal{D}_{kt} \quad = \quad & \left(\frac{\tilde{P}_{kt}^{EXP}}{\tilde{Q}_t} - \tilde{P}_t^X \right) \left(\frac{\tilde{P}_{kt}^{EXP}}{\tilde{P}_t^{EXP}} \right)^{-\frac{\mu_t^X}{\mu_t^X - 1}} \tilde{X}_t \qquad \text{(A.86)} \\
- \quad & \frac{\phi_X}{2\left(\mu^X - 1\right)} \left(\frac{\tilde{P}_{kt}^{EXP}}{\tilde{P}_{kt-1}^{EXP} \left(\Pi^{EXP}\right)^{1-\xi^X} \left(\Pi_{t-1}^{EXP}\right)^{\xi^X}} - 1 \right)^2 \left(\frac{\tilde{P}_t^{EXP}}{\tilde{Q}_t} \right) \tilde{X}_t \\
+ \quad & \mathrm{E}_t \left[\Gamma_{t+1}^H \frac{\Theta_{t+1}}{\Theta t} \frac{\tilde{\Lambda}_{t+1}^C}{\tilde{\Lambda}_t^C} \left\{ \begin{array}{c} \left(\frac{\tilde{P}_{kt+1}^{EXP}}{\tilde{Q}_{t+1}} - \tilde{P}_{t+1}^X \right) \left(\frac{\tilde{P}_{kt+1}^{EXP}}{\tilde{P}_{t+1}^{EXP}} \right)^{-\frac{\mu_{t+1}^X}{\mu_{t+1}^X - 1}} \tilde{X}_{t+1} \\ -\frac{\phi_X}{2(\mu^X - 1)} \left(\frac{\tilde{P}_{kt+1}^{EXP}}{\tilde{P}_{kt}^{EXP} \left(\Pi^{EXP}\right)^{1-\xi^X} \left(\Pi_t^{EXP}\right)^{\xi^X}} - 1 \right)^2 \\ \times \left(\frac{\tilde{P}_{t+1}^{EXP}}{\tilde{Q}_{t+1}} \right) \tilde{X}_{t+1} \end{array} \right\} \right] \\
+ \quad & \ldots
\end{aligned}$$

The first-order condition for export pricing is as follows:

$$\begin{aligned}
0 \quad = \quad & \frac{1}{\tilde{Q}_t} \left(\frac{\tilde{P}_{kt}^{EXP}}{\tilde{P}_t^{EXP}} \right)^{-\frac{\mu_t^X}{\mu_t^X - 1}} \tilde{X}_t - \frac{\mu_t^X}{\mu_t^X - 1} \left(\frac{\tilde{P}_{kt}^{EXP}}{\tilde{Q}_t} - \tilde{P}_t^X \right) \left(\frac{\tilde{P}_{kt}^{EXP}}{\tilde{P}_t^{EXP}} \right)^{-\frac{\mu_t^X}{\mu_t^X - 1} - 1} \frac{\tilde{X}_t}{\tilde{P}_t^{EXP}} \\
- \quad & \frac{\phi_X}{\left(\mu^X - 1\right)} \left(\frac{\tilde{P}_{kt}^{EXP}}{\tilde{P}_{kt-1}^{EXP} \left(\Pi^{EXP}\right)^{1-\xi^X} \left(\Pi_{t-1}^{EXP}\right)^{\xi^X}} - 1 \right) \left(\frac{\tilde{P}_t^{EXP}}{\tilde{Q}_t} \right) \\
& \times \frac{\tilde{X}_t}{\tilde{P}_{kt-1}^{EXP} \left(\Pi^{EXP}\right)^{1-\xi^X} \left(\Pi_{t-1}^{EXP}\right)^{\xi^X}} \\
+ \quad & \mathrm{E}_t \left[\Gamma_{t+1}^H \frac{\Theta_{t+1}}{\Theta_t} \frac{\tilde{\Lambda}_{t+1}^C}{\tilde{\Lambda}_t^C} \left\{ \begin{array}{c} \frac{\phi_X}{(\mu^X - 1)} \left(\frac{\tilde{P}_{kt+1}^{EXP}}{\tilde{P}_{kt}^{EXP} \left(\Pi^{EXP}\right)^{1-\xi^X} \left(\Pi_t^{EXP}\right)^{\xi^X}} - 1 \right) \left(\frac{\tilde{P}_{t+1}^{EXP}}{\tilde{Q}_{t+1}} \right) \tilde{X}_{t+1} \\ \times \frac{\tilde{P}_{kt+1}^{EXP}}{\left(\tilde{P}_{kt}^{EXP}\right)^2 \left(\Pi^{EXP}\right)^{1-\xi^X} \left(\Pi_t^{EXP}\right)^{\xi^X}} \end{array} \right\} \right]
\end{aligned}$$

$$0 = \frac{1}{\tilde{Q}_t}\left(\frac{\tilde{P}^{EXP}_{kt}}{\tilde{P}^{EXP}_t}\right)^{-\frac{\mu^X_t}{\mu^X_t-1}}\tilde{X}_t - \frac{\mu^X_t}{\mu^X_t-1}\left(\frac{\tilde{P}^{EXP}_{kt}}{\tilde{Q}_t}-\tilde{P}^X_t\right)\left(\frac{\tilde{P}^{EXP}_{kt}}{\tilde{P}^{EXP}_t}\right)^{-\frac{\mu^X_t}{\mu^X_t-1}-1}\frac{\tilde{X}_t}{\tilde{P}^{EXP}_t}$$
$$- \frac{\phi_X \tilde{P}^{EXP}_t \tilde{X}_t}{(\mu^X-1)\tilde{Q}_t\tilde{P}^{EXP}_{kt}}\left\{\begin{array}{l}\left(\zeta^X_{kt}-1\right)\zeta^X_{kt}- \\ \mathrm{E}_t\left[\frac{\Gamma^H_{t+1}\Theta_{t+1}\tilde{\Lambda}^C_{t+1}}{\Theta_t\tilde{\Lambda}^C_t}\left(\zeta^X_{kt+1}-1\right)\zeta^X_{kt+1}\frac{\tilde{Q}_t\tilde{P}^{EXP}_{t+1}\tilde{X}_{t+1}}{\tilde{Q}_{t+1}\tilde{P}^{EXP}_t\tilde{X}_t}\right]\end{array}\right\} \quad \text{(A.87)}$$

Retailers

There is a continuum of perfectly competitive retailers defined on the unit interval, who buy final output goods and convert them into differentiated goods representing each expenditure component. Retailer h in sector N converts goods using the following linear technology:

$$N_{ht} = \tilde{\chi}^N_t \tilde{Z}^N_{ht}, \text{ for } N = \tilde{C}, \tilde{I}, \tilde{G}, \tilde{X}, \tilde{I}^O \quad \text{(A.88)}$$

where $\tilde{Z}^N_{ht}$ is the per capita amount of the final good bundle $\tilde{Z}_t$ used by firm h in sector N and where the final good bundle, $\tilde{Z}_t$, is defined by equation (A.44). The term $\tilde{\chi}^N_t$, common to all retailers in sector N, is the rate at which the final good bundle can be converted into the expenditure component, N. Hence, $\tilde{\chi}^N_t$ is a measure of sector-specific productivity in sector N. Below we will normalise so that $\tilde{\chi}^C_t = 1$, implying that the final output production function could be interpreted as a production function for a 'generic final consumption good', which could be used directly for consumption, or which could be transformed by 'retailers' into investment, government spending or export goods.

Each retailer h in sector N chooses input $\tilde{Z}^N_{ht}$ to maximise profits, taking the price of its output, $\tilde{P}^N_t$, and the price of final output, $\tilde{P}^Z_t$, as given. They solve:

$$\max_{\tilde{Z}^N_{ht}} \tilde{P}^N_t \tilde{\chi}^N_t \tilde{Z}^N_{ht} - \tilde{P}^Z_t \tilde{Z}^N_{ht} \quad \text{(A.89)}$$

with first-order condition given by:

$$\tilde{P}^N_t \tilde{\chi}^N_t = \tilde{P}^Z_t \quad \text{(A.90)}$$

for $N = \tilde{C}, \tilde{I}, \tilde{G}, \tilde{X}, \tilde{I}^O$.

A.1.4 Government

The government purchases goods from retailers and finances its expenditure by raising lump-sum taxes from optimising households.[195] Real government spending growth follows an exogenous rule, where, for simplicity, growth in government spending is assumed to exhibit 'error correction' to its long-run trend:

$$\frac{\tilde{G}_t}{\tilde{G}_{t-1}} = \left(\frac{\tilde{G}_{t-1}}{\tilde{\chi}^Z_{t-1}\tilde{\chi}^G_{t-1}}\right)^{\rho_G-1}\varepsilon^G_t \quad \text{(A.91)}$$

[195] As described above, the government also makes transfers between optimising and rule-of-thumb households to ensure that per capita consumption in the two groups are equalised in steady state.

where $\tilde{G}_t$ is real per capita government spending, $\tilde{\chi}^Z_{t-1}\tilde{\chi}^G_{t-1}$ is the trend in government spending (discussed below) and ε^G_t is a disturbance to government spending that follows the following process:

$$\begin{aligned} \log \varepsilon^G_t &= \left(1-\rho_G^2\right)^{1/2} \sigma_G \eta^G_t \\ \eta^G_t &\sim N(0,1) \end{aligned} \tag{A.92}$$

The government budget constraint is given by:

$$\tilde{P}^G_t \tilde{G}_t + \frac{\tilde{B}_{t-1}R_{t-1} + \tilde{\mathcal{M}}_{t-1}}{\Gamma^H_t} = \tilde{T}_t + \tilde{B}_t + \tilde{\mathcal{M}}_t \tag{A.93}$$

which shows that the government finances government spending and net debt using the lump-sum tax. Since the lump sum is levied on optimising households, the model exhibits so-called 'Ricardian equivalence' in the sense that, in equilibrium, the present value of the lump-sum tax payments offsets the value of the government debt in optimising households' lifetime budget constraints. As a result, the debt issuance (and tax financing) decisions of the government have no effect on the consumption decisions of households or any other variables. In light of this observation, we choose the simplest possible assumptions for government debt issuance and tax financing. Specifically, we assume that debt issuance is zero in each period $\tilde{B}_t = 0$ and that the lump sum tax adjusts to ensure that equation (A.93) holds.

A.1.5 Monetary policy

The monetary policy maker follows a simple rule for the nominal interest rate in which it responds to persistent deviations of annual CPI inflation, $\Pi^{C,annual}_t$, from its target, $\Pi^{*,annual}$, and a measure of the output gap, $\hat{Y}_t$. This gives the following rule:

$$R_t = R^{1-\theta_R} R^{\theta_R}_{t-1} \left(\frac{\Pi^{C,annual}_t}{\Pi^{*,annual}}\right)^{\frac{(1-\theta_R)\theta_\Pi}{4}} \left(\hat{Y}_t\right)^{(1-\theta_R)\theta_Y} \varepsilon^R_t \tag{A.94}$$

with $\Pi^{C,annual}_t = \tilde{P}^C_t / \tilde{P}^C_{t-4}$, $\Pi^{*,annual} = (\Pi^*)^4$, $\hat{Y}_t \equiv \tilde{V}_t / \tilde{V}^{flex}_t$ where $\tilde{V}^{flex}_t$ is the level of value added that would be observed if all prices and wages were flexible (to be defined below), R is the steady state nominal interest rate consistent with steady-state inflation being at target, and ε^R_t is an interest rate shock, given by:

$$\begin{aligned} \log \varepsilon^R_t &= \sigma_R \eta^R_t \\ \eta^R_t &\sim N(0,1)\,. \end{aligned} \tag{A.95}$$

Given the interest rate rule, the central bank will supply any quantity of money demanded at that rate. Money supply therefore equals money demand, given by equation (A.26).

A.1.6 The external sector

Demand for the bundle of domestic exports depends on the foreign currency price of domestic exports relative to the world export price, $\frac{\tilde{P}^{EXP}_t}{\tilde{P}^{X^F}_t}$, and on world output:

$$\tilde{X}_t = \left(\frac{\tilde{P}^{EXP}_t}{\tilde{P}^{X^F}_t}\right)^{-\epsilon_F} \tilde{Z}^F_t \varepsilon^{\kappa^F}_t \frac{\tilde{\chi}^{H^F}_t}{\tilde{\chi}^H_t} \tilde{\chi}^X_t \tag{A.96}$$

where the parameter ϵ_F is the elasticity of substitution between differentiated export goods in the rest of the world. Total world demand for exports is made up of four terms: $\tilde{Z}_t^F$ is total world output expressed in foreign per capita; $\varepsilon_t^{\kappa^F}$ is a disturbance to world demand for domestic exports; $\tilde{\chi}_t^{H^F}/\tilde{\chi}_t^H$ converts world output in foreign per capita terms into a measure expressed in domestic per capita terms; $\tilde{\chi}_t^X$ is the trend productivity growth in the export retail sector, which is assumed to be mirrored in the world economy consistent with a balanced growth path (as discussed below). The export demand disturbance represents exogenous shifts in foreign preferences for domestic output and satisfies:

$$\begin{aligned} \log \varepsilon_t^{\kappa^F} &= (1-\rho_{\kappa^F}) \log \varepsilon^{\kappa^F} + \rho_{\kappa^F} \log \varepsilon_{t-1}^{\kappa^F} + \left(1-\rho_{\kappa^F}^2\right)^{1/2} \sigma_{\kappa^F} \eta_t^{\kappa^F} \\ \eta_t^{\kappa^F} &\sim N(0,1) \end{aligned} \tag{A.97}$$

The levels of the non-stationary foreign variables are exogenously given, but we assume that there is cointegration between the domestic and foreign trends (discussed below). World output, $\tilde{Z}_t^F$, is modelled as an error correction to its long-run trend and the price of world exports relative to the price of world output, $P^{X^F} \equiv \frac{\tilde{P}^{X^F}}{\tilde{P}^{Z^F}}$, is assumed to follow a simple autoregressive process:

$$\frac{\tilde{Z}_t^F}{\tilde{Z}_{t-1}^F} = \left(\frac{\tilde{Z}_{t-1}^F}{\tilde{\chi}_{t-1}^{Z^F}}\right)^{\rho_{Z^F}-1} \varepsilon_t^{Z^F} \tag{A.98}$$

$$P_t^{X^F} = \left(P^{X^F}\right)^{1-\rho_{PX^F}} \left(P_{t-1}^{X^F}\right)^{\rho_{PX^F}} \varepsilon_t^{PX^F} \tag{A.99}$$

where:

$$\begin{aligned} \log \varepsilon_t^{Z^F} &= \left(1-\rho_{Z^F}^2\right)^{1/2} \sigma_{Z^F} \eta_t^{Z^F} \\ \eta_t^{Z^F} &\sim N(0,1) \end{aligned} \tag{A.100}$$

$$\begin{aligned} \log \varepsilon_t^{PX^F} &= \left(1-\rho_{PX^F}^2\right)^{1/2} \sigma_{PX^F} \eta_t^{PX^F} \\ \eta_t^{PX^F} &\sim N(0,1) \end{aligned} \tag{A.101}$$

For simplicity, we assume that the world nominal and real interest rates are constant such that $R_t^F \equiv R^F$ and $\Pi_t^{Z^F} = \Pi^{Z^F}$.

A.1.7 Potential output & the flexible-price economy

We define potential output as the level of output that would prevail if all prices were flexible and only a subset of shocks (detailed below) affect the economy.[196] In this economy, only real factors affect the paths for real variables. Although relative prices move in response to the real shocks, the absence of nominal rigidities implies that, as long as policy-makers set interest rates appropriately, inflation is always at target.

So potential output is defined using a variant of the model in which there are no costs of adjusting prices or wages and where nominal shocks do not exist. This means that it is a version of the model where the monetary policy shock and mark-up shocks do not exist (so that $\mu_t^J = \mu^J$ for $J = W, Z, V, M, X$). As a result, the equations of the flexible-price

[196] The flexible-price model is derived under the assumption that prices have always been flexible, and will remain flexible in the future.

model look the same as those just presented with the exception of those associated with price- and wage-setting and the Taylor rule. For the price- and wage-setting equations, the flexible-price model equations are obtained by setting the pricing adjustment cost to zero, $\phi_J = 0$ for $J = W, Z, V, M$, and X, as in equation (A.56) for final output prices. There is no explicit monetary policy rule in the flexible-price model. Implicitly, the rule is that inflation is at target in all periods.

A.1.8 Aggregation and market-clearing conditions

In this sub-section the various market-clearing conditions which need to hold in equilibrium are presented, first for firms and then households. Since all firms and households in a particular group are identical *ex ante*, we impose a symmetric equilibrium in which the *ex post* decisions of each member of a particular group are identical. The symmetric equilibrium of the flexible price model can be obtained in an analogous way, by setting the price and wage adjustment cost parameters to zero in the relevant equations.

Firms

Aggregating over equation (A.88) gives the following expression for aggregate per capita supply of expenditure component N:

$$\tilde{N}_t = \tilde{\chi}_t^N \tilde{Z}_t^N, \text{ for } N = C, I, G, X, I^O \tag{A.102}$$

Summing this expression over sectors obtains the aggregate per capita real resource constraint:

$$\begin{aligned} \tilde{Z}_t &= \tilde{Z}_t^C + \tilde{Z}_t^I + \tilde{Z}_t^{I^O} + \tilde{Z}_t^G + \tilde{Z}_t^X \\ \tilde{Z}_t &= \frac{1}{\tilde{\chi}_t^C}\tilde{C}_t + \frac{1}{\tilde{\chi}_t^I}\tilde{I}_t + \frac{1}{\tilde{\chi}_t^{I^O}}\tilde{I}_t^O + \frac{1}{\tilde{\chi}_t^G}\tilde{G}_t + \frac{1}{\tilde{\chi}_t^X}\tilde{X}_t \end{aligned} \tag{A.103}$$

The nominal per capita resource constraint is

$$\tilde{P}_t^Z \tilde{Z}_t = \tilde{P}_t^C \tilde{C}_t + \tilde{P}_t^I \tilde{I}_t + \tilde{P}_t^{I^O} \tilde{I}_t^O + \tilde{P}_t^G \tilde{G}_t + \tilde{P}_t^X \tilde{X}_t \tag{A.104}$$

In equilibrium, all that is produced by retailers will be demanded by consumers $(\tilde{C}_t, \tilde{I}_t, \tilde{I}_t^O)$, the government $(\tilde{G}_t)$ and exporters $(\tilde{X}_t)$. In a symmetric equilibrium all final output firms set the same price, $\tilde{P}_{nt}^Z = \tilde{P}_t^Z$, choose the same level of inputs, $\tilde{M}_{nt} = \tilde{M}_t$ & $\tilde{V}_{nt} = \tilde{V}_t$, produce the same level of output, $\tilde{Z}_{nt} = \tilde{Z}_t$, and face the same marginal cost, $\widetilde{MC}_{nt}^Z = \widetilde{MC}_t^Z$. Given this, the production function, demand for value added, the demand for imports, the optimal price and profits of final output firms are given by:

$$\tilde{Z}_t = \tilde{V}_t^{\alpha_V} \tilde{M}_t^{1-\alpha_V} \tag{A.105}$$

$$\widetilde{MC}_t^Z = \frac{\tilde{P}_t^V \tilde{V}_t}{\alpha_V \tilde{Z}_t} \tag{A.106}$$

$$\widetilde{MC}_t^Z = \frac{\tilde{P}_t^M \tilde{M}_t}{(1-\alpha_V)\tilde{Z}_t}\varepsilon_t^M \tag{A.107}$$

$$\begin{aligned} \tilde{P}_t^Z &= \mu_t^Z \widetilde{MC}_t^Z \\ &- \frac{\phi_Z(\mu_t^Z - 1)\tilde{P}_t^Z}{(\mu^Z - 1)} \left\{ \begin{array}{c} (\zeta_t^Z - 1)\zeta_t^Z - \\ \mathrm{E}_t\left[\frac{\Gamma_{t+1}^H \Theta_{t+1} \tilde{\Lambda}_{t+1}^C}{\Theta_t \tilde{\Lambda}_t^C} \frac{\Pi_{t+1}^Z \tilde{Z}_{t+1}}{\tilde{Z}_t} (\zeta_{t+1}^Z - 1)\zeta_{t+1}^Z\right] \end{array} \right\} \end{aligned} \tag{A.108}$$

$$\tilde{\Xi}_t^Z = \tilde{P}_t^Z \tilde{Z}_t \left(1 - \frac{\phi_Z(\zeta_t^Z - 1)^2}{2(\mu^Z - 1)}\right) - \tilde{P}_t^V \tilde{V}_t - \tilde{P}_t^M \tilde{M}_t \varepsilon_t^M \tag{A.109}$$

where:

$$\zeta_t^Z = \frac{\Pi_t^Z}{(\Pi^Z)^{1-\xi_Z}\left(\Pi_{t-1}^Z\right)^{\xi_Z}} \tag{A.110}$$

Similarly, in a symmetric equilibrium, where all value added firms set the same price, $\tilde{P}_{jt}^V = \tilde{P}_t^V$, employ the same amount of labour, $\tilde{L}_{jt} = \tilde{L}_t$, produce the same level of output, $\tilde{V}_{jt} = \tilde{V}_t$, and face the same marginal cost, $\widetilde{MC}_t^V = \widetilde{MC}_{jt}^V$. Market clearing for capital implies that the total amount of capital previously accumulated by optimising households is rented to firms so that $\tilde{K}_{jt} = \tilde{K}_{t-1}$, where the right hand side of this equation represents the per capita capital stock in the economy (since it is all held by optimising households).

The production function, demand for labour and capital services, the optimal value-added price and profits are:

$$\tilde{V}_t = \varepsilon_t^{TFP}\tilde{K}_{t-1}^{1-\alpha_L}\left(\tilde{\chi}_t^{LAP}L_t\right)^{\alpha_L} \tag{A.111}$$

$$\widetilde{MC}_t^V = \frac{\tilde{W}_t L_t}{\alpha_L \tilde{V}_t} \tag{A.112}$$

$$\widetilde{MC}_t^V = \frac{\tilde{R}_t^K \tilde{K}_{t-1}}{(1-\alpha_L)\tilde{V}_t} \tag{A.113}$$

$$\begin{aligned}\tilde{P}_t^V &= \mu_t^V \widetilde{MC}_t^V \\ &- \frac{\phi_V\left(\mu_t^V - 1\right)\tilde{P}_t^V}{(\mu^V-1)}\left\{\begin{array}{c}\left(\zeta_t^V - 1\right)\zeta_t^V - \\ \mathrm{E}_t\left[\frac{\Gamma_{t+1}^H\Theta_{t+1}\tilde{\Lambda}_{t+1}^C}{\Theta_t\tilde{\Lambda}_t^C}\frac{\Pi_{t+1}^V\tilde{V}_{t+1}}{\tilde{V}_t}\left(\zeta_{t+1}^V - 1\right)\zeta_{t+1}^V\right]\end{array}\right\}\end{aligned} \tag{A.114}$$

$$\tilde{\Xi}_t^V = \tilde{P}_t^V\tilde{V}_t\left(1 - \frac{\phi_V}{2(\mu^V-1)}\left(\zeta_t^V - 1\right)^2\right) - \tilde{W}_t L_t - \tilde{R}_t^K\tilde{K}_{t-1} \tag{A.115}$$

where:

$$\zeta_t^V = \frac{\Pi_t^V}{(\Pi^V)^{1-\xi_V}\left(\Pi_{t-1}^V\right)^{\xi_V}} \tag{A.116}$$

And in a symmetric equilibrium where $\tilde{P}_{ft}^M = \tilde{P}_t^M$, the optimal import price decision and import sector profits are:

$$\begin{aligned}\tilde{P}_t^M &= \mu_t^M \frac{\tilde{P}_t^{X^F}}{\tilde{Q}_t} \\ &- \frac{\phi_M\left(\mu_t^M - 1\right)\tilde{P}_t^M}{(\mu^M-1)}\left\{\begin{array}{c}\left(\zeta_t^M - 1\right)\zeta_t^M - \\ \mathrm{E}_t\left[\frac{\Gamma_{t+1}^H\Theta_{t+1}\tilde{\Lambda}_{t+1}^C}{\Theta_t\tilde{\Lambda}_t^C}\frac{\Pi_{t+1}^M\tilde{M}_{t+1}}{\tilde{M}_t}\left(\zeta_{t+1}^M - 1\right)\zeta_{t+1}^M\right]\end{array}\right\}\end{aligned} \tag{A.117}$$

$$\tilde{\Xi}_t^M = \tilde{P}_t^M\tilde{M}_t\left(1 - \frac{\phi_M}{2(\mu^M-1)}\left(\zeta_t^M - 1\right)^2\right) - \frac{\tilde{P}_t^{X^F}\tilde{M}_t}{\tilde{Q}_t} \tag{A.118}$$

where:

$$\zeta_t^M = \frac{\Pi_t^M}{(\Pi^M)^{1-\xi_M}\left(\Pi_{t-1}^M\right)^{\xi_M}} \tag{A.119}$$

Similarly, in a symmetric equilibrium where $\tilde{P}_{kt}^{EXP} = \tilde{P}_t^{EXP}$, the optimal price and profits for export firms are:

$$\begin{aligned}
\tilde{P}_t^{EXP} &= \mu_t^X \tilde{P}_t^X \tilde{Q}_t \\
&- \frac{\phi_X \left(\mu_t^X - 1\right) \tilde{P}_t^{EXP}}{\left(\mu^X - 1\right)} \left\{ \begin{array}{c} \left(\zeta_t^X - 1\right) \zeta_t^X - \\ \mathrm{E}_t \left[\frac{\Gamma_{t+1}^H \Theta_{t+1} \tilde{\Lambda}_{t+1}^C}{\Theta_t \tilde{\Lambda}_t^C} \frac{\Pi_{t+1}^{EXP} \tilde{Q}_t \tilde{X}_{t+1}}{\tilde{Q}_{t+1} \tilde{X}_t} \left(\zeta_{t+1}^X - 1\right) \zeta_{t+1}^X \right] \end{array} \right\} \quad \text{(A.120)} \\
\tilde{\Xi}_t^X &= \frac{\tilde{P}_t^{EXP} \tilde{X}_t}{\tilde{Q}_t} \left(1 - \frac{\phi_X}{2\left(\mu^X - 1\right)} \left(\zeta_t^X - 1\right)^2 \right) - \tilde{P}_t^X \tilde{X}_t \quad \text{(A.121)}
\end{aligned}$$

where:

$$\zeta_t^X = \frac{\Pi_t^{EXP}}{\left(\Pi^{EXP}\right)^{1-\xi^X} \left(\Pi_{t-1}^{EXP}\right)^{\xi^X}} \quad \text{(A.122)}$$

Households

Aggregate per capita consumption is obtained by integrating over the two types of households. In a symmetric equilibrium, the aggregate quantity of any variable $\tilde{J}$ held by optimising households is $\int_0^{\omega_o} \tilde{J}_{it}^o di \equiv \omega_o \tilde{J}_t^o$, while for rule-of-thumb households it is $\int_{\omega_o}^1 \tilde{J}_{it}^{rot} di \equiv (1-\omega_o) \tilde{J}_t^{rot}$. This implies that aggregate consumption can be defined as:

$$\tilde{C}_t = \omega_o \tilde{C}_t^o + (1-\omega_o) \tilde{C}_t^{rot} \quad \text{(A.123)}$$

Aggregating over rule-of-thumb households we get:

$$\tilde{P}_t^C \tilde{C}_t^{rot} = \tilde{W}_t \tilde{L}_t + \tilde{P}_t^C \tilde{\chi}_t^Z \tilde{T}rans^{rot} \quad \text{(A.124)}$$

where the assumption that rule-of-thumb households supply the same amount of labour at the same wage as optimising households implies that $\tilde{L}_{it}^o = \tilde{L}_{it}^{rot} = \tilde{L}_t, \forall i$ and $\tilde{W}_{it}^o = \tilde{W}_{it}^{rot} = \tilde{W}_t, \forall i$.

