BANK OF ENGLAND

Working Paper No. 526

A joint affine model of commodity futures and US Treasury yields

Michael Chin and Zhuoshi Liu

March 2015

BANK OF ENGLAND

Working Paper No. 526

A joint affine model of commodity futures and US Treasury yields

Michael Chin(1) and Zhuoshi Liu(2)

Abstract

We derive a general joint affine term structure model of US government bond yields and the convenience yields on physical commodities. We apply this framework separately to oil and gold. Our results show clear links between bond and commodity markets, since bond factors play a significant role in the pricing of the convenience yield term structure. Our framework allows us to decompose the term structure of futures prices into expectations of future spot prices and risk premia components. We estimate that the risk premium in oil futures has been negative over the 1980s and 1990s, and turned positive in the mid-2000s, consistent with a declining role for supply shocks in the oil market over this period. In contrast, we estimate that the gold risk premium is mostly positive throughout the sample period.

Key words: Commodity futures, gold, oil, risk premium, convenience yields, affine term structure model, Treasury yields.

JEL classification: E43, G13, Q02, Q40.

---

(1) Macro Financial Analysis Division, Bank of England. Email: michael.chin@bankofengland.co.uk
(2) Macro Financial Analysis Division, Bank of England. (Author has since joined CIC International (Hong Kong).)
Email: liuzs@china-inv.hk

The views expressed in this paper are those of the authors, and do not necessarily reflect those of the Bank of England, the Monetary/Financial Policy Committees or the Prudential Regulation Authority Board. We thank participants at the Empirical Modelling of Financial Markets Workshop at Brunel University on 28 May 2014 and the Asset Pricing Workshop at the University of York on 30 June 2014 for their useful comments, in particular John Hunter for his discussion. We also thank an anonymous referee for extensive and valuable comments. This paper was finalised on 26 January 2015.

The Bank of England's working paper series is externally refereed.

Information on the Bank's working paper series can be found at
www.bankofengland.co.uk/research/Pages/workingpapers/default.aspx

Publications Team, Bank of England, Threadneedle Street, London, EC2R 8AH
Telephone +44 (0)20 7601 4030 Fax +44 (0)20 7601 3298 email publications@bankofengland.co.uk

ISSN 1749-9135 (on-line)

Summary

Commodity price fluctuations can have a significant effect on inflation, and as a result central banks are interested in where commodity prices currently are, and in where they might go in the future. It is possible to obtain commodity price forecasts from financial markets, where prices of commodity ‘futures’ contracts - agreements to buy or sell a commodity at a future date - reflect market participants’ forecasts of commodity prices at various points in the future. Central banks often use these futures prices directly in forecasting commodity prices. Unfortunately, however, previous work has found that futures prices do not accurately predict future commodity prices. In a sense, this result is not too surprising, as in theory there are factors other than expectations of future commodity prices that can determine the prices of futures contracts. In particular, investors may require additional compensation or a ‘risk premium’ for the uncertainty around future prices, and prices of futures contracts will reflect this. We can think of a futures price as being made up of two components, the expected future price and the risk premium. These components cannot be observed separately.

In this paper, we develop an econometric model to estimate the expected future price and risk premium components embedded in futures prices. Specifically, we jointly model the US yield curve (i.e. the interest rates on US government bonds of different maturities), and the futures prices of two commodities: oil and gold, respectively. Until recently, models of interest rate and commodity markets have mostly been developed in isolation, and this separation may have been increasingly unjustified over time, as over the past decade or so, financial institutions have become more involved in commodity markets while maintaining a significant presence in interest rate markets. An attractive feature of our framework is that it allows for the potential interactions between these markets, that may result from this ‘financialization’ of commodity markets. We statistically test whether it is better to model these markets in isolation, or whether one should indeed allow for the markets to interact, and find evidence strongly in favour of a joint model of interest rates and commodities.

Within our model, ‘no-arbitrage’ relationships are enforced, meaning there are no risk-free profits that can be made in the bond and futures markets. The advantage of using this assumption is that it allows us to identify the risk premium and forecast components of futures prices in a robust theoretical framework. We find that there is a significant difference between the risk premium in oil futures and the premium in gold futures. On average, the risk premium is negative for oil contracts, while it is positive for gold. This suggests, as one might expect, that over time oil and gold have been perceived rather differently in financial markets. While the oil risk premium is negative over large parts of the period covered by our data, it follows an upward trend during the 2000s and recently turns positive. This behaviour could reflect the changing nature of the oil market over time, where the relative importance of demand and supply factors may have changed. In general, positive demand shocks are more likely to be associated with oil price increases, whereas negative supply shocks can imply oil price increases that put downward pressure on economic growth. To the extent that there have been large supply shocks in the oil market, an investor might actually have benefitted from holding oil, since the value of their holding would have increased in difficult times. Since investors prefer to hold assets that pay off in situations when their income is low (for example, a recession), they are willing to pay a

premium for this type of asset, and this would imply a negative risk premium. Previous evidence suggests that supply influences on the price of oil were more important in the past, though have diminished over time, where demand shocks have been more prominent recently. This is consistent with the value of holdings co-varying positively with the economic cycle in more recent times, and with market participants requiring a positive risk premium as a result. We estimate that the gold premium is mostly positive, indicating that in general investors require additional compensation for holding gold relative to US government bonds. This suggests that gold holdings are not perceived as providing better protection against economic downturns relative to US government bonds, where the value of gold generally falls in bad states of the world. This contrasts with the common portrayal of gold as an asset that offers a high level of protection against bad states of the world.

According to our estimates, the risk premium components of oil and gold futures prices can be relatively large. The importance of this component does appear to change over time, however, and this suggests that there are periods when futures prices may be a better forecast measure, and other times when they are not so reliable. Within our paper, we explore the behaviour of the premia further; by examining how risk premia change depending on different states of the economy, and under different financial market conditions. We find that risk premia vary depending on the level of economic activity and inflation. We also find that both oil and gold risk premia depend on the types of market participants holding futures contracts, where the balance of participants hedging risks, and speculating on commodity price movements, explains movements in premia over time.

1 Introduction

Commodity prices play an important role in the world economy, and can cause significant fluctuations in output and inflation. As a result, policymakers frequently examine developments in commodities markets, and consider prospects for the future path of commodity prices when forecasting output and inflation. Within this analysis, commodity futures contracts are often used to measure financial market expectations of commodity price movements. Central banks and other policy institutions commonly use the futures curve as a forecast of future oil prices (e.g. Nixon and Smith (2012)), and these contracts provide timely information about expected commodity price movements. However, the futures curve embodies a time-varying risk premium, which implies that the futures price does not represent an unbiased expectation of future spot prices.

In this paper, we jointly model the term structures of interest rates and commodity futures, and provide estimates of the risk premium in commodity futures contracts, using a framework originally developed for jointly pricing cross-country interest rates and foreign exchange forward rates (Backus, Foresi and Telmer (2001)). We extend this framework to commodities by considering synthetic oil and gold 'bonds', which have payoffs denominated in oil and gold units, respectively, analogous to considering government bonds denominated in different currencies. The advantage of modelling commodity prices in this way is that is allows us to use recently developed techniques in the term structure model literature. A similar methodology implemented alongside but independently of our work is Alquist, Bauer and Diez de los Rios (2014). They derive a similar set up for jointly modelling crude oil futures and the interest rate term structure, where they examine excess returns on bonds and oil, rather than the risk premium embedded in futures contracts, which is the focus of our study. Le and Zhu (2013) similarly apply term structure modelling techniques to the term structure of gold lease rates to estimate gold risk premia. A key difference of our study is that our model includes survey forecast data for interest rates, oil and gold prices in our estimation.[1] As shown in Kim and Wright (2005) and Kim and Orphanides (2012), using surveys can help anchor the model dynamics and provide a reliable way to obtain robust decompositions. Our model tests suggest that the joint model specifications that include survey data outperform models that do not include these data.

Traditionally, attempts to model interest rates and commodities have evolved separately

---

[1] Alquist, Bauer and Diez de los Rios (2014) model the term structure of interest rates and net convenience yields by following an approach similar to Adrian, Crump and Moench (2013). Despite the computational ease and tractability of this estimation approach, it is not straightforward to incorporate survey data within this framework.

from one another. Standard models of commodities typically assign only a minor role to interest rates, while models of the interest rate term structure tend not to consider the role of commodities at all. Modelling commodity markets in isolation may have become harder to justify over time, where there is evidence to suggest that the financialization of commodity markets may have induced co-movement with other financial markets (e.g. (Büyükşahin and Robe (2014)). While maintaining a standard assumption of no-arbitrage, our model introduces the explicit interaction between government bond and commodity markets. Our model can be applied to any commodity, though we focus on oil and gold.

Our results show clear links between US government bond and commodity markets, where we strongly reject models in which bond and commodity factors are unrelated, in favour of an unrestricted model. In particular, factors extracted from the US Treasury market play a significant role in the pricing of the commodity futures term structure, which is consistent with other empirical studies (e.g. Frankel (2008) and Rosa (2013)). We obtain estimates of expected future spot prices and risk premia (the difference between the futures price and the expected future spot price) and find that the risk premium in oil futures is negative for much of our sample, and turns positive during the 2000s. The change in the oil premium is consistent with changing roles of demand and supply shocks in the oil market, where supply shocks (which tend to push up on the price of oil and down on economic growth) have been less prominent in recent years (Kilian (2009), Baumeister and Peersman (2013)). The gold premium is mostly positive, and appears to display counter-cyclical behaviour. This indicates that investors require additional compensation for holding gold relative to risk-free government bonds, and suggests that gold is not seen as providing protection against bad states of the world, relative to US Treasuries. Our findings contrast with the common view that gold offers a high degree of insurance, in the sense that returns are countercyclical. Using simple regression analysis, we show that the oil and gold risk premium estimates vary over the business cycle and under changing market conditions. We also find a significant relationship between risk premia and a measure of hedging pressure, suggesting that the balance of long and short hedging activity is also an important determinant of commodity futures premia.

Crude oil is an important factor of production and can have a significant impact on global output and inflation. Oil market shocks and volatile oil prices have had significant economic effects and have influenced policy making (e.g. Hamilton (2011)). Gold has also been a closely followed commodity over time, not only due to its historical role in the international monetary system through the gold standard, but also due to its position in modern financial system as an important global reserve asset, financial collateral and investment asset (Le and Zhu (2013)). There is a vast literature on the oil market, where often the focus is on modelling the links between the oil market and real economic activity and price levels (e.g. Saporta, Trott and Tudela (2009), Kilian and Hicks (2013)). On the other hand, the theoretical and empirical literature on gold is small relative to the oil literature, but is growing. The gold literature mainly focuses on the role of gold as a hedge against inflation, and as an asset uncorrelated with financial assets in downturns (e.g. Erb and Harvey (2013), Baur and Lucey (2010) and Baur and McDermott (2010)).

There is a large literature that attempts to model spot and futures commodity prices, where the models can be broadly defined as either structural models or reduced-form time series models. The structural models are usually based on theories such as the 'theory of storage' (Working (1949) and Brennan (1958)) and 'hedging pressure' theory (Keynes (1923), Hicks (1939), Hirshleifer (1988)). The theory of storage argues that the slope of the term structure of commodity futures prices is mainly determined by the level of commodity inventories. According to this theory, a downward sloping futures curve (i.e. a backwardated market) is an indication of tight inventories, and an upward-sloping futures curve (i.e. a contango market) reflects a high level of inventories. Hedging pressure theory, on the other hand, focuses on the determination of optimal hedging and speculative positions for hedgers and speculators within the futures market. The hypothesis predicts futures prices should be lower than the expected spot price (i.e. a positive risk premium) if hedgers have net short positions within the market, and that futures prices should be higher than the expected spot price (i.e. a negative risk premium) if hedgers are net long.