The government raises a lump-sum tax $\tilde{T}_{it}^o$ and a transfer $\tilde{T}rans_i^o$ from each optimising household, so total revenue equals:

$$\int_0^{\omega_o} \left(\tilde{T}_{it}^o + \tilde{\chi}_t^Z \tilde{P}_t^C \tilde{T}rans_i^o \right) di = \omega_o \left(\tilde{T}_t^o + \tilde{\chi}_t^Z \tilde{P}_t^C \tilde{T}rans^o \right) \quad \text{(A.125)}$$

A transfer is paid to each rule-of-thumb households of size $Trans_i^{rot}$, which aggregates to:

$$\int_{\omega_o}^1 \tilde{\chi}_t^Z \tilde{P}_t^C \tilde{T}rans_i^{rot} di = (1-\omega_o) \tilde{\chi}_t^Z \tilde{P}_t^C \tilde{T}rans^{rot} \quad \text{(A.126)}$$

and it is further assumed that the transfer to rule-of-thumb households is fully financed by optimising households such that:

$$\omega_o \tilde{T}rans^o = (1-\omega_o) \tilde{T}rans^{rot} \quad \text{(A.127)}$$

Money balances, which are held by optimising households only, aggregate to:

$$\tilde{\mathcal{M}}_t = \omega_o \tilde{\mathcal{M}}_t^o \quad \text{(A.128)}$$

By aggregating over optimising households, the first order conditions for consumption, labour supply, money, deposits, investment and capital are:

$$\frac{\tilde{U}_t^{\tilde{C}^o}}{\tilde{P}_t^C} = \tilde{\Lambda}_t^C \tag{A.129}$$

$$\frac{\widetilde{MRS}_t}{\tilde{W}_t} = \frac{\mu_t^W - 1}{\mu_t^W\left(\mu^W - 1\right)}\left\{\begin{array}{l} \frac{\mu^W-1}{\mu_t^W-1}\left(1-\frac{\phi_W\left(\zeta_t^W-1\right)^2}{2(\mu^W-1)}\right)+\phi_W\left(\zeta_t^W-1\right)\zeta_t^W \\ -\mathrm{E}_t\left[\frac{\Gamma_{t+1}^H\Theta_{t+1}\tilde{\Lambda}_{t+1}^C\tilde{W}_{t+1}L_{t+1}}{\Theta_t\tilde{\Lambda}_t^C\tilde{W}_tL_t}\phi_W\left(\zeta_{t+1}^W-1\right)\zeta_{t+1}^W\right]\end{array}\right\} \tag{A.130}$$

$$\tilde{U}_t^{\tilde{\mathcal{M}}^o} = \frac{R_t^A - 1}{R_t^A}\tilde{\Lambda}_t^C \tag{A.131}$$

$$\tilde{\Lambda}_t^C = \mathrm{E}_t\left[\frac{\Theta_{t+1}}{\Theta_t}\tilde{\Lambda}_{t+1}^C R_t^A\right] \tag{A.132}$$

$$\begin{aligned}\tilde{P}_t^I &= \frac{\tilde{\Lambda}_t^K}{\tilde{\Lambda}_t^C}\varepsilon_t^I\left[\Psi_I\left(\tilde{\zeta}_t^{I,o},\varepsilon_t^I\right)-\psi_I\left(\zeta_t^{I,o}-\Gamma^H\Gamma^Z\Gamma^I\right)\zeta_t^{I,o}\right] \\ &+ \mathrm{E}_t\left[\frac{\Theta_{t+1}\tilde{\Lambda}_{t+1}^C}{\Theta_t\tilde{\Lambda}_t^C}\frac{\tilde{\Lambda}_{t+1}^K}{\tilde{\Lambda}_{t+1}^C}\psi_I\left(\zeta_{t+1}^{I,o}-\Gamma^H\Gamma^Z\Gamma^I\right)\left(\zeta_{t+1}^{I,o}\right)^2\varepsilon_{t+1}^I\right]\end{aligned} \tag{A.133}$$

$$\widetilde{TQ}_t = \mathrm{E}_t\left[\frac{\Theta_{t+1}}{\Theta_t}\frac{\tilde{\Lambda}_{t+1}^C}{\tilde{\Lambda}_t^C}\left\{\tilde{R}_{t+1}^K+\widetilde{TQ}_{t+1}\left(1-\delta_K\right)\right\}\right] \tag{A.134}$$

where:

$$\zeta_t^W = \frac{\Pi_t^W}{\left(\Pi^W\right)^{1-\xi^W}\left(\Pi_{t-1}^W\right)^{\xi^W}} \tag{A.135}$$

$$\zeta_t^{I,o} = \frac{\Gamma_t^H\tilde{I}_t^o}{\tilde{I}_{t-1}^o} \tag{A.136}$$

and where the marginal utilities of consumption, labour and real money balances are:

$$\tilde{U}_t^{\tilde{C}^o} = \left(\frac{\tilde{C}_t^o}{\tilde{\chi}_t^Z}-\psi_C\frac{\tilde{C}_{t-1}^o}{\tilde{\chi}_{t-1}^Z}\right)^{-\epsilon_C}\frac{1}{\tilde{\chi}_t^Z} \tag{A.137}$$

$$\tilde{U}_t^{L^o} = -\nu_L\varepsilon_t^L\left(L_t^o\right)^{\epsilon_L} \tag{A.138}$$

$$\tilde{U}_t^{\tilde{\mathcal{M}}^o} = \nu_M\left(\frac{\tilde{\mathcal{M}}_t^o}{\tilde{\chi}_t^Z\tilde{P}_t^C}\right)^{-\epsilon_C}\frac{1}{\tilde{\chi}_t^Z\tilde{P}_t^C} \tag{A.139}$$

and aggregate capital accumulation is:

$$\Gamma_{t+1}^H\tilde{K}_t^o = \left(1-\delta_K\right)\tilde{K}_{t-1}^o - \Psi_I\left(\tilde{\zeta}_t^{I,o},\varepsilon_t^I\right)\tilde{I}_t^o \tag{A.140}$$

Finally, aggregating over optimising households also gives an aggregate process for 'other' investment of:

$$\frac{\tilde{I}_t^{O,o}}{\tilde{I}_{t-1}^{O,o}} = \left(\frac{\tilde{I}_{t-1}^{O,o}}{\tilde{\chi}_{t-1}^Z}\right)^{1-\rho_{IO}}\varepsilon_t^{IO} \tag{A.141}$$

A.2 Balanced growth path and detrending

The model has a well-defined balanced growth path along which variables grow at constant rates in the steady state. This sub-section defines the steady state balanced growth path, discusses the assumption underpinning it, derives the trends and then presents the detrended first-order conditions.

Growth in the model arises from labour augmenting productivity growth, which is stochastic, as well as determinstic population growth and deterministic retail sector-specific productivity growth. As detailed above, labour augmenting productivity growth, LAP, follows the following stochastic process (defined in equations (A.59) & (A.60) and repeated here for convenience):

$$\begin{aligned}\tilde{\chi}_t^{LAP} &= \Gamma^{LAP}\tilde{\chi}_{t-1}^{LAP}\exp\left(\hat{\varepsilon}_t^{LAP}\right)\\ \hat{\varepsilon}_t^{LAP} &= \rho_{LAP}\hat{\varepsilon}_{t-1}^{LAP} + \left(1-\rho_{LAP}^2\right)^{1/2}\sigma_{LAP}\eta_t^{LAP}\end{aligned}$$

where Γ^{LAP} defines the steady state growth rate of LAP. The retail sector-specific tehnological trends and the population trends are deterministic, which means that the following holds:

$$\tilde{\chi}_t^N = \Gamma^N\tilde{\chi}_{t-1}^N \tag{A.142}$$

for $N = C, I, G, X, I^O, H$ and where Γ^N is the steady state sector/population growth rate (as applicable).

A.2.1 Steady state balanced growth path

This sub-section defines the steady state balanced growth path and the assumption necessary to deliver it given the exogenous drivers of growth outlined above. Consistent with the description of the model, we use Γ to denote growth rates and Π to denote inflation rates. In addition to the processes for the exogenous sources of growth defined above (and the assumptions inherent in the description of the model), we make use of the following assumptions and results:

1. Sector-specific productivity growth in the consumption and 'other' investment retail sectors is assumed to be zero so that $\Gamma^C = \Gamma^{I^O} = 1$. This means that the prices of these expenditure components relative to final output are stationary (as are their real ratios to final output) such that $\tilde{P}_t^C \equiv \tilde{P}_t^{I^O} \equiv \tilde{P}_t^Z$ & $\Pi_t^C \equiv \Pi_t^{I^O} \equiv \Pi_t^Z$.

2. Sector-specific productivity growth in the investment, government and export sectors is assumed to be non-zero. This means that the relative prices and real ratios (to final output) of these expenditure components have deterministic trends.

3. The nominal side of the model is pinned down by the Taylor rule. The monetary authority ensures that money growth delivers its inflation target along the balanced growth path.

4. Population growth in the rest of the world, Γ^{H^F}, is assumed to be the same as that of the domestic economy. We also assume that (per capita) world final output grows at the same rate as domestic final output along a balanced growth path, $\Gamma^Z = \Gamma^{Z^F}$.[197]

[197] Rather than assume that population growth and final output growth are the same in the domestic and the rest of the world, a less restrictive assumption can be made that: $\Gamma^Z\Gamma^H = \Gamma^{H^F}\Gamma^{Z^F}$.

5. We make the additional assumptions that the rest of the world shares the same trend in LAP such that $\tilde{\chi}_t^{Z^F} = \tilde{\chi}_t^Z$ and that export-specific productivity growth is the same in the rest of the world as in the domestic economy. These assumptions ensure that imports and exports have the same trend.

6. The assumptions about domestic and world trends, together with an assumption that the rest of the world has the same inflation target as the domestic economy implies that the real exchange rate is constant along the balanced growth path.

Given those assumptions and the exogenous sources of growth outlined above, the rest of this sub-section derives the growth rates of the variables in the model along the steady state balanced growth path.

Expenditure components

The Cobb-Douglas production technologies imply that the ratios of the nominal expenditure components to final output are constant *along the steady state balanced growth path* such that the following is true:

$$\frac{\tilde{P}_t^N \tilde{N}_t}{\tilde{P}_t^Z \tilde{Z}_t} = \varpi^N, \; N = C, I, G, X, I^O \tag{A.143}$$

where ϖ^N are constants. Using the first order condition for retailers' pricing decisions (A.90) implies that:

$$\frac{\tilde{P}_t^N \tilde{N}_t}{\tilde{P}_t^Z \tilde{Z}_t} = \frac{\tilde{N}_t}{\tilde{Z}_t \tilde{\chi}_t^N} = \varpi^N, \; N = C, I, G, X, I^O \tag{A.144}$$

and so:

$$\frac{\tilde{N}_t}{\tilde{N}_{t-1}} = \frac{\tilde{Z}_t \tilde{\chi}_t^N}{\tilde{Z}_{t-1} \tilde{\chi}_{t-1}^N} \tag{A.145}$$

It follows from this equation and the assumptions outlined above that $\tilde{N}_t$ grows at rate $\Gamma^Z \Gamma^N$ along the balanced growth path. For example, sector-specific productivity in the investment sector, $\tilde{\chi}_t^I$, grows at rate Γ^I, so real investment per capita grows at rate $\Gamma^Z \Gamma^I$.

Expressing equation (A.90) in growth rates, the rate of growth in the price of expenditure component N along the balanced growth path is:

$$\frac{\tilde{P}_t^N}{\tilde{P}_{t-1}^N} \equiv \Pi^N = \frac{\Pi^Z}{\Gamma^N}, \; N = C, I, G, X, I^O \tag{A.146}$$

Since the price of consumption, other investment, and final output goods are equivalent (such that $\Pi^{I^O} \equiv \Pi^C \equiv \Pi^Z \equiv \Pi^*$ on the balanced growth path), this implies that:

$$\Pi^N = \frac{\Pi^*}{\Gamma^N}, \; N = I, G, X, I^O \tag{A.147}$$

Labour & capital

Per capita labour supply is stationary along the balanced growth path such that overall labour supply is growing in line with the population. Taking the capital accumulation equation (A.140), dividing both sides by the capital stock and rearranging:

$$\frac{\tilde{I}_t}{\tilde{K}_{t-1}} = \frac{\Gamma^H \frac{\tilde{K}_t}{\tilde{K}_{t-1}} - \left(1-\delta^K\right)}{\Psi_I\left(\zeta_t^I, \varepsilon_t^I\right)} \tag{A.148}$$

which implies that along a balanced growth path (where $\Psi_I(.) = 1$), the investment to capital stock ratio is constant, so the per-capita capital stock grows at the same rate as investment, $\Gamma^Z\Gamma^I$.

From the first-order condition for investment (A.133), the following is true (using the fact that $\Psi_I(.) = 1$ on the balanced growth path):

$$\frac{\widetilde{TQ}_t}{\widetilde{TQ}_{t-1}} = \frac{\tilde{P}_t^I}{\tilde{P}_{t-1}^I} = \frac{\Pi^*}{\Gamma^I} \tag{A.149}$$

so that Tobin's Q grows in line with the relative price of investment, which is defined above. It follows that the first-order condition with respect to capital (A.134) that the growth rate of the rental rate of capital is equal to the growth rate of Tobin's Q:

$$\frac{\tilde{R}_t^K}{\tilde{R}_{t-1}^K} = \frac{\widetilde{TQ}_t}{\widetilde{TQ}_{t-1}} = \frac{\tilde{P}_t^I}{\tilde{P}_{t-1}^I} = \frac{\Pi^*}{\Gamma^I} \tag{A.150}$$

Value-added

Using the first order condition for the demand for value-added by final output sector firms (A.52), we can define the following:

$$\frac{\widetilde{MC}_t^Z}{\widetilde{MC}_{t-1}^Z} = \frac{\tilde{P}_t^V \tilde{V}_t / \left(\tilde{P}_{t-1}^V \tilde{V}_{t-1}\right)}{\tilde{Z}_t / \tilde{Z}_{t-1}} \equiv \frac{\Pi^V \Gamma^V}{\Gamma^Z} \tag{A.151}$$

Given the optimal pricing decision of final output producers, (A.108), we have that:

$$\begin{aligned} 1 \quad &= \quad \mu_t^Z \frac{\widetilde{MC}_t^Z}{\tilde{P}_t^Z} \\ &- \quad \frac{\phi_Z\left(\mu_t^Z - 1\right)}{\left(\mu^Z - 1\right)} \left\{ \left(\zeta_t^Z - 1\right)\zeta_t^Z - \mathrm{E}_t \left[\Gamma^H \frac{\Theta_{t+1}\tilde{\Lambda}_{t+1}^C \tilde{P}_{t+1}^Z \tilde{Z}_{t+1}}{\Theta_t \tilde{\Lambda}_t^C \tilde{P}_t^Z \tilde{Z}_t} \left(\zeta_{t+1}^Z - 1\right)\zeta_{t+1}^Z \right] \right\} \end{aligned}$$

where $\zeta_t^Z = 1$ along the balanced growth path so that:

$$\mu_t^Z \frac{\widetilde{MC}_t^Z}{\tilde{P}_t^Z} = 1$$

and:

$$\frac{\widetilde{MC}_t^Z}{\widetilde{MC}_{t-1}^Z} = \frac{\tilde{P}_t^Z}{\tilde{P}_{t-1}^Z} = \Pi^Z = \Pi^* \tag{A.152}$$

which implies that:

$$\Pi^V = \frac{\Pi^Z \Gamma^Z}{\Gamma^V} \tag{A.153}$$

Expressing the value-added production function (A.111) in growth rates implies that:

$$\frac{V_t}{V_{t-1}} = \frac{\varepsilon_t^{TFP} \left(\tilde{K}_t\right)^{1-\alpha_L} \left(\tilde{\chi}_t^{LAP} L_t\right)^{\alpha_L}}{\varepsilon_{t-1}^{TFP} \left(\tilde{K}_{t-1}\right)^{1-\alpha_L} \left(\tilde{\chi}_{t-1}^{LAP} L_{t-1}\right)^{\alpha_L}} \tag{A.154}$$

which means that:

$$\Gamma^V = \left(\Gamma^Z \Gamma^I\right)^{1-\alpha_L} \left(\Gamma^{LAP}\right)^{\alpha_L} \tag{A.155}$$

Wages

Using logic analogous to that used for value-added pricing (specifically, that along the balanced growth path $\zeta_t^W = 1$ and using the labour supply decision (A.130), noting that labour supply per capita and the marginal disutility of supplying labour are constant along the balanced growth path):

$$\frac{\tilde{W}_t}{\tilde{W}_{t-1}} = \frac{\widetilde{MRS}_t}{\widetilde{MRS}_{t-1}} = \frac{\tilde{P}_t^C \tilde{U}_{t-1}^{C^o}}{\tilde{P}_{t-1}^C \tilde{U}_t^{C^o}} \tag{A.156}$$

where:

$$\frac{\tilde{U}_t^{C^o}}{\tilde{U}_{t-1}^{C^o}} = \frac{\tilde{\chi}_{t-1}^Z}{\tilde{\chi}_t^Z} = \frac{1}{\Gamma^Z} \tag{A.157}$$

so that wage growth along the balanced growth path is:

$$\frac{\tilde{W}_t}{\tilde{W}_{t-1}} = \Pi^* \Gamma^Z \tag{A.158}$$

Trade

By virtue of the assumptions about the rest-of-the-world outlined above, the nominal net trade position is constant along the balanced growth path. This implies that nominal imports must grow at the same rate as nominal exports. The growth rate of exports and (domestic currency) export prices along the balanced growth path were determined above. In order to determine the growth rate of imports and import prices, we must determine how $\tilde{P}_t^{XF}$, $\tilde{Q}_t$ and $\tilde{P}_t^{EXP}$ evolve. The growth rate of exports is $\frac{\tilde{X}_t}{\tilde{X}_{t-1}} = \Gamma^Z \Gamma^X$ and export prices is $\frac{\tilde{P}_t^X}{\tilde{P}_{t-1}^X} = \frac{\Pi^Z}{\Gamma^X}$. Taking growth rates from the aggregate optimal pricing decision of export firms (and using that $\zeta_t^X = 1$), we have that:

$$\Pi^{EXP} = \Pi^X \frac{\tilde{Q}_t}{\tilde{Q}_{t-1}} = \frac{\Pi^Z}{\Gamma^X} \frac{\tilde{Q}_t}{\tilde{Q}_{t-1}} \tag{A.159}$$

where, given the symmetry between the domestic economy and the rest of the world, $\frac{\tilde{Q}_t}{\tilde{Q}_{t-1}} \equiv 1$ along the balanced growth path. This implies that:

$$\Pi^{EXP} = \Pi^X = \frac{\Pi^Z}{\Gamma^X} \tag{A.160}$$

Next, consider the growth rate of export demand along the balanced growth path using equation (A.96):

$$\frac{X_t}{X_{t-1}} = \left(\frac{\Pi^{EXP}}{\Pi^{X^F}}\right)^{-\epsilon_F} \frac{\Gamma^{H^F}}{\Gamma^H}\Gamma^{Z^F}\Gamma^X \tag{A.161}$$

By using that, along a balanced growth path, $\frac{X_t}{X_{t-1}} = \Gamma^Z\Gamma^X$, $\Gamma^{H^F} \equiv \Gamma^H$ and $\Gamma^{Z^F} \equiv \Gamma^Z$, we get the condition that domestic export prices and world export prices are growing at the same rate:[198]

$$\Pi^{EXP} = \Pi^{X^F} \tag{A.162}$$

Import growth along the balanced growth path can be derived given the expression for nominal net trade, which in growth rates can be expressed as:

$$\frac{\tilde{M}_t}{\tilde{M}_{t-1}} = \frac{\Pi^X\Gamma^Z\Gamma^X}{\Pi^{X^F}} \tag{A.163}$$

which, given that $\Pi^X = \Pi^{X^F}$, means that imports grow at the same rate as exports:

$$\frac{\tilde{M}_t}{\tilde{M}_{t-1}} = \Gamma^Z\Gamma^X \tag{A.164}$$

Final output

Taking growth rates of the final output production function (A.105) and using the definitions of the balanced path growth rates of value-added and imports from above:

$$\frac{\tilde{Z}_t}{\tilde{Z}_{t-1}} = \Gamma^Z = \left(\Gamma^V\right)^{\alpha_V}\left(\Gamma^Z\Gamma^X\right)^{1-\alpha_V} \tag{A.165}$$

This means that along the balanced growth path:

$$\Gamma^Z = \Gamma^V\left(\Gamma^X\right)^{\frac{1-\alpha_V}{\alpha_V}} \tag{A.166}$$

or, equivalently:

$$\Gamma^V = \Gamma^Z\left(\Gamma^X\right)^{\frac{\alpha_V-1}{\alpha_V}} \tag{A.167}$$

Interest rates & money

Using the aggregated first order condition for household deposits (A.132), we have the following (given that $\frac{\Theta_{t+1}}{\Theta_t} = \beta$ along the balanced growth path):

$$R_t^A = R_t = \beta\frac{\tilde{\Lambda}_t^C}{\tilde{\Lambda}_{t+1}^C} = \frac{\tilde{P}_{t+1}^C\tilde{U}_t^{C^o}}{\tilde{P}_t^C\tilde{U}_{t+1}^{C^o}} \tag{A.168}$$

[198] Another way to obtain the same result is to consider the demand for imports by final output producers. Taking the growth rate of equation (A.107) along the balanced growth path (and using that $\frac{\widetilde{MC}_t^Z}{\widetilde{MC}_{t-1}^Z} = \Pi^Z$), we get that $\Pi^Z = \Pi^M = \frac{\tilde{M}_t}{\tilde{M}_{t-1}}\Gamma^Z$. Using the optimal pricing decision of importers (A.117) and that, on the balanced growth path, $\zeta_t^M = 1$, $\Pi^M = \Pi^{X^F} = \Pi^{EXP} = \frac{\Pi^Z}{\Gamma^X}$, so $\frac{\tilde{M}_t}{\tilde{M}_{t-1}} = \Gamma^X\Gamma^Z$.

which is a constant along the balanced growth path. From the money demand equation (A.131) and the first order condition for consumption (A.129) we have the following along the balanced growth path (given that the interest rate is constant):

$$\frac{\tilde{U}_t^{\tilde{\mathcal{M}}^o}}{\tilde{U}_{t-1}^{\tilde{\mathcal{M}}^o}} = \frac{\tilde{\Lambda}_t^{\tilde{C}^o}}{\tilde{\Lambda}_{t-1}^{\tilde{C}^o}} = \frac{1}{\Gamma^Z \Pi^*} \tag{A.169}$$

using the expression for the marginal utility of money (A.139):

$$\left(\frac{\tilde{\mathcal{M}}_t^o}{\tilde{\mathcal{M}}_{t-1}^o}\right)^{-\epsilon_C} \left(\Gamma^Z \Pi^*\right)^{\epsilon_C - 1} = \frac{1}{\Gamma^Z \Pi *} \tag{A.170}$$

so along the balanced growth path:

$$\frac{\tilde{\mathcal{M}}_t^o}{\tilde{\mathcal{M}}_{t-1}^o} = \Gamma^Z \Pi^* \tag{A.171}$$

that is, nominal money balances grow in line with nominal final output or, equivalently, real money balances grow in line with real final output.

A.2.2 Detrending

The derivation of the balanced growth path above abstracted from the permanent labour augmenting productivity shock. Since this shock has permanent effects on the levels of many variables, the detrending of the model must take it into account. In this sub-section, we derive the trends for value-added and final output, which form the basis of a complete set of detrending factors that can be used to convert the model described above into a set of equations for stationary variables.

To derive the detrending factors, we take the per capita value added and final output production functions and use what we have learned about the growth rates of the factor inputs along a steady state balanced growth path. We start with the value added production function, which states that value added is a function of capital services and effective labour. Since these grow at the rate of growth of investment and labour-augmenting productivity, we have that:

$$\tilde{\chi}_t^V = \left(\tilde{\chi}_{t-1}^Z \tilde{\chi}_{t-1}^I\right)^{1-\alpha_L} \left(\tilde{\chi}_t^{LAP}\right)^{\alpha_L} \tag{A.172}$$

Next consider the final output production function, which depends on value added and imports. The detrending factor for final output therefore satifies:

$$\tilde{\chi}_t^Z = \left(\tilde{\chi}_t^V\right)^{\alpha_V} \left(\tilde{\chi}_t^Z \tilde{\chi}_t^X\right)^{1-\alpha_V}$$

which implies:

$$\tilde{\chi}_t^Z = \tilde{\chi}_t^V \left(\tilde{\chi}_t^X\right)^{\frac{1-\alpha_V}{\alpha_V}} \tag{A.173}$$

Substituting equation (A.172) into equation (A.173) yields the detrending factor for Z:

$$\tilde{\chi}_t^Z = \left(\tilde{\chi}_{t-1}^Z \tilde{\chi}_{t-1}^I\right)^{1-\alpha_L} \left(\tilde{\chi}_t^{LAP}\right)^{\alpha_L} \left(\tilde{\chi}_t^X\right)^{\frac{1-\alpha_V}{\alpha_V}} \tag{A.174}$$

Note that this implies that the growth rate of final output is given by:

$$\Gamma_t^Z = \left(\Gamma_{t-1}^Z \Gamma^I\right)^{1-\alpha_L} \left(\Gamma^{LAP} \exp\left(\hat{\varepsilon}_t^{LAP}\right)\right)^{\alpha_L} \left(\Gamma^X\right)^{\frac{1-\alpha_V}{\alpha_V}} \tag{A.175}$$

And the growth rate of value added is given by:

$$\Gamma_t^V = \left(\Gamma_{t-1}^Z \Gamma^I\right)^{1-\alpha_L} \left(\Gamma^{LAP} \exp\left(\hat{\varepsilon}_t^{LAP}\right)\right)^{\alpha_L} \tag{A.176}$$

Table 5 shows how each of the non-stationary variables in COMPASS is detrended given the processes for the exogenous trends and the value-added and final-output trends derived above.

Table 5: Detrending factors

Trending variable	Detrended	Trending variable	Detrended
Value added	$V_t \equiv \frac{\tilde{V}_t}{\tilde{\chi}_t^V}$	Marginal cost of value added	$MC_t^V \equiv \frac{\widetilde{MC}_t^V}{\tilde{P}_t^V}$
Final output	$Z_t \equiv \frac{\tilde{Z}_t}{\tilde{\chi}_t^Z}$	Marginal cost of final output	$MC_t^Z \equiv \frac{\widetilde{MC}_t^Z}{\tilde{P}_t^Z}$
Capital Stock	$K_t \equiv \frac{\tilde{K}_t}{\tilde{\chi}_t^Z \tilde{\chi}_t^I}$	Rental rate of capital	$R_t^K \equiv \frac{\tilde{R}_t^K \tilde{\chi}_t^I}{\tilde{P}_t^Z}$
Wages	$W_t \equiv \frac{\tilde{W}_t}{\tilde{P}_t^Z \tilde{\chi}_t^Z}$	Imports	$M_t \equiv \frac{\tilde{M}_t}{\tilde{\chi}_t^Z \tilde{\chi}_t^X}$
Exports	$X_t \equiv \frac{\tilde{X}_t}{\tilde{\chi}_t^Z \tilde{\chi}_t^X}$	Value-added prices	$P_t^V \equiv \frac{\tilde{P}_t^V \tilde{\chi}_t^V}{\tilde{P}_t^Z \tilde{\chi}_t^Z}$
Consumption	$C_t \equiv \frac{\tilde{C}_t}{\tilde{\chi}_t^Z}$	Exchange rate	$Q_t \equiv \frac{\tilde{Q}_t \tilde{P}_t^Z}{\tilde{P}_t^{Z^F}}$
Investment	$I_t \equiv \frac{\tilde{I}_t}{\tilde{\chi}_t^Z \tilde{\chi}_t^I}$	Consumer prices	$P_t^C \equiv \frac{\tilde{P}_t^C}{\tilde{P}_t^Z}$
Government spending	$G_t \equiv \frac{\tilde{G}_t}{\tilde{\chi}_t^Z \tilde{\chi}_t^G}$	Relative prices	$P_t^N \equiv \frac{\tilde{P}_t^N}{\tilde{P}_t^Z \tilde{\chi}_t^N}$
Other investment	$I_t^O \equiv \frac{\tilde{I}_t^O}{\tilde{\chi}_t^Z}$	Tobin's Q	$TQ_t \equiv \frac{\widetilde{TQ}_t \tilde{\chi}_t^I}{\tilde{P}_t^Z}$
Money	$\mathcal{M}_t \equiv \frac{\tilde{\mathcal{M}}_t}{\tilde{\chi}_t^Z \tilde{P}_t^Z}$	Lump-sum taxes	$T_t \equiv \frac{\tilde{T}_t}{\tilde{P}_t^Z \tilde{\chi}_t^Z}$
Marginal utility	$U_t^{C^o} \equiv \tilde{U}_t^{C^o} \tilde{\chi}_t^Z$	Domestic bonds	$B_t \equiv \frac{\tilde{B}_t}{\tilde{P}_t^Z \tilde{\chi}_t^Z}$
Foreign bonds	$B_t^F \equiv \frac{\tilde{B}_t^F}{\tilde{P}_t^{Z^F} \tilde{\chi}_t^Z}$	Marginal utility of money	$U_t^{\mathcal{M}^o} \equiv \tilde{U}_t^{\tilde{\mathcal{M}}^o} \tilde{\chi}_t^Z \tilde{P}_t^Z$
World demand	$Z_t^F \equiv \frac{\tilde{Z}_t^F}{\tilde{\chi}_t^Z}$	Marginal rate of substitution	$MRS_t \equiv \frac{\widetilde{MRS}_t}{\tilde{\chi}_t^Z \tilde{P}_t^Z}$

A.2.3 Stationary model equations

The detrending factors defined above can be used to produce a complete set of stationary model equations, which are as follows:

Final output production function:

$$Z_t = V_t^{\alpha_V} M_t^{1-\alpha_V} \tag{A.177}$$

Value-added demand:

$$MC_t^Z = \frac{P_t^V V_t}{\alpha_V Z_t} \tag{A.178}$$

Import demand:

$$MC_t^Z = \frac{P_t^M M_t}{(1-\alpha_V) Z_t} \varepsilon_t^M \tag{A.179}$$

Value-added production function:

$$V_t = \varepsilon_t^{TFP} K_{t-1}^{1-\alpha_L} L_t^{\alpha_L} \tag{A.180}$$

Labour demand:

$$MC_t^V = \frac{W_t L_t}{\alpha_L P_t^V V_t} \tag{A.181}$$

Capital demand:

$$MC_t^V = \frac{R_t^K K_{t-1}}{(1-\alpha_L)\Gamma^I P_t^V V_t \Gamma_t^Z} \tag{A.182}$$