Unlike structural models, reduced-form models focus on modelling the time series behaviour of commodity prices and their futures prices, and our models fall into this category. In contrast to many reduced-form time series models, however, our model is based on a dynamic stochastic framework where no-arbitrage conditions are imposed. Over the past few decades, various dynamic stochastic no-arbitrage models of commodities have been developed, most of which have focused on nominal commodity prices, as we do in this paper, rather than real commodity prices.[2] Earlier models focusing on oil (e.g. Brennan and Schwartz (1985)) made strong assumptions: (1) the convenience yield (the net benefit from holdings of a physical commodity) is a deterministic function of the oil spot price only; and (2) the risk-free short rate is constant. Gibson and Schwartz (1990) later relaxed the first assumption, modelling the convenience yield as a stationary stochastic variable. And Schwartz (1997) relaxed the second assumption by allowing the short rate to follow a stochastic process. Casassus and

---

[2] See Alquist, Kilian and Vigfusson (2013) for examples of time series econometric models for both nominal and real oil prices.

Collin-Dufresne (2005) advanced further still by introducing a more general model of commodities where convenience yields can be affected by both spot prices and interest rates, and risk premia are allowed to be time varying. However, all of these models assume convenience yields are affected by the level of spot prices, which has been an increasingly difficult assumption to justify given the non-stationary nature of commodity prices over the past decade alongside relatively stable convenience yields. The reduced-form model proposed in our paper removes the link between convenience yields and the level of spot prices.

The paper proceeds as follows. Section 2 sets outs out the theoretical foundations of the model, and describes the general affine term structure setup. Section 3 describes the data set and describes some preliminary analysis. Sections 4 and 5 respectively describe the estimation and empirical results from our preferred models.

2 A Joint Model of Bonds and Commodities

2.1 Pricing commodity futures using synthetic commodity bonds

The framework we use is analogous to the pricing of foreign exchange where we consider the relationship between asset prices denominated in two different currencies (see Backus, Foresi and Telmer (2001)). In our case, the two 'currencies' are US dollars and the physical unit of a commodity, where the spot dollar price of commodity is considered to be the 'exchange rate' between the two 'currencies'. We first consider the problem of pricing US dollar-denominated assets. The fundamental asset pricing equation is:

$$1 = E_t \left[M_{t+1}^{\$} R_{t+1}^{\$} \right]$$

where $R_{t+1}^{\$}$ is the gross nominal return from holding the asset between times t and $t+1$; and $M_{t+1}^{\$}$ is the nominal stochastic discount factor (SDF). Following the standard bond pricing literature, the n-period dollar zero-coupon bond price at time t is given as:

$$P_{t,n}^{\$} = E_t \left[\prod_{i=1}^{n} M_{t+i}^{\$} \right] = E_t \left[M_{t+1}^{\$} P_{t+1,n-1}^{\$} \right]$$

where the spot yield, $y_{t,n}^{\$}$, of a n-period dollar zero-coupon Treasury bond is given by:

$$y_{t,n}^{\$} = -\frac{1}{n} \ln P_{t,n}^{\$}$$

An agent invests in the commodity via a synthetic commodity bond, which promises to pay one unit of commodity at the end of the investment period, and the price of the commodity bond is denoted in commodity units. By investing in a commodity bond for one period, the agent will receive a gross return of R^*_{t+1} (again, in commodity units). From the point of view of a US investor, who can invest in both commodity and standard bonds, the gross return of this commodity bond must satisfy the following equation to rule out arbitrage opportunities:

$$1 = E_t \left[M^\$_{t+1} \frac{S_{t+1}}{S_t} R^*_{t+1} \right]$$

where S_t is the dollar price of the commodity at time t. We can rewrite this as:

$$1 = E_t \left[M^*_{t+1} R^*_{t+1} \right]$$

where the SDF for the commodity bond is given as:

$$M^*_{t+1} = M^\$_{t+1} \frac{S_{t+1}}{S_t} \tag{1}$$

The n-period commodity bond price, expressed in units of commodity, at time t is given as:

$$P^*_{t,n} = E_t \left[\prod_{i=1}^{n} M^*_{t+i} \right] = E_t \left[M^*_{t+1} P^*_{t+1,n-1} \right]$$

Equation (1) suggests that the dynamics of the dollar commodity price S_t are driven by both the dollar SDF and the commodity SDF. This is analogous to the pricing of foreign exchange as developed in Backus, Foresi and Telmer (2001). We can re-arrange (1), and solve forward for the spot oil price at time $t+n$:

$$S_{t+n} = S_t \prod_{i=1}^{n} \frac{M^*_{t+i}}{M^\$_{t+i}}$$

The synthetic commodity bond can be constructed using a nominal bond and a nominal forward contract, both of the same maturity. The exact trade is as follows: the agent sells one unit of commodity (with price S_t) and invests the proceeds in $(S_t/P^\$_{t,n})$ units of an n-period nominal zero-coupon bond (with price $P^\$_{t,n}$). At the same time (i.e. time t), the agent enters a forward contract with the same n-period maturity, which allows the agent to buy the commodity at a pre-determined price ($F_{t,n}$). In n-periods time, the agent will receive $S_t/(F_{t,n} \times P^\$_{t,n})$ units of the commodity. By entering the trade, the agent is effectively buying an n-period synthetic commodity bond which promises to pay one unit of commodity for every $(F_{t,n} \times P^{\$}_{t,n})/S_t$ units of the commodity at the onset of the bond. The current price of the synthetic commodity bond that pays one unit of the commodity is therefore given by:

$$P^{*}_{t,n} = (F_{t,n} \times P^{\$}_{t,n})/S_t$$

which gives the following expression for the forward price:

$$F_{t,n} = S_t \frac{P^{*}_{t,n}}{P^{\$}_{t,n}}$$

The yield on this commodity bond (denoted as $y^{*}_{t,n}$) follows the well-known definition of the 'convenience yield' on a commodity futures contract:

$$y^{*}_{t,n} = -\frac{1}{n} \ln P^{*}_{t,n} = y^{\$}_{t,n} - (\ln F_{t,n} - \ln S_t)/n$$

The convenience yield represents the net benefit associated with physical holdings of a given commodity, and if this yield is positive, it discounts the forward price relative to the current spot price. An example of this is where an oil refiner would prefer to hold oil physically than to wait for oil to be delivered in the future, because holding stocks of oil protects against increases in demand and the risk of shortages. If the benefits of owning physical oil exceed storage and financing costs, the forward price must be lower to encourage the refinery plant to enter into the contract.

Following Fama (1984), the commodity forward-spot basis (or 'forward discount'), i.e. the log difference between the forward price and the current spot price, can be decomposed into the expected change in the spot price (the first term on the right-hand-side of (2)) and a risk premium (the second term):

$$f_{t,n} - s_t = (E_t[s_{t+n}] - s_t) - (E_t[s_{t+n}] - f_{t,n}) \tag{2}$$

where $f_{t,n} = \ln F_{t,n}$ and $s_t = \ln S_t$. The annualised expected oil price change over n periods (using 1-month periods, as in our application below) at time t is denoted by $q_{t,n}$, and annualised risk premium for the n-period forward-spot basis at time t denoted by $\rho_{t,n}$:

$$\begin{aligned} q_{t,n} &= \frac{12}{n}(E_t[s_{t+n}] - s_t) \\ \rho_{t,n} &= \frac{12}{n}(E_t[s_{t+n}] - f_{t,n}) \end{aligned}$$

Our interpretation and discussions of risk premia are based on the inverse of Fama's definition of the risk premium. This definition has the more natural interpretation as the expected excess return from the transaction where an investor is short US dollars forward and long the commodity price, and aligns with the hedging pressure theory literature (e.g. Hirshleifer (1988) and Bessembinder (1992)).[3] While this section derives the price of a commodity forward contract, we use futures contracts within our empirical implementation. There are conceptual differences between commodity forwards and futures, though the differences in price have been shown to be economically small (see Chow, McAleer and Sequeira (2000)).

2.2 Joint affine term structure model of Treasury bonds and commodity futures

We model dollar US Treasury yields and convenience yields using a joint affine term structure model. As is standard in these models, we assume that the one-period dollar risk-free interest rate ($r_t^\$$) is an affine function of a $K^\$ \times 1$ vector of unobserved factors $\mathbf{x}_t^\$$:

$$r_t^\$ = \mathbf{j}^\$ \mathbf{x}_t^\$, where : \mathbf{j}^\$ = [1, 1, ..., 1]$$

Similarly we assume that the convenience yield on a one-period commodity bond (r_t^*) is affine in a $K^* \times 1$ vector $\mathbf{x}_t^*$:

$$r_t^* = \mathbf{j}^* \mathbf{x}_t^* where : \mathbf{j}^* = [1, 1, ..., 1]$$

We stack $\mathbf{x}_t^\$$ and $\mathbf{x}_t^*$ to form $\mathbf{x}_t = [\mathbf{x}_t^{\$\prime}, \mathbf{x}_t^{*\prime}]'$, which follows real-world dynamics described by:

$$\begin{aligned} \mathbf{x}_{t+1} &= \boldsymbol{\kappa} + \boldsymbol{\Phi} \mathbf{x}_t + \boldsymbol{\Sigma} \boldsymbol{\varepsilon}_{t+1} \\ \boldsymbol{\varepsilon}_{t+1} &\sim \mathcal{N}(\mathbf{0}, \mathbf{I}) \end{aligned} \tag{3}$$

where $\boldsymbol{\kappa}$ is a $K \times 1$ matrix, $\boldsymbol{\Phi}$ is a $K \times K$ matrix and $\boldsymbol{\Sigma}$ is a $K \times K$ lower triangular matrix. The risk-neutral ($\mathbb{Q}$) dynamics are given by:

$$\begin{aligned} \mathbf{x}_{t+1} &= \boldsymbol{\kappa}^{\mathbb{Q}} + \boldsymbol{\Phi}^{\mathbb{Q}} \mathbf{x}_t + \boldsymbol{\Sigma} \boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}} \\ \boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}} &\sim \mathcal{N}(\mathbf{0}, \mathbf{I}) \end{aligned} \tag{4}$$

Following Joslin, Singleton and Zhu (2011), we set

---

[3] The definition of the risk premium was similarly reversed for interpreting FX forward premia in Hodrick and Srivastava (1986).

$$\begin{aligned}\boldsymbol{\kappa}^{\mathbb{Q}} &= [\kappa_{\infty}^{\$\mathbb{Q}},\ \underbrace{0...0}_{(K^{\$}-1)},\kappa_{\infty}^{*\mathbb{Q}},\ \underbrace{0...0}_{(K^{*}-1)}]' \\ \mathbf{\Phi}^{\mathbb{Q}} &= diag[(1-\xi_1),(1-\xi_2),...(1-\xi_{K^{\$}+K^{*}})].\end{aligned}$$

Following Duffee (2002), we specify the nominal and commodity SDFs as

$$\begin{aligned}M_{t+1}^{\$} &= \exp\left(-r_t^{\$}-\frac{1}{2}\mathbf{\Lambda}_t'\mathbf{\Sigma}\mathbf{\Sigma}'\mathbf{\Lambda}_t-\mathbf{\Lambda}_t'\mathbf{\Sigma}\boldsymbol{\varepsilon}_{t+1}\right) \\ M_{t+1}^{*} &= \exp\left(-r_t^{*}-\frac{1}{2}\mathbf{\Lambda}_t'\mathbf{\Sigma}\mathbf{\Sigma}'\mathbf{\Lambda}_t-\mathbf{\Lambda}_t'\mathbf{\Sigma}\boldsymbol{\varepsilon}_{t+1}\right)\end{aligned}$$

where $\mathbf{\Lambda}_t$ is the price of risk for the innovation term $\boldsymbol{\varepsilon}_{t+1}$. As shown earlier, under the assumption of no arbitrage, the price of an n-period zero-coupon dollar bond at time t must be equal to the expected present value of an $(n-1)$-period bond at time $t+1$, discounted using $M_{t+1}^{\$}$. To simplify the calculation, we rewrite the no-arbitrage relation for both nominal Treasury and commodity bonds under the risk-neutral measure $\mathbb{Q}$:

$$P_{t,n}^{\$}=E_t^{\mathbb{Q}}\left[\exp\left(-r_t^{\$}\right)P_{t+1,n-1}^{\$}\right]$$

$$P_{t,n}^{*}=E_t^{\mathbb{Q}}\left[\exp\left(-r_t^{*}\right)P_{t+1,n-1}^{*}\right]$$

The vector of factors, $\mathbf{x}_t$, that drives both $r_t^{\$}$ and r_t^{*}, is assumed to follow a first-order Gaussian VAR under the risk-neutral measure as shown in (4). Given these assumptions, we can show that the price of an n-period bond is an exponential-affine function of $\mathbf{x}_t$:

$$P_{t,n}^{\$}=\exp\left(a_n^{\$}+\mathbf{b}_n^{\$}\mathbf{x}_t\right)$$

where the scalar $a_n^{\$}$, and the $1\times K$ vector $\mathbf{b}_n^{\$}$, follow the recursive equations:

$$a_n^{\$} = a_{n-1}^{\$}+\mathbf{b}_{n-1}^{\$}\boldsymbol{\kappa}^{\mathbb{Q}}+\frac{1}{2}\mathbf{b}_{n-1}^{\$}\mathbf{\Sigma}\mathbf{\Sigma}'\mathbf{b}_{n-1}^{\$} \qquad (5)$$

$$\mathbf{b}_n^{\$} = -\boldsymbol{\delta}^{\$}+\mathbf{b}_{n-1}^{\$}\mathbf{\Phi}^{\mathbb{Q}} \qquad (6)$$

where $\boldsymbol{\delta}^{\$}=[\mathbf{j}^{\$},0,...,0]$ is a $1\times$ K vector. Given that the price of a zero-coupon bond at maturity is equal to one, we can start these recursions with the boundary conditions $a_0^{\$}=0$ and $\mathbf{b}_0^{\$}=\mathbf{0}$. Given that $\mathbf{\Phi}^{\mathbb{Q}}$ is a diagonal matrix and $\boldsymbol{\delta}^{\$}$ only has non-zero values for the first $K^{\$}$ elements, the $\mathbf{b}_n^{\$}$ vector will have non-zero values only for the first $K^{\$}$ elements and zeros for the rest. Therefore, the US bond price $P_{t,n}^{\$}$ is only affected by $\mathbf{x}_t^{\$}$. The yield-to-maturity of an n-period dollar bond is defined as:

$$y^{\$}_{t,n} = -\frac{1}{n}\log P^{\$}_{t,n} = -\frac{1}{n}\left(a^{\$}_{n} + \mathbf{b}^{\$}_{n}\mathbf{x}_{t}\right) \tag{7}$$

Similarly the price of an n-period commodity bond is also an exponential affine function of $\mathbf{x}_t$:

$$P^{*}_{t,n} = \exp\left(a^{*}_{n} + \mathbf{b}^{*}_{n}\mathbf{x}_{t}\right)$$

where the scalar a^{*}_{n} and $1\times\ K$ vector $\mathbf{b}^{*}_{n}$ can be defined recursively in an analogous way to (5) and (6), where we simply replace $\boldsymbol{\delta}^{\$}$ with $\boldsymbol{\delta}^{*} = [0, ..., 0, \mathbf{j}^{*}]$. The n-period convenience yield is defined as:

$$y^{*}_{t,n} = -\frac{1}{n}\log P^{*}_{t,n} = -\frac{1}{n}\left(a^{*}_{n} + \mathbf{b}^{*\prime}_{n}\mathbf{x}_{t}\right) \tag{8}$$

where $\mathbf{b}^{*}_{n}$ will have non-zero values only for the last K^{*} elements and zeros for the rest, since $\mathbf{x}^{\$}_{t}$ does not enter the short convenience yield equation and has no impact on future $\mathbf{x}^{*}_{t}$ under the risk neutral measure. Therefore, the commodity bond price $P^{*}_{t,n}$ is only affected by $\mathbf{x}^{*}_{t}$. Hence equations (7) and (8) show that the bond factors $(\mathbf{x}^{\$}_{t})$ are unspanned factors for the commodity convenience yield model while the commodity factors $(\mathbf{x}^{*}_{t})$ are unspanned factors for the bond yield model.

2.3 Joslin, Singleton and Zhu (2011) transformation

This subsection carries out the transformation proposed by Joslin, Singleton and Zhu (2011). The purpose of the transformation is to separate the estimation of real world and risk-neutral dynamics, so that the risk-neutral parameters can be expressed as functions of the parameters in a more parsimonious risk-neutral model. Let $\mathbf{z}_t$ denote the portfolio factors which include the first $K^{\$}$ principal components (PCs) of the nominal Treasury yield curves, and the first K^{*} principal components of the convenience yield curves. Given the principal component loadings, we have

$$\mathbf{z}_t = \begin{pmatrix} \mathbf{z}^{\$}_{t} \\ \mathbf{z}^{*}_{t} \end{pmatrix} = \begin{pmatrix} \mathbf{G}^{\$} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}^{*} \end{pmatrix} \begin{pmatrix} \mathbf{y}^{\$}_{t} \\ \mathbf{y}^{*}_{t} \end{pmatrix} = \mathbf{G}\mathbf{y}_t = \begin{pmatrix} \mathbf{G}^{\$}\mathbf{y}^{\$}_{t} \\ \mathbf{G}^{*}\mathbf{y}^{*}_{t} \end{pmatrix}$$

where $\mathbf{G}^{\$}$ is the loading matrix for computing the Treasury yield curve PCs $(\mathbf{z}^{\$}_{t})$; $\mathbf{G}^{*}$ is the loading matrix for the convenience yield PCs $(\mathbf{z}^{*}_{t})$; $\mathbf{y}^{\$}_{t}$ is a vector of the US government bond yields of different maturities at time t; $\mathbf{y}^{*}_{t}$ is a vector of the convenience yields of different maturities at time t. Given that $\mathbf{z}^{\$}_{t} = \mathbf{G}^{\$}\mathbf{y}^{\$}_{t}$ and $\mathbf{y}^{\$}_{t}$ is only affected by $\mathbf{x}^{\$}_{t}$, we can write $\mathbf{z}^{\$}_{t}$ as a function of $\mathbf{x}^{\$}_{t}$. Similarly, we can write $\mathbf{z}^{*}_{t}$ as a function of $\mathbf{x}^{*}_{t}$. We can therefore write

$\mathbf{z}_t$ as an affine function of the state vector $\mathbf{x}_t$:

$$\mathbf{z}_t = \mathbf{a}_g + \mathbf{B}_g \ \mathbf{x}_t$$

where $\mathbf{a}_g = \mathbf{G} \cdot \mathbf{a}$, $\mathbf{B}_g = \mathbf{G} \cdot \mathbf{B}$ and $\mathbf{a}$ and $\mathbf{B}$ are given as:

$$\begin{aligned} \mathbf{a} &= \left(\frac{-1}{n_1} a^{\$}_{n_1}, ..., \frac{-1}{n_{K^{\$}}} a^{\$}_{n_{K^{\$}}}, \frac{-1}{n_1} a^{*}_{n_1}, ..., \frac{-1}{n_{K^{*}}} a^{*}_{n_{K^{*}}} \right)' \\ \mathbf{B} &= \left(\frac{-1}{n_1} \mathbf{b}^{\$\prime}_{n_1}, ..., \frac{-1}{n_{K^{\$}}} \mathbf{b}^{\$\prime}_{n_{K^{\$}}}, \frac{-1}{n_1} \mathbf{b}^{*\prime}_{n_1}, ..., \frac{-1}{n_{K^{*}}} \mathbf{b}^{*\prime}_{n_{K^{*}}} \right)' \end{aligned}$$

The real-world dynamics of $\mathbf{z}_t$ are given as:

$$\begin{aligned} \mathbf{z}_{t+1} &= \boldsymbol{\kappa}^{z} + \boldsymbol{\Phi}^{z} \mathbf{z}_t + \mathbf{w}^{z}_{t+1} \\ \mathbf{w}^{z}_{t+1} &\sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}^{z} \boldsymbol{\Sigma}^{z\prime}\right) \end{aligned}$$

where the parameters are linked to those in (3) as follows:

$$\boldsymbol{\kappa}^{z} = \mathbf{B}_g \boldsymbol{\kappa} + \mathbf{a}_g - \boldsymbol{\Phi}^{z} \mathbf{a}_g; \boldsymbol{\Phi}^{z} = \mathbf{B}_g \boldsymbol{\Phi} \mathbf{B}_g^{-1}; \boldsymbol{\Sigma}^{z} = \mathbf{B}_g \boldsymbol{\Sigma}$$

The dynamics of $\mathbf{z}_t$ under $\mathbb{Q}$ are given as:

$$\begin{aligned} \mathbf{z}_{t+1} &= \boldsymbol{\kappa}^{z\mathbb{Q}} + \boldsymbol{\Phi}^{z\mathbb{Q}} \mathbf{z}_t + \mathbf{w}^{z\mathbb{Q}}_{t+1} \\ \mathbf{w}^{z}_{t+1} &\sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}^{z} \boldsymbol{\Sigma}^{z\prime}\right) \end{aligned}$$

where the parameters can be inferred from those in (4).

$$\boldsymbol{\kappa}^{z\mathbb{Q}} = \mathbf{B}_g \boldsymbol{\kappa}^{\mathbb{Q}} + \mathbf{a}_g - \boldsymbol{\Phi}^{z\mathbb{Q}} \mathbf{a}_g; \boldsymbol{\Phi}^{z\mathbb{Q}} = \mathbf{B}_g \boldsymbol{\Phi}^{\mathbb{Q}} \mathbf{B}_g^{-1}; \boldsymbol{\Sigma}^{z} = \mathbf{B}_g \boldsymbol{\Sigma}$$