Final output pricing:

$$\begin{aligned} 1 &= \mu_t^Z MC_t^Z \\ &- \frac{\phi_Z\left(\mu_t^Z - 1\right)}{\left(\mu^Z - 1\right)}\left\{\left(\zeta_t^Z - 1\right)\zeta_t^Z - \mathbb{B}\left(\frac{C_t^o}{C^o}\right)\Gamma^H \mathrm{E}_t\left[\frac{\Lambda_{t+1}^C}{\Lambda_t^C}\frac{Z_{t+1}}{Z_t}\left(\zeta_{t+1}^Z - 1\right)\zeta_{t+1}^Z\right]\right\} \end{aligned} \tag{A.183}$$

with

$$\zeta_t^Z \equiv \frac{\Pi_t^Z}{\left(\Pi^Z\right)^{1-\xi_Z}\left(\Pi_{t-1}^Z\right)^{\xi_Z}}$$

Value-added pricing:

$$\begin{aligned} 1 &= \mu_t^V MC_t^V \\ &- \frac{\phi_V\left(\mu_t^V - 1\right)}{\left(\mu^V - 1\right)}\left\{\left(\zeta_t^V - 1\right)\zeta_t^V - \mathbb{B}\left(\frac{C_t^o}{C^o}\right)\Gamma^H \mathrm{E}_t\left[\frac{\Lambda_{t+1}^C}{\Lambda_t^C}\frac{\Pi_{t+1}^V}{\Pi_{t+1}^Z}\frac{\Gamma_{t+1}^V}{\Gamma_{t+1}^Z}\frac{V_{t+1}}{V_t}\left(\zeta_{t+1}^V - 1\right)\zeta_{t+1}^V\right]\right\} \end{aligned} \tag{A.184}$$

with

$$\zeta_t^V \equiv \frac{\Pi_t^V}{\left(\Pi^V\right)^{1-\xi_V}\left(\Pi_{t-1}^V\right)^{\xi_V}}$$

Import pricing:

$$\begin{aligned} 1 &= \mu_t^M \frac{P_t^{X^F}}{Q_t P_t^M} \\ &- \frac{\phi_M\left(\mu_t^M - 1\right)}{\left(\mu^M - 1\right)}\left\{\left(\zeta_t^M - 1\right)\zeta_t^M - \mathbb{B}\left(\frac{C_t^o}{C^o}\right)\Gamma^H \mathrm{E}_t\left[\frac{\Lambda_{t+1}^C}{\Lambda_t^C}\frac{\Pi_{t+1}^M}{\Pi_{t+1}^Z}\frac{M_{t+1}}{M_t}\Gamma^X\left(\zeta_{t+1}^M - 1\right)\zeta_{t+1}^M\right]\right\} \end{aligned} \tag{A.185}$$

with

$$\zeta_t^M \equiv \frac{\Pi_t^M}{\left(\Pi^M\right)^{1-\xi_M}\left(\Pi_{t-1}^M\right)^{\xi_M}}$$

Export pricing:

$$\begin{aligned} 1 &= \mu_t^X \frac{Q_t P_t^X}{P_t^{EXP}} \\ &- \frac{\phi_X\left(\mu_t^X - 1\right)}{\left(\mu^X - 1\right)}\left\{\left(\zeta_t^X - 1\right)\zeta_t^X - \mathbb{B}\left(\frac{C_t^o}{C^o}\right)\Gamma^H \mathrm{E}_t\left[\frac{\Lambda_{t+1}^C}{\Lambda_t^C}\frac{\Gamma^X \Pi_{t+1}^{EXP}}{\Pi^{Z^F}}\frac{Q_t}{Q_{t+1}}\frac{X_{t+1}}{X_t}\left(\zeta_{t+1}^X - 1\right)\zeta_{t+1}^X\right]\right\} \end{aligned} \tag{A.186}$$

with

$$\zeta_t^X \equiv \frac{\Pi_t^{EXP}}{\left(\Pi^{EXP}\right)^{1-\xi^X}\left(\Pi_{t-1}^{EXP}\right)^{\xi^X}}$$

Consumption Euler condition:

$$\Lambda_t^C = \mathbb{B}\left(\frac{C_t^o}{C^o}\right)\mathrm{E}_t\left[\Lambda_{t+1}^C\frac{R_t^A}{\Pi_{t+1}^Z\Gamma_{t+1}^Z}\right] \tag{A.187}$$

Consumption Lagrange multiplier:

$$\Lambda_t^C = U_t^{C^o} \tag{A.188}$$

Marginal utility of consumption:

$$U_t^{C^o} = \left(C_t^o - \psi_C C_{t-1}^o\right)^{-\epsilon_C} \tag{A.189}$$

Labour supply:

$$\begin{aligned}\frac{MRS_t}{W_t} &= \frac{\mu_t^W - 1}{\mu_t^W}\left\{\frac{1}{(\mu_t^W - 1)}\left(1 - \frac{\phi_W\left(\zeta_t^W - 1\right)^2}{2\left(\mu^W - 1\right)}\right)\right.\\ &+ \left.\frac{\phi_W}{(\mu^W - 1)}\left\{\left(\zeta_t^W - 1\right)\zeta_t^W - \mathbb{B}\left(\frac{C_t^o}{C^o}\right)\Gamma^H\mathrm{E}_t\left[\frac{\Lambda_{t+1}^C}{\Lambda_t^C}\frac{W_{t+1}L_{t+1}}{W_tL_t}\left(\zeta_{t+1}^W - 1\right)\zeta_{t+1}^W\right]\right\}\right\}\end{aligned} \tag{A.190}$$

where:

$$MRS_t \equiv \frac{U_t^L}{\Lambda_t^C}$$

$$\zeta_t^W \equiv \frac{\Pi_t^W}{\left(\Pi^W\right)^{1-\xi^W}\left(\Pi_{t-1}^W\right)^{\xi^W}}$$

Marginal disutility of labour:

$$U_t^{L^o} = -\nu_L\varepsilon_t^L\left(L_t\right)^{\epsilon_L} \tag{A.191}$$

Money demand:

$$U_t^{\mathcal{M}^o} = \frac{R_t^A - 1}{R_t^A}\Lambda_t^C \tag{A.192}$$

Marginal utility of real money balances

$$U_t^{\mathcal{M}^o} = \nu_M\left(\mathcal{M}_t\right)^{-\epsilon_C} \tag{A.193}$$

Aggregate money holdings:

$$\mathcal{M}_t = \omega_o\mathcal{M}_t^o \tag{A.194}$$

Investment equation:

$$\begin{aligned}1 &= TQ_t\varepsilon_t^I\left[1 - \frac{\psi_I}{2}\left(\zeta_t^I - \Gamma^H\Gamma^Z\Gamma^I\right)^2 - \psi_I\left(\zeta_t^I - \Gamma^H\Gamma^Z\Gamma^I\right)\zeta_t^I\right]\\ &+ \mathbb{B}\left(\frac{C_t^o}{C^o}\right)\psi_I\mathrm{E}_t\left[TQ_{t+1}\frac{\Lambda_{t+1}^C}{\Gamma^I\Lambda_t^C\Gamma_{t+1}^Z}\varepsilon_{t+1}^I\left(\zeta_{t+1}^I - \Gamma^H\Gamma^Z\Gamma^I\right)\left(\zeta_{t+1}^I\right)^2\right]\end{aligned} \tag{A.195}$$

Tobin's Q:

$$TQ_t = \mathbb{B}\left(\frac{C_t^o}{C^o}\right)\mathrm{E}_t\left\{\frac{\Lambda_{t+1}^C}{\Gamma^I\Lambda_t^C\Gamma_{t+1}^Z}\left[R_{t+1}^K + TQ_{t+1}\left(1 - \delta^K\right)\right]\right\} \tag{A.196}$$

Capital accumulation:

$$\Gamma^H K_t = \left(1 - \delta^K\right)\frac{K_{t-1}}{\Gamma_t^Z\Gamma^I} + \varepsilon_t^I\left(1 - \frac{\psi_I}{2}\left(\zeta_t^I - \Gamma^H\Gamma^Z\Gamma^I\right)^2\right)I_t \tag{A.197}$$

where:

$$\zeta_t^I \equiv \frac{\Gamma^I \Gamma^H \Gamma_t^Z I_t}{I_{t-1}}$$

Other investment:

$$\frac{I_t^O}{I_{t-1}^O}\Gamma_t^Z = \left(I_{t-1}^O\right)^{\rho_{IO}-1} \varepsilon_t^{IO} \tag{A.198}$$

Export demand:

$$X_t = \kappa_t^F \left(\frac{P_t^{EXP}}{P_t^{XF}}\right)^{-\epsilon_F} Z_t^F \tag{A.199}$$

World output:

$$\frac{Z_t^F}{Z_{t-1}^F}\Gamma_t^Z = Z_{t-1}^{\rho_Z^F - 1} \varepsilon_t^{Z^F} \tag{A.200}$$

World export prices:

$$P_t^{X^F} = \left(P^{X^F}\right)^{1-\rho_{PX}^F} \left(P_{t-1}^{X^F}\right)^{\rho_{PX}^F} \varepsilon_t^{PX^F} \tag{A.201}$$

Rule-of-thumb consumption:

$$C_t^{rot} = \frac{W_t}{P_t^C} L_t + Trans^{rot} \tag{A.202}$$

Aggregate consumption:

$$C_t = \omega_o C_t^o + (1-\omega_o)\, C_t^{rot} \tag{A.203}$$

Resource constraint:

$$Z_t = C_t + I_t + G_t + X_t + I_t^O \tag{A.204}$$

Relative price of consumption and other investment (to final output price):

$$P_t^C = P_t^{I^O} = 1 \tag{A.205}$$

Relative price of value-added (to final output price):

$$P_t^V = \frac{\Gamma_t^V \Pi_t^V}{\Gamma_t^Z \Pi_t^Z} P_{t-1}^V \tag{A.206}$$

Relative price of exports (to foreign final output price):

$$P_t^{EXP} = \frac{\Gamma^X \Pi_t^{EXP}}{\Pi_t^{Z^F}} P_{t-1}^{EXP} \tag{A.207}$$

Relative price of imports (to final output price):

$$P_t^M = \frac{\Gamma^X \Pi_t^M}{\Pi_t^Z} P_{t-1}^M \tag{A.208}$$

Relative wage (to final output price):

$$W_t = \frac{\Pi_t^W}{\Gamma_t^Z \Pi_t^Z} W_{t-1} \tag{A.209}$$

Monetary policy rule:

$$R_t = R^{1-\theta_R} R_{t-1}^{\theta_R} \left(\frac{\Pi_t^{C,annual}}{\Pi^{*,annual}} \right)^{\frac{(1-\theta_R)\theta_\Pi}{4}} \left(\hat{Y}_t \right)^{(1-\theta_R)\theta_Y} \varepsilon_t^R \tag{A.210}$$

CPI inflation:

$$\Pi_t^C = \Pi_t^Z \tag{A.211}$$

Definition of the output gap:

$$\hat{Y}_t = \frac{V_t}{V_t^{flex}} \tag{A.212}$$

Definition of the return on deposits:

$$R_t^A = R_t \varepsilon_t^B \tag{A.213}$$

Real government spending:

$$\frac{G_t}{G_{t-1}} \Gamma_t^Z = G_{t-1}^{\rho_G - 1} \varepsilon_t^G \tag{A.214}$$

Government budget constraint:

$$G_t + \frac{(B_{t-1} R_{t-1} + \mathcal{M}_{t-1})}{\Pi_t^Z \Gamma_t^Z \Gamma^H} = T_t + B_t + \mathcal{M}_t \tag{A.215}$$

UIP condition:

$$\frac{R_t}{R^F} = \mathrm{E}_t \left[\frac{\varepsilon_t^{B^F} Q_t}{Q_{t+1}} \frac{\Pi_{t+1}^Z}{\Pi^{Z^F}} \right] \tag{A.216}$$

Final output growth:

$$\Gamma_t^Z = \left(\Gamma_{t-1}^Z \Gamma^I \right)^{1-\alpha_L} \left(\Gamma^{LAP} \exp\left(\hat{\varepsilon}_t^{LAP} \right) \right)^{\alpha_L} \left(\Gamma^X \right)^{\frac{1-\alpha_V}{\alpha_V}} \tag{A.217}$$

Value-added growth:

$$\Gamma_t^V = \left(\Gamma_{t-1}^Z \Gamma^I \right)^{1-\alpha_L} \left(\Gamma^{LAP} \exp\left(\hat{\varepsilon}_t^{LAP} \right) \right)^{\alpha_L} \tag{A.218}$$

In addition, the stationary model equations comprise the following forcing processes for the stochastic disturbances to the model, where all shocks (denoted η) have standard normal distributions:
Exogenous process for the labour supply shock:

$$\log \varepsilon_t^L = (1-\rho_L) \log \varepsilon^L + \rho_L \log \varepsilon_{t-1}^L + \left(1 - \rho_L^2 \right)^{1/2} \sigma_L \eta_t^L \tag{A.219}$$

Exogenous process for the import shock:

$$\log \varepsilon_t^M = (1-\rho_M) \log \varepsilon^M + \rho_M \log \varepsilon_{t-1}^M + \left(1 - \rho_M^2 \right)^{1/2} \sigma_M \eta_t^M \tag{A.220}$$

Exogenous process for TFP:

$$\log \varepsilon_t^{TFP} = (1-\rho_{TFP}) \varepsilon^{TFP} + \rho_{TFP} \varepsilon_{t-1}^{TFP} + \left(1 - \rho_{TFP}^2 \right)^{1/2} \sigma_{TFP} \eta_t^{TFP} \tag{A.221}$$

Exogenous process for labour-augmenting productivity shock:

$$\hat{\varepsilon}_t^{LAP} = \rho_{LAP}\hat{\varepsilon}_{t-1}^{LAP} + \left(1-\rho_{LAP}^2\right)^{1/2}\sigma_{LAP}\eta_t^{LAP} \tag{A.222}$$

Exogenous process for the domestic risk premium shock:

$$\log \varepsilon_t^B = (1-\rho_B)\log \varepsilon^B + \rho_B \log \varepsilon_{t-1}^B + \left(1-\rho_B^2\right)^{1/2}\sigma_B\eta_t^B \tag{A.223}$$

Exogenous process for the UIP shock:

$$\log \varepsilon_t^{B^F} = (1-\rho_{B^F})\log \varepsilon^{B^F} + \rho_{B^F} \log \varepsilon_{t-1}^{B^F} + \left(1-\rho_{B^F}^2\right)^{1/2}\sigma_{B^F}\eta_t^{B^F} \tag{A.224}$$

Exogenous process for the investment shock:

$$\log \varepsilon_t^I = (1-\rho_I)\log \varepsilon^I + \rho_I \log \varepsilon_{t-1}^I + \left(1-\rho_I^2\right)^{1/2}\sigma_I\eta_t^I \tag{A.225}$$

Exogenous process for the government spending shock:

$$\log \varepsilon_t^G = \left(1-\rho_G^2\right)^{1/2}\sigma_G\eta_t^G \tag{A.226}$$

Exogenous process for the world preference for domestic exports shock:

$$\log \varepsilon_t^{\kappa^F} = (1-\rho_{\kappa^F})\log \varepsilon^{\kappa^F} + \rho_{\kappa^F} \log \varepsilon_{t-1}^{\kappa^F} + \left(1-\rho_{\kappa^F}^2\right)^{1/2}\sigma_{\kappa^F}\eta_t^{\kappa^F} \tag{A.227}$$

Exogenous process for the other investment shock:

$$\log \varepsilon_t^{IO} = \left(1-\rho_{IO}^2\right)^{1/2}\sigma_{IO}\eta_t^{IO} \tag{A.228}$$

Exogenous process for the monetary policy shock:

$$\log \varepsilon_t^R = \sigma_R\eta_t^R \tag{A.229}$$

Exogenous process for the world output shock:

$$\log \varepsilon_t^{Z^F} = \left(1-\rho_{Z^F}^2\right)\sigma_{Z^F}\eta_t^{Z^F} \tag{A.230}$$

Exogenous process for world export prices shock:

$$\log \varepsilon_t^{PX^F} = \left(1-\rho_{PX^F}^2\right)\sigma_{PX^F}\eta_t^{PX^F} \tag{A.231}$$

Exogenous process for the final output price mark-up:

$$\log \mu_t^Z = \sigma_{\mu^Z}\eta_t^{\mu^Z}$$

Exogenous process for the value-added price mark-up:

$$\log \mu_t^V = \sigma_{\mu^V}\eta_t^{\mu^V}$$

Exogenous process for the wage mark-up:

$$\log \mu_t^W = \sigma_{\mu^W}\eta_t^{\mu^W} \tag{A.232}$$

Exogenous process for the import price mark-up:

$$\log \mu_t^M = \sigma_{\mu^M}\eta_t^{\mu^M} \tag{A.233}$$

Exogenous process for the export price mark-up:

$$\log \mu_t^X = \sigma_{\mu^X}\eta_t^{\mu^X} \tag{A.234}$$

A.3 Steady state

Having obtained the stationary version of the model we outline its steady state in this subsection. This is the characterisation of the system obtained when there are no shocks and so with the time dimension removed. The steady system has a recursive representation and can be solved analytically.

There are several different ways of obtaining numerical values for the steady states in this recursive system. We choose to calibrate (treat as parameters) the ratios of business investment, other investment, exports, imports and government spending to final output (ω_{IZ}, ω_{I^OZ}, ω_{XZ}, ω_{MZ} and ω_{GZ}). This is not necessary to obtain a steady state solution, but is useful in mapping the model to the data as described in the main text.

Given these shares, then from the market clearing condition (A.204) we obtain the consumption to final output ratio by residual:

$$\omega_{CZ} = 1 - \omega_{IZ} - \omega_{I^OZ} - \omega_{XZ} - \omega_{GZ} \tag{A.235}$$

From the consumption Euler equation (A.187) and the portfolio packager's first order condition (A.213) we know that:

$$R^A = R = \frac{\Pi^Z \Gamma^Z}{\beta} \tag{A.236}$$

From the capital accumulation equation (A.197), we obtain the capital to final output ratio:

$$\begin{aligned} \frac{K}{Z} &= \frac{1-\delta^K}{\Gamma^H \Gamma^Z \Gamma^I} \frac{K}{Z} + \frac{I}{\Gamma^H Z} \\ \omega_{KZ} &= \frac{\omega_{IZ}}{\left(1 - \frac{1-\delta^K}{\Gamma^H \Gamma^Z \Gamma^I}\right) \Gamma^H} \end{aligned} \tag{A.237}$$

We can use the equation for Tobin's Q (A.196) to derive an expression for the rental rate of capital:

$$R^K = \frac{\Gamma^I \Gamma^Z}{\beta} - \left(1 - \delta^K\right) \tag{A.238}$$

From the demand for value added (A.178) and the final output pricing equation we know (A.183):

$$\frac{P^V V}{Z} = \frac{\alpha_V}{\mu^Z} \tag{A.239}$$

And combining the labour demand equation (A.181) with the value-added pricing equation (A.184):

$$\frac{1}{\mu^V} = \frac{WL}{\alpha_L P^V V} \tag{A.240}$$

If we multiply and divide the right-hand-side of this expression by the steady-state value for final output and use the expression for the steady state value-added mark-up (A.240), we get:

$$\frac{W}{Z} = \omega_{WZ} = \frac{\alpha_L \alpha_V}{\mu^Z \mu^V} L \tag{A.241}$$

The value-added share, $\frac{P^V V}{Z} = \frac{\alpha_V}{\mu^Z}$, and the labour share, $\frac{WL}{P^V V} = \frac{\alpha_L}{\mu^V}$, are calibrated, implying that equation (A.241) delivers the steady-state wage to final output ratio. Using expressions (A.237), (A.238), (A.239), (A.241) and the capital demand equation (A.182) we obtain:

$$
\begin{aligned}
MC^V &= \frac{R^K}{(1-\alpha_L)\Gamma^I\Gamma^Z}\frac{K}{Z}\frac{Z}{P^V V} \\
1 &= \frac{R^K}{\frac{1-\alpha_L}{\alpha_L}\Gamma^I\Gamma^Z}\frac{K}{Z}\frac{\mu^Z\mu^V}{\alpha_L\alpha_V} \\
\alpha_L &= \frac{1}{1+\frac{R^K}{\Gamma^I\Gamma^Z}\frac{K}{Z}\frac{Z}{WL}}
\end{aligned} \tag{A.242}
$$

Given values for α_L and the labour share, $\frac{\alpha_L}{\mu^V}$, we obtain the steady-state value of the value-added mark-up, μ^V. In addition, from the valued-added pricing equation we also know that:

$$
MC^V = \frac{1}{\mu^V} \tag{A.243}
$$

which pins down the steady-state value of value-added marginal cost. Using the final output production function (A.177), we obtain the value added to final output ratio:

$$
\omega_{VZ} = \left(\frac{1}{\omega_{MZ}^{1-\alpha_V}}\right)^{\frac{1}{\alpha_V}} \tag{A.244}
$$

The value-added production function (A.180) can then be used to define the steady-state value of final output:

$$
\begin{aligned}
\frac{V}{Z} &= \left(\frac{K}{Z}\right)^{1-\alpha_L}\left(\frac{L}{Z}\right)^{\alpha_L} \\
Z &= \frac{\omega_{KZ}^{\frac{1-\alpha_L}{\alpha_L}}}{\omega_{VZ}^{\frac{1}{\alpha_L}}}L
\end{aligned} \tag{A.245}
$$

which defines steady-state final output given a value for L, which we calibrate to 1, backing out the vallue for the parameter, ν_L, which delivers that (see below). From this point onwards all other steady-state values of the model arise naturally:

$$C = \omega_{CZ}Z \tag{A.246}$$
$$I = \omega_{IZ}Z \tag{A.247}$$
$$I^O = \omega_{I^O Z}Z \tag{A.248}$$
$$K = \omega_{KZ}Z \tag{A.249}$$
$$W = \omega_{WZ}Z \tag{A.250}$$
$$V = \omega_{VZ}Z \tag{A.251}$$
$$M = \omega_{MZ}Z \tag{A.252}$$
$$X = \omega_{XZ}Z \tag{A.253}$$

We normalise the price of final output to one, both at home and in the rest of the world. Productivity is equal across the different retail sectors in the detrended steady state. The relative prices of the expenditure components are therefore equalised:

$$
1 = P^Z = P^C = P^I = P^X = P^G = P^{I^O} = P^{X^F} \tag{A.254}
$$

As discussed above, transfers paid to rule-of-thumb households, as a device to ensure that optimising and rule-of-thumb households share the same steady-state level of consumption. This implies that:

$$\begin{aligned} C^{rot} &= C^{o} = C = WL + Trans^{rot} \\ Trans^{rot} &= C - WL \end{aligned} \tag{A.255}$$

Given the steady state value of consumption, the steady-state value of the marginal utility of consumption is given by:

$$U^{C^{o}} = \Lambda^{C} = \frac{1}{\left(\left(1-\psi_{C}\right)C\right)^{\epsilon_{C}}} \tag{A.256}$$

From the money demand equation (A.192) we obtain the steady-state value of $\mathcal{M}^{o}$:

$$\mathcal{M}^{o} = \left(\frac{\left(R^{A}-1\right)\Lambda^{C}}{\nu_{M}R^{A}}\right)^{-\frac{1}{\epsilon_{C}}} \tag{A.257}$$

So:

$$\mathcal{M} = \omega_{o}\mathcal{M}^{o} \tag{A.258}$$

We can use the government's budget constraint (A.215) to derive the steady-state value of lump-sum taxes:

$$T = G + B\left(\frac{R}{\Pi^{Z}\Gamma^{Z}\Gamma^{H}} - 1\right) + \mathcal{M}\left(\frac{1}{\Pi^{Z}\Gamma^{Z}\Gamma^{H}} - 1\right) \tag{A.259}$$

From the labour supply equation (A.190) we know that:

$$\nu_{L}\varepsilon^{L}\left(L\right)^{\epsilon_{L}}\left(\left(1-\psi_{C}\right)C\right)^{\epsilon_{C}} = \frac{W}{\mu^{W}} \tag{A.260}$$

As discussed above, we calibrate ν_{L} to ensure that $L = 1$ in the steady state, so:

$$\nu_{L} = \frac{W}{\mu^{W}\varepsilon^{L}\left(\left(1-\psi_{C}\right)C\right)^{\epsilon_{C}}} \tag{A.261}$$

The final steady state value to derive is that of the real exchange rate. The steady state ratio of net trade to final output can be defined as:

$$NT = \frac{P^{EXP}}{Q}\omega_{XZ} - \frac{P^{X^{F}}}{Q}\omega_{MZ} \tag{A.262}$$

where NT is a calibrated steady-state level for the nominal net trade ratio. As we will see, this calibration corresponds to an implicit assumption about the relative preference of foreign consumers for domestic output (the following steps show how to derive the required assumption for the parameter κ_{F} to deliver the desired net trade ratio). From the import and export pricing equations, (A.185) & (A.186), and given that prices are normalised to 1 in the steady state, we know that:

$$\frac{P^{EXP}}{Q} = \mu^{X}P^{X} = \mu^{X} \tag{A.263}$$

$$\frac{P^{X^{F}}}{Q} = \frac{P^{M}}{\mu^{M}} \tag{A.264}$$

In addition, we know from the import demand equation (A.179) that:

$$\begin{aligned} MC^{Z} &= \frac{P^{M}M}{(1-\alpha_{V})\,M} \\ (1-\alpha_{V})\,\mu^{Z} &= P^{M}\frac{M}{Z} \\ \frac{(1-\alpha_{V})\,\mu^{Z}}{\mu^{M}} &= \frac{P^{X^{F}}}{Q}\omega_{MZ} \end{aligned} \tag{A.265}$$

From the export demand equation (A.186) we have:

$$\begin{aligned} X &= \kappa^{F}\left(\frac{P^{EXP}}{P^{X^{F}}}\right)^{-\epsilon_{F}} Z^{F} \\ \omega_{XZ} &= \kappa^{F}\left(\mu^{X}Q\right)^{-\epsilon^{F}}\frac{1}{Z} \end{aligned}$$

where $P^{X^{F}} = 1$ and $Z^{F} = 1$ are normalised. Combining these expressions with the expression for net trade from above yields:

$$NT = \left(\mu^{X}\right)^{1-\epsilon_{F}}\kappa^{F}Q^{-\epsilon_{F}}\frac{1}{Z} - \frac{(1-\alpha_{V})\,\mu^{Z}}{\mu^{M}}$$

We can use this expression to solve for the steady-state value of the exchange rate:

$$Q = \left[\frac{\left(NT + \frac{(1-\alpha_{V})\mu^{Z}}{\mu^{M}}\right)Z}{\left(\mu^{X}\right)^{1-\epsilon^{F}}\kappa^{F}}\right]^{-\frac{1}{\epsilon^{F}}}$$

We calibrate μ^{M}, μ^{X} and Q, backing out the parameter κ^{F} that delivers the calibrated steady state real exchange rate. As long as $\frac{(1-\alpha_{V})\mu^{Z}}{\mu^{M}} > -NT$, then $Q > 0$.

A.4 Dynamics around the steady state

The stationary model equations are linearised or log-linearised around the steady state. These equations will then determine how the economy responds to a temporary shock that drives the economy away from its equilibrium. Log deviations of variables from steady state are denoted using small letters and are defined as $j_t \equiv \log J_t - \log J$. For example, the percentage deviation of consumption from steady state is $c_t \equiv \log C_t - \log C$.[199] The only exceptions to this rule are for forcing processes. We use the notation $\hat{\epsilon}_t^{D} \equiv \log \epsilon_t^{D}$ to denote log-deviations of forcing processes from their steady-state values of unity. We also use the notation $\hat{\mu}_t^{D} \equiv \log \mu_t^{D} - \log \mu^{D}$ to denote log-deviations of mark-up processes from their steady-state values.

This sub-section presents the log-linearised model. It first presents a log-linearisation of the stationary model equations around the steady state derived above. It then presents two amendments made to that log-linearised model prior to estimation.

[199] Note that the following relationship holds: $J_t - J \simeq J j_t$.

A.4.1 Log-linearised model equations

The following represents a complete set of log-linearised, stationary model equations.

Final output production function:

$$z_t = \alpha_V v_t + (1 - \alpha_V) m_t \tag{A.266}$$

Value-added demand:

$$mc_t^Z = p_t^V + v_t - z_t \tag{A.267}$$

Import demand:

$$mc_t^Z = p_t^M + m_t - z_t + \hat{\varepsilon}_t^M \tag{A.268}$$

Value-added production function:

$$v_t = \hat{\varepsilon}_t^{TFP} + (1 - \alpha_L) k_{t-1} + \alpha_L l_t \tag{A.269}$$

Labour demand:

$$mc_t^V = w_t + l_t - p_t^V - v_t \tag{A.270}$$

Capital demand:

$$mc_t^V = r_t^K + k_{t-1} - p_t^V - v_t - \gamma_t^Z \tag{A.271}$$

Final output inflation:

$$\pi_t^Z = \hat{\mu}_t^Z + \frac{1}{\phi_Z (1 + \beta \Gamma^H \xi_Z)} mc_t^Z + \frac{\xi_Z}{1 + \beta \Gamma^H \xi_Z} \pi_{t-1}^Z + \frac{\beta \Gamma^H}{1 + \beta \Gamma^H \xi_Z} \mathrm{E}_t \pi_{t+1}^Z \tag{A.272}$$

Value-added inflation:

$$\pi_t^V = \hat{\mu}_t^V + \frac{1}{\phi_V (1 + \beta \Gamma^H \xi_V)} mc_t^V + \frac{\xi_V}{1 + \beta \Gamma^H \xi_V} \pi_{t-1}^V + \frac{\beta \Gamma^H}{1 + \beta \Gamma^H \xi_V} \mathrm{E}_t \pi_{t+1}^V \tag{A.273}$$

Import price inflation:

$$\pi_t^M = \hat{\mu}_t^M + \frac{p_t^{X^F} - q_t - p_t^M}{\phi_M (1 + \beta \Gamma^H \xi_M)} + \frac{\xi_M}{1 + \beta \Gamma^H \xi_M} \pi_{t-1}^M + \frac{\beta \Gamma^H}{1 + \beta \Gamma^H \xi_M} \pi_{t+1}^M \tag{A.274}$$

Export price inflation:

$$\pi_t^{EXP} = \hat{\mu}_t^X + \frac{p_t^X + q_t - p_t^{EXP}}{\phi_X (1 + \beta \Gamma^H \xi_X)} + \frac{\xi_X}{1 + \beta \Gamma^H \xi_X} \pi_{t-1}^{EXP} + \frac{\beta \Gamma^H}{1 + \beta \Gamma^H \xi_X} \mathrm{E}_t \pi_{t+1}^{EXP} \tag{A.275}$$

Consumption Euler equation:

$$\begin{aligned} c_t^o = {} & \frac{\psi_C}{1 + \psi_C - \frac{\epsilon_\beta (1 - \psi_C)}{\epsilon_C}} c_{t-1}^o + \frac{1}{1 + \psi_C - \frac{\epsilon_\beta (1 - \psi_C)}{\epsilon_C}} c_{t+1}^o \\ & - \frac{1 - \psi_C}{(1 + \psi_C) \epsilon_C - \epsilon_\beta (1 - \psi_C)} \mathrm{E}_t \left(r_t - \pi_{t+1}^Z + \hat{\varepsilon}_t^B - \gamma_{t+1}^Z \right) \end{aligned} \tag{A.276}$$

Labour supply (wage Phillips curve):

$$\pi_t^W = \hat{\mu}_t^W + \frac{\xi_W}{1 + \beta \Gamma^H \xi_W} \pi_{t-1}^W + \frac{\beta \Gamma^H}{1 + \beta \Gamma^H \xi_W} \mathrm{E}_t \pi_{t+1}^W + \frac{mrs_t - w_t}{\phi_W (1 + \beta \Gamma^H \xi_W)} \tag{A.277}$$

Marginal rate of substitution:

$$mrs_t = \epsilon_L l_t + \frac{\epsilon_C}{1-\psi_C}\left(c_t^o - \psi_C c_{t-1}^o\right) + \hat{\varepsilon}_t^L \tag{A.278}$$

Money demand:

$$mon_t = -\frac{1}{(R-1)\,\epsilon_C}\left(r_t + \hat{\varepsilon}_t^B\right) + \frac{1}{1-\psi_C}\left(c_t^o - \psi_C c_{t-1}^o\right) \tag{A.279}$$

Investment Euler equation:

$$i_t = \frac{\beta\Gamma^H}{1+\beta\Gamma^H}\left(i_{t+1} + \gamma_{t+1}^Z\right) + \frac{1}{1+\beta\Gamma^H}\left(i_{t-1} - \gamma_t^Z\right) + \frac{tq_t + \hat{\varepsilon}_t^I}{\psi_I\left(1+\beta\Gamma^H\right)\left(\Gamma^H\Gamma^Z\Gamma^I\right)^2} \tag{A.280}$$

Tobin's Q:

$$tq_t = \frac{R^K}{R^K + \left(1-\delta^K\right)}\mathrm{E}_t r_{t+1}^K - \left(r_t - \mathrm{E}_t\pi_{t+1}^Z + \hat{\varepsilon}_t^B\right) + \frac{1-\delta^K}{R^K + \left(1-\delta^K\right)}\mathrm{E}_t tq_{t+1} \tag{A.281}$$

Capital accumulation:

$$k_t = \frac{1-\delta^K}{\Gamma^H\Gamma^Z\Gamma^I}\left(k_{t-1} - \gamma_t^Z\right) + \frac{I}{\Gamma^H K}\left(i_t + \hat{\varepsilon}_t^I\right) \tag{A.282}$$

Other investment:

$$i_t^O - i_{t-1}^O + \gamma_t^Z = \left(\rho_{I^O} - 1\right) i_{t-1}^O + \hat{\varepsilon}_t^{I^O} \tag{A.283}$$

Export demand:

$$x_t = z_t^F + \hat{\varepsilon}_t^{\kappa^F} - \epsilon_F\left(p_t^{EXP} - p_t^{X^F}\right) \tag{A.284}$$

World output:

$$z_t^F - z_{t-1}^F + \gamma_t^Z = \left(\rho_{Z^F} - 1\right) z_{t-1}^F + \hat{\varepsilon}_t^{Z^F} \tag{A.285}$$

World export prices:

$$p_t^{X^F} = \rho_{PX^F} p_{t-1}^{X^F} + \hat{\varepsilon}_t^{PX^F} \tag{A.286}$$

Rule-of-thumb consumption:

$$c_t^{rot} = \frac{WL}{C}\left(w_t + l_t\right) \tag{A.287}$$

Aggregate consumption:

$$c_t = \omega_o c_t^o + \left(1-\omega_o\right) c_t^{rot} \tag{A.288}$$

Resource constraint:

$$z_t = \frac{C}{Z}c_t + \frac{I}{Z}i_t + \frac{G}{Z}g_t + \frac{X}{Z}x_t + \frac{I^O}{Z}i_t^O \tag{A.289}$$

Price of value-added (relative to final output):

$$p_t^V = \pi_t^V - \pi_t^Z + p_{t-1}^V \tag{A.290}$$

Price of exports (relative to final output):

$$p_t^{EXP} = \pi_t^{EXP} + p_{t-1}^{EXP} \tag{A.291}$$

Price of imports (relative to final output):

$$p_t^M = \pi_t^M - \pi_t^Z + p_{t-1}^M \tag{A.292}$$

Wage (relative to price of final output):

$$w_t = \pi_t^W - \gamma_t^Z - \pi_t^Z + w_{t-1} \tag{A.293}$$

Monetary policy rule:

$$r_t = \theta_R r_{t-1} + (1-\theta_R)\left(\frac{\theta_\Pi}{4}\pi_t^{C,annual} + \theta_Y \hat{y}_t\right) + \hat{\varepsilon}_t^R \tag{A.294}$$

Consumer price inflation:

$$\pi_t^C = \pi_t^Z \tag{A.295}$$

Annual consumer price inflation:

$$\pi_t^{C,annual} = \pi_t^C + \pi_{t-1}^C + \pi_{t-2}^C + \pi_{t-3}^C \tag{A.296}$$

Output gap:

$$\hat{y}_t = v_t - v_t^{flex} \tag{A.297}$$

Government spending:

$$g_t - g_{t-1} + \gamma_t^Z = (\rho_G - 1)\, g_{t-1} + \hat{\varepsilon}_t^G \tag{A.298}$$

UIP condition:

$$r_t = \mathrm{E}_t \pi_{t+1}^Z - \mathrm{E}_t q_{t+1} + q_t + \hat{\varepsilon}_t^{B^F} \tag{A.299}$$

Final output growth:

$$\gamma_t^Z = \alpha_L \hat{\varepsilon}_t^{LAP} + (1-\alpha_L)\gamma_{t-1}^Z \tag{A.300}$$

In addition, the log-linearised model equations comprise the following forcing processes for the stochastic disturbances to the model, where all shocks (denoted η) have standard normal distributions:

Exogenous process for TFP:

$$\hat{\varepsilon}_t^{TFP} = \rho_{TFP}\hat{\varepsilon}_{t-1}^{TFP} + \left(1-\rho_{TFP}^2\right)^{1/2}\sigma_{TFP}\eta_t^{TFP} \tag{A.301}$$

Exogenous process for labour-augmenting productivity shock:

$$\hat{\varepsilon}_t^{LAP} = \rho_{LAP}\hat{\varepsilon}_{t-1}^{LAP} + \left(1-\rho_{LAP}^2\right)^{1/2}\sigma_{LAP}\eta_t^{LAP} \tag{A.302}$$

Exogenous process for the domestic risk premium shock:

$$\hat{\varepsilon}_t^{B} = \rho_{B}\hat{\varepsilon}_{t-1}^{B} + \left(1-\rho_{B}^2\right)^{1/2}\sigma_{B}\eta_t^{B} \tag{A.303}$$

Exogenous process for the UIP shock:

$$\hat{\varepsilon}_t^{B^F} = \rho_{B^F}\hat{\varepsilon}_{t-1}^{B^F} + \left(1-\rho_{B^F}^2\right)^{1/2}\sigma_{B^F}\eta_t^{B^F} \tag{A.304}$$

Exogenous process for investment shock:

$$\hat{\varepsilon}_t^I = \rho_I \hat{\varepsilon}_{t-1}^I + \left(1 - \rho_I^2\right)^{1/2} \sigma_I \eta_t^I \tag{A.305}$$

Exogenous process for government spending shock:

$$\hat{\varepsilon}_t^G = \left(1 - \rho_G^2\right)^{1/2} \sigma_G \eta_t^G \tag{A.306}$$

Exogenous process for world preference for domestic exports shock:

$$\hat{\varepsilon}_t^{\kappa^F} = \rho_{\kappa^F} \hat{\varepsilon}_{t-1}^{\kappa^F} + \left(1 - \rho_{\kappa^F}^2\right)^{1/2} \sigma_{\kappa^F} \eta_t^{\kappa^F} \tag{A.307}$$

Exogenous process for other investment shock:

$$\hat{\varepsilon}_t^{I^O} = \left(1 - \rho_{I^O}^2\right)^{1/2} \sigma_{I^O} \eta_t^{I^O} \tag{A.308}$$

Exogenous process for monetary policy shock:

$$\hat{\varepsilon}_t^R = \sigma_R \eta_t^R \tag{A.309}$$

Exogenous process for world final output shock:

$$\hat{\varepsilon}_t^{Z^F} = \left(1 - \rho_{Z^F}^2\right)^{1/2} \sigma_{Z^F} \eta_t^{Z^F} \tag{A.310}$$

Exogenous process for world export prices shock:

$$\hat{\varepsilon}_t^{PX^F} = \left(1 - \rho_{PX^F}^2\right)^{1/2} \sigma_{PX^F} \eta_t^{PX^F} \tag{A.311}$$

Exogenous process for the final output price mark-up:

$$\hat{\mu}_t^Z = \sigma_{\mu^Z} \eta_t^{\mu^Z} \tag{A.312}$$

Exogenous process for the wage mark-up:

$$\hat{\mu}_t^W = \sigma_{\mu^W} \eta_t^{\mu^W} \tag{A.313}$$

Exogenous process for the import price mark-up:

$$\hat{\mu}_t^M = \sigma_{\mu^M} \eta_t^{\mu^M} \tag{A.314}$$

Exogenous process for the export price mark-up:

$$\hat{\mu}_t^X = \sigma_{\mu^X} \eta_t^{\mu^X} \tag{A.315}$$

Exogenous process for the value-added mark-up:

$$\hat{\mu}_t^V = \sigma_{\mu^V} \eta_t^{\mu^V} \tag{A.316}$$

A.4.2 Amendments to log-linearised equations in the estimated model

We make two minor amendments to the equations listed above in the estimated model described in the main text. The first amendment we make is to decouple the investment adjustment cost parameters, ψ_I, from the forcing process for the investment adjustment cost shock. This aids separate identification of the standard deviation of the adjustment cost shock and the structural parameter and can be interpreted as the implementation of a 'hierarchical prior' on the standard deviation of the investment adjustment cost shock conditional on the value of ψ_I. The investment Euler equation (A.280) becomes:

$$\begin{aligned} i_t = &\frac{\beta\Gamma^H}{1+\beta\Gamma^H}\left(i_{t+1}+\gamma^Z_{t+1}\right)+\frac{1}{1+\beta\Gamma^H}\left(i_{t-1}-\gamma^Z_t\right) \\ &+\frac{1}{\left(1+\beta\Gamma^H\right)\left(\Gamma^H\Gamma^Z\Gamma^I\right)^2}\left(\frac{tq_t}{\psi_I}+\hat{\varepsilon}^I_t\right) \end{aligned} \tag{A.317}$$

The second amendment we make is designed to soften the implications of the (extreme) cointegration assumption between the rest of the world and the domestic economy. This assumption made it possible to define a balanced growth path, but has the implication that labour augmenting permanent productivity shocks in the domestic economy are inherited or mirrored in world output at a rate determined by ρ_{Z^F}, which is the same parameter that determines the persistence of shocks to world output. In order to decouple the speed of export adjustment to LAP shocks from the persistence of world output shocks, while retaining the cointegration assumption, we amend the world output process in the following way:

$$z^F_t = \omega^F_t + \rho_{Z^F} z^F_{t-1} + \hat{\varepsilon}^{Z^F}_t \tag{A.318}$$

where:

$$\omega^F_t = -\gamma^Z_t + \left(1-\zeta_{\omega_F}\right)\omega^F_{t-1} \tag{A.319}$$

where $0 < \zeta_{\omega_F} \leq 1$ is the parameter that controls the speed with which permanant domestic labour productivity shocks are mirrored in world output.

B COMPASS model properties

In this appendix, we document some properties of COMPASS by describing the impulse responses to each of the eighteen shocks in the model. Impulse responses are a popular diagnostic for examining model properties as they allow the model user to focus on the thought experiment of how the model would respond if perturbed by a single shock, starting from steady state. Although a fairly abstract experiment, it provides a common framework for comparing the behaviour of different models, partly because the rules of computing impulse response functions are clearly defined. Moreover, for linear models, the marginal responses of the model to a shock that hits the model when it is in steady state are the same as the marginal responses to that shock when the model is away from steady state. So impulse responses can be used as the building blocks for understanding how a model responds to combinations of shocks applied to a baseline forecast or to revisions in past data. Linearity also means that the responses of the model to a negative shock are the mirror image of the responses to a positive shock.

The responses we plot are expressed in terms of percentage deviations from steady state or percentage points (denoted 'pp') where specifically noted. Because the steady state of COMPASS features balanced growth, our shock responses are measured *relative* to the steady-state balanced growth path. So if a particular shock causes consumption to "fall", this means that the path of consumption in response to the shock lies below the original balanced growth path that consumption would have followed in the absence of the shock. For small shocks, this means that consumption would continue to grow, albeit at a temporarily slower rate than is consistent with the balanced growth path. Of course, for large shocks, the level of consumption (in terms of the chained volume measure appearing in the national accounts) may indeed fall. To simplify the exposition, however, we focus on the model responses relative to the balanced growth that would have prevailed in the absence of the shock. This is a standard convention.

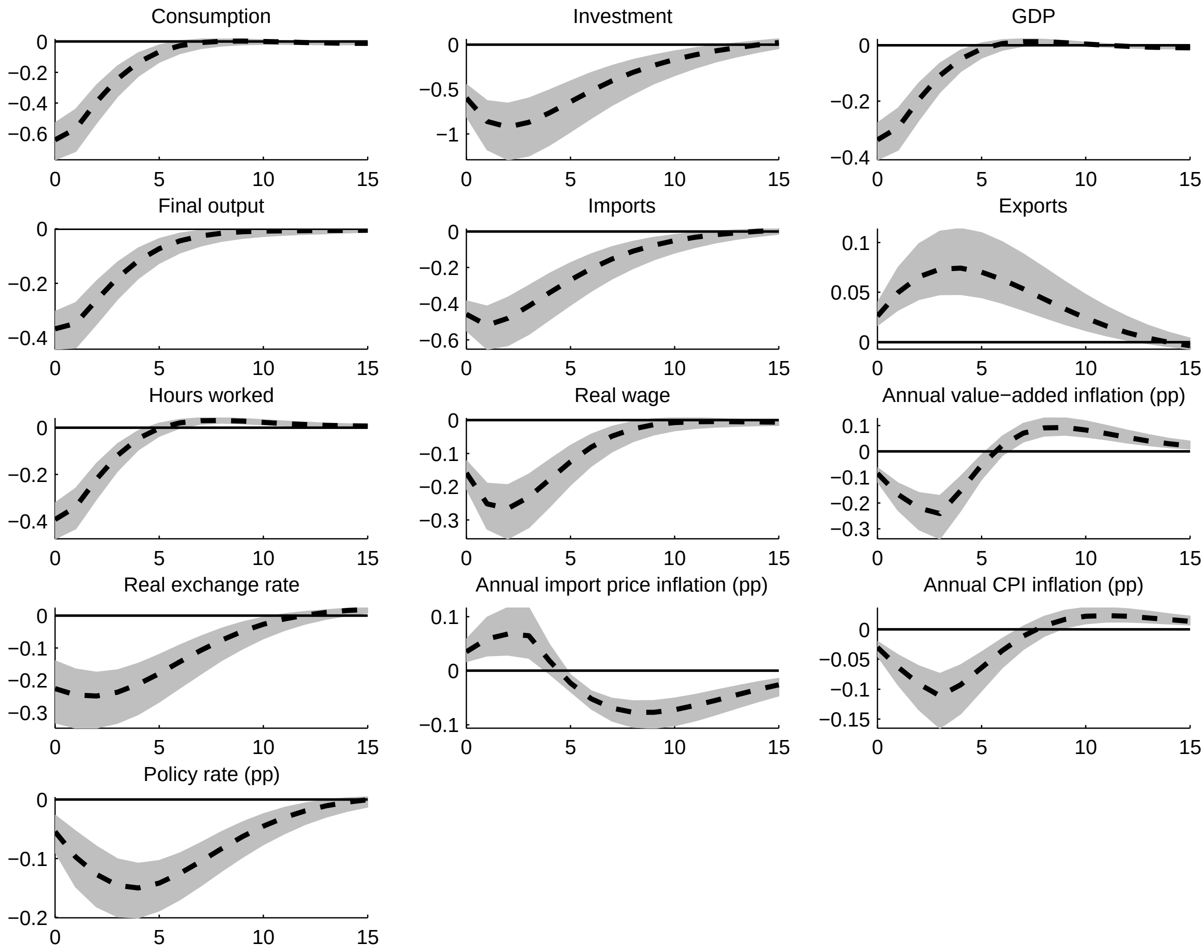
Bayesian estimation of the model provides us with an estimate of the posterior uncertainty surrounding the values of the model parameters. This uncertainty is represented by plotting a 'swathe' of impulse responses which records the range spanned by the set of impulse responses lying between the 5-th and 95-th percentiles of the distribution (generated by random draws from the posterior distribution of the parameters). The dashed lines represent the responses of COMPASS with parameter values set equal to the mean of the posterior distribution.

In terms of the economic transmission mechanisms in COMPASS, many of the shocks affect the endogenous variables in broadly similar ways. For example, shocks that primarily impact expenditure components of final demand tend to increase or reduce inflation through their effect on the demand for factors of production, the prices of those factors and hence production costs. For example, a shock of this type that leads to a fall in production would tend to reduce the real wage and the return to capital, reducing the marginal cost of production and hence lead to lower inflation.

Shocks that more directly affect the relative prices of goods and services or factors of production (for example, shocks to desired mark-ups of producers in particular sectors) can have slightly more nuanced effects. These shocks tend to lead to substitution between expenditure components (or factors of production) which have implications for output and inflation that depend on the net effects of the frictions governing the reallocation of spending (or of factors of production).

B.1 Domestic risk premium shock

Figure B.1: Risk premium shock



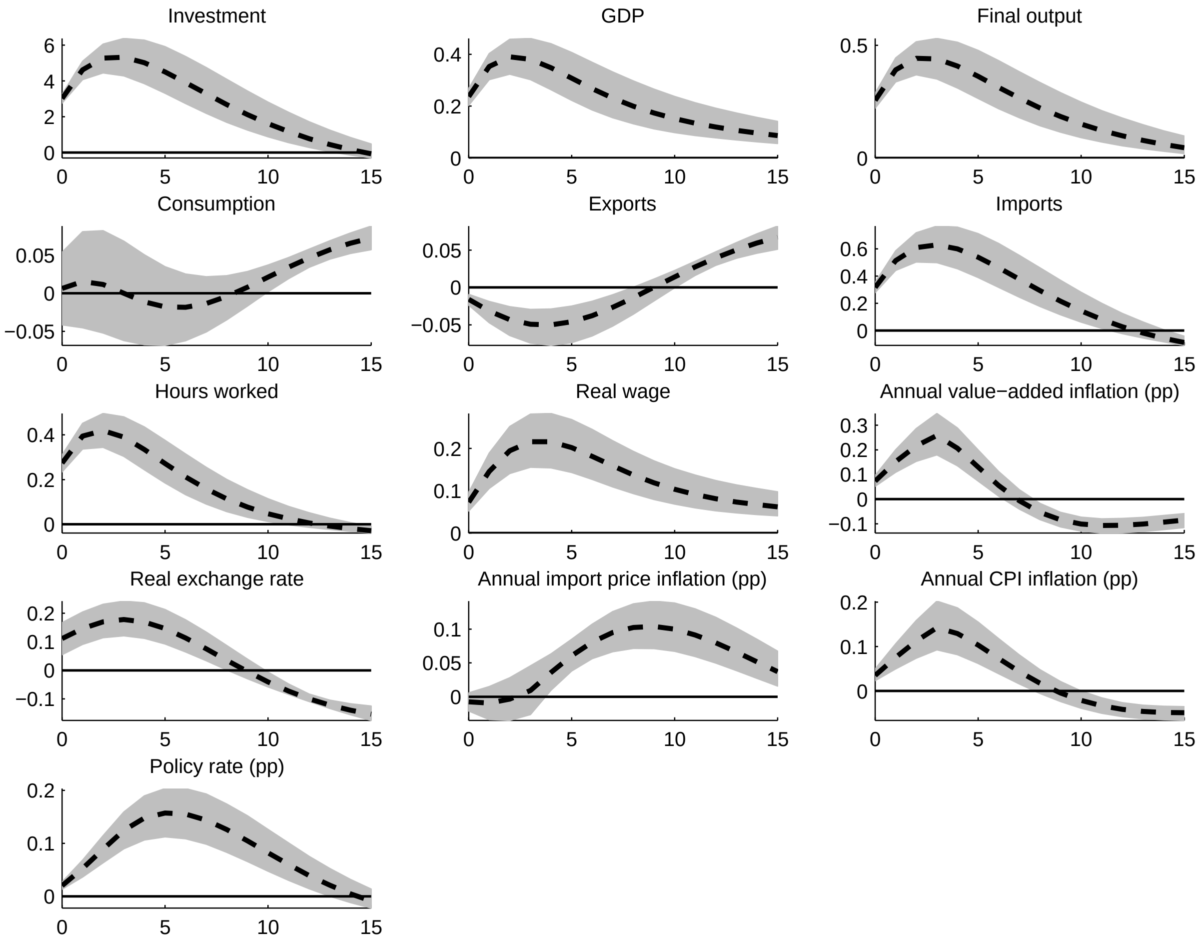
The charts show responses of COMPASS variables to a one standard deviation risk premium shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

A positive domestic risk premium shock temporarily increases the effective cost of borrowing for a given real interest rate. This induces households to postpone consumption (in favour of saving) and firms to postpone investment. The short-run reduction in private domestic demand reduces final output and GDP.

The fall in production lowers firms' demand for factor inputs, including imports. This reduces the prices of domestic factor inputs and the marginal cost of firms producing value added. As a result, value added inflation declines leading to a fall in CPI inflation. CPI inflation falls despite a rise in import prices brought about by a depreciation of the exchange rate. The depreciation reflects a reduction in interest rates as the monetary policy reaction function responds to weaker inflation and activity. The depreciation is sufficient to generate a modest increase in exports, which partially offsets the reduction in private domestic demand brought about by the shock.

B.2 Investment adjustment cost shock

Figure B.2: Investment adjustment cost shock



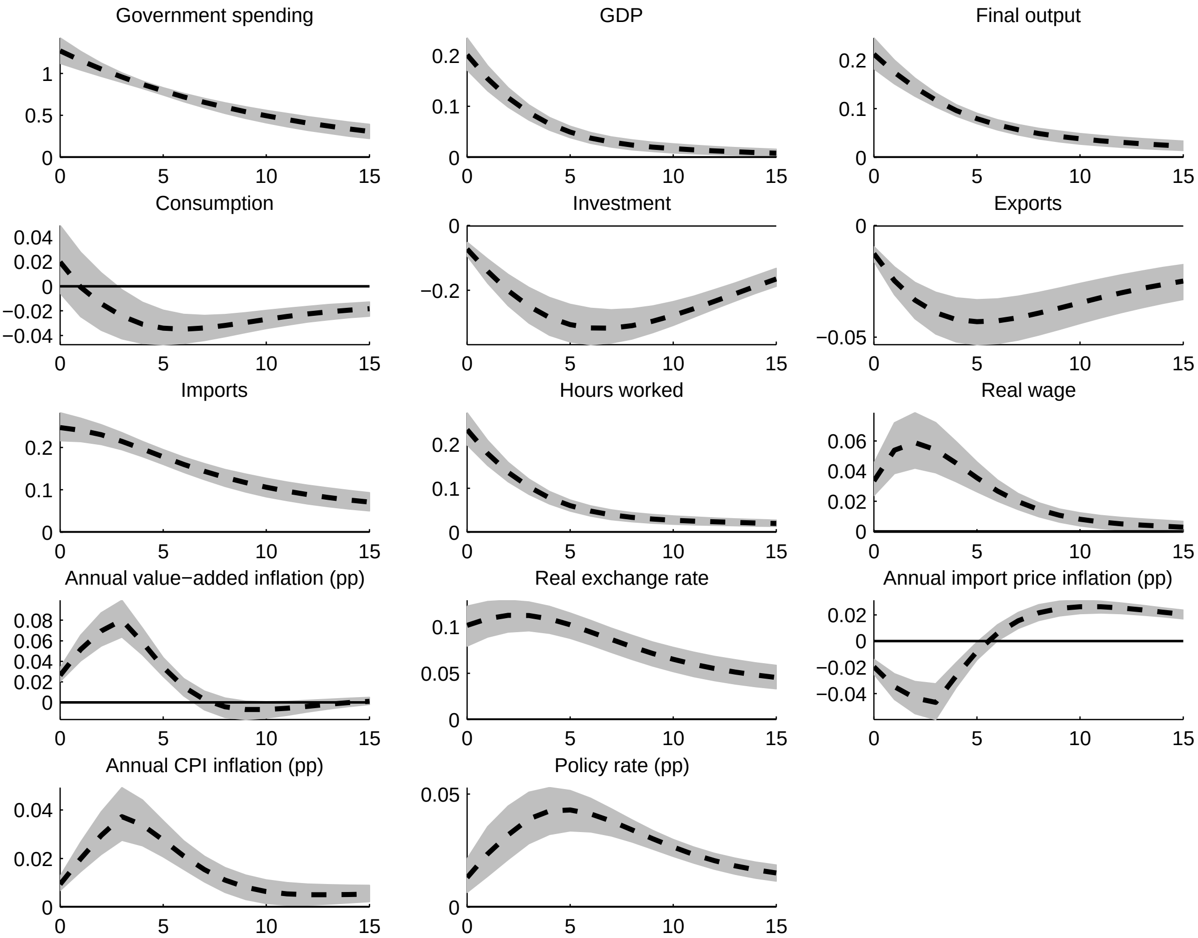
The charts show responses of COMPASS variables to a one standard deviation investment adjustment cost shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

An investment adjustment cost shock temporarily reduces the cost of increasing the capital stock, thereby increasing the returns to investment. This induces firms to temporarily increase investment which in turn increases GDP and final output. Activity expands despite a modest fall in exports (generated by a small appreciation). Consumption of constrained households is supported by the rise in wages and employment generated by the increase in production. Consumption of unconstrained households is relatively unaffected, since real interest rates move by little in response to the shock.

The increase in demand for factors of production increases their prices, so that the marginal cost of value added production rises. This leads to a rise in value added inflation. The increase in value added price inflation leads to an increase in the marginal cost of final output production and hence a rise in CPI inflation. Monetary policy responds to the rise in inflation and activity by increasing the policy rate. The response of the policy rate is greater than the increase in inflation and the real interest rate increases (though not on impact) leading to an appreciation to the real exchange rate.