The new short rate and short convenience yield equations are given as:

$$\begin{aligned} r^{\$}_t &= \delta^{z\$}_0 + \boldsymbol{\delta}^{z\$}_1 \mathbf{z}_t \\ r^{*}_t &= \delta^{z*}_0 + \boldsymbol{\delta}^{z*}_1 \mathbf{z}_t \end{aligned}$$

where the $\boldsymbol{\delta}^{z\$}_1$ vector has non-zero values only for the first $K^{\$}$ elements and zeros for the rest while the $\boldsymbol{\delta}^{z*}_1$ vector has non-zero values only for the last K^{*} elements and zeros for the rest. Given the no-arbitrage conditions, we can derive the following general pricing model for Treasury bonds and convenience yields with regard to $\mathbf{z}_t$:

$$y^{(i)}_{t,n} = -\frac{1}{n}\left(a^{z(i)}_n + \mathbf{b}^{z(i)\prime}_n \mathbf{z}_t\right)$$

where

$$\begin{aligned}
a^{z(i)}_n &= a^{z(i)}_{n-1} + \mathbf{b}^{z(i)}_{n-1}\boldsymbol{\kappa}^{z\mathbb{Q}} + \frac{1}{2}\mathbf{b}^{z(i)}_{n-1}\boldsymbol{\Sigma}^{z}\boldsymbol{\Sigma}^{z\prime}\mathbf{b}^{z(i)}_{n-1} - \delta^{z(i)}_0 \\
\mathbf{b}^{z(i)}_n &= \mathbf{b}^{z(i)}_{n-1}\boldsymbol{\Phi}^{z\mathbb{Q}} - \boldsymbol{\delta}^{z(i)}_1
\end{aligned}$$

with $a^{z(i)}_0 = 0$, and $\mathbf{b}^{z(i)}_0 = \mathbf{0}$. We replace (i) with $(\$)$ for bond yields and $(*)$ for convenience yields. For the purposes of estimation, we assume that both bond yields and convenience yields are measured with independent and Normally distributed errors:

$$\begin{aligned}
y^{\$}_{t,n} &= -\frac{1}{n}\left(a^{z\$}_n + \mathbf{b}^{z\$\prime}_n \mathbf{z}_t\right) + w^{z\$}_{t,n}, w^{z\$}_{t,n} \sim \mathcal{NID}\left(0, \sigma^2_{n,\$}\right) && (9) \\
y^{*}_{t,n} &= -\frac{1}{n}\left(a^{z*}_n + \mathbf{b}^{z*\prime}_n \mathbf{z}_t\right) + w^{z*}_{t,n}, w^{z*}_{t,n} \sim \mathcal{NID}\left(0, \sigma^2_{n,*}\right) && (10)
\end{aligned}$$

2.4 Commodity spot prices and survey forecasts

Equation (1) implies that the log change in the spot oil price is given by:

$$\Delta s_{t+1} = \ln(S_{t+1}/S_t) = \ln M^{*}_{t+1} - \ln M^{\$}_{t+1} = r^{\$}_t - r^{*}_t \tag{11}$$

which can be rewritten as an affine function of the factors:

$$\Delta s_{t+1} = \delta^{z\$}_0 + \boldsymbol{\delta}^{z\$}_1\mathbf{z}_t - (\delta^{z*}_0 + \boldsymbol{\delta}^{z*}_1\mathbf{z}_t) \tag{12}$$

For the purposes of estimation, we assume that changes in the commodity spot price are observed with independent and Normally distributed errors:

$$\begin{aligned}
\Delta s_{t+1} &= \delta^{z\$}_0 - \delta^{z*}_0 + (\boldsymbol{\delta}^{z\$}_1 - \boldsymbol{\delta}^{z*}_1)\mathbf{z}_t + w^{s}_t \\
w^{s}_t &\sim \mathcal{NID}\left(0, \sigma^2_s\right)
\end{aligned}$$

In our benchmark model, we also include survey data on expectations of future bond yields and oil spot prices. Using (11) and (12), the expected change in the commodity spot price in h periods is given as

$$S^{e}_{t,h} = E_t(S_{t+h}/S_t) = E_t\left[S^{e}_{t+1,h-1}\exp\left(\delta^{z\$}_0 - \delta^{z*}_0 + (\boldsymbol{\delta}^{z\$}_1 - \boldsymbol{\delta}^{z*}_1)\mathbf{z}_t\right)\right]$$

Following the standard bond formula, we derive the log of the expected commodity price changes as

$$\ln S^{e}_{t,h} = a^{z,s}_{h} + \mathbf{b}^{z,s\prime}_{h}\mathbf{z}_{t}$$

where

$$\begin{aligned} a^{z,s}_{h} &= a^{z,s}_{h-1} + \mathbf{b}^{z,s}_{h-1}\boldsymbol{\kappa}^{z} + \frac{1}{2}\mathbf{b}^{z,s}_{h-1}\boldsymbol{\Sigma}^{z}\boldsymbol{\Sigma}^{z\prime}\mathbf{b}^{z,s}_{h-1} - (\delta^{z*}_{0} - \delta^{z\$}_{0}) \\ \mathbf{b}^{z,s}_{h} &= \mathbf{b}^{z,s}_{h-1}\boldsymbol{\Phi}^{z} - (\boldsymbol{\delta}^{z*}_{1} - \boldsymbol{\delta}^{z\$}_{1}) \end{aligned} \tag{13}$$

with $a^{z,s}_{0} = 0$, and $\mathbf{b}^{z,s}_{0} = \mathbf{0}$. We use survey data to approximate the expected commodity spot price in the future, which are observed with independent and Normally distributed errors:

$$\begin{aligned} \ln S^{e}_{t,h} &= a^{z,s}_{h} + \mathbf{b}^{z,s\prime}_{h}\mathbf{z}_{t} + w^{s}_{t,h} \\ w^{s}_{t,h} &\sim \mathcal{NID}\left(0, \sigma^{2}_{s,h}\right) \end{aligned}$$

For bond yields, we include survey expectations of future three-month yields (further details on the data set used are provided in Section 3). It is straightforward to show that the h-period ahead expectation of the three-month yield is the affine function of $\mathbf{z}_{t}$:

$$y^{e}_{t+h|t,3} = E_{t}\left[y_{t+h,3}\right] = -\frac{1}{3}\left(a^{z\$}_{3} + \mathbf{b}^{z\$\prime}_{3}\left(\mathbf{I} - \boldsymbol{\Phi}^{z}\right)^{-1}\left(\mathbf{I} - \left(\boldsymbol{\Phi}^{z}\right)^{h}\right)\boldsymbol{\kappa}^{z} + \mathbf{b}^{z\$\prime}_{3}\left(\boldsymbol{\Phi}^{z}\right)^{h}\mathbf{z}_{t}\right)$$

which can be rewritten as

$$y^{e}_{t+h|t,3} = a^{e\$}_{h} + \mathbf{b}^{e\$\prime}_{h}\mathbf{z}_{t} + w^{ey}_{t,h}$$

where

$$\begin{aligned} a^{e\$}_{h} &= -\frac{1}{3}\left(a^{z\$}_{3} + \mathbf{b}^{z\$\prime}_{3}\left(\mathbf{I} - \boldsymbol{\Phi}^{z}\right)^{-1}\left(\mathbf{I} - \left(\boldsymbol{\Phi}^{z}\right)^{h}\right)\boldsymbol{\kappa}^{z}\right) \\ \mathbf{b}^{e\$\prime}_{h} &= -\frac{1}{3}\left(\mathbf{b}^{z\$\prime}_{3}\left(\boldsymbol{\Phi}^{z}\right)^{h}\right) \end{aligned}$$

where we assume that the survey forecasts of 3-month yields over h horizon, $y^{e}_{t+h|t,3}$, are measured with a Normal and independently distributed error term $w^{ey}_{t,h} \sim \mathcal{NID}\left(0, \sigma^{2}_{ey,h}\right)$.

2.5 Model specification

We can summarise the models for US Treasury yields, commodity convenience yields, commodity prices and survey expectations into the following state-space system:

$$\begin{aligned} \mathbf{z}_{t+1} &= \boldsymbol{\kappa}^{z} + \boldsymbol{\Phi}^{z}\mathbf{z}_{t} + \mathbf{w}^{z}_{t+1}, \mathbf{w}^{z}_{t+1} \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}^{z}\boldsymbol{\Sigma}^{z\prime}\right) \\ \mathbf{y}_{t} &= \mathbf{a} + \mathbf{B}\mathbf{z}_{t} + \mathbf{w}^{y}_{t}, \mathbf{w}^{y}_{t}\tilde{\ }\mathcal{N}(\mathbf{0}, \boldsymbol{\Omega}) \end{aligned}$$

where

$$\mathbf{y}_t = \left(y^{\$}_{t,n\_1}...y^{\$}_{t,n\_N^{\$}}, y^{*}_{t,n\_1}...y^{*}_{t,n\_N^{*}}, \Delta s_{t+1}, y^{e}_{t+h\_1|t,3}...y^{e}_{t+h\_Hy|t,3}, \ln S^{e}_{t,h\_1}... \ln S^{e}_{t,h\_Hs}\right)$$

$$\mathbf{a}{=}\left(-\tfrac{1}{n\_1}a^{z\$}_{n\_1}... - \tfrac{1}{n\_N^{\$}}a^{z\$}_{n\_N^{\$}}, -\tfrac{1}{n\_1}a^{z*}_{n\_1}, -\tfrac{1}{n\_N^{*}}a^{z*}_{n\_N^{*}}...\delta^{z\$}_{0} - \delta^{z*}_{0}, a^{e\$}_{h\_1}...a^{e\$}_{h\_Hy}, a^{z,s}_{h\_1}...a^{z,s}_{h\_Hs}\right)$$

$$\boldsymbol{\Omega}{=}diag\left(\sigma^{2}_{n\_1,\$}...\sigma^{2}_{n\_N^{\$},\$}, \sigma^{2}_{n\_1,*}...\sigma^{2}_{n\_N^{*},*}, \sigma^{2}_{s}, \sigma^{2}_{ey,h\_1}...\sigma^{2}_{ey,h\_Hy}, \sigma^{2}_{s,h\_1}...\sigma^{2}_{s,h\_Hs}\right)$$

$$\mathbf{B}{=}\left(-\tfrac{1}{n\_1}\mathbf{b}^{z\$\prime}_{n\_1}... - \tfrac{1}{n\_N^{\$}}\mathbf{b}^{z\$\prime}_{n\_N^{\$}}, -\tfrac{1}{n\_1}\mathbf{b}^{z*\prime}_{n\_1}... - \tfrac{1}{n\_N^{*}}\mathbf{b}^{z*\prime}_{n\_N^{*}}, (\boldsymbol{\delta}^{z\$}_{1} - \boldsymbol{\delta}^{z*}_{1}), \mathbf{b}^{e\$\prime}_{h\_1}...\mathbf{b}^{e\$\prime}_{h\_Hy}, \mathbf{b}^{z,s\prime}_{h\_1}...\mathbf{b}^{z,s\prime}_{h\_Hs}\right)$$

$$\mathbf{w}^{y}_{t} = \left(w^{z\$}_{t,n\_1}...w^{z\$}_{t,n\_N^{\$}}, w^{z*}_{t,n\_1}...w^{z*}_{t,n\_N^{*}}, w^{s}_{t}, w^{ey}_{t,h\_1}...w^{ey}_{t,h\_Hy}, w^{s}_{t,h\_1}...w^{s}_{t,h\_Hs}\right)$$

$$\boldsymbol{\Omega}{=}diag\left(\sigma^{2}_{z\_n1,\$}...\sigma^{2}_{z\_nN^{\$},\$}, \sigma^{2}_{z,*}...\sigma^{2}_{z,*}, \sigma^{2}_{s}, \sigma^{2}_{ey,h\_1}...\sigma^{2}_{ey,h\_Hy}, \sigma^{2}_{s,h\_1}...\sigma^{2}_{s,h\_Hs}\right)$$

Here, the number of dollar yields is denoted as $N^{\$}$, the number of convenience yields denoted by N^{*}, the number of 3-month yield surveys is Hy and the number of commodity surveys is Hs. We define the parameter set of the above system as Θ, which includes the following parameters: $\boldsymbol{\kappa}^{z}$, $\boldsymbol{\Phi}^{z}$, $\boldsymbol{\kappa}^{\mathbb{Q}}$, $\boldsymbol{\Phi}^{\mathbb{Q}}$, $\boldsymbol{\Sigma}$ and $\boldsymbol{\Omega}$. The risk-neutral parameters $\boldsymbol{\kappa}^{\mathbb{Q}}$, $\boldsymbol{\Phi}^{\mathbb{Q}}$, and $\boldsymbol{\Sigma}$ as defined in (4) are already modelled in a parsimonious manner and do not need any further constraints. For the real-world dynamics, the parameter $\boldsymbol{\Phi}^{z}$ matrix can be further constrained to reflect different model specifications. Recall that $\mathbf{z}_t$ consists of two types of factors: the bond factor vector $\mathbf{z}^{\$}_{t}$ and the commodity factor vector $\mathbf{z}^{\$}_{t}$. The $\boldsymbol{\Phi}^{z}$ matrix can be partitioned into four parts:

$$\boldsymbol{\Phi}^{z} = \begin{pmatrix} \boldsymbol{\Phi}^{z}_{11} & \boldsymbol{\Phi}^{z}_{12} \\ \boldsymbol{\Phi}^{z}_{21} & \boldsymbol{\Phi}^{z}_{22} \end{pmatrix}$$

where $\boldsymbol{\Phi}^{z}_{11}$ shows how the lagged bond factors affect the current bond factors, $\boldsymbol{\Phi}^{z}_{12}$ shows how the lagged commodity factors affect the current bond factors, $\boldsymbol{\Phi}^{z}_{21}$ shows how the lagged bond factors affect the current commodity factors, and $\mathbf{\Phi}^{z}_{22}$ shows how the lagged commodity factors affect current commodity factors. The benchmark model also includes survey forecasts, where we later perform tests comparing models with and without survey data.