B.3 Government spending shock

Figure B.3: Government spending shock



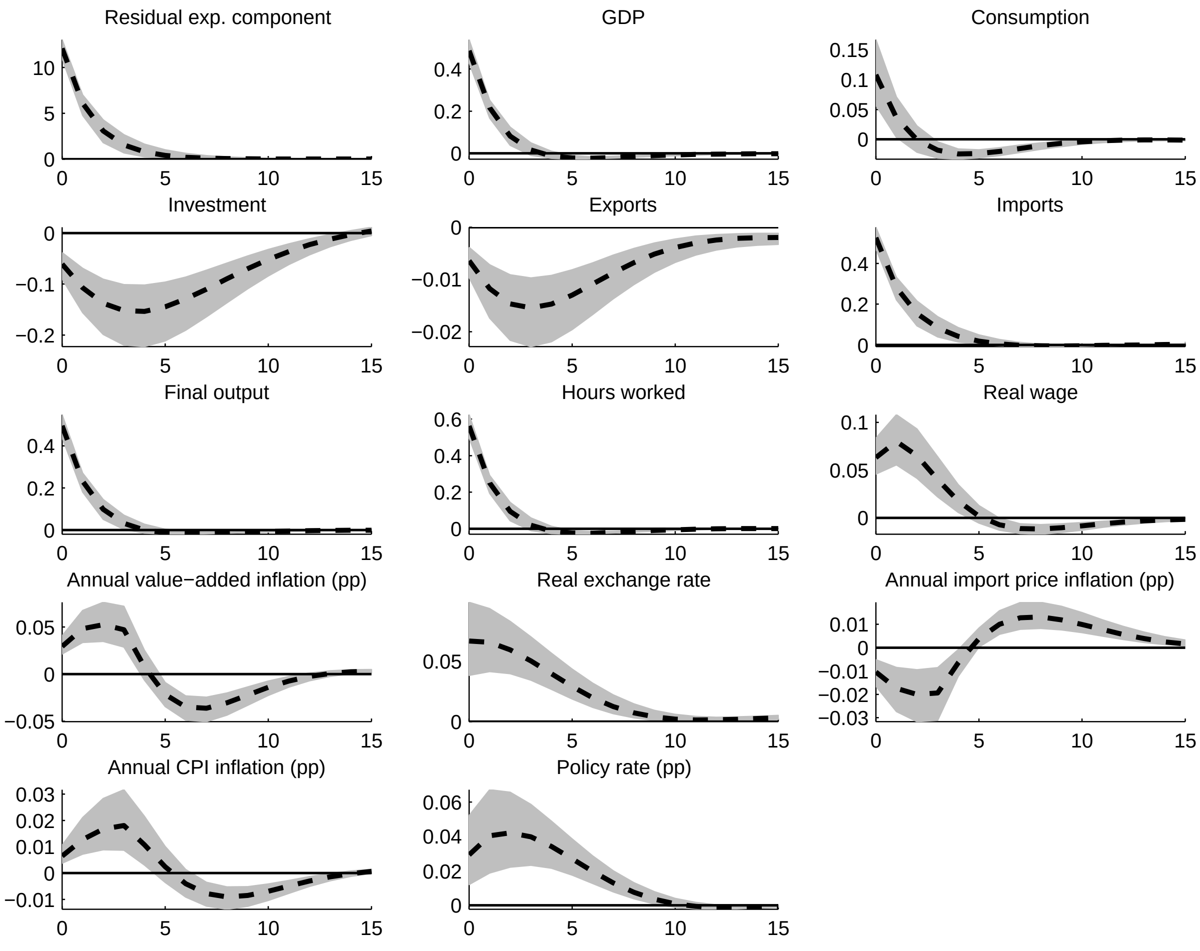
The charts show responses of COMPASS variables to a one standard deviation government spending shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

A positive shock to government expenditure raises real government spending. The rise in government spending increases GDP and total final output. Consumption rises initially as the consumption of constrained households increases in light of higher real labour income. However, unconstrained households reduce consumption in response to higher real interest rates. Over time, total consumption declines as the effects of higher wage income diminish. The increase in real interest rates also induces households to reduce investment spending and generates a real exchange rate appreciation. The appreciation reduces exports and increases imports.

To meet the increased demand for output from the government, production must increase. As a result, firms' demand for factors of production also increases. Total hours and imports increase, though investment is weaker. Increased demand for the factors of production pushes up factor prices and increases the marginal costs of production, which in turn lead to a rise in CPI inflation. The increase in prices of domestic factors of production is mitigated somewhat by a reduction in import price inflation driven by the appreciation of the exchange rate, though this effect is small. The monetary policy reaction function responds to the higher level of inflation and the output gap by increasing the policy rate. The response of the policy rate is greater than the increase in inflation and the real interest rate increases.

B.4 Residual expenditure component shock

Figure B.4: Residual component of total final expenditure shock



The charts show responses of COMPASS variables to a one standard deviation 'residual component of total final expenditure' shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

As described in Appendix A, COMPASS includes an exogenous 'residual' or 'other' investment GDP component to ensure that all of GDP is accounted for by expenditure categories in the model. The residual component is modelled as a simple exogenous process. A positive shock to the residual expenditure component increases final output and GDP. However, since this expenditure component must (ultimately) be financed by consumers and is not a productive resource, it crowds out other expenditure components. Consumption initially rises because the expansion in GDP increases labour income and hence consumption of constrained households. But as the effect of the shock wanes, real labour income falls back and total consumption is lower because unconstrained households face higher real interest rates.

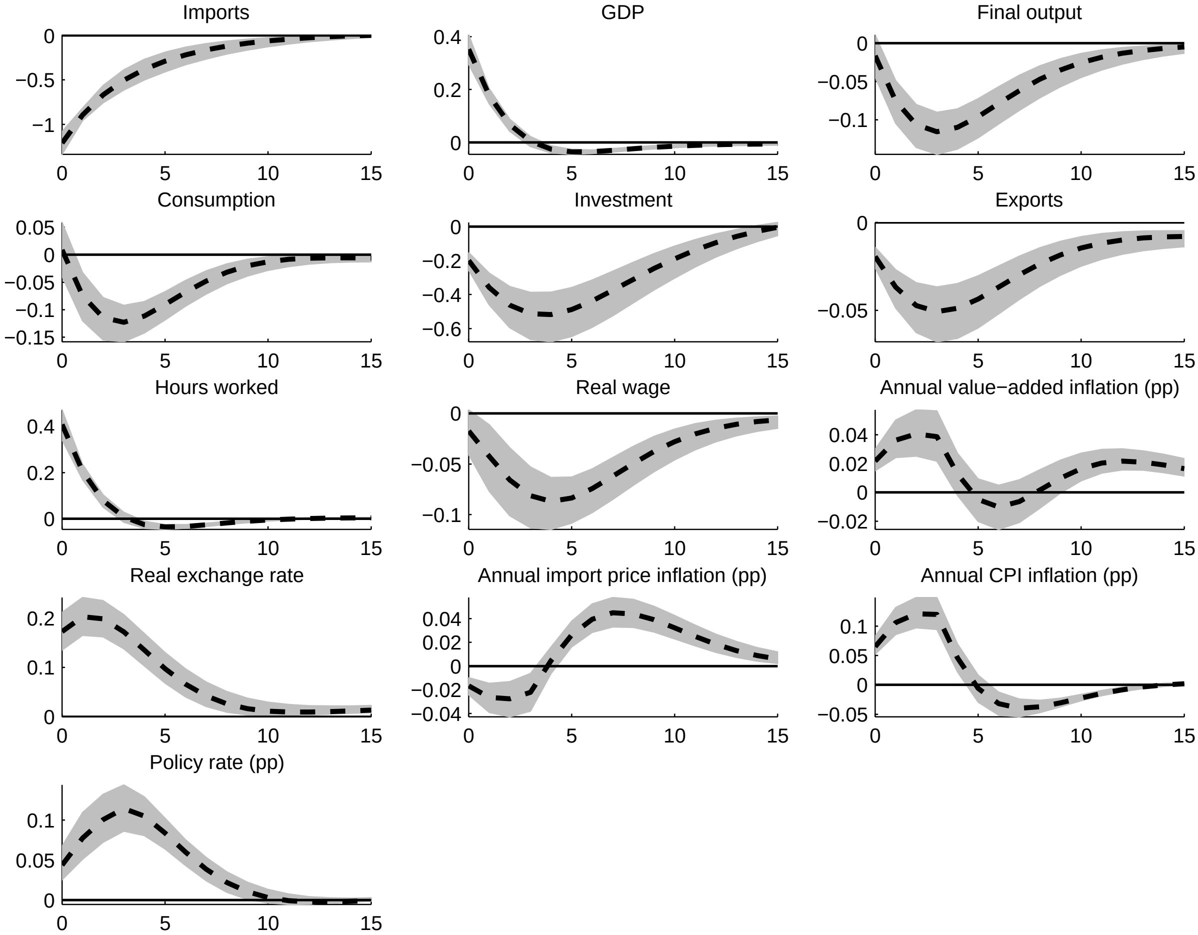
The rise in real interest rates also reduces investment and induces an appreciation of the real exchange rate, which reduces exports. To meet the increase in demand, production is expanded, bidding up the prices of factors of production. The marginal cost of producing value added rises. The exchange rate appreciation induces a modest fall in import price inflation. But the net effect is an increase in the marginal cost of final output production which leads to a rise in CPI inflation. The policy rate increases in response to the rise in inflation and the output gap.

B.5 Import preference shock

A negative import preference shock temporarily reduces the desired share of imports in final output production by temporarily increasing the effective cost of using imports (independently of the import price). The shock leads producers to substitute away from imports, increasing the demand for value added in final output production. The rise in GDP is brought about by an increase in total hours worked. This substitution mitigates the effects of the large reduction in imports on final output.

Nevertheless, total final expenditure across all private final demand components (consumption, investment and exports) falls somewhat. In order to bring about the fall in expenditure, the real interest rate increases. This causes unconstrained households to postpone consumption and firms to postpone investment. The increase in the real interest rate generates a real appreciation, which reduces foreign demand for exports. The import preference shock leads to a direct increase in the marginal cost of final output production, which generates an increase CPI inflation despite a small initial reduction in import price inflation. In response to the rise in CPI inflation and the output gap, the policy rate is increased.

Figure B.5: Import preference shock

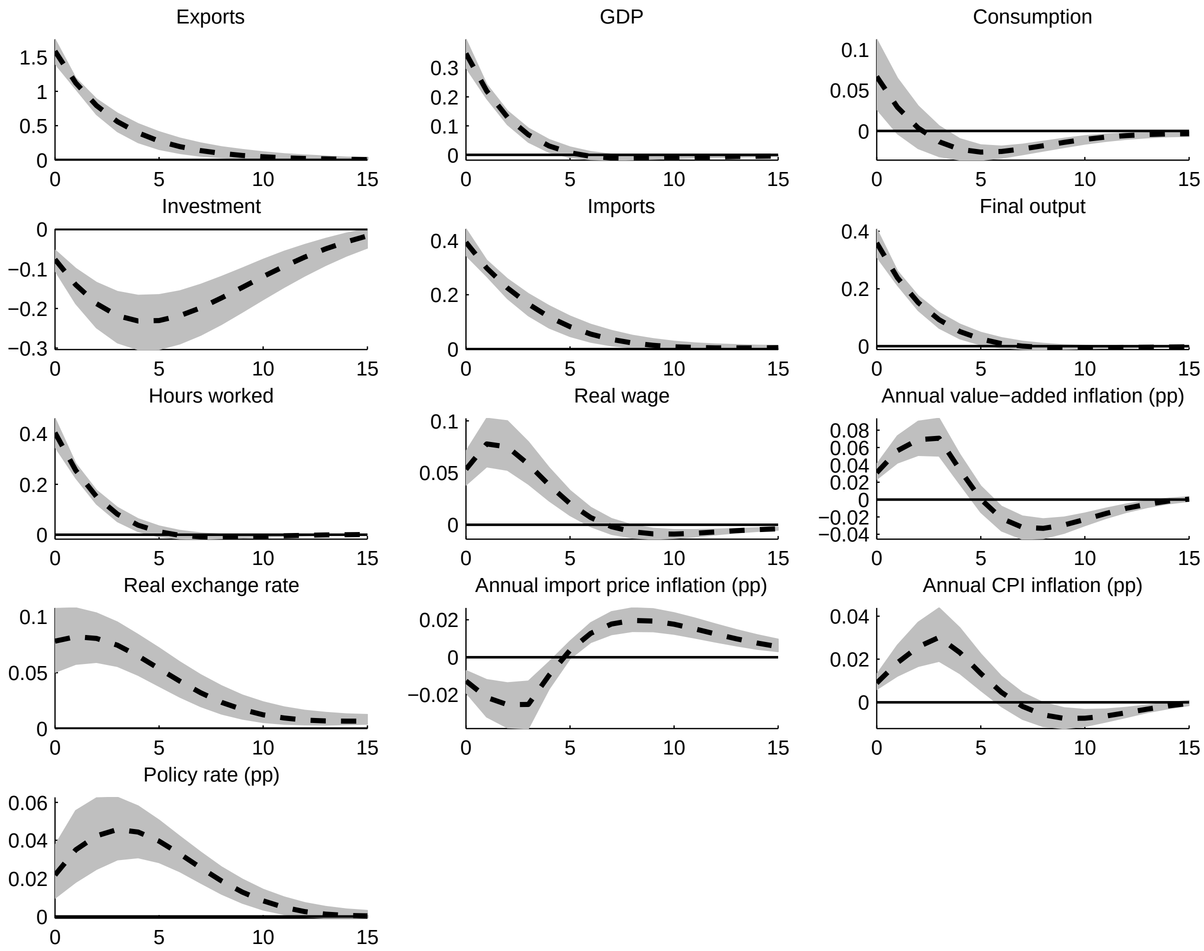


The charts show responses of COMPASS variables to a one standard deviation import preference shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

B.6 Export preference shock

A temporary increase in world preferences for domestic exports increases export demand for given levels of world demand and world export prices. This leads to an increase in exports and thus final output. Consumption rises in the short run as constrained households' consumption is supported by a rise in real labour income. But in later periods consumption is weaker as unconstrained household consumption falls in response to higher real interest rates. Higher real interest rates also reduce investment spending and generate a real exchange rate appreciation. As a result, import price inflation falls initially and the demand for imports increases. As final output production increases, GDP and hours worked rise. This pushes up wages and increases the marginal costs of production for value added producers. Although import price inflation falls slightly, the marginal cost of final output production increases, leading to a rise in CPI inflation. The rise in GDP increases the output gap which, together with higher inflation, leads to an increase in the policy rate.

Figure B.6: World preferences for UK exports shock

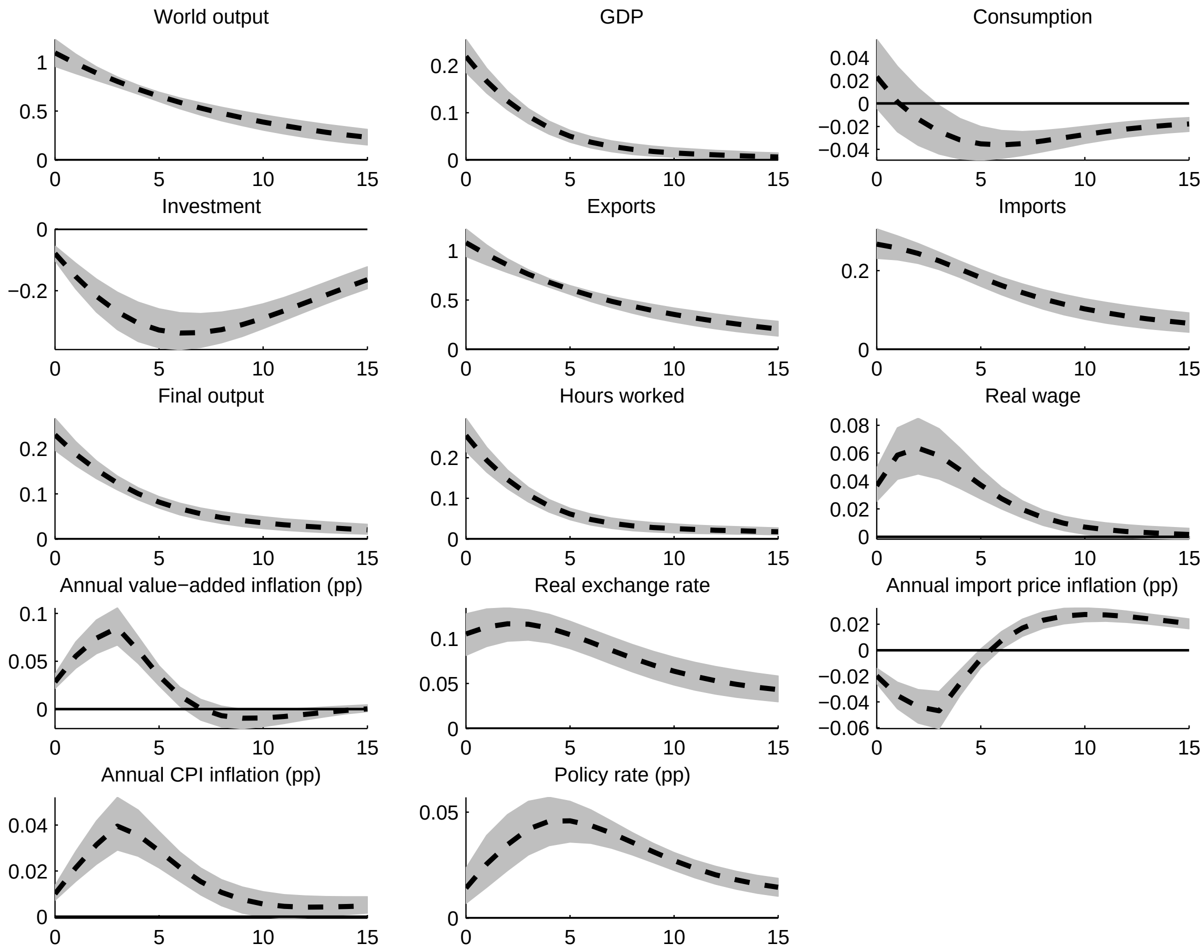


The charts show responses of COMPASS variables to a one standard deviation shock to world preferences for UK exports. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

B.7 World output shock

A temporary increase in world output leads to an increase in exports and final output. Consumption increases in the short run, supported by a rise in real labour income. But in later periods consumption is weaker, as unconstrained household consumption falls in response to higher real interest rates. The real exchange rate appreciates as a result of higher real interest rates. The appreciation of the real exchange rate reduces import price inflation and increases import demand, leading to a rise in final output. Final output is also produced with with value added which in turn is produced with labour and capital. The rise in GDP pushes up wages and the rental rate of capital, increasing the marginal costs of value added production and hence value added prices. But the appreciation to the exchange rate reduces import price inflation, which mitigates somewhat the increase in the marginal costs for final output producers and thus the extent of rise in CPI inflation. The higher level of GDP increases the output gap which, together with higher CPI inflation, leads to an increase in the policy rate.

Figure B.7: World output shock



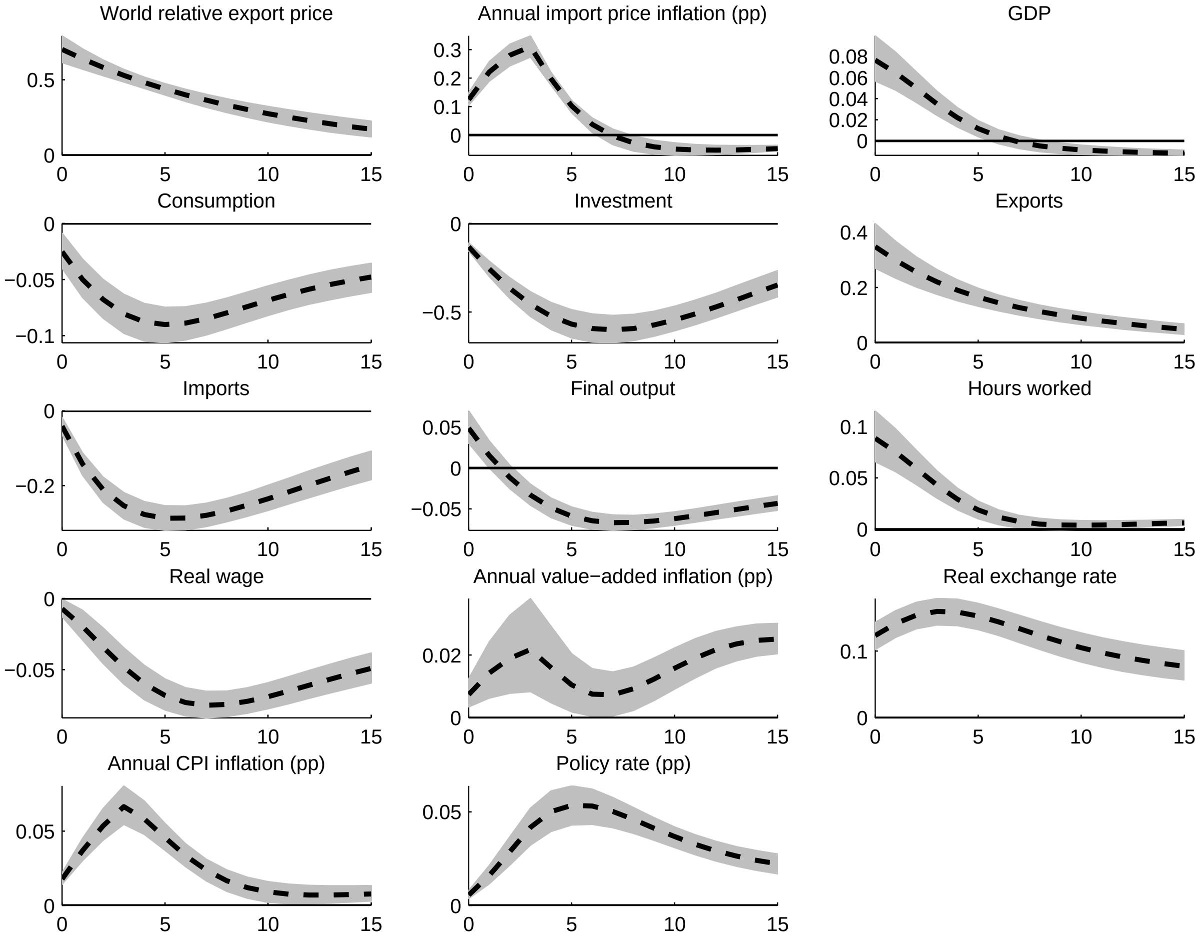
The charts show responses of COMPASS variables to a one standard deviation world output shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

B.8 World export price shock

A temporary increase in world export prices leads to a rise in costs for importers. This generates a rise in import price inflation and thus a fall in imports. The increase in world export prices leads to a fall in the relative price of domestic output and so exports increase. There is substitution of value added for imports and value added increases, thereby supporting final output in the short run. Final output eventually falls as the increase in exports is more than offset by weak domestic demand.

The increase in value added is produced via an increase in hours worked. The increase in the demand for the domestic factors of production leads to an increase in nominal wages and the marginal cost of producing value added. However, value added prices do not increase as quickly as final output prices and the relative price to final output falls. The rise in import prices increases the marginal cost of final output production and hence CPI inflation. Monetary policy responds by increasing the nominal interest rate leading to a persistent increase in the real interest rate, thereby inducing a real exchange rate

Figure B.8: World export price shock



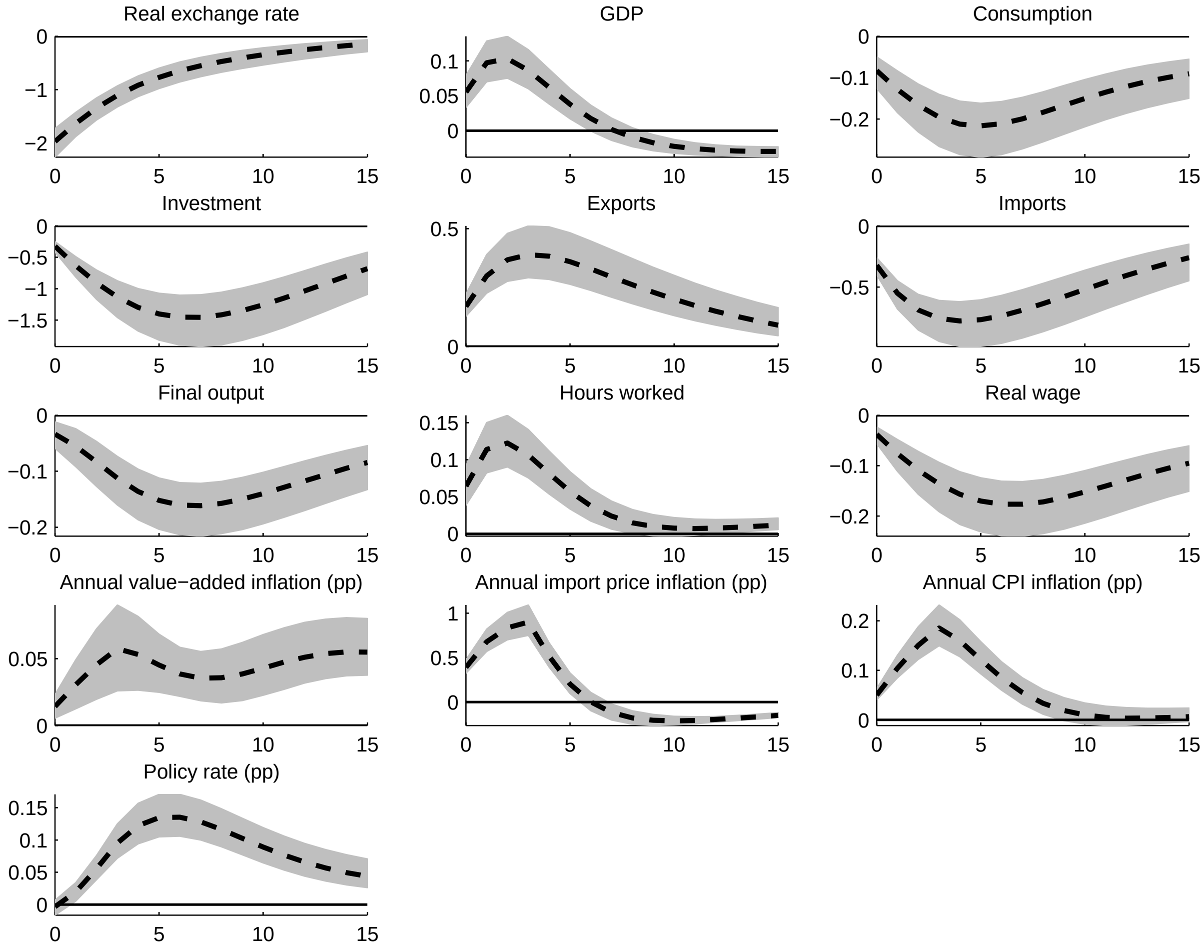
The charts show responses of COMPASS variables to a one standard deviation world export price shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

appreciation. Higher real interest rates induce households to postpone consumption and firms to postpone investment.

B.9 Exchange rate risk premium shock

A temporary increase in the exchange rate risk premium reduces the non-pecuniary return on domestic currency assets, leading to a depreciation of the exchange rate. The depreciation reduces export prices but increases import prices, acting to increase the demand for exports and reduce the demand for imports. The increase in exports crowds out investment and consumption, but the net effect on GDP is positive. The increase in final output comes entirely from an increase in value added production, generated by an increase in hours worked. This increase in the demand for the inputs of production acts to increase the nominal prices of domestic factor inputs pushing up marginal costs of producing value added. Together with the sharp increase in import price inflation, costs

Figure B.9: Exchange rate risk premium shock



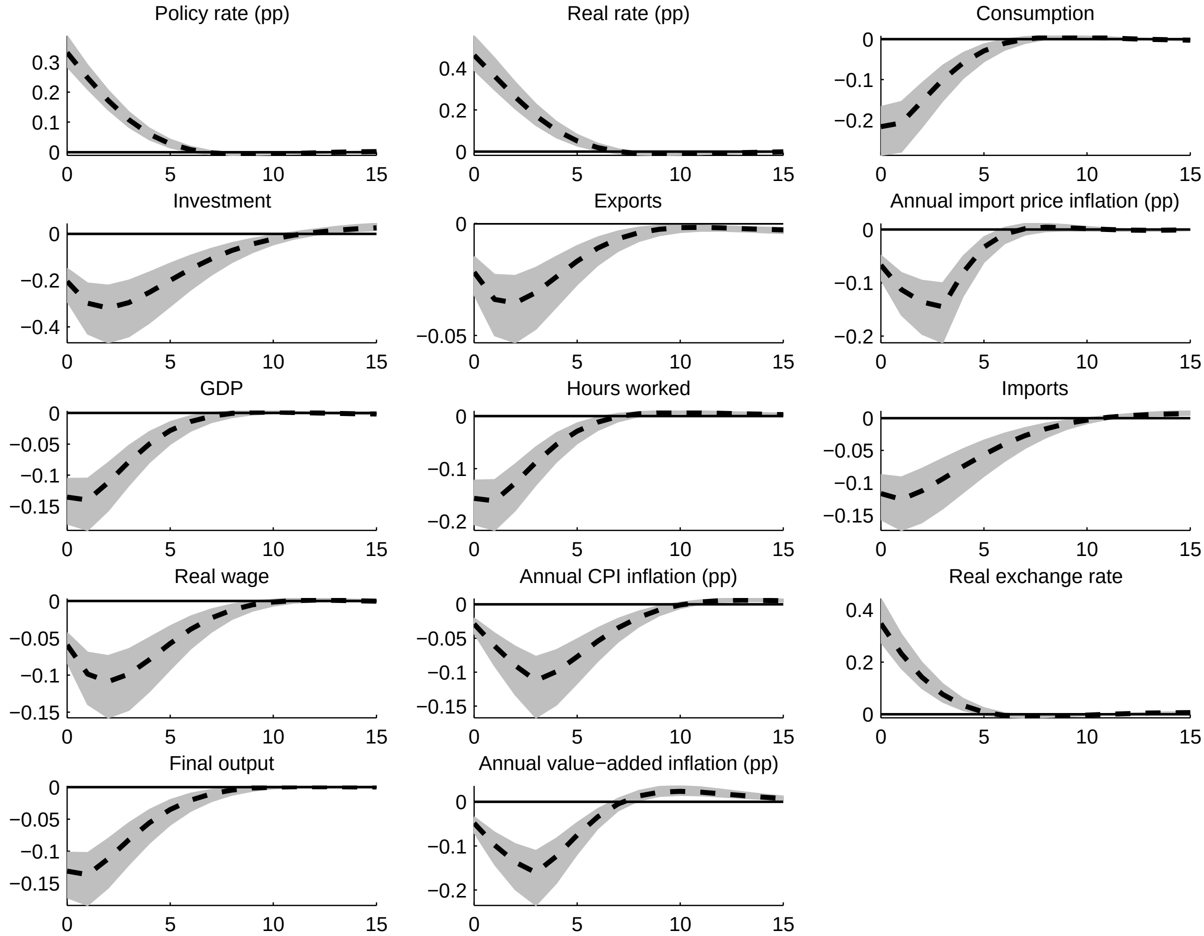
The charts show responses of COMPASS variables to a one standard deviation exchange rate risk premium shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

of producing final output rise, leading to an increase in CPI inflation. The higher level of GDP increases the output gap which, together with higher inflation, leads to an increase in the nominal interest rate and a persistent increase in the real interest rate. The rise in the real interest rate generates the crowding out of consumption and investment as households and firms are induced to postpone spending. Consumption is also weak because of a persistent decline in labour income generated by the rise in CPI relative to nominal wages.

B.10 Monetary policy shock

A temporary positive shock to the monetary policy reaction function raises the policy rate. The real interest rate rises because of the presence of nominal rigidities (sticky prices and wages). The increase in the real interest rate has two important effects. First, it acts to depress private domestic demand as the increased return to saving induces

Figure B.10: Monetary policy shock

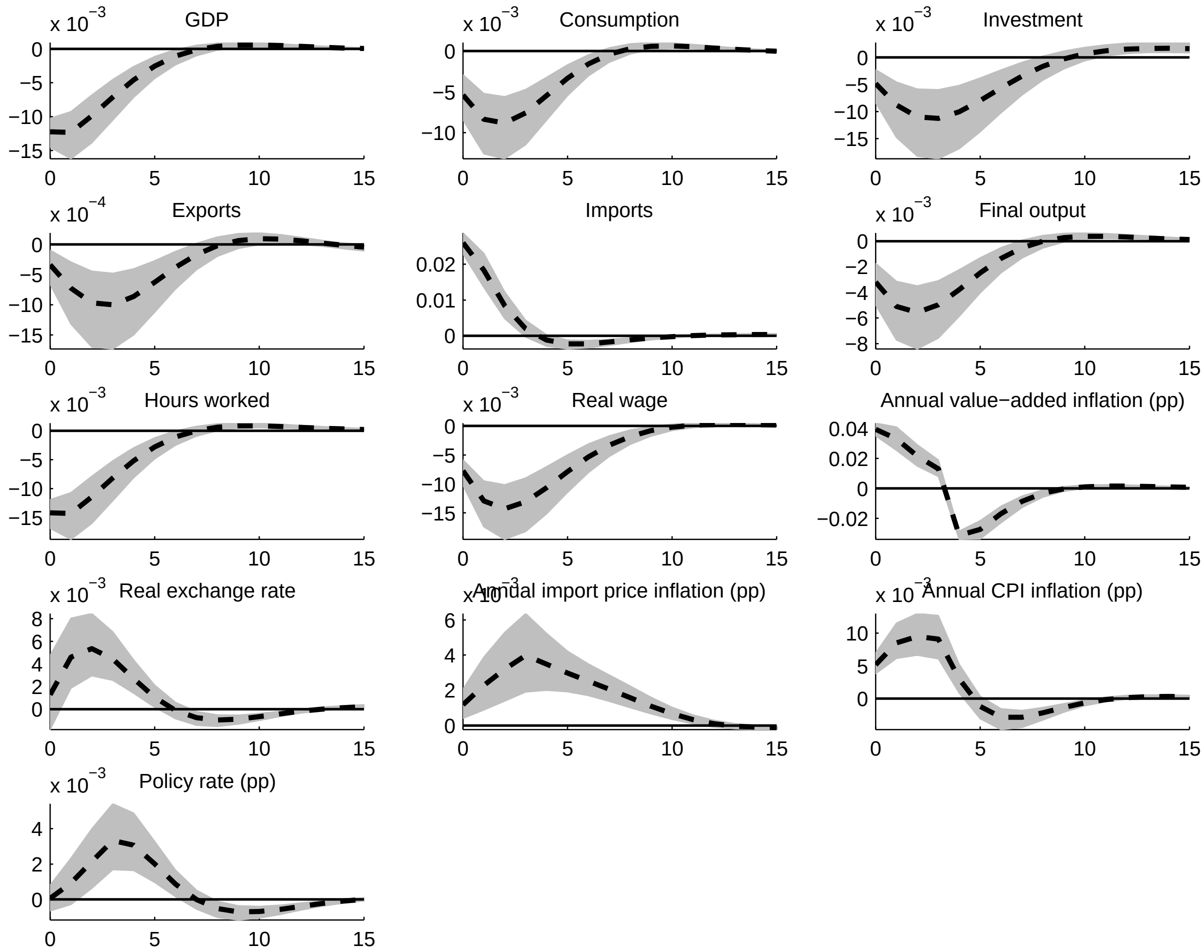


The charts show responses of COMPASS variables to a one standard deviation monetary policy shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

households to postpone consumption spending and investment in physical capital. The second effect of a persistent increase in the real interest rate is that the real exchange rate appreciates (via the UIP condition), reducing export demand and import price inflation. Taken together, these effects reduce GDP.

The fall in output reduces demand for labour, capital services and imports and so the relative prices of the factors of production fall. This acts to lower the marginal cost of producing final output, which in turn reduces inflation. Although the shock is not persistent, interest rate smoothing in the monetary policy reaction function keeps the nominal interest rate above steady state for a number of periods, though the fall in inflation and the output gap pulls the policy rate towards equilibrium.

Figure B.11: Value added price markup shock



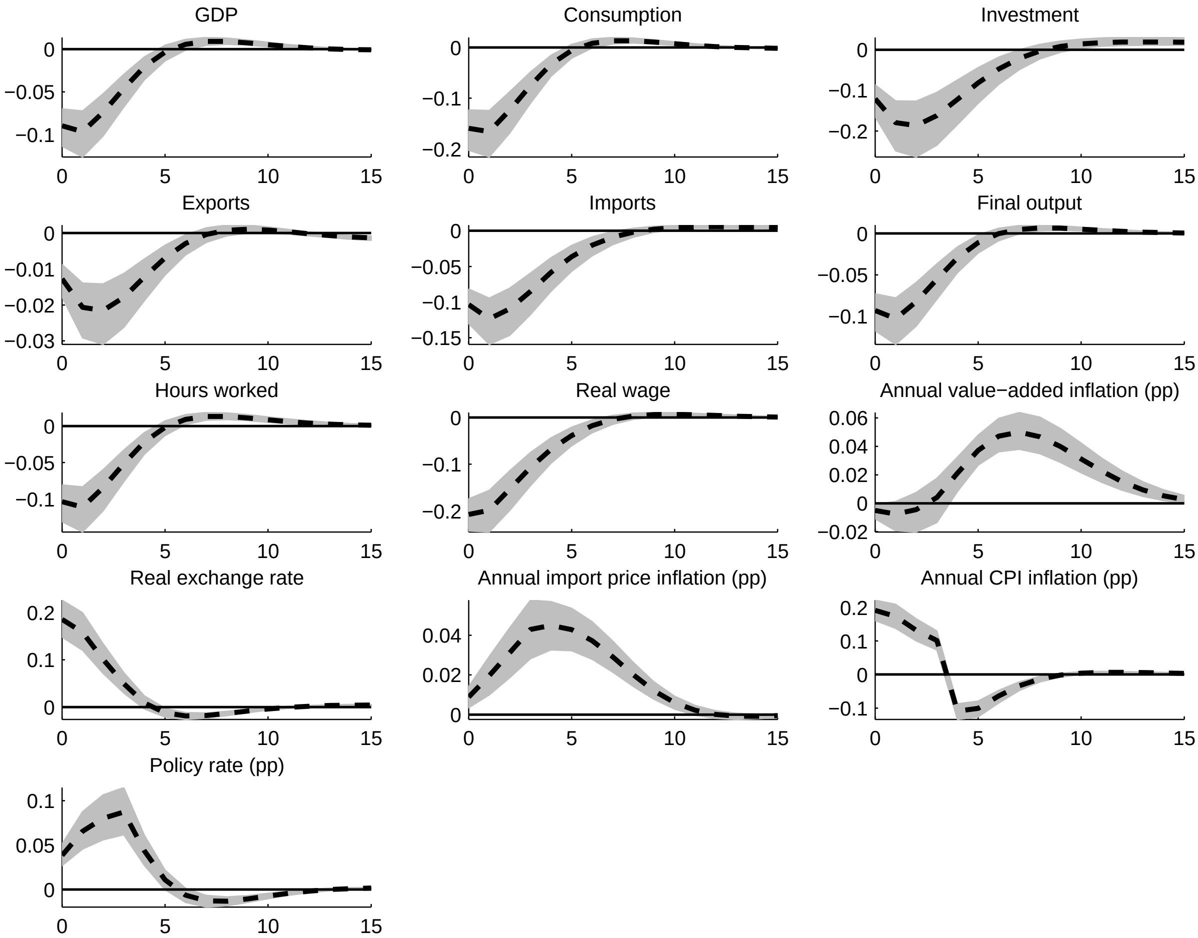
The charts show responses of COMPASS variables to a one standard deviation value added price markup shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

B.11 Value added mark-up shock

A positive shock temporarily raises the desired mark-ups of firms in the value added output sector. So the direct effect of the shock induces value added producers to raise their prices. The rise in the relative price of value added induces final output producers to substitute imports for value added, which increase. The reduction in demand for value added reduces GDP and hence hours worked fall. The main expenditure components all fall. Consumption declines because reduced labour income reduces the consumption of constrained households and (slightly) higher real interest rates reduce the consumption of unconstrained households. The small rise in real interest rates prompts firms to postpone investment spending and generates a small appreciation in the real exchange rate, which reduces exports. The decline in value added production reduces the demand for labour and capital and hence input prices. Despite the resulting fall in the marginal cost of value added production, value added price inflation rises temporarily as the direct effect of higher desired mark-ups dominates. Higher value added prices feed into final output production costs, increasing CPI inflation. The increase in CPI inflation tends to increase the policy rate, but the fall in the output gap generated by the fall in GDP partially offsets this.[B1]

B.12 Final output mark-up shock

Figure B.12: Final output price markup shock



The charts show responses of COMPASS variables to a one standard deviation final output price markup shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

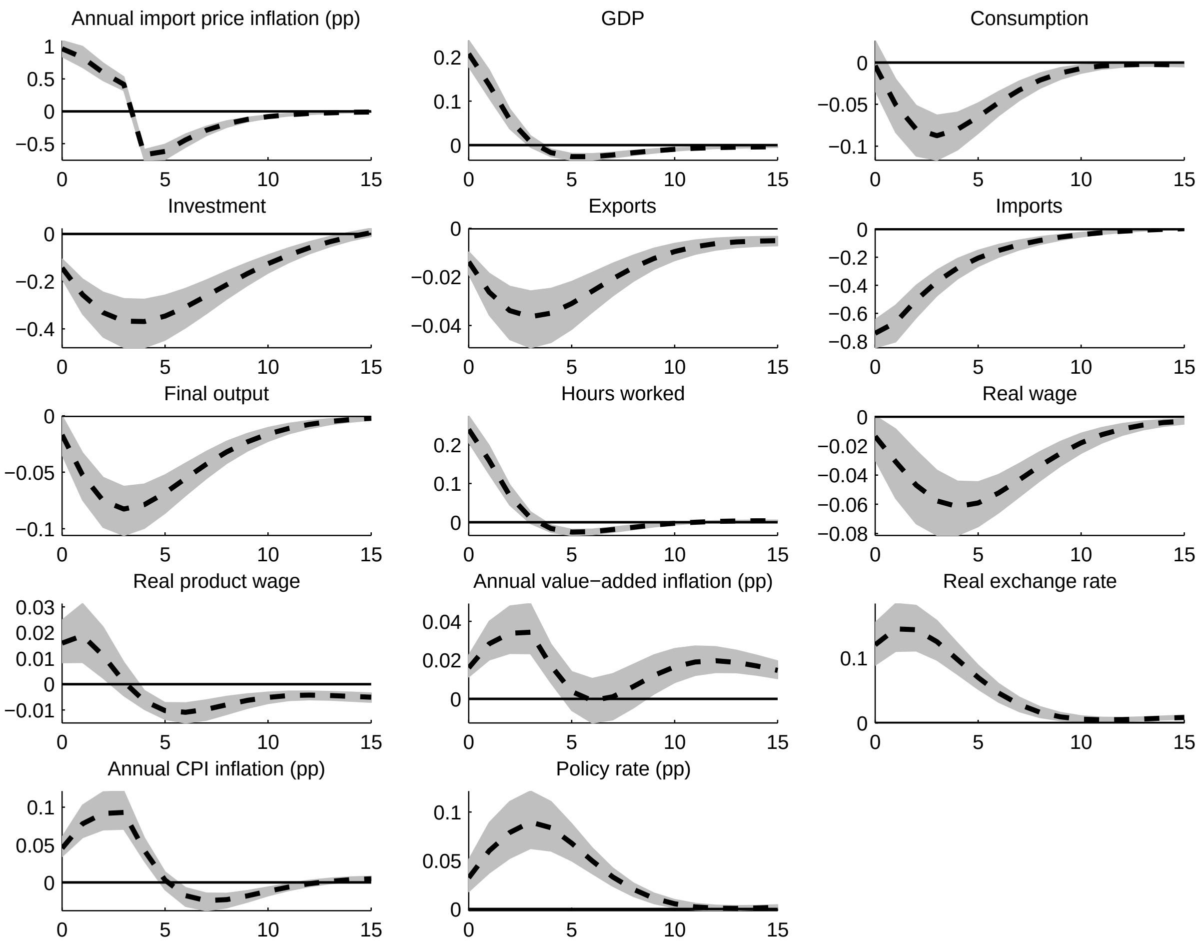
A positive shock temporarily raises the desired mark-ups of firms in the final output sector: the direct effect of the shock induces final output producers to raise their prices and so CPI inflation increases. The temporary increase in final output prices reduces expenditure on final output and the main expenditure components all fall. Consumption declines because the rise in CPI reduces real labour income and so the consumption of constrained households falls. Slightly higher real interest rates reduce the consumption of

[B1]The responses to this shock are quantitatively small because the standard deviation was calibrated to a relatively small value: see the discussion in Section 4.3 for more information.

unconstrained households. The small rise in real interest rates prompts firms to postpone investment spending and generates an appreciation in the real exchange rate, which reduces exports. Despite the appreciation, imports also fall, because the reduced demand for final output results in lower demand for factors of production. The decline in output reduces the demand for labour and capital and hence factor prices. This pushes down on production costs and hence CPI inflation after the initial effect of the mark-up shock has worn off.

B.13 Import price mark-up shock

Figure B.13: Import price markup shock



The charts show responses of COMPASS variables to a one standard deviation import price markup shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

A positive shock raises the desired mark-up of importers over the world export price (measured in domestic currency). The direct effect is to increase import prices. The rise in import prices passes along the supply chain and generates an increase in CPI inflation.

The rise in import prices leads firms to substitute value added for imports, generating an increase in GDP. As value added production expands, employment increases, putting upward pressure on real *producer* wages. This leads to an increase in the marginal cost of value added production and thus to a modest increase in value added inflation (though the relative price of value added price falls as final output inflation increases more than the increase in value added prices).

Despite the increase in GDP, the main final expenditure components fall. Consumption of constrained households is supported by stronger employment, though this is slightly offset by a reduction in the real (consumption) wage. This is not sufficient to deliver a rise in aggregate consumption, however, given a fall in the consumption of unconstrained households in light of a small but persistent increase in the real interest rate. The higher real interest rate also depresses investment spending and generates an appreciation of the real exchange rate. The real exchange rate appreciation generates a small fall in exports. Monetary policy tightens in light of the expansion in GDP and the rise in CPI inflation.

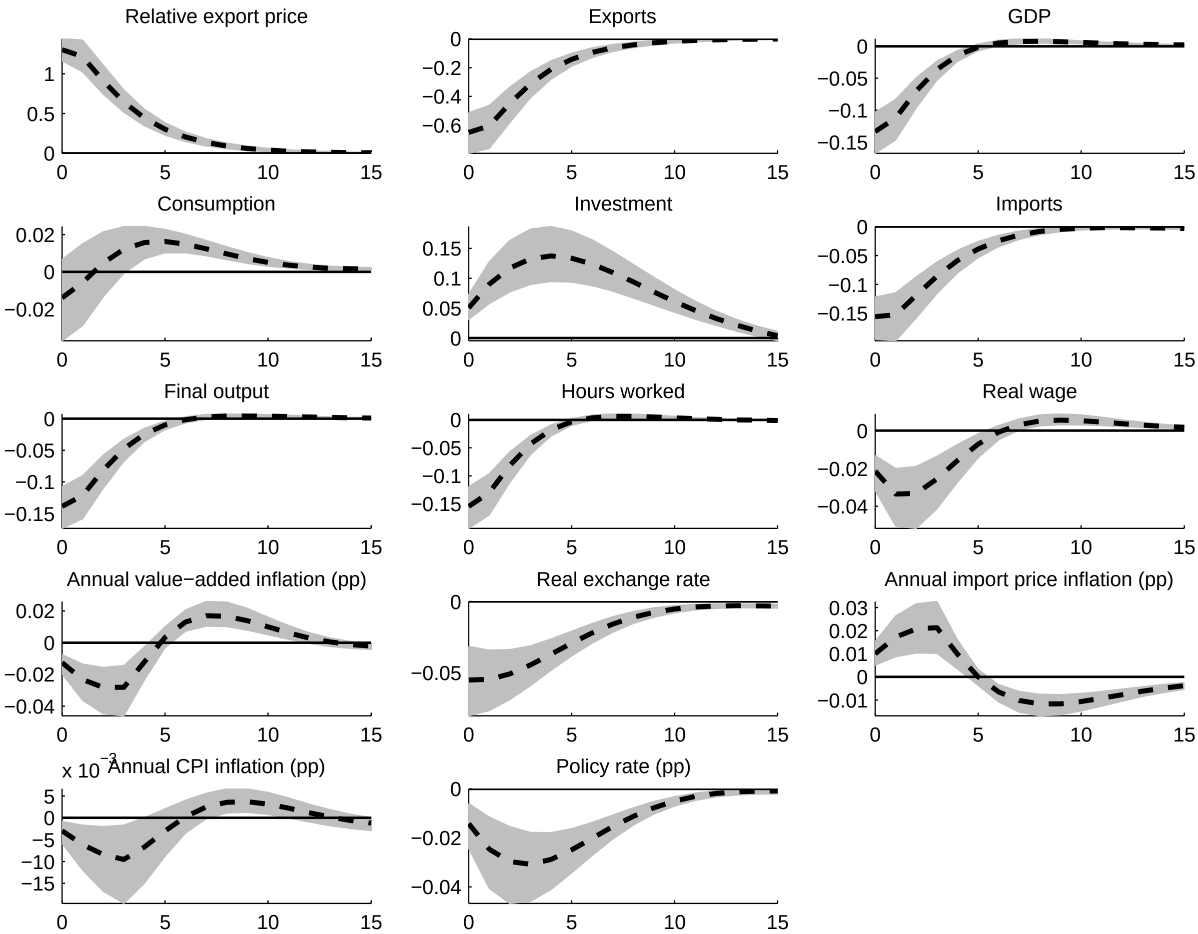
B.14 Export price mark-up shock

The direct effect of a temporary increase in the desired mark-ups of exporters is an increase in export prices. This generates a fall in demand for exports which reduces final output production and GDP. The fall in activity reduces the demand for factors of production (labour, capital and imports) and thereby factor prices and the marginal cost of producing GDP.

The real exchange rate depreciates slightly in response to a small reduction in real interest rates. The depreciation also helps to mitigate the effect of the fall in export demand on the current account, but reducing demand for imports via an increase in import prices. The rise in import prices partially offsets the fall in value added price inflation on the marginal cost of final output producers. As a result CPI inflation falls, though only slightly.

The fall in GDP depresses the output gap which, together with the fall in CPI inflation, leads to a small loosening of monetary policy. As noted, this is sufficient to reduce the real interest rate slightly, which induces firms to bring forward investment spending. Beyond the near term, the reduction in real interest rates generates a small increase in the consumption of unconstrained households. But consumption falls initially, driven by the lower labour income of constrained households.

Figure B.14: Export price markup shock



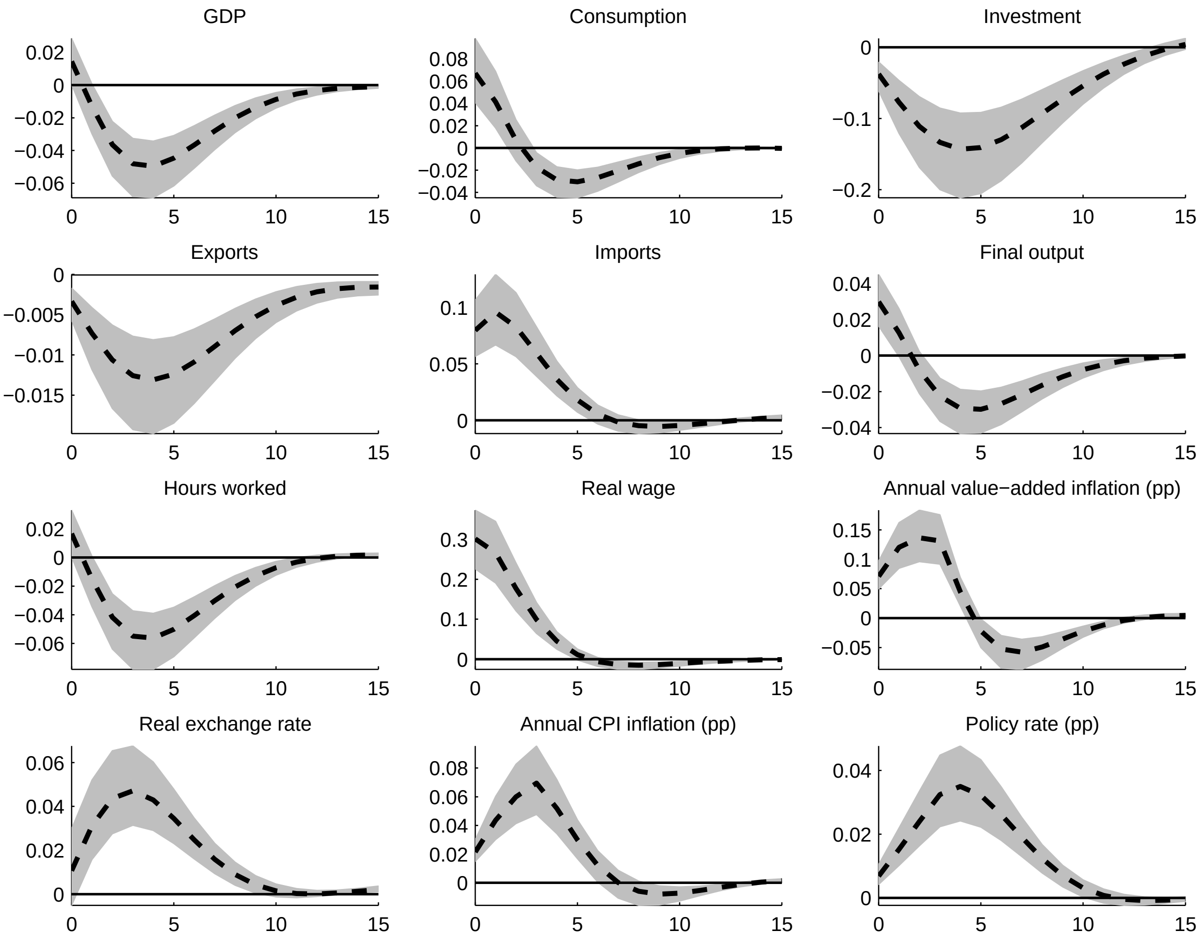
The charts show responses of COMPASS variables to a one standard deviation export price markup shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

B.15 Wage mark-up shock

A positive shock temporarily raises households' desired mark-up of the wage over the marginal rate of substitution between leisure and consumption. The direct effect is to raise nominal wage inflation and thus the real wage. The increase in the real wage temporarily increases labour income and consumption of constrained households, which is sufficient to increase total consumption in the near term. This effect means that GDP does not fall immediately, despite the increase in production costs generated by the rise in wage costs.

The increase in real wages results in an increase in the marginal cost of value added production and thus an increase in value added inflation which is passed along the supply chain and generates a small increase in CPI inflation and a small rise in the policy rate in response. The small but persistent increase in the real interest rate is sufficient to generate a fall in investment and (beyond the first few quarters) consumption. Exports also decline as a result of the real exchange rate appreciation.

Figure B.15: Wage markup shock



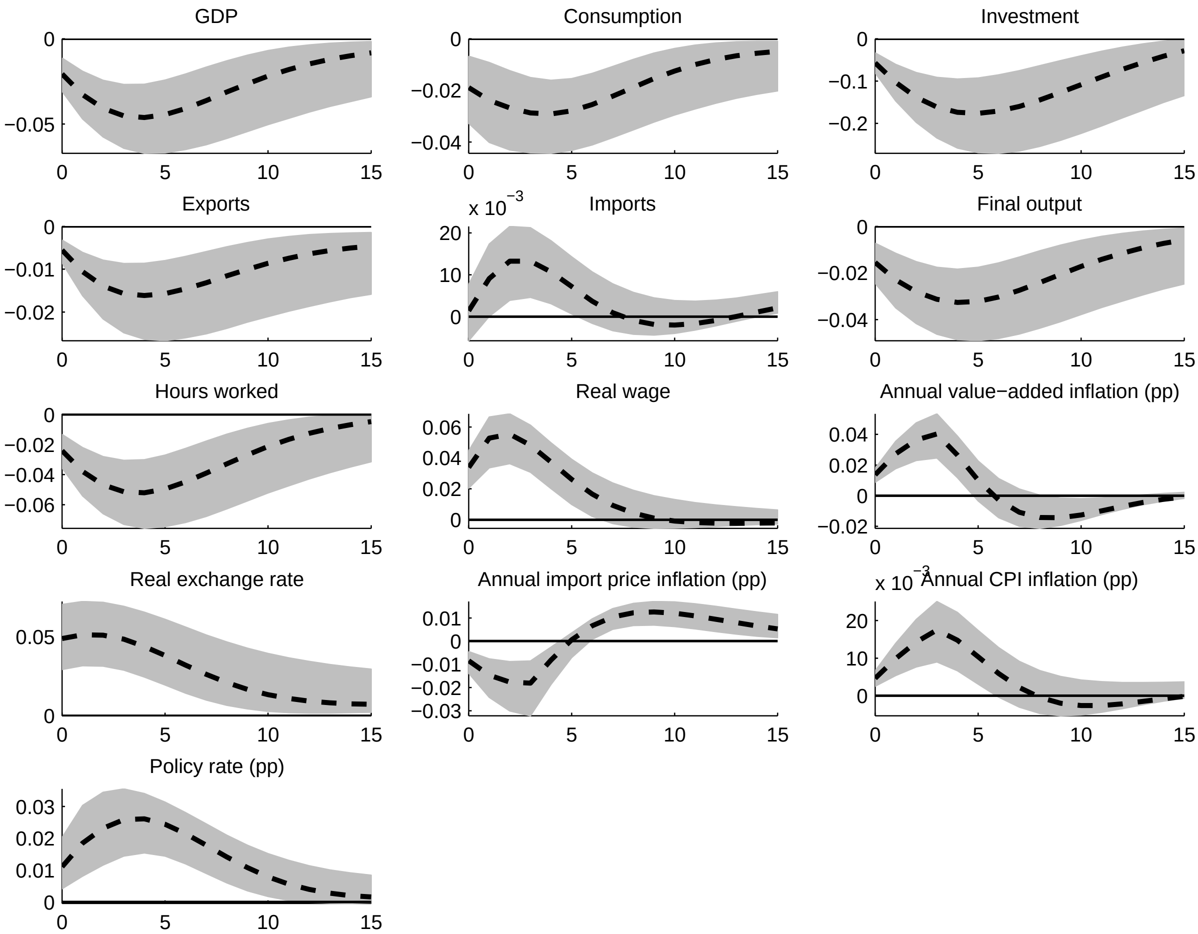
The charts show responses of COMPASS variables to a one standard deviation wage markup shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

B.16 Labour supply shock

A negative labour supply shock increases the value of leisure in the household utility function so that households temporarily become less willing to supply labour at a given real wage. This shifts the labour supply curve inwards, putting upward pressure on the real consumption wage and real product wage. The direct effect of the shock elicits a reduction in hours worked, but this is partially offset by the fact that consumption falls (which tends to generate a rise in labour supply as an endogenous response given that consumption and leisure are substitutes in the household utility function).