3 Data and Preliminary Analysis

We use monthly observations of zero-coupon US Treasury yields from Gurkaynak, Sack and Wright (2007), for maturities of 12, 18, 24, 36, 60, 84, and 120 months. Spot yields for constant maturity bonds for 1, 3, 6, and 9 months are from the U.S. Department of the Treasury website. The sample covers April 1983 to October 2013. All yields are annualised.

For oil, we use West Texas Intermediate (WTI) futures contracts provided by the Chicago Mercantile Exchange (CME). These contracts represent agreements to buy or sell 1000 barrels of WTI crude oil at a pre-determined date in the future. We use mid-month futures prices to account for trading, settlement and delivery conventions.[4] The sample length varies according to contract maturity – contracts up to twelve months are available from 1984 onwards, while longer dated contracts up to 24 months are available from 1990 onwards. For gold, futures prices are also provided by CME. Each contract represents the agreement to buy or sell 100 troy ounces of gold. As with the WTI futures contracts, there are certain trading and settlement conventions associated with the gold futures contracts, and we again use mid-month prices to minimise these issues. Futures contracts with maturities up to 24 months are available from April 1983 onwards. We use contracts for both oil and gold futures with maturities of 1, 3, 6, 9, 12, 18, and 24 months. Oil and gold futures prices are shown in Figure 1.

Our baseline models also include survey data provided by Consensus Economics, who survey over 250 professional forecasters views on a range of macroeconomic variables. These include the 3-month and 1-year ahead forecasts of the 3-month Treasury bill rate, available at monthly frequency from 1989. And for oil and gold, we use forecasts of the 3-, 12- and 24-month ahead spot prices, available at monthly frequency from 1998.[5] We use the mean forecast values, which refer to the average forecast across survey respondents.

---

[4] For example, WTI futures contracts cease trading in the month prior to the listed delivery month (e.g. the June 2014 contract stops trading in May 2014). Taking futures prices at mid-month ensures that contracts are still being traded. For consistency, we also take mid-month values for Treasury yields.

[5] The shorter sample period for the surveys affects the estimation of the factors in the model, but does not impact the estimation of their dynamics.

3.1 Convenience Yields

We calculate convenience yields implied by the standard no-arbitrage relationship between spot and futures prices as shown in (8), which shows that the futures price (F_t) is equal to the spot price (S_t) adjusted for the cost of borrowing ($y^{\$}_{t,n}$), and the convenience yield ($y^{*}_{t,n}$) associated with holding the physical commodity over the life of the futures contract.

In order to calculate implied convenience yields as described by equation (8), we need spot prices of oil and gold. However, at least for the oil market, spot prices are essentially unobserved.[6] A common approach in related studies is to use the price of the futures contract closest to expiry as the spot price. We instead follow Gibson and Schwartz (1990) and proxy the commodity spot price by assuming that the convenience yield is constant across the shorter maturity futures contracts. Dropping the t subscripts on yields and future prices for ease of notation, we denote the one-period forward convenience yield n periods ahead as f^{*}_{n}, and the one-period forward Treasury yield n periods ahead as $f^{\$}_{n}$:

$$f^{*}_{n} = f^{\$}_{n} - 12. \ln \left[\frac{F_n}{F_{n-1}} \right] \tag{14}$$

Using (14), we can obtain f^{*}_{2} (the one-month convenience yield, one-month forward) from $f^{\$}_{2}$, F_2 and F_1. Given f^{*}_{2} and assuming it is equal to y^{*}_{1} (i.e. the one-period convenience yield), we can use the above equation to solve for a 'synthetic' spot price (i.e. F_0) as $f^{\$}_{1}$ and F_1 are already known. For consistency, we calculate a synthetic spot price for both oil and gold.

3.2 Descriptive Statistics

Descriptive statistics for bond, oil and gold yields are provided in Table 1. The average convenience yield for oil is substantially higher than for gold, a difference of almost 7 percentage points at the 3-month horizon. The higher oil convenience yield is consistent with a higher production value associated with oil and the value of holding oil inventories. The lower gold convenience yield is consistent with its lesser role in production, and that the fact that the total stock of gold is very large compared to its consumption. The oil convenience yield is also far more volatile than for gold.

---

[6] Logistical issues in the transportation of oil prevent spot transactions for immediate delivery. 'Spot' transactions often more closely resemble forward contracts. See Fattouh (2011).

3.3 Principal Component Analysis

We use a preliminary principal component analysis to identify the number of factors that explain the bond, oil and gold term structures. Table 2 shows the percentage of variance explained by the first five principal components, for the three term structures. The results for bond yields are well-known: the first three principal components describe around 99.95 percent of the variation in yields, and the first component captures a large part of this explained variation. In the literature that focuses on the bond yield term structure, is it common practice to include three or more factors in the models. Since our model specification is somewhat broader, we need to strike a balance between model fit and ease of estimation. The term structures of oil and gold convenience yields are almost entirely explained by three factors. For oil convenience yields, the first two components explain over 99.8 percent of the variance of the term structure, whereas for gold convenience yields, around 98 percent of the variance is captured by two factors. Two factors therefore seem adequate to explain the majority of the variation in bond and convenience yields, and with this in mind, we include four factors in each model version: two bond factors and two commodity factors.

4 Model Estimation and Specification Tests

We estimate the two models, bond-oil and bond-gold, using Kalman filter techniques and Maximum Likelihood estimation. The factors are estimated using a Kalman filter, which feed into the following log-likelihood function:

$$\log \mathcal{L}(\Theta; \mathbf{y}_{t=1,\dots,T}) = \sum_{t=1}^{T} \log f(\mathbf{y}_t | \mathbf{y}_{t-1}, \Theta)$$

Having chosen to include four factors, we also consider several possible model specifications, using Likelihood Ratio (LR) tests. We test several versions of the model, imposing restrictions on the $\mathbf{\Phi}^z$ parameter matrix defined earlier. Specifically, we explore the following range of alternative parameter restrictions in the physical factor dynamics:

$$\begin{array}{lcl}
\textbf{Model M0} & : & \textit{The unconstrained model} : \Phi^z \ \textit{unconstrained} \\
\textbf{Model M1} & : & \textit{Commodity factors do not affect bond factors} : \Phi^z_{12} = 0 \\
\textbf{Model M2} & : & \textit{Bond factors do not affect commodity factors} : \Phi^z_{21} = 0 \\
\textbf{Model M3} & : & \textit{Commodity and bond factors independent} : \Phi^z_{21} = 0 \ \textit{and} \ \ \Phi^z_{12} = 0
\end{array}$$

The parameter set $\Theta = \{\, \boldsymbol{\kappa}^z(4), \boldsymbol{\Phi}^z(16), \boldsymbol{\kappa}^{\mathbb{Q}}(2), \boldsymbol{\Phi}^{\mathbb{Q}}(4), \boldsymbol{\Sigma}(10) \text{ and } \boldsymbol{\Omega}(7)\}$[7] has 43 parameters in total to estimate for the benchmark M0 model. The total numbers of parameters to be estimated for M1, M2 and M3 are 39, 39 and 35 respectively. The Likelihood Ratio test results are shown in Table 3. For both the oil and gold models, the alternative hypothesis of a restricted model is tested against the null hypothesis of the unrestricted model. In all cases the restricted specifications are strongly rejected at a 1% significance level. The tests therefore suggest a clear role for the full set of commodity and bond factors in the pricing of both the bond and convenience yield term structures. These results provide a strong motivation for the joint modelling of bonds and commodities, and we adopt the unrestricted model as our preferred specification as a result. As noted in the introduction, models of interest rates and models of commodities have largely evolved in isolation of one another, and our findings suggest that a more general approach incorporating links between the two markets is justified.

We also consider the gains from the inclusion of surveys within the models. Table 4 compares the performance of two versions of the model, one that includes surveys, and another that does not. The models are compared based on their in-sample forecasting ability, measured by mean absolute forecast errors (MAE), and root mean squared errors (RMSE), at the 12 and 24 month horizons. The results indicate a mild reduction in forecast errors with the inclusion of surveys. For oil, the model with surveys improves upon the model without by all measures except RMSE at the 24 month horizon. Similarly for gold, the model with surveys outperforms in most cases, except for RMSE at the 12 month horizon. As mentioned earlier, a key difference between our specification and those presented in Alquist, Bauer and Diez de los Rios (2014) and Le and Zhu (2013) is the inclusion of survey forecasts within our model. The results in this section point to improved forecast performance with the inclusion of surveys, and suggest that the identification of expectation and risk premia components may be improved under our specification.

5 Results

This section reports the results from the preferred models for both oil and gold. The parameter estimates from the preferred models are shown in Table 5. The fitted bond and convenience yields, along with their fitting errors, are shown in Figures 2, 3 and 4. The standard deviation of the pricing errors is around 97 basis points for the 1-year oil convenience yield, which is small relative to the large magnitude of oil convenience yields. And

---

[7] The numbers in parentheses indicate the number of individual scalar parameters to be estimated for the vector/matrix.

the standard deviation of the gold convenience yield errors is around 23 basis points, which is also economically small. The model fits bond yields closely, with a 9 basis point standard deviation of the pricing errors.

5.1 Analysis of Commodity and Bond Factors

Figure 5 shows the estimated commodity and bond factors (i.e. $\mathbf{z}_t$) for both the joint bond-oil model and joint bond-gold model. The bond factors shown are taken from the bond-oil model, which are almost identical to those from the bond-gold model. The loadings of bond and convenience yields on these factors are shown in Figure 6, which shows the values of the coefficients of the bond and commodity factors in equations (9) and (10), plotted against their maturity (n).[8] In both models, the pattern of bond yield loadings on the two bond factors are in line with estimates from other models of the government bond term structure. The loadings on the first and second bond factors have the standard ‘level’ and ‘slope’ interpretation, indicated by the relatively constant loadings across maturities on the first factor, and increasing loadings with maturity on the second factor. The loadings of the oil and gold convenience yields on their respective factors have a similar level and slope interpretation.

To explore the interlinkages between bond and commodity markets further, we examine the role of commodity and bond factor innovations in accounting for convenience yield variation. Figure 7 shows forecast error variance decompositions of 6, 12 and 24 month oil and gold convenience yields. For each convenience yield, the chart shows the proportion of its variance that can be explained by innovations in the bond and commodity factors (where the innovations are identified using a Cholesky factorization). Across both versions of the model, a large part of the variation in convenience yields is attributed to commodity factor innovations. But in the oil-bond model, a reasonable amount of variation in oil convenience yields is attributed to bond factor innovations, where their contribution increases with forecast horizon (to 20% for the 12 month oil convenience yield). And bond factors also drive gold convenience yields to an extent, where the bond factor contributions are larger than for oil, and also increase with horizon (to 24% for the 12 month gold convenience yield). Changes in the yield factors are indicative of general macroeconomic conditions affecting the oil market, which likely explains the role for bond factors in the oil variance decompositions. Gold is often considered alongside currencies, and their associated interest rates, in investment decisions. With this in mind, the role of US Treasury curve in gold price

---

[8] Recall the loadings of commodity factors in yield models are all zero, and the loadings of bond factors in convenience models are all zero.

variation seems intuitive given the role of the yield curve in exchange rate determination. The oil market is also subject to potentially significant supply shocks, while the supply of gold does not fluctuate substantially, and the total stock of gold is very large compared to its consumption.[9] This might explain the slightly larger role for bond factor innovations in the gold market relative to oil.

5.2 Expectations and Risk Premia Decomposition

Following Section 2.1, we are able to decompose the annualised commodity futures-spot basis (i.e. the log difference between futures and spot prices) at a given maturity into two components: an annualised expected change in the spot commodity price; and an annualised risk premium. The risk premium components of the 12- and 24-month futures-spot basis are shown in Figure 8. The shaded areas indicate periods of slowing economic growth, using NBER recession dates. There is substantial time variation in both risk premium estimates, though the magnitude of the oil risk premium is somewhat larger than the gold premium for the majority of the sample period.

The oil risk premium is negative for much of the sample, following an upward trend during the 2000s, and sharply increasing during the 2008 financial crisis. The change in the oil premium over time potentially reflects changes in the oil market over this period. As outlined in Kilian (2009), the relationship between oil price changes and the global economic cycle can be very different depending on whether demand or supply shocks are driving oil market fluctuations. Positive (negative) demand shocks are more likely to be associated with oil price increases (falls) during an upturn (downturn) in the global economic cycle, whereas negative (positive) supply shocks can imply oil price increases (falls) that put downward (upward) pressure on economic growth. Hypothetically, if the oil market was entirely driven by demand shocks, the positive correlation between oil prices and economic output would be consistent with investors requiring a positive risk premium for holding oil. Similarly, in a market driven by supply shocks, investors may be willing to pay a premium (that is, receive a negative risk premium) in order to hold the commodity, since they would benefit from oil price increases in difficult times. The change in the oil premium could therefore reflect the changing composition of oil market shocks over time. Indeed, the evidence presented in Kilian (2009) suggests that the relative role of demand and supply shocks has changed materially over time. According to the demand and supply shocks identified using a structural VAR, he shows that the role for supply shocks has diminished over time (which is also in line with

---

[9] Oil supply may be related to the yield curve, though appears to be relatively inelastic in the short run (see Kilian (2009)).

the evidence in Baumeister and Peersman (2013) and Hamilton (2011)). This is consistent with the relationship between oil price changes and the economic environment also changing, and the risk premium turning positive in the 2000s.

The gold premium is mostly positive, indicating that investors require compensation for holding gold relative to risk-free government bonds. The premium also appears to display counter-cyclical behaviour, where the premium increases during recession periods. The level and variation of our estimate is consistent with Le and Zhu (2013), who similarly estimate positive and counter-cyclical expected returns from gold lease rates (at one week horizons). While gold is often portrayed as providing protection against economic downturns, the positive sign of our estimate suggests the opposite. Our findings suggest that financial markets expect gold to co-vary positively with the business cycle, and for the value of gold holdings to decline in bad states of the world. It seems plausible that, to the extent that the supply of gold is relatively large and stable, the positive premium reflects gold demand shocks that are positively correlated with the business cycle.

As outlined in Bessembinder (1992) and de Roon, Nijman and Veld (2000), as well as reflecting systematic risk of futures positions, futures risk premia can also reflect speculative and hedging behaviour of futures market participants. This is the well-known hedging pressure theory, where futures premia also reflect compensation to speculators for taking on residual risk from hedgers. The theory predicts a positive risk premium in futures prices (implying the futures price lies below the expected future spot price) if hedgers have net short positions within the market, and a negative risk premium (futures price above the expected future spot price) if hedgers are net long. To the extent that hedging pressure effects are important determinants of futures risk premia, the level and changes of our estimates could reflect variation in the net hedging positions in oil and gold futures markets. The negative oil premium would suggest that oil market hedgers (in the aggregate) are exposed to oil price increases, and protect against upside risks by taking long positions in the futures contracts. In order for a premium to accrue to speculators, the futures price would therefore lie above the expected future spot price. In this case, the change in the sign of the oil premium could reflect a change in the nature of hedgers' exposures. Similarly, according to the hedging pressure hypothesis, the positive gold premium would reflect net short hedger positions, which implies that future prices lie below the expected future spot price.

The first two charts in Figure 9 shows the decomposition of the futures basis into the expectation and risk premium components. The line and bars represent the average values over the whole sample of the futures basis, expected spot price changes and risk premia, respectively. As outlined in Section 2.1, the forward basis is equal to the expected spot price change less the risk premium. For oil, the average basis for longer maturity contracts is negative, consistent with the stylised market view that the oil futures curve tends to be in 'backwardation', in the sense that the futures price lies below the *current* spot price (the definition commonly used by market practitioners). The decomposition indicates that, on average, there is substantial negative risk premium component in the forward basis. For gold, the futures price sits above the spot price on average, shown by the positive black line, consistent with the gold futures curve being in 'contango'. The size of the gold premium is smaller relative to the expected price change component.

For both oil and gold, the expectations component accounts for a large proportion of the futures basis, particularly in the case of gold. The smaller risk premium component for gold futures could suggest that using the futures price is a less biased forecast, at least relative to oil futures prices. However, our results do not necessarily suggest that gold futures will predict gold prices better than oil futures do oil prices. For example, Chinn and Coibion (2014) find that precious metals futures, including gold, tend to perform relatively poorly in forecasting future spot prices, and that oil futures perform better. Our results suggest that the forecasting ability of oil futures can be further improved when adjusting for the risk premium, and that the forecast improvement from adjusting gold futures is potentially more limited. The decompositions look very similar for the period between 1991 and 2008. However, the decompositions based on data after 2008, shown in the bottom half of Figure 9, show a very different picture. For oil, the expectation component is much larger relative to the risk premium, which suggests that futures prices may perform better at forecasting future spot prices over this period. On the other hand, the risk premium has become more important in the gold basis over this period, as the magnitude of the expectation component has declined, while the risk premium has remained a similar size.

5.3 Further Risk Premia Analysis

We further explore the oil and gold risk premium estimates through multiple regression analysis. We regress the risk premia estimates on contemporaneous macroeconomic and market factors, with the aim of understanding how they vary over the business cycle and under different market conditions. For both oil and gold, we regress the 12- and 24- month risk premia on global output and inflation variables, and market uncertainty and market activity measures. We estimate the following model using OLS, where the $h = 12, 24$ month risk premium, ρ, is regressed on the set of explanatory variables:

$$\rho_{t,h}^{i} = \beta_{h}^{i} x_t + \varepsilon_{t,h}^{i}$$

$$i = \left[\begin{array}{cc} Oil & Gold \end{array} \right] \qquad x_t = \left[\begin{array}{ccccc} ACT_t & INF_t & TS_t & VIX_t & HP_t \end{array} \right]'$$

Given the international nature of oil and gold markets, where possible we consider the ability of global rather than country specific macroeconomic variables to explain variation in risk premia. Our measures of global economic activity and inflation are both provided by Global Financial Data (GFD). Global activity (ACT_t) is measured by annual world industrial production growth (GFD code 'NDWWLDM'), and for global inflation (INF_t) we use annual inflation measured by the 12-month growth in a consumer price index of OECD member countries ('CPOECDM'). We include a 'term spread' measure defined as the difference between the 10-year and 1-year zero-coupon US Treasury yields (TS_t), which captures the slope of the interest rate term structure, and include a measure of market stress, proxied by the VIX (VIX_t), a measure of equity volatility implied from option prices (provided by Datastream). We also construct a measure of hedging pressure using the CFTC Aggregated Commitment of Traders Report. This report summarises positions in a range of commodity futures contracts, where they provide a breakdown of the outstanding open interest into short and long positions of 'hedgers' and 'speculators'. Our measure of hedging pressure (HP_t) is defined as the number of short hedger positions less the number of long hedger positions, divided by the total number of hedger positions (following de Roon, Nijman and Veld (2000)).

Table 6 shows the regression results for the 12- and 24-month oil risk premia. The set of explanatory variables is able to capture a significant amount of variation in oil risk premia, with R^2 values greater than 0.4 for both regressions. As outlined earlier, theoretical models (e.g. Hirshleifer (1988)) predict that both systematic and residual risks can influence the sign and variation of commodity futures risk premia. The inflation measure and term spread, both indicators of business cycle conditions, significantly explain variation in oil premia, suggesting that the risk premia estimates vary systematically with macroeconomic conditions. The coefficient on the hedging pressure measure is also significant, indicating that net hedger positions also play a role in determining risk premia. Since hedging pressure is defined as short hedger positions less long positions, we might expect an increase in this measure to be associated with an increase in risk premia, rather than the negative relationship we have estimated. The interpretation of the regression coefficients is difficult due to the change in sign in the oil risk premium over the sample. To address this, we re-estimate the regression model with the sample restricted to the period pre-2000, the period after which the relative role of demand and supply shocks in the oil market may have changed materially. The regression results are shown in Table 7. These results show a positive coefficient on hedging pressure, in line with the theory. In addition, when using the restricted sample, the coefficient on the VIX also changes to negative (and significant for the 12 month estimate), indicating that the premium becomes more negative under stressed market conditions. The activity variable is also statistically significant in this model, again highlighting that variation in oil premia can be explain by macroeconomic variables.

The results for gold are shown in Table 8. In general, the set of explanatory variables are also able to account for a reasonable amount of the variation in gold risk premia, where the R^2 values for the 12- and 24-month regressions are 0.29 and 0.24 respectively. In line with the oil regressions, there appears to be a role for both the business cycle and hedging pressure components in determining gold risk premia. There is evidence that business cycle variables significantly explain variation in gold risk premia, where inflation and the term spread are statistically significant. The positive relationship with the term spread, which tends to increase in economic downturns, indicates that gold risk premia tend to vary counter-cyclically. The coefficients on the activity and VIX variables are also in line with this interpretation, though they are insignificant. Finally, the hedging pressure coefficient is positive and statistically significant, consistent with hedging pressure theory, where a greater number of short hedger positions puts upward pressure on the gold risk premium.

6 Conclusion

We apply techniques from the literature pricing foreign exchange forwards, to modelling government bond yields and convenience yields from commodity futures. We jointly model the interest rate and commodity futures term structures, where we consider the relationship between government bonds and synthetic commodity 'bonds' (where the yield on a commodity bond is the well-known convenience yield). We estimate four-factor affine term structure models for oil and gold convenience yields, and find a substantial role for bond factors in accounting for variation in convenience yields. In particular, our results suggest that there are strong influences of bond factors in commodity markets, and that it is important to incorporate the term structure of interest rates in commodity pricing models. Our models allow us to obtain estimates of risk premia, which we find vary substantially over time. The risk premium in oil futures is negative for much of our sample, turning positive during the 2000s, which may reflect the declining role of supply shocks in the oil market. The gold premium is mostly positive, which contrasts with the common portrayal of gold as an asset that protects against economic downturns. Consistent with hedging pressure theory, both risk premia estimates are significantly related to net hedger positions in futures markets, suggesting that commodity futures risk premia are determined by both business cycle and hedging pressure components.