In response to the rise in real wages, there is some substitution of imports for labour by final-output producers, though this is not enough to offset the effects of the reduction in hours worked. As a result, both GDP and final output fall. Despite the fall in hours worked, the increase in real wages leads to an increase in the labour share and hence the marginal cost of value added production increases. This is passed along the supply chain so that the marginal cost of final output production and hence CPI inflation rises. This

Figure B.16: Labour supply shock



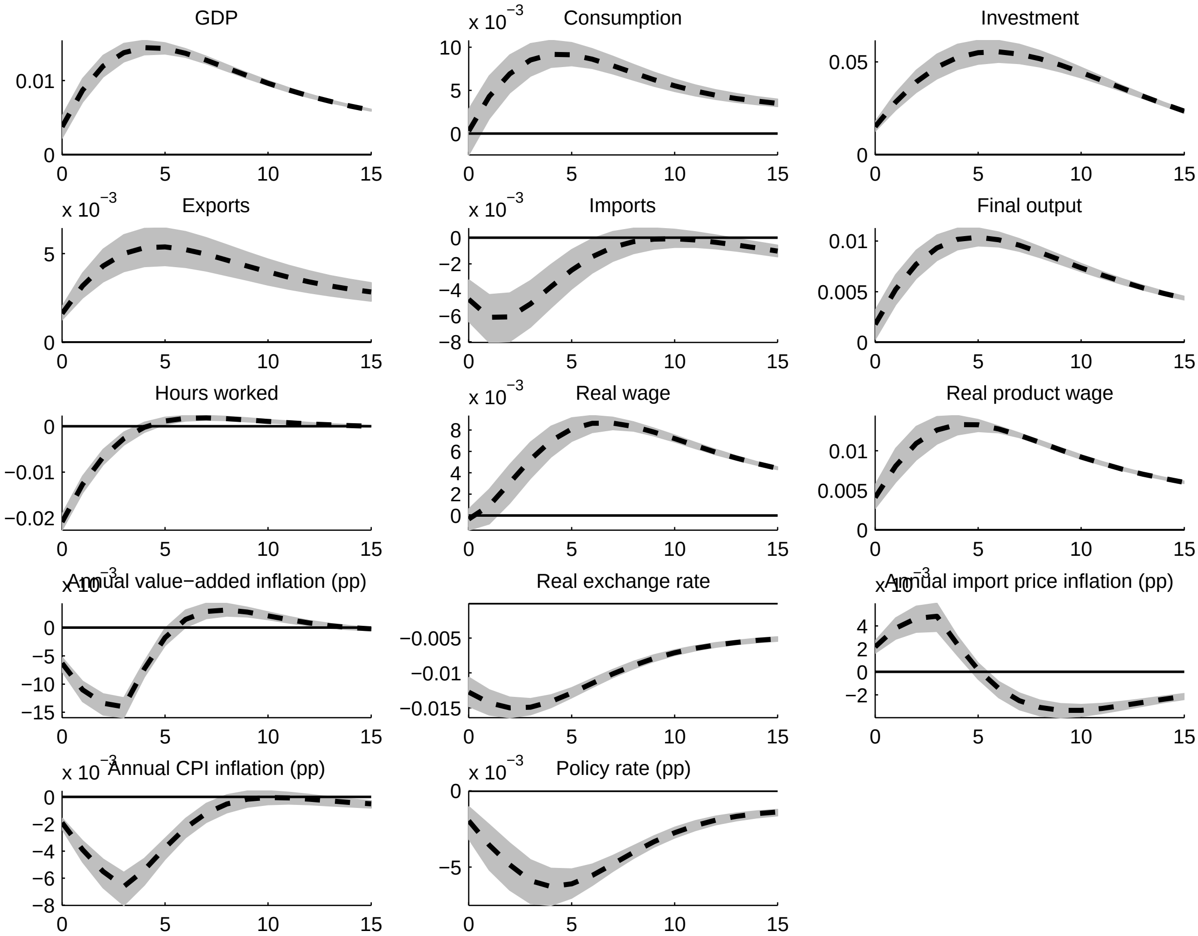
The charts show responses of COMPASS variables to a one standard deviation labour supply shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

induces an increase in the nominal interest rate which leads to higher real rates and an appreciation of the real exchange rate. The appreciation of the real exchange rate reduces import prices, which partially offsets some of the effects of the increase in wages on final output production costs. Higher real rates and the appreciation of the real exchange rate lead to a fall in demand across all expenditure components.

B.17 Total factor productivity shock

A positive TFP shock generates a temporary increase in total factor productivity, so that value added firms can produce more output for a given level of inputs. The direct result is an increase in GDP. Hours worked initially fall in response to the shock, because sticky prices and wages prevent demand from increasing in proportion to the expansion

Figure B.17: TFP shock



The charts show responses of COMPASS variables to a one standard deviation TFP shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

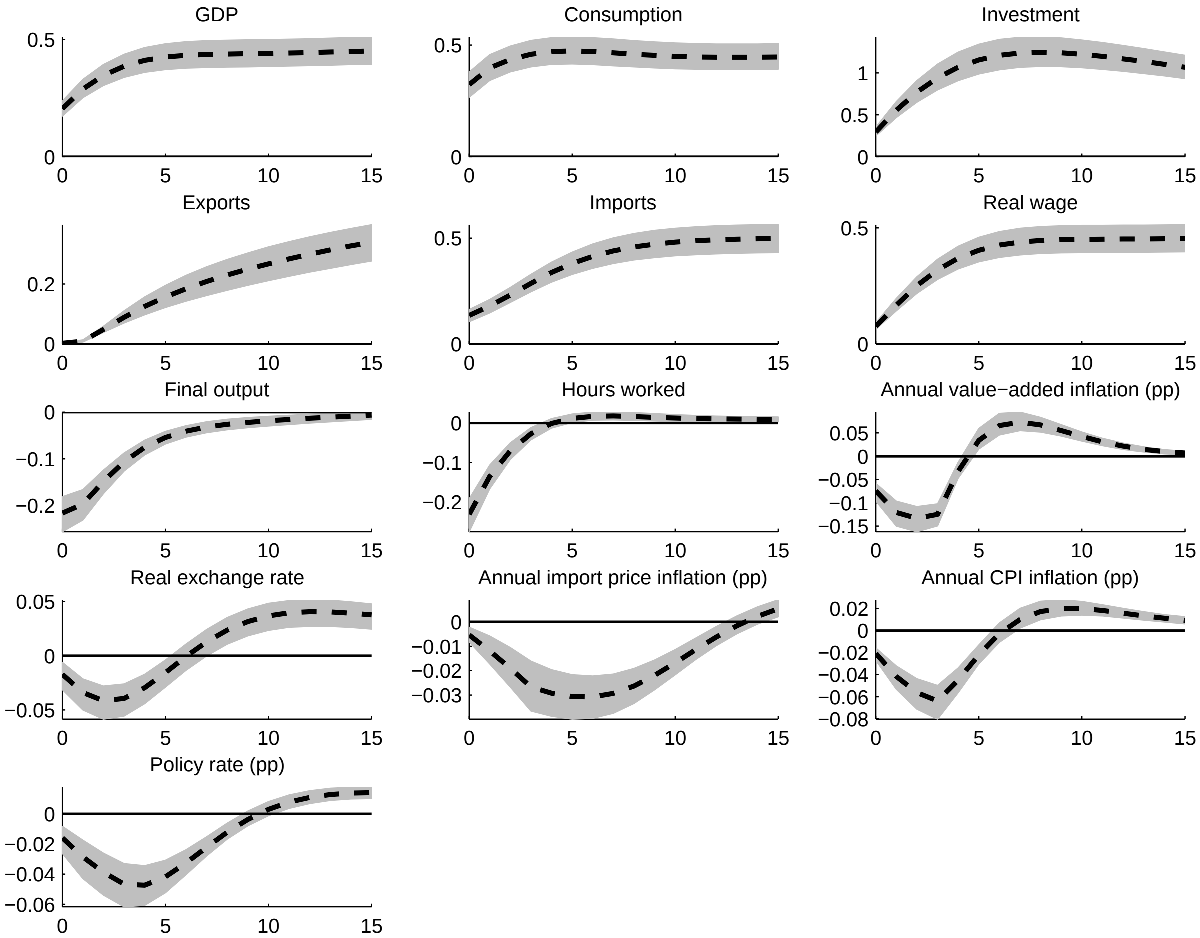
in supply capacity.[B2] The increase in productivity lowers the marginal cost of value added production and leads to a fall in value added prices. It also leads to an increase in the real product wage. The fall in value added prices leads to a substitution away from imports and is sufficient to reduce the marginal cost of final output. This in turn reduces CPI inflation. There is further substitution away from imports as import prices increase in response to a depreciation of the real exchange rate. In response to lower inflation, the policy maker reduces the policy rate. The real interest rate is persistently lower, leading to a depreciation of the real exchange rate. Demand increases across all expenditure components, supported by lower real interest rates and the depreciation of the real exchange rate.[B3]

[B2]If prices and wages were flexible, factor inputs would increase because aggregate demand would expand more than proportionally to the increase in productivity. This is because the persistence of the shock gives rise to positive wealth effects.

[B3]The responses to this shock are quantitatively small because the standard deviation was calibrated to a relatively small value: see the discussion in Section 4.3 for more information.

B.18 Labour augmenting productivity shock

Figure B.18: Labour augmenting productivity growth shock



The charts show responses of COMPASS variables to a one standard deviation labour augmenting productivity growth shock. Responses are measured in percentage changes from steady state or percentage points (pp) where specifically indicated. The x-axis shows the number of quarters following the shock. The shaded area shows the range between the 5-th percentile and the 95-th percentile of the distribution of responses associated with the posterior parameter uncertainty. Dashed lines show the responses when parameter values are set to the mean of the posterior distribution.

This shock temporarily raises the growth rate of the economy's supply capacity as well as the growth rates of the expenditure components, while pushing down on domestic inflation. Most importantly, this shock has a permanent effect on the levels of variables along the balanced growth path, by permanently increasing the *level* of productivity. By doing so, it shifts up the balanced growth path for the levels of real variables in the model.

Value added prices, measured relative to final output prices, fall in response to the shock, reflecting increased productivity and a reduction in marginal cost. The reduction in production costs leads to a fall in CPI inflation. The fall in the marginal cost for final output producers is mitigated by higher relative import prices associated with a depreciation in the real exchange rate, which results from the policy maker reducing the nominal interest rate in response to weaker inflation. Employment falls initially in response to the shock: sticky prices and wages prevent demand from increasing as much as potential supply.

C MAPS inversion algorithm

This appendix details the inversion algorithm employed in MAPS. This inversion algorithm makes it possible to impose judgements on endogenous variables directly and vary the assumptions underpinning those judgements in a flexible way. As explained in Section 6.2.4 and illustrated in Section 8, the imposition of judgement is a key part of the forecast process at the Bank and so the inversion algorithm is an important part of the MAPS toolkit.

The starting point for thinking about a judgemental projection in MAPS is equation (45) from Section 6.2.4. This is repeated below for convenience.

$$x_{T+h} = Bx_{T+h-1} + \Phi u_{T+h} + \sum_{s=h+1}^{H} F^{s-h} \Phi a_{T+s|T+h}$$

The equation states that a projection for the endogenous variables of a linear state space model in period h of the forecast depends on the values of the endogenous variables in period $h-1$, the impact of contemporaneous unanticipated shocks, u, in period h, and the impact of future anticipated shocks, a in periods $h+1$ to H. Any judgements made to the model-based projection are embodied in (non-zero) values for those unanticipated and anticipated structural disturbances. The inversion problem boils down to finding numerical values for the shocks that support numerical judgements made to the endogenous variables, sometimes known as "fixes" or "conditioning paths".

C.1 Statement of the problem

Before more formally stating the problem, we first make two additional assumptions necessary for identification of the anticipated shocks in the inversion algorithm outlined below:

- We assume that there is a single set of anticipated shocks over the forecast horizon such that $a_{T+s|T+h}$ is identical for all h (and so can be written more compactly as a_{T+s}). In addition, and for convenience, we allow for anticipated shocks to be realised contemporaneously (and so, in effect, partition the vector of unanticipated shocks into two parts in each forecast period).[C1]
- We assume that the horizon up to which judgements can be made to the endogenous variables coincides with the horizon up to which shocks can be anticipated (and that that horizon is denoted H).[C2]

These identification assumptions imply that a judgemental projection in MAPS can be written as:

$$x_{T+h} = Bx_{T+h-1} + \Phi u_{T+h} + \sum_{s=h}^{H} F^{s-h} \Phi a_{T+s} \tag{C.1}$$

Given equation (C.1), the inversion problem can be described in the following manner. Suppose that the forecasts for a subset of the endogenous variables, $\{x^i_{T+h}\}_{h=1}^{H}$, are

[C1] This is convenient in both MAPS and EASE because it means that the vector of anticipated and unanticipated shocks are of the same length.

[C2] The forecast horizon typically used to construct the MPC's forecast is 12 quarters (see Section 8 for examples). This horizon is not hard-wired into the MAPS toolkit, which permits any forecast horizon to be specified.

conditioned by a set of values, $\{\overline{x}^i_{T+h}\}^H_{h=1}$, using a chosen subset of the shocks under the assumption that those shocks are anticipated, $\{a^i_{T+h}\}^H_{h=1}$, and a chosen subset of the shocks under the assumption that those shocks are unanticipated, $\{u^i_{T+h}\}^H_{h=1}$ (where the subsets of variables conditioned and shocks used could be different in each time period).[C3]

Then the solution to the inversion problem is the sequence of shocks $\{a^i_{T+h}\}^H_{h=1}$ and $\{u^i_{T+h}\}^H_{h=1}$ that satisfies the conditioning information $\{\overline{x}^i_{T+h}\}^H_{h=1}$ given initial conditions, $x_T = \overline{x}_T$[C4], and values for the shocks not being used in the inversion, $\{a^n_{T+h} = \overline{a}^n_{T+h}\}^H_{h=1}$ & $\{u^n_{T+h} = \overline{u}^n_{T+h}\}^H_{h=1}$.

MAPS caters for two different sorts of inversion. If the number of anticipated and unanticipated shocks used as instruments is greater than or equal to the number of variables being conditioned, then the solution is such that the variables being conditioned are 'fixed' to those conditioning paths, $\{x^i_{T+h} = \overline{x}^i_{T+h}\}^H_{h=1}$. Alternatively, if the number of shock instruments is fewer than the number of variables being conditioned, then the solution involves finding values for the shocks that come closest to delivering the conditioning information (the precise definition of "closest" is outlined in detail below). In both cases, the inversion is subject to feasibility conditions. Most of those feasibility conditions can only be assessed with reference to the particular inversion in question given the structure of the model being used. Those conditions are discussed as part of the discussion of the solution in Section C.2.5. There are two conditions, however, which can be assessed independently of the model. In particular, the inversion is only valid if the following two conditions are met:

- There must exist at least one shock instrument that can affect each of the variables conditioned in every forecast period in which they are conditioned.[C5]
- Unanticipated shocks cannot be used as instruments beyond the horizon up to which any of the variables is being conditioned because in such cases there is no way of identifying them.

Note that the statement of the problem implies that all conditioning information is treated as 'hard', meaning that it is imposed on the model as if it were known with certainty. The implication is that we do not account for uncertainty around the conditioning information regardless of its source.[C6] Future work on MAPS may include extending the inversion toolkit to allow for some form of uncertainty around the conditioning information being incorporated. See, for example, Maih (2010), who allows for a (range of) uncertainty around the conditioning information with 'hard' conditioning and the unconditional forecast as limiting cases, and Waggoner and Zha (1999).

[C3]Note that the empty set applies as a special case, meaning that some inversions may exclude anticipated and/or unanticipated shocks and conditioning paths in one or more time periods. For example, a single inversion could include conditioning a single variable using an unanticipated shock in a single time period, say $T+1$, in which case $\{a^i_{T+h}\}^H_{h=1}$, $\{x^i_{T+h}\}^H_{h=2}$ & $\{u^i_{T+h}\}^H_{h=2}$ would be empty sets.

[C4]As discussed in Section 6.2.1, the initial condition for projection in LSS models is uncovered using the Kalman filter and smoother conditional on data for the observables and the model.

[C5]This means that unanticipated shocks are only effective instruments up to and including the period in which the variable in question is being conditioned, while anticipated shocks can be effective instruments in any period because, by definition, future anticipated shocks matter for current behaviour.

[C6]In particular, the misspecification approach discussed in 7 and applied in 8 advocates applying judgement using alternative models. The MAPS inversion algorithm does not allow for information about the distribution of possible outcomes from these alternative models to be taken into account.

C.2 Solution

The solution to the inversion problem described above involves several steps, which are explained in detail below. The first step is to reorder the system so that it can be partitioned into distinct blocks in the second step. Third, the partitioned system of equations is iterated backwards to express the blocked endogenous variables as a function of the initial conditions for the projection and the shocks. Fourth, this set of equations is stacked together across time periods to deliver a set of simultaneous equations. And fifth, that system of equations is inverted under an appropriate identification scheme to deliver a unique solution. Finally, the judgemental projection is computed by recovering the original ordering of the system and then projecting the initial conditions forward. The rest of this appendix describes each of those steps in detail.

C.2.1 Reordering

The aim of this first step is to reorder the system so that it can be partitioned. Specifically, the system of equations is reordered separately in each time period so that: the endogenous variables being conditioned are stacked on top of the endogenous variables that are not being conditioned; the shocks used in the inversion are stacked on top of the shocks not used, separately for both anticipated and unanticipated shocks; the matrices, B, Φ & F, are reordered to be consistent with the reordered variables and shocks.

Denote the reordered vectors with superscript $\mathbb{T}$, such that $x^{\mathbb{T}}_{T+h} = \left[\begin{array}{cc} x^i_{T+h} & x^n_{T+h} \end{array}\right]'$, $a^{\mathbb{T}}_{T+h} = \left[\begin{array}{cc} a^i_{T+h} & a^n_{T+h} \end{array}\right]'$ & $u^{\mathbb{T}}_{T+h} = \left[\begin{array}{cc} u^i_{T+h} & u^n_{T+h} \end{array}\right]'$, where each block in the partitioned vectors is defined above apart from $\{x^n_{T+h}\}^H_{h=1}$, which is the subset of endogenous variables not being conditioned. Denote also the transformation matrices that deliver those reordered vectors, $\mathbb{T}^x_{T+h}$, $\mathbb{T}^a_{T+h}$ & $\mathbb{T}^u_{T+h}$, where, for example:

$$x^{\mathbb{T}}_{T+h} = \mathbb{T}^x_{T+h} x_{T+h}$$

Element i, j of $\mathbb{T}^x_{T+h}$ is unity if the variable in position i of vector x_{T+h} is to take position j in vector $x^{\mathbb{T}}_{T+h}$ and zero otherwise. Based on that logic, it is straightforward to reorder the system described in equation (C.1) in the following way to deliver equation (C.2) (noting that one property of the transformation matrices is that $(\mathbb{T}^x_{T+h})'\mathbb{T}^x_{T+h} = I$):

$$x_{T+h} = Bx_{T+h-1} + \sum_{s=h}^{H} F^{s-h}\Phi a_{T+s} + \Phi u_{T+h}$$

$$\mathbb{T}^x_{T+h} x_{T+h} = \mathbb{T}^x_{T+h} B x_{T+h-1} + \mathbb{T}^x_{T+h} \sum_{s=h}^{H} F^{s-h}\Phi a_{T+s} + \mathbb{T}^x_{T+h}\Phi u_{T+h}$$

$$\begin{aligned} \mathbb{T}^x_{T+h} x_{T+h} = {} & \mathbb{T}^x_{T+h} B (\mathbb{T}^x_{T+h-1})'\mathbb{T}^x_{T+h-1} x_{T+h-1} + \mathbb{T}^x_{T+h} \sum_{s=h}^{H} F^{s-h}\Phi (\mathbb{T}^a_{T+s})'\mathbb{T}^a_{T+s} a_{T+s} \\ & + \mathbb{T}^x_{T+h}\Phi(\mathbb{T}^u_{T+h})'\mathbb{T}^u_{T+h} u_{T+h} \end{aligned}$$

$$x^{\mathbb{T}}_{T+h} = \mathbb{T}^x_{T+h} B (\mathbb{T}^x_{T+h-1})' x^{\mathbb{T}}_{T+h-1} + \mathbb{T}^x_{T+h} \sum_{s=h}^{H} F^{s-h}\Phi (\mathbb{T}^a_{T+s})' a^{\mathbb{T}}_{T+s} + \mathbb{T}^x_{T+h}\Phi(\mathbb{T}^u_{T+h})' u^{\mathbb{T}}_{T+h}$$

$$x^{\mathbb{T}}_{T+h} = \mathbb{T}^x_{T+h} B(\mathbb{T}^x_{T+h-1})' x^{\mathbb{T}}_{T+h-1} + \mathbb{T}^x_{T+h} \sum_{s=h}^{H} F^{s-h} (\mathbb{T}^x_{T+h})' \mathbb{T}^x_{T+h} \Phi (\mathbb{T}^a_{T+s})' a^{\mathbb{T}}_{T+s}$$
$$+ \mathbb{T}^x_{T+h} \Phi (\mathbb{T}^u_{T+h})' u^{\mathbb{T}}_{T+h}$$

$$x^{\mathbb{T}}_{T+h} = B^{\mathbb{T}\mathbb{T}}_{T+h|T+h-1} x^{\mathbb{T}}_{T+h-1} + \sum_{s=h}^{H} \left(F^{\mathbb{T}\mathbb{T}}_{T+h}\right)^{s-h} \Phi^{\mathbb{T}\mathbb{T}^a}_{T+h|T+s} a^{\mathbb{T}}_{T+s} + \Phi^{\mathbb{T}\mathbb{T}^u}_{T+h} u^{\mathbb{T}}_{T+h} \tag{C.2}$$

Where:

$$B^{\mathbb{T}\mathbb{T}}_{T+h|T+h-1} = \mathbb{T}^x_{T+h} B(\mathbb{T}^x_{T+h-1})'$$
$$\Phi^{\mathbb{T}\mathbb{T}^a}_{T+h|T+s} = \mathbb{T}^x_{T+h} \Phi (\mathbb{T}^a_{T+h+s})'$$
$$\Phi^{\mathbb{T}\mathbb{T}^u}_{T+h} = \mathbb{T}^x_{T+h} \Phi (\mathbb{T}^u_{T+h})'$$
$$F^{\mathbb{T}\mathbb{T}}_{T+h} = \mathbb{T}^x_{T+h} F (\mathbb{T}^x_{T+h})'$$

With the final of those equations following from:

$$\left(F^{\mathbb{T}\mathbb{T}}_{T+h}\right)^0 = \mathbb{I}$$

$$\left(F^{\mathbb{T}\mathbb{T}}_{T+h}\right)^1 = \mathbb{T}^x_{T+h} F (\mathbb{T}^x_{T+h})'$$
$$\left(F^{\mathbb{T}\mathbb{T}}_{T+h}\right)^2 = \mathbb{T}^x_{T+h} F (\mathbb{T}^x_{T+h})' \mathbb{T}^x_{T+h} F (\mathbb{T}^x_{T+h})' = \mathbb{T}^x_{T+h} F^2 (\mathbb{T}^x_{T+h})'$$
$$\left(F^{\mathbb{T}\mathbb{T}}_{T+h}\right)^3 = \mathbb{T}^x_{T+h} F (\mathbb{T}^x_{T+h})' \mathbb{T}^x_{T+h} F (\mathbb{T}^x_{T+h})' \mathbb{T}^x_{T+h} F (\mathbb{T}^x_{T+h})' = \mathbb{T}^x_{T+h} F^3 (\mathbb{T}^x_{T+h})'$$

And so on.

Finally, note that it is strightforward to recover the original vector orderings by applying the transpose operator to the transformation matrices. For example, it is straightforward to recover x_{T+s} from $x^{\mathbb{T}}_{T+s}$ using:

$$x_{T+h} = (\mathbb{T}^x_{T+h})' x^{\mathbb{T}}_{T+h}$$

C.2.2 Partitioning

Having reordered the system, the next step in the algorithm is to partition it into distinct blocks. The partitioned version of equation (C.2) is given by equation (C.3), where: x^i_{T+h} is the subset of variables being conditioned in period $T+h$ (with x^n_{T+h} analogous for variables not being conditioned); x^i_{T+h-1} is the (potentially different) subset of variables conditioned in period $T+h-1$ (with x^n_{T+h-1} analogous again); $B^{ii}_{T+h|T+h-1}$ is the upper left block of matrix $B^{\mathbb{T}\mathbb{T}}_{T+h|T+h-1}$ that dictates the effect of x^i_{T+h-1} on x^i_{T+h}, while $B^{in}_{T+h|T+h-1}$ is the upper right block of $B^{\mathbb{T}\mathbb{T}}_{T+h|T+h-1}$ that dictates the effect of x^n_{T+h-1} on x^i_{T+h} and so on; A^{ii}_{T+h} captures the effect of anticipated shocks used in the inversion on x^i_{T+h}, while A^{ni}_{T+h} captures the effect of anticipated shocks used in the inversion on x^n_{T+h}; U^{ii}_{T+h} captures the effect of unanticipated shocks used in the inversion on x^i_{T+h} (with U^{ni}_{T+h} analogous); A^{in}_{T+h} captures the effect of anticipated shocks not used in the inversion on x^i_{T+h} (with A^{nn}_{T+h} analogous); U^{in}_{T+h} captures the effect of unanticipated shocks not used in the inversion on x^i_{T+h} (with U^{nn}_{T+h} analogous).

$$\begin{bmatrix} x^i_{T+h} \\ x^n_{T+h} \end{bmatrix} = \begin{bmatrix} B^{ii}_{T+h|T+h-1} & B^{in}_{T+h|T+h-1} \\ B^{ni}_{T+h|T+h-1} & B^{nn}_{T+h|T+h-1} \end{bmatrix} \begin{bmatrix} x^i_{T+h-1} \\ x^n_{T+h-1} \end{bmatrix}$$
$$+ \begin{bmatrix} A^{ii}_{T+h} \\ A^{ni}_{T+h} \end{bmatrix} + \begin{bmatrix} U^{ii}_{T+h} \\ U^{ni}_{T+h} \end{bmatrix} + \begin{bmatrix} A^{in}_{T+h} \\ A^{nn}_{T+h} \end{bmatrix} + \begin{bmatrix} U^{in}_{T+h} \\ U^{nn}_{T+h} \end{bmatrix} \tag{C.3}$$

Where:[C7]

$$\begin{bmatrix} A^{ii}_{T+h} \\ A^{ni}_{T+h} \end{bmatrix} = \begin{bmatrix} \Phi^{ia^i}_{T+h|T+h} & [F\Phi]^{ia^i}_{T+h|T+h+1} & \cdots & [F^{H-h}\Phi]^{ia^i}_{T+h|T+H} \\ \Phi^{na^i}_{T+h|T+h} & [F\Phi]^{na^i}_{T+h|T+h+1} & \cdots & [F^{H-h}\Phi]^{na^i}_{T+h|T+H} \end{bmatrix} \begin{bmatrix} a^i_{T+h} \\ \cdots \\ a^i_{T+H} \end{bmatrix}$$

$$\begin{bmatrix} U^{ii}_{T+h} \\ U^{ni}_{T+h} \end{bmatrix} = \begin{bmatrix} \Phi^{iu^i}_{T+h} \\ \Phi^{nu^i}_{T+h} \end{bmatrix} u^i_{T+h}$$

$$\begin{bmatrix} A^{in}_{T+h} \\ A^{nn}_{T+h} \end{bmatrix} = \begin{bmatrix} \Phi^{ia^n}_{T+h|T+h} & [F\Phi]^{ia^n}_{T+h|T+H+1} & \cdots & [F^{H-h}\Phi]^{ia^n}_{T+h|T+H} \\ \Phi^{na^n}_{T+h|T+h} & [F\Phi]^{na^n}_{T+h|T+H+1} & \cdots & [F^{H-h}\Phi]^{na^n}_{T+h|T+H} \end{bmatrix} \begin{bmatrix} a^n_{T+h} \\ \cdots \\ a^n_{T+H} \end{bmatrix}$$

$$\begin{bmatrix} U^{in}_{T+h} \\ U^{nn}_{T+h} \end{bmatrix} = \begin{bmatrix} \Phi^{iu^n}_{T+h} \\ \Phi^{nu^n}_{T+h} \end{bmatrix} u^n_{T+h}$$

C.2.3 Backward induction

The aim of partitioning the system in the way outlined above is to separate blocks of the system that are being conditioned, x^i_{T+h}, from those that are not, x^n_{T+h}, and to separate shocks that are known and that can be treated as constants, $u^n_{T+h} = \overline{u}^n_{T+h}$ & $\{a^n_{T+s} = \overline{a}^n_{T+s}\}^H_{s=h}$, from those that are unknown and for which we are trying to solve, u^i_{T+h} & $\{a^i_{T+s}\}^H_{s=h}$. Consider the upper block of equation (C.3):

$$x^i_{T+h} = B^{ii}_{T+h|T+h-1}x^i_{T+h-1} + B^{in}_{T+h|T+h-1}x^n_{T+h-1} + A^{ii}_{T+h} + U^{ii}_{T+h} + A^{in}_{T+h} + U^{in}_{T+h}$$

Combining the terms for the impact of the lagged model variables together, this can be written as (where $B^{i\mathbb{T}}_{T+h|T+h-1} = \begin{bmatrix} B^{ii}_{T+h|T+h-1} & B^{in}_{T+h|T+h-1} \end{bmatrix}$ is the upper row in the partitioned $B^{\mathbb{T}\mathbb{T}}_{T+h|T+h-1}$ matrix):

$$x^i_{T+h} = B^{i\mathbb{T}}_{T+h|T+h-1}x^{\mathbb{T}}_{T+h-1} + A^{ai}_{T+h} + U^{ai}_{T+h} + A^{an}_{T+h} + U^{an}_{T+h}$$

The lagged term $x^{\mathbb{T}}_{T+h-1}$ can be written as (which is valid for $h \geq 2$):

$$x^{\mathbb{T}}_{T+h-1} = B^{\mathbb{T}\mathbb{T}}_{T+h-1|T+h-2}x^{\mathbb{T}}_{T+h-2} + A^{\mathbb{T}i}_{T+h-1} + U^{\mathbb{T}i}_{T+h-1} + A^{\mathbb{T}n}_{T+h-1} + U^{\mathbb{T}n}_{T+h-1}$$

With $A^{\mathbb{T}i}_{T+h-1} = \begin{bmatrix} A^{ii}_{T+h-1} & A^{ni}_{T+h-1} \end{bmatrix}'$, $U^{\mathbb{T}i}_{T+h-1} = \begin{bmatrix} U^{ii}_{T+h-1} & U^{ni}_{T+h-1} \end{bmatrix}'$, $A^{\mathbb{T}n}_{T+h-1} = \begin{bmatrix} A^{in}_{T+h-1} & A^{nn}_{T+h-1} \end{bmatrix}'$ & $U^{\mathbb{T}n}_{T+h-1} = \begin{bmatrix} U^{in}_{T+h-1} & U^{nn}_{T+h-1} \end{bmatrix}'$. This expression for

[C7] Note that these definitions apply for an arbitrary forecast period, h, but where $h < H$. If $h = H$, then it is straightforward to amend the definitions in the text in an appropriate way.