References

Alquist, R., Bauer, G. H. and Diez de los Rios, A. (2014). "What does the Convenience Yield Curve Tell Us about the Crude Oil Market", *Bank of Canada Working Paper 2014–42*

Alquist, R., Kilian, L. and Vigfusson, R. J. (2013). "Forecasting the Price of Oil", *Handbook of Economic Forecasting*, 2(A), pp. 427–507

Adrian, T., Crump, R. and Moench, E. (2013). "Pricing the Term Structure with Linear Regressions", *Journal of Financial Economics*, 110(1), pp. 110–138

Backus, D. K., Foresi, S. and Telmer, C. I. (2001). "Affine Term Structure Models and the Forward Premium Anomaly", *Journal of Finance*, 56(1), pp. 279–304

Baumeister, C. and Peersman, G. (2013). "Time-Varying Effects of Oil Supply Shocks on the US Economy", *American Economic Journal: Macroeconomics*, 5(4), pp. 1–28

Baur, D. G. and Lucey, B. M. (2010). "Is Gold a Hedge or a Safe Haven? An Analysis of Stocks, Bonds and Gold", *The Financial Review*, 45(2), pp. 217–229

Baur, D. G. and McDermott, T. K. J. (2010). "Is Gold a Safe Haven? International Evidence", *Journal of Banking and Finance*, 34(8), pp.1886–1898

Bessembinder, H. (1992). "Systematic Risk, Hedging Pressure, and Risk Premiums in Futures Markets", *Review of Financial Studies*, 5(4), pp. 637–667

Brennan, M. (1958). "The Supply of Storage", *American Economic Review,* 48(1), pp. 50–72

Brennan, M. J. and Schwartz, E. S. (1985). "Evaluating Natural Resource Investments", *Journal of Business*, 58(2), pp. 135–157

Büyükşahin, B. and Robe, M. A. (2014). "Speculators, Commodities and Cross-Market Linkages", *Journal of International Money and Finance*, 42, pp. 38–70

Casassus, J. and Collin-Dufresne, P. (2005). "Stochastic Convenience Yield Implied from Commodity Futures and Interest Rates", *Journal of Finance*, 60(5), pp. 2283–2331

Chinn, M. D. and Coibion, O. (2014). "The Predictive Content of Commodity Futures", *Journal of Futures Markets*, 34(7), pp. 607–636

Chow, Y., McAleer, M. and Sequeira, J. M. (2000). “Pricing of Forward and Futures Contracts”, *Journal of Economic Surveys*, 14(2), pp. 215–253

de Roon, F. A., Nijman, T. E. and Veld, C. (2000). “Hedging Pressure Effects in Futures Markets”, *Journal of Finance*, 55(3), pp. 1437–1456

Duffee, G. R. (2002). “Term Premia and Interest Rate Forecasts in Affine Models”, *Journal of Finance*, 57(1), pp. 405–443

Erb, C. B. and Harvey, C. R. (2013). “The Golden Dilemma”, *NBER Working Paper 18706*

Fama, E. (1984). “Forward and Spot Exchange Rates”, *Journal of Monetary Economics*, 14(3), pp. 319–338

Fattouh, B. (2011). “An Anatomy of the Crude Oil Pricing System”, *Oxford Institute for Energy Studies. WPM40*

Frankel, J. A. (2008). “The Effect of Monetary Policy on Real Commodity Prices”, *Asset Prices and Monetary Policy*

Gibson, R. and Schwartz, E. S. (1990). “Stochastic Convenience Yield and the Pricing of Oil Contingent Claims”, *Journal of Finance*, 45(3), pp. 959–976

Gürkaynak, R. S., Sack, B. & Wright, J. H. (2007). “The U.S. Treasury Yield Curve: 1961 to the Present”, *Journal of Monetary Economics,* 54(8), pp. 2291–2304

Hamilton, J. (2011). “Historical Oil Shocks”, *Routledge Handbook of Major Events in Economic History*, pp. 239–265

Hamilton, J. and Wu, C. (2014). “Risk Premia in Crude Oil Futures Prices”, *Journal of International Money and Finance*, 42, pp. 9–37

Hicks, J. (1939). “Value and Capital: An Inquiry into Some Fundamental Principles of Economic Theory”, *Oxford Clarendon Press.*

Hirshleifer, D. (1988). “Residual Risk, Trading Costs, and Commodity Futures Risk Premia”, *Review of Financial Studies*, 1(2), pp. 173–193

Hodrick, R. J. and Srivastava, S. (1986). “The Covariation of Risk Premiums and Expected Future Spot Exchange Rates”, *Journal of International Money and Finance*, 5(1), pp. 5–21

Joslin, S., Singleton, K. J. and Zhu, H. (2011). "A New Perspective on Gaussian Dynamic Term Structure Models", *Review of Financial Studies*, 24(3), pp. 926–970

Keynes, J. M. (1923). "Some Aspects of Commodity Markets", *Manchester Guardian Commercial*, European Reconstruction Series, Section 13, pp. 784–786

Kilian, L. (2009). "Not All Oil Price Shocks Are Alike: Disentangling Demand and Supply Shocks in the Crude Oil Market", *American Economic Review*, 99(3), pp. 1053–1069

Kilian, L. and Hicks, B. (2013). "Did Unexpectedly Strong Economic Growth Cause the Oil Price Shock of 2003–2008?", *Journal of Forecasting*, 32(5), pp. 385–394

Kim, D. and Orphanides, A. (2012). "Term Structure Estimation with Survey Data on Interest Rate Forecasts", *Journal of Financial and Quantitative Analysis*, 47(1), pp. 241–272

Kim, D. and Wright, J. (2005). "An Arbitrage-Free Three-Factor Term Structure Model and the Recent Behavior of Long-Term Yields and Distant-Horizon Forward Rates", *Finance and Economics Discussion Series* 2005-33, Federal Reserve Board

Le, A. and Zhu, H. (2013). "Risk Premia in Gold Leasing Markets", *UNC* W*orking Paper*

Nixon, D. and Smith, T. (2012). "What Can the Oil Futures Curve Tell Us About the Outlook for Oil Prices?", *Bank of England Quarterly Bulletin* 2012 Q1

Rosa, C. (2013). "The High-Frequency Response of Energy Prices to Monetary Policy: Understanding the Empirical Evidence", *FRBNY Staff Reports*, 598

Saporta, V., Trott, M. and Tudela, M. (2009). "What Can Be Said About the Rise and Fall in Oil Prices?", *Bank of England Quarterly Bulletin* 2009 Q3

Schwartz, E. S. (1997). "The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging", *Journal of Finance*, 52(3), pp. 923–973

Working, H. (1949). "The Theory of Price of Storage", *American Economic Review,* 39(6), pp. 1254–1262

7 Appendix:

Table 1: Data Summary Statistics

Summary statistics are in annualised percentage points.

Variable	Maturity (months)	Mean	Standard Deviation
Oil Convenience Yields	3	7.14	29.02
	12	7.44	14.66
	24	5.47	9.72
Gold Convenience Yields	3	-0.04	1.01
	12	0.38	0.87
	24	0.53	0.90
Bond Yields	3	4.24	2.82
	12	4.62	2.92
	60	5.57	2.79
	120	6.24	2.49

Table 2: Principal Component Analysis: Proportion of Variance Explained

PC	Bond Yields		Oil conv. yields		Gold conv. yields	
	(%)	Cumulative (%)	(%)	Cumulative (%)	(%)	Cumulative (%)
1	97.65	97.65	96.16	96.16	92.96	92.96
2	2.17	99.82	3.67	99.83	5.55	98.51
3	0.13	99.95	0.14	99.97	0.77	99.28
4	0.02	99.97	0.02	99.99	0.45	99.73
5	0.02	99.99	0.00	99.99	0.18	99.91

Table 3: Likelihood Ratio Tests

Table 3a shows the four model specifications we consider and their associated likelihood values, $LogL$, for the oil and gold applications. Table 3b shows the likelihood ratio tests, where the alternative H_1 restricted models (M_1,M_2,M_3) are tested against the null H_0 of an unrestricted model (M_0). The table shows the likelihood ratio test statistic $LRStat$, associated p-value, and degrees of freedom df.

Table 3a

Model	Description	$LogL$ (Oil)	$LogL$ (Gold)
M_0	Unconstrained	43761.8	50632.8
M_1	Oil/gold factors do not affect bond factors	43755.6	50612.9
M_2	Bond factors do not affect oil/gold factors	43741.2	50605.1
M_3	Bond and Oil/Gold independent	43737.3	50589.9

Table 3b

Oil					Gold				
H_0	H_1	$LRStat.$	$p-value$	df	H_0	H_1	$LRStat.$	$p-value$	df
M_3	M_0	49.07	0.00	8	M_3	M_0	53.15	0.00	8
M_2	M_0	41.12	0.00	4	M_2	M_0	15.44	0.00	4
M_1	M_0	12.46	0.01	4	M_1	M_0	48.71	0.00	4

Table 4: In-sample Forecasting

Oil	Mean Absolute Error		Root Mean Squared Error	
Forecast Horizon	12	24	12	24
Model - with surveys	9.09	11.66	15.84	19.67
Model - without surveys	9.65	13.17	16.52	19.50
Futures	9.29	12.25	15.59	18.36
Random Walk	10.42	13.88	17.08	19.83
Hotelling's Rule	10.69	14.39	17.14	19.59

Gold	Mean Absolute Error		Root Mean Squared Error	
Forecast Horizon	12	24	12	24
Model - with surveys	80.18	135.94	119.71	187.63
Model - without surveys	80.49	136.81	120.53	191.40
Futures	80.56	136.88	120.70	191.04
Random Walk	78.55	134.99	123.35	202.21
Hotelling's Rule	82.34	142.03	121.94	195.05

Table 5a: Oil Model Parameter Estimates

Parameter	Estimate	Standard Error	Parameter	Estimate	Standard Error
κ^{z}_{11}	0.0001*	0.00003	$\Sigma_{11}*10^4$	7.25*	0.27
κ^{z}_{21}	0.0001*	0.00004	$\Sigma_{21}*10^4$	0.66*	0.18
κ^{z}_{31}	−0.001	0.002	$\Sigma_{22}*10^4$	2.83*	0.12
κ^{z}_{41}	−0.0004	0.0009	$\Sigma_{31}*10^4$	20.15	14.46
Φ^{z}_{11}	0.9896*	0.002	$\Sigma_{32}*10^4$	35.13*	16.24
Φ^{z}_{12}	0.023*	0.009	$\Sigma_{33}*10^4$	262.13*	9.75
Φ^{z}_{13}	0.0004	0.0003	$\Sigma_{41}*10^4$	−8.51*	3.81
Φ^{z}_{14}	−0.002	0.002	$\Sigma_{42}*10^4$	−1.24	4.30
Φ^{z}_{21}	−0.0008	0.002	$\Sigma_{43}*10^4$	35.85*	3.43
Φ^{z}_{22}	0.9789*	0.011	$\Sigma_{44}*10^4$	57.80*	2.32
Φ^{z}_{23}	0.0001	0.0003	$\kappa^{\$\mathbb{Q}}_{\infty}*10^4$	1.11	2.78
Φ^{z}_{24}	0.0002	0.002	$\kappa^{*\mathbb{Q}}_{\infty}*10^4$	9.56*	0.61
Φ^{z}_{31}	0.2279*	0.097	$\Phi^{\mathbb{Q}}_{11}$	0.9844*	0.04
Φ^{z}_{32}	−0.082	0.488	$\Phi^{\mathbb{Q}}_{22}$	0.9841*	0.04
Φ^{z}_{33}	0.7774*	0.018	$\Phi^{\mathbb{Q}}_{33}$	0.8775*	0.006
Φ^{z}_{34}	−1.0438*	0.104	$\Phi^{\mathbb{Q}}_{44}$	0.7934*	0.008
Φ^{z}_{41}	−0.0552	0.046			
Φ^{z}_{42}	−0.1210	0.271			
Φ^{z}_{43}	−0.0314*	0.008			
Φ^{z}_{44}	0.6364*	0.041			