$x^{\mathbb{T}}_{T+h-1}$ can be iterated backwards to derive the following where $x_T = \overline{x}_T$ is the vector of initial conditions for the projection and is known:[C8]

$$
\begin{aligned}
x^{\mathbb{T}}_{T+h-1} = & \prod_{s=1}^{h-1} B^{\mathbb{T}\mathbb{T}}_{T+h-s|T+h-s-1} x_T \\
& + \sum_{s=1}^{h-2} \prod_{k=s}^{h-2} B^{\mathbb{T}\mathbb{T}}_{T+h-1+s-k|T+h-2+s-k} (A^{\mathbb{T}i}_{T+s} + U^{\mathbb{T}i}_{T+s} + A^{\mathbb{T}n}_{T+s} + U^{\mathbb{T}n}_{T+s}) \\
& + A^{\mathbb{T}i}_{T+h-1} + U^{\mathbb{T}i}_{T+h-1} + A^{\mathbb{T}n}_{T+h-1} + U^{\mathbb{T}n}_{T+h-1}
\end{aligned}
$$

This equation states that, in any given period, the projection for the vector of (reordered) endogenous variables can be decomposed into the impact of initial conditions, the impact of lagged realisations of anticipated and unanticipated shocks, and the impact of contemporaneous realisations of anticipated and unanticipated shocks. This expression can be substituted into the equation for x^i_{T+h} from above to give the following (valid for $h \geq 3$):[C9]

$$
\begin{aligned}
x^i_{T+h} = & B^{i\mathbb{T}}_{T+h|T+h-1} \prod_{s=1}^{h-1} B^{\mathbb{T}\mathbb{T}}_{T+h-s|T+h-s-1} x_T \\
& + B^{i\mathbb{T}}_{T+h|T+h-1} \sum_{s=1}^{h-2} \prod_{k=s}^{h-2} B^{\mathbb{T}\mathbb{T}}_{T+h-1+s-k|T+h-2+s-k} (A^{\mathbb{T}i}_{T+s} + U^{\mathbb{T}i}_{T+s} + A^{\mathbb{T}n}_{T+s} + U^{\mathbb{T}n}_{T+s}) \\
& + B^{i\mathbb{T}}_{T+h|T+h-1} (A^{\mathbb{T}i}_{T+h-1} + U^{\mathbb{T}i}_{T+h-1} + A^{\mathbb{T}n}_{T+h-1} + U^{\mathbb{T}n}_{T+h-1}) \\
& + A^{ii}_{T+h} + U^{ii}_{T+h} + A^{in}_{T+h} + U^{in}_{T+h}
\end{aligned}
\tag{C.4}
$$

The complete set of equations (C.4) from periods $h = 1...H$ form a system in the unknowns $\{u^i_{T+h}\}^H_{h=1}$, $\{a^i_{T+h}\}^H_{h=1}$ given the known initial condition $x_T = \overline{x}_T$ and the known values for the shocks not being used in the inversion $\{u^n_{T+h} = \overline{u}^n_{T+h}\}^H_{h=1}$ & $\{a^n_{T+h} = \overline{a}^n_{T+h}\}^H_{h=1}$. In the special case in which the only instruments in the inversion are unanticipated shocks (i.e. where $\{A^{\mathbb{T}i}_{T+h}\}^H_{h=1}$ and, by extension, $\{A^{ii}_{T+h}\}^H_{h=1}$ are empty sets) and where the number of unanticipated shocks used as instruments in the inversion is equal the number of endogenous variables being conditioned in every period (i.e. exact identification), it is straightforward to solve equation (C.4) for u^i_{T+h} recursively from $h = 1, ..., H$. In more general cases, where anticipated shocks are used as instruments in the inversion or where the number of shock instruments in not exactly equal the number of conditioning paths, it is not possible to solve the system of equations period by period using simple inversions. In particular, the inclusion of anticipated shocks precludes a recursive solution (even if identification is exact) because, by definition, the endogenous variables are functions of all current and future anticipated shocks. For example, the final set of shocks in the sequence $\{a^i_{T+1}, ..., a^i_{T+H}\}$ affects all endogenous variables simultaneously across the whole horizon, $h = 1, ..., H$. Following that logic, the system of equations must be solved across all periods of the inversion simultaneously.

[C8] Note that this expression is valid for all $h \geq 3$. When $h = 2$, $x^{\mathbb{T}}_{T+h-1} = B^{\mathbb{T}\mathbb{T}}_{T+1|T} x_T + A^{\mathbb{T}i}_{T+1} + U^{\mathbb{T}i}_{T+1} + A^{\mathbb{T}n}_{T+1} + U^{\mathbb{T}n}_{T+1}$ and when $h = 1$, $x^{\mathbb{T}}_{T+h-1} = x_T$.

[C9] It is straightforward to derive expressions when $h = 1$ and $h = 2$ by substituting in the relevant expressions from footnote C8.

C.2.4 Stacking

Stacking equation (C.4) across all forecast periods and grouping like terms together yields the following:

$$\begin{bmatrix} x^i_{T+1} \\ x^i_{T+2} \\ \cdots \\ x^i_{T+H-1} \\ x^i_{T+H} \end{bmatrix} = \begin{bmatrix} A^{ii}_{T+1} \\ B^{i\mathbb{T}}_{T+2|T+1}A^{\mathbb{T}i}_{T+1} + A^{ii}_{T+2} \\ \cdots \\ \begin{bmatrix} B^{i\mathbb{T}}_{T+H-1|T+H-2}(\sum_{s=1}^{H-3}\prod_{k=s}^{H-3} B^{\mathbb{T}\mathbb{T}}_{T+H-2+s-k|T+H-3+s-k}A^{\mathbb{T}i}_{T+s} \\ +A^{\mathbb{T}i}_{T+H-2}) + A^{ii}_{T+H-1} \end{bmatrix} \\ \begin{bmatrix} B^{i\mathbb{T}}_{T+H|T+H-1}(\sum_{s=1}^{H-2}\prod_{k=s}^{H-2} B^{\mathbb{T}\mathbb{T}}_{T+H-1+s-k|T+H-2+s-k}A^{\mathbb{T}i}_{T+s} \\ +A^{\mathbb{T}i}_{T+H-1}) + A^{ii}_{T+H} \end{bmatrix} \end{bmatrix}$$

$$+ \begin{bmatrix} U^{ii}_{T+1} \\ B^{i\mathbb{T}}_{T+2|T+1}U^{\mathbb{T}i}_{T+1} + U^{ii}_{T+2} \\ \cdots \\ \begin{bmatrix} B^{i\mathbb{T}}_{T+H-1|T+H-2}(\sum_{s=1}^{H-3}\prod_{k=s}^{H-3} B^{\mathbb{T}\mathbb{T}}_{T+H-2+s-k|T+H-3+s-k}U^{\mathbb{T}i}_{T+s} \\ +U^{\mathbb{T}i}_{T+H-2}) + U^{ii}_{T+H-1} \end{bmatrix} \\ \begin{bmatrix} B^{i\mathbb{T}}_{T+H|T+H-1}(\sum_{s=1}^{H-2}\prod_{k=s}^{H-2} B^{\mathbb{T}\mathbb{T}}_{T+H-1+s-k|T+H-2+s-k}U^{\mathbb{T}i}_{T+s} \\ +U^{\mathbb{T}i}_{T+H-1}) + U^{ii}_{T+H} \end{bmatrix} \end{bmatrix}$$

$$+ \begin{bmatrix} B^{i\mathbb{T}}_{T+1|T}x_T + A^{in}_{T+1} + U^{in}_{T+1} \\ B^{i\mathbb{T}}_{T+2|T+1}(B^{\mathbb{T}\mathbb{T}}_{T+1|T}x_T + A^{\mathbb{T}n}_{T+1} + U^{\mathbb{T}n}_{T+1}) + A^{in}_{T+2} + U^{in}_{T+2} \\ \cdots \\ \begin{bmatrix} B^{i\mathbb{T}}_{T+H-1|T+H-2}(\prod_{s=1}^{H-2} B^{\mathbb{T}\mathbb{T}}_{T+H-1-s|T+H-2-s}x_T \\ +\sum_{s=1}^{H-3}\prod_{k=s}^{H-3} B^{\mathbb{T}\mathbb{T}}_{T+H-2+s-k|T+H-3+s-k}(A^{\mathbb{T}n}_{T+s} + U^{\mathbb{T}n}_{T+s}) \\ +A^{\mathbb{T}n}_{T+H-2} + U^{\mathbb{T}n}_{T+H-2}) + A^{in}_{T+H-1} + U^{in}_{T+H-1} \\ B^{i\mathbb{T}}_{T+H|T+H-1}(\prod_{s=1}^{H-1} B^{\mathbb{T}\mathbb{T}}_{T+H-s|T+H-1-s}x_T \\ +\sum_{s=1}^{H-2}\prod_{k=s}^{H-2} B^{\mathbb{T}\mathbb{T}}_{T+H-1+s-k|T+H-2+s-k}(A^{\mathbb{T}n}_{T+s} + U^{\mathbb{T}n}_{T+s}) \\ +A^{\mathbb{T}n}_{T+H-1} + U^{\mathbb{T}n}_{T+H-1}) + A^{in}_{T+H} + U^{in}_{T+H} \end{bmatrix} \end{bmatrix}$$

Separating out the loadings in the first two (block) vector terms on the right-hand-side (which are given by the relevant blocks of the model solution matrices and are known) from the shocks, $a^i = \begin{bmatrix} a^i_{T+1} & a^i_{T+2} & \dots & a^i_{T+H-1} & a^i_{T+H} \end{bmatrix}'$ & $u^i = \begin{bmatrix} u^i_{T+1} & u^i_{T+2} & \dots & u^i_{T+H-1} & u^i_{T+H} \end{bmatrix}'$, this stacked set of equations can be written in the following way (where $x^i = \begin{bmatrix} x^i_{T+1} & x^i_{T+2} & \dots & x^i_{T+H-1} & x^i_{T+H} \end{bmatrix}'$):

$$x^i = W^a a^i + W^u u^i + C \tag{C.5}$$

The matrices W^a, W^u & C can be defined as (where $\overline{a}^n = \begin{bmatrix} \overline{a}^n_{T+1} & \overline{a}^n_{T+2} & \dots & \overline{a}^n_{T+H-1} & \overline{a}^n_{T+H} \end{bmatrix}'$ & $\overline{u}^n = \begin{bmatrix} \overline{u}^n_{T+1} & \overline{u}^n_{T+2} & \dots & \overline{u}^n_{T+H-1} & \overline{u}^n_{T+H} \end{bmatrix}'$):

$$W^a = \mathbb{B}^{i\mathbb{T}}\mathbb{B}\mathbb{B}^{\mathbb{T}\mathbb{T}}\mathbb{A}^{\mathbb{T}i} + \mathbb{A}^{ii}$$

$$W^u = \mathbb{B}^{i\mathbb{T}}\mathbb{B}\mathbb{B}^{\mathbb{T}\mathbb{T}}\mathbb{U}^{\mathbb{T}i} + \mathbb{U}^{ii}$$

$$C = \mathbb{B}^{i\mathbb{T}}(\mathbb{B}^{\mathbb{T}\mathbb{T}}\overline{x}_T + \mathbb{B}\mathbb{B}^{\mathbb{T}\mathbb{T}}(\mathbb{A}^{\mathbb{T}n}\overline{a}^n + \mathbb{U}^{\mathbb{T}n}\overline{u}^n)) + \mathbb{A}^{in}\overline{a}^n + \mathbb{U}^{in}\overline{u}^n$$

Where:

$$\mathbb{BB}^{\mathbb{TT}} = \begin{bmatrix} 0 & 0 & ... & 0 & 0 & 0 \\ I & 0 & ... & 0 & 0 & 0 \\ B^{\mathbb{TT}}_{T+2|T+1} & I & ... & 0 & 0 & 0 \\ \prod_{k=1}^{2} B^{\mathbb{TT}}_{T+4-k|T+3-k} & B^{\mathbb{TT}}_{T+3|T+2} & ... & 0 & 0 & \\ ... & ... & ... & ... & ... & ... \\ \prod_{k=1}^{H-3} B^{\mathbb{TT}}_{T+H-1-k|T+H-2-k} & \prod_{k=1}^{H-4} B^{\mathbb{TT}}_{T+H-1-k|T+H-2-k} & ... & I & 0 & 0 \\ \prod_{k=1}^{H-2} B^{\mathbb{TT}}_{T+H-k|T+H-1-k} & \prod_{k=1}^{H-3} B^{\mathbb{TT}}_{T+H-s|T+H-1-k} & ... & B^{\mathbb{TT}}_{T+H-1|T+H-2} & I & 0 \end{bmatrix}$$

$$\mathbb{B}^{\mathbb{TT}} = \begin{bmatrix} I \\ B^{\mathbb{TT}}_{T+1|T} \\ ... \\ \prod_{k=1}^{H-2} B^{\mathbb{TT}}_{T+H-k-1|T+H-k-2} \\ \prod_{k=1}^{H-1} B^{\mathbb{TT}}_{T+H-k|T+H-k-1} \end{bmatrix}$$

$$\mathbb{A}^{\mathbb{T}i} = \begin{bmatrix} \Phi^{\mathbb{T}a^i}_{T+1|T+1} & [F\Phi]^{\mathbb{T}a^i}_{T+1|T+2} & ... & \left[F^{H-2}\Phi\right]^{\mathbb{T}a^i}_{T+1|T+H-1} & \left[F^{H-1}\Phi\right]^{\mathbb{T}a^i}_{T+1|T+H} \\ 0 & [\Phi]^{\mathbb{T}a^i}_{T+2|T+2} & ... & \left[F^{H-3}\Phi\right]^{\mathbb{T}a^i}_{T+2|T+H-1} & \left[F^{H-2}\Phi\right]^{\mathbb{T}a^i}_{T+2|T+H} \\ 0 & 0 & ... & \left[F^{H-4}\Phi\right]^{\mathbb{T}a^i}_{T+3|T+H-1} & \left[F^{H-3}\Phi\right]^{\mathbb{T}a^i}_{T+3|T+H} \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \Phi^{\mathbb{T}a^i}_{T+H-1|T+H-1} & [F\Phi]^{\mathbb{T}a^i}_{T+H-1|T+H} \\ 0 & 0 & ... & 0 & \Phi^{\mathbb{T}a^i}_{T+H|T+H} \end{bmatrix}$$

$$\mathbb{U}^{\mathbb{T}i} = \begin{bmatrix} \Phi^{\mathbb{T}u^i}_{T+1} & 0 & ... & 0 & 0 \\ 0 & \Phi^{\mathbb{T}u^i}_{T+2} & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & \Phi^{\mathbb{T}u^i}_{T+H-1} & 0 \\ 0 & 0 & ... & 0 & \Phi^{\mathbb{T}u^i}_{T+H-1} \end{bmatrix}$$

$$\mathbb{B}^{i\mathbb{T}} = \begin{bmatrix} B^{i\mathbb{T}}_{T+1|T} & 0 & ... & 0 & 0 \\ 0 & B^{i\mathbb{T}}_{T+2|T+1} & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & B^{i\mathbb{T}}_{T+H-1|T+H-2} & 0 \\ 0 & 0 & ... & 0 & B^{i\mathbb{T}}_{T+H|T+H-1} \end{bmatrix}$$

$$\mathbb{A}^{ii} = \begin{bmatrix} \Phi^{ia^i}_{T+1|T+1} & [F\Phi]^{ia^i}_{T+1|T+2} & ... & \left[F^{H-2}\Phi\right]^{ia^i}_{T+1|T+H-1} & \left[F^{H-1}\Phi\right]^{ia^i}_{T+1|T+H} \\ 0 & \Phi^{ia^i}_{T+2|T+2} & ... & \left[F^{H-3}\Phi\right]^{ia^i}_{T+2|T+H-1} & \left[F^{H-2}\Phi\right]^{ia^i}_{T+2|T+H} \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \Phi^{ia^i}_{T+H-1|T+H-1} & [F\Phi]^{ia^i}_{T+H-1|T+H} \\ 0 & 0 & ... & 0 & \Phi^{ia^i}_{T+H|T+H} \end{bmatrix}$$

$$\mathbb{U}^{ii} = \begin{bmatrix} \Phi^{iu^i}_{T+1} & 0 & ... & 0 & 0 \\ 0 & \Phi iu^i{}_{T+2} & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \Phi^{iu^i}_{T+H-1} & 0 \\ 0 & 0 & ... & 0 & \Phi^{iu^i}_{T+H} \end{bmatrix}$$

And where $\mathbb{A}^{\mathbb{T}n}$, $\mathbb{U}^{\mathbb{T}n}$, $\mathbb{A}^{in}$ & $\mathbb{U}^{in}$ can be defined by analogy. Note that the dimension of these matrices are the following, where n denotes the number of variables or shocks in a particular set (and where the number of endogenous variables, n^x, is the same in each period): $\mathbb{BB}^{\mathbb{TT}}$ is $H \times n^x$ by $H \times n^x$; $\mathbb{BB}^{\mathbb{T}}$ is $H \times n^x$ by n^x; $\mathbb{A}^{\mathbb{T}i}$ is $H \times n^x$ by $\sum_{h=1}^{H} n^{a^i_{T+h}}$; $\mathbb{A}^{\mathbb{T}n}$ is $H \times n^x$ by $\sum_{h=1}^{H} n^{a^n_{T+h}}$; $\mathbb{U}^{\mathbb{T}i}$ is $H \times n^x$ by $\sum_{h=1}^{H} n^{u^i_{T+h}}$; $\mathbb{U}^{\mathbb{T}n}$ is $H \times n^x$ by $\sum_{h=1}^{H} n^{u^n_{T+h}}$; $\mathbb{B}^{i\mathbb{T}}$ is $\sum_{h=1}^{H} n^{x^i_{T+h}}$ by $H \times n^x$; $\mathbb{A}^{ii}$ is $\sum_{h=1}^{H} n^{x^i_{T+h}}$ by $\sum_{h=1}^{H} n^{a^i_{T+h}}$; $\mathbb{A}^{in}$ is $\sum_{h=1}^{H} n^{x^i_{T+h}}$ by $\sum_{h=1}^{H} n^{a^n_{T+h}}$; $\mathbb{U}^{ii}$ is $\sum_{h=1}^{H} n^{x^i_{T+h}}$ by $\sum_{h=1}^{H} n^{u^i_{T+h}}$; $\mathbb{U}^{in}$ is $\sum_{h=1}^{H} n^{x^i_{T+h}}$ by $\sum_{h=1}^{H} n^{u^n_{T+h}}$.

C.2.5 Inversion

In the special case where the number of endogenous variables being conditioned ('targets') is equal the number of shocks being used to deliver the inversion ('instruments'), the system in equation (C.5) can be solved by direct inversion and substitution to find the values for the shock instruments, $\{a^i_{T+h}\}_{h=1}^{H}$ & $\{u^i_{T+h}\}_{h=1}^{H}$, that fix the endogenous variables at the targets, $\{x^i_{T+h} = \overline{x}^i_{T+h}\}_{h=1}^{H}$, conditional on known values for the shocks not being used as instruments in the inversion, $\{\overline{a}^n_{T+h}\}_{h=1}^{H}$ & $\{\overline{u}^n_{T+h}\}_{h=1}^{H}$, and known initial conditions for the endogenous variables, $\overline{x}_T$.[C10]

In more general cases, where the number of instruments may not equal the number of targets in one or more time periods, it is necessary to make additional identification assumptions. There are two alternative cases. First, if the number of instruments exceeds the number of targets ('over-identification'), then an identification assumption must be used because there are an infinite number of candidate solutions that would fix the endogenous variables to the targets. In such cases, MAPS contains two alternative identification schemes, which are explained in more detail below. Second, if the number of targets exceeds the number of instruments ('under-identification'), then it is not possible to deliver the targets exactly and another type of identification scheme must be employed. These two cases are discussed in turn below

In both cases, the anticipated and unanticipated shocks in equation (C.5) are stacked together to give the following:

$$x^i = Wz^i + C \tag{C.6}$$

Where:

$$W = \begin{bmatrix} W^a & W^u \end{bmatrix}$$

$$z^i = \begin{bmatrix} a^i \\ u^i \end{bmatrix}$$

Over-identification

As discussed above, one possible case is that of over-identification. In such cases, the variables being conditioned can be fixed to the targets, but in an infinite number of ways,

[C10]This is demonstrated as a special case for both over- and under- identification below.

which manifests itself in the matrix W not being square and so not being invertible. In that general case, we need to employ an appropriately chosen identification scheme to deliver a unique set of values for the shock instruments in the inversion. The MAPS toolkit includes two such identification schemes outlined in turn below.

One candidate identification scheme is one that minimises the size of the shocks necessary to implement the inversion. Put differently, this scheme selects the **minimum variance** combination of shocks. It is worth noting that this approach makes more sense when the model being used is an estimated model like COMPASS, where the relevant elements of the Φ matrix (the standard deviations of the shocks in particular) have been estimated.[C11] The minimum variance combination of shocks can be delivered as the solution to the following problem:[C12]

$$z^{*i} = \min_{z^i}(z^i)'z^i \quad st : Wz^i = \overline{x}^i - C$$

The above minimisation problem can be written as a Lagrangian, where λ is a column vector of Lagrange multipliers of the same dimension as the (stacked) vector of variables being fixed:

$$L = (z^i)'z^i - 2\lambda' \left(Wz^i + C - \overline{x}^i\right)$$

The first order conditions associated with this problem are as follows:

$$\frac{dL}{dz^i} : 2z^i - 2\left(\lambda'W\right)' = 0$$

$$\frac{dL}{d\lambda} : 2\left(Wz^i + C - \overline{x}^i\right) = 0$$

The first of these first order conditions can be rearranged to give:

$$z^i = W'\lambda$$

And this can be substituted into the second first order condition and rearranged to give the following expression for λ:

$$\lambda = (WW')^{-1}\left(\overline{x}^i - C\right)$$

Finally, this can be substituted back into the rearranged first of the first order conditions to give the solution that delivers the minimum variance set of inversion shocks:

$$z^{*i} = W'(WW')^{-1}\left(\overline{x}^i - C\right) \tag{C.7}$$

Note that the solution in equation (C.7) requires that WW' is of full rank and therefore non-singular (noting that WW' is guaranteed to be square by construction). The satisfaction or otherwise of this rank condition is determined by the interaction between the

[C11]Note, though, that even in those cases an assumption must be made about the standard deviations of the anticipated shocks. The assumption employed here is that the anticipated shocks have identical standard deviations to their unanticipated counterparts, which is reflected in the pre-multiplication of the anticipated shocks in equation (C.1) by the same matrix Φ as pre-multiplies the unanticipated shocks.

[C12]This setup follows from the standardisation assumption used in MAPS' linear state space modelling framework (see Section 6.2.1), whereby all disturbances have unitary standard deviations with the size of the impact of those disturbances reflected in the coefficients of the Ψ and hence Φ matrices. This means that there is no need to include the covariance matrix of the shocks as a weighting matrix because the shock variances will be taken account of automatically through the coefficients in the W matrix.

choice of shock instruments given the variables being conditioned and the structure of the model. Failure to meet the full rank condition often has an economic interpretation. For example, world demand in COMPASS is assumed to follow an AR(1) process (see equation (A.98) in Section 4). Therefore, any attempt to fix world demand in COMPASS using any shock other than the world demand disturbance (the driving force of the AR(1)) would fail the rank condition and the inversion would be infeasible.[C13]

One feature of the minimum variance solution is that it does not use information about the existing values of the shocks being used as instruments. This may not be desirable if those shock values embody a story for pre-existing judgements. In that case, it may instead be appropriate to minimise the sum of squared changes in the shock values.[C14] The minimisation problem and associated Lagrangian that deliver the **minimum change** is as follows, where $\overline{z}^i$ are the existing values for the shocks being used as instruments in the inversion (i.e. values that have arisen from previous judgements imposed either directly via those shocks or indirectly via inversion):

$$z^{*i} = \min_{z^i} \left(z^i - \overline{z}^i\right)' \left(z^i - \overline{z}^i\right) \quad st : Wz^i = \overline{x}^i - C$$

$$L = \left(z^i - \overline{z}^i\right)' \left(z^i - \overline{z}^i\right) - 2\lambda' \left(Wz^i + C - \overline{x}^i\right)$$

This has the following first order conditions:

$$\frac{dL}{dz^i} : 2\left(z^i - \overline{z}^i\right) - 2\left(\lambda' W\right)' = 0$$

$$\frac{dL}{d\lambda} : 2\left(Wz^i + C - \overline{x}^i\right) = 0$$

The first of these first order conditions can be rearranged to give:

$$z^i = \overline{z}^i + W'\lambda$$

And this can be substituted into the second first order condition and rearranged to give the following expression for λ:

$$\lambda = \left(WW'\right)^{-1} \left(\overline{x}^i - C - W\overline{z}^i\right)$$

Finally, this can be substituted back into the rearranged first of the first order conditions to give a formula that delivers the minimum sum of squared change in the shocks (subject to the same rank condition as discussed in the minimum variance case being met):

$$z^{*i} = \overline{z}^i + W'\left(WW'\right)^{-1} \left(\overline{x}^i - C - W\overline{z}^i\right) \tag{C.8}$$

It is worth noting that both identification schemes deliver the same solution in two special cases. First and most obviously from inspection of equation (C.8), if the existing values for the shock instruments being used in the inversion are all zero in every time period the two solutions are identical. Second, if identification is exact across the whole inversion horizon, then $W'\left(WW'\right)^{-1} W\overline{z}^i = \overline{z}^i$, and the minimum change solution in equation (C.8) collapses to the minimum variance solution in equation (C.7). In that special case, both solutions collapse to a straightforward inversion based on equation (C.6) to give $z^{*i} = W^{-1}\left(\overline{x}^i - C\right)$.

[C13] In such cases, MAPS returns an informative error message describing the combination of fixes and instruments that are responsible for the rank condition not being met.

[C14] This is particularly relevant given the iterative nature of the forecast process at the Bank described in Section 8.

Under-identification

The over-identification solutions rely on knowledge that the variables being conditioned could be fixed exactly at the target values, allowing us to treat those variables as constants in the minimisation problem. If the number of shock instruments used is smaller than the number of variables being conditioned, then it is no longer possible to fix those variables exactly. In those circumstances, an obvious identification scheme is one that delivers values for the shock instruments that minimise the sum of squared deviations of the variables being conditioned from their targets:

$$z^{*i} = \min_{z^i} \left(x^i - \overline{x}^i\right)' \left(x^i - \overline{x}^i\right)$$

$$z^{*i} = \min_{z^i} \left(Wz^i + C - \overline{x}^i\right)' \left(Wz^i + C - \overline{x}^i\right)$$

This has the following first order condition:

$$2W' \left(Wz^i + C - \overline{x}^i\right) = 0$$

This can be rearranged to give:

$$z^{*i} = \left(W'W\right)^{-1} W'(\overline{x}^i - C) \tag{C.9}$$

As in the discussion of over-identified inversions above, it is worth noting the special case of exact identification in which case the solution collapses to the same direct inversion of the W matrix. It is also worth noting that the solution requires a similar rank condition to hold as for the over-identification case. In this case, the matrix $W'W$ must be invertible.[C15]

C.2.6 Recovering the judgemental projection

Finally, note that the vector of shocks, z^{*i} can be separated into their constituent anticipated and unanticipated parts, a^i & u^i, and combined with the vectors of each type of shock not used as instruments in the inversion, a^n & u^n. They can then be separated across time periods and reordered using the transpose of the relevant reordering operators to recover a complete, correctly ordered set of shocks emobodying the forecast judgements. It is then straightforward to use the definition for a judgemental projection in equation (C.1) to update the judgemental forecast for the endogenous variables such that a sub-set of the forecasts for the endogenous variables have been conditioned using a choice of shocks with all the other endogenous variables having responded endogenously.

[C15] Failure to meet this rank condition has many of the same economic interpretations as discussed above including, for example, attempting to use a shock as an instrument without having a conditioning variable that that shock can affect.