*Significant at the 5% level

Table 5b: Gold Model Parameter Estimates

Parameter	Estimate	Standard Error	Parameter	Estimate	Standard Error
κ^z_{11}	0.0001	0.00004	$\Sigma_{11}*10^4$	7.20*	0.265
κ^z_{21}	0.0001*	0.00004	$\Sigma_{21}*10^4$	0.67*	0.180
κ^z_{31}	−0.001	0.00009	$\Sigma_{22}*10^4$	2.87*	0.124
κ^z_{41}	−0.0001*	0.00005	$\Sigma_{31}*10^4$	−0.10	0.393
Φ^z_{11}	0.9941*	0.002	$\Sigma_{32}*10^4$	−0.68	0.439
Φ^z_{12}	0.0031	0.012	$\Sigma_{33}*10^4$	6.95*	0.262
Φ^z_{13}	−0.035*	0.008	$\Sigma_{41}*10^4$	−0.06*	0.217
Φ^z_{14}	−0.208*	0.04	$\Sigma_{42}*10^4$	−0.93*	0.241
Φ^z_{21}	−0.0039*	0.002	$\Sigma_{43}*10^4$	2.26*	0.197
Φ^z_{22}	0.9788*	0.012	$\Sigma_{44}*10^4$	2.77*	0.138
Φ^z_{23}	0.0125	0.008	$\kappa^{\$\mathbb{Q}}_{\infty}*10^4$	1.11	3.35
Φ^z_{24}	−0.086*	0.04	$\kappa^{*\mathbb{Q}}_{\infty}*10^4$	0.074*	0.03
Φ^z_{31}	0.0128*	0.005	$\Phi^{\mathbb{Q}}_{11}$	0.9843*	0.048
Φ^z_{32}	−0.0733*	0.028	$\Phi^{\mathbb{Q}}_{22}$	0.9841*	0.048
Φ^z_{33}	0.8957*	0.018	$\Phi^{\mathbb{Q}}_{33}$	1.00*	0.000003
Φ^z_{34}	−0.0372	0.084	$\Phi^{\mathbb{Q}}_{44}$	0.827*	0.010
Φ^z_{41}	0.0019	0.003			
Φ^z_{42}	−0.0400*	0.014			
Φ^z_{43}	−0.0451*	0.009			
Φ^z_{44}	0.7136*	0.043			

*Significant at the 5% level

Table 6: Oil Risk Premium Regression Analysis

OLS regressions of 12- and 24-month risk premia on 12-month world industrial production growth (ACT_t), 12-month growth in OECD member consumer price index (INF_t), the term spread defined as the difference between the 10-year and 1-year US Treasury yields (TS_t), the VIX implied volatility measure (VIX_t) and a measure of hedging pressure (HP_t). p-values based on t-tests using Newey-West standard errors are shown in parentheses.

	c	ACT_t	INF_t	TS_t	VIX_t	HP_t	R^2
$RP^{12}_{t,OIL}$	−5.57	−0.10	−1.21	1.35	0.03	−0.18	0.43
	(0.00)	(0.29)	(0.03)	(0.00)	(0.38)	(0.00)	
$RP^{24}_{t,OIL}$	−7.61	−0.18	−2.09	1.71	0.07	−0.29	0.44
	(0.00)	(0.23)	(0.02)	(0.00)	(0.20)	(0.00)	

Table 7: Oil Risk Premium Regression Analysis - Pre-2000 Sample

OLS regressions of 12- and 24-month risk premia on 12-month world industrial production growth (ACT_t), 12-month growth in OECD member consumer price index (INF_t), the term spread defined as the difference between the 10-year and 1-year US Treasury yields (TS_t), the VIX implied volatility measure (VIX_t) and a measure of hedging pressure (HP_t). p-values based on t-tests using Newey-West standard errors are shown in parentheses.

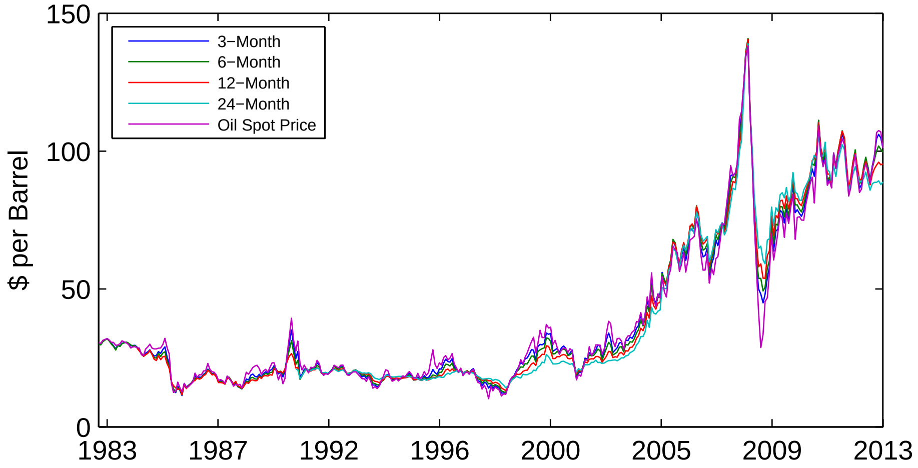
	c	ACT_t	INF_t	TS_t	VIX_t	HP_t	R^2
$RP^{12}_{t,OIL}$	−0.24	−0.54	−2.25	0.26	−0.08	−0.10	0.63
	(0.79)	(0.00)	(0.00)	(0.21)	(0.02)	(0.00)	
$RP^{24}_{t,OIL}$	−2.69	−0.66	−3.48	0.55	0.03	0.14	0.73
	(0.07)	(0.00)	(0.00)	(0.01)	(0.41)	(0.00)	

Table 8: Gold Risk Premium Regression Analysis

OLS *regressions of 12- and 24-month risk premia on 12-month world industrial production growth (ACT_t), 12-month growth in OECD member consumer price index (INF_t), the term spread defined as the difference between the 10-year and 1-year US Treasury yields (TS_t), the VIX implied volatility measure (VIX_t) and a measure of hedging pressure (HP_t). p-values based on* t*-tests using Newey-West standard errors are shown in parentheses.*

	c	ACT_t	INF_t	TS_t	VIX_t	HP_t	R^2
$RP^{12}_{t,GOLD}$	0.84	−0.004	−0.10	0.10	0.003	0.01	0.29
	(0.00)	(0.66)	(0.00)	(0.00)	(0.47)	(0.00)	
$RP^{24}_{t,GOLD}$	0.64	−0.01	−0.18	0.07	0.01	0.01	0.24
	(0.00)	(0.62)	(0.00)	(0.05)	(0.40)	(0.00)	

Figure 1: Oil and Gold Futures Prices



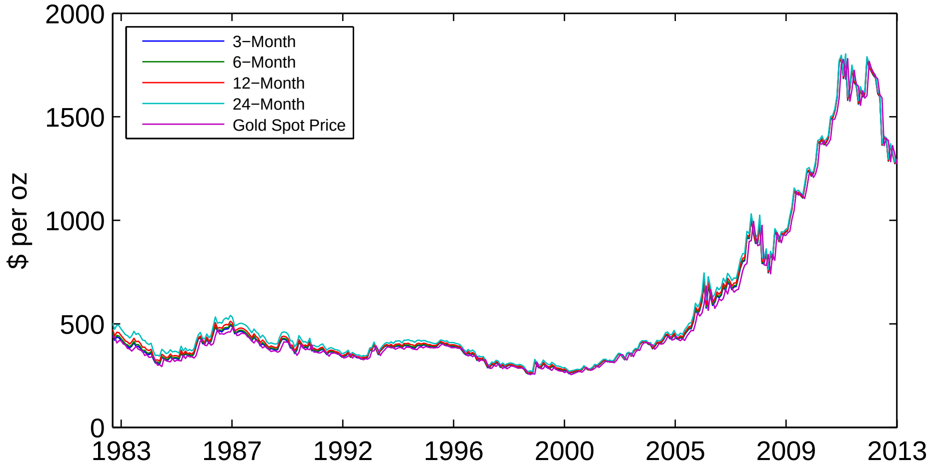


Figure 2: Actual vs. Fitted 1-Year Bond Yields

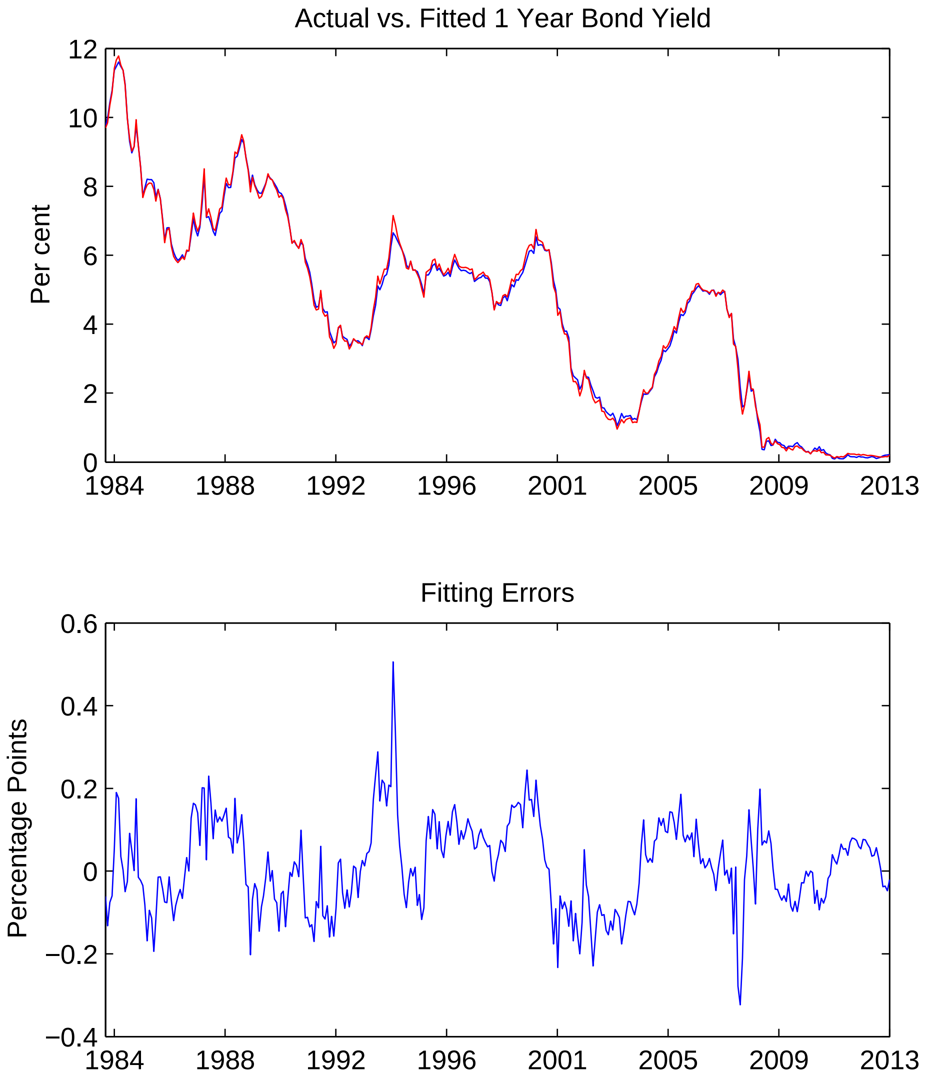


Figure 3: Actual vs. Fitted 1-Year Convenience Yields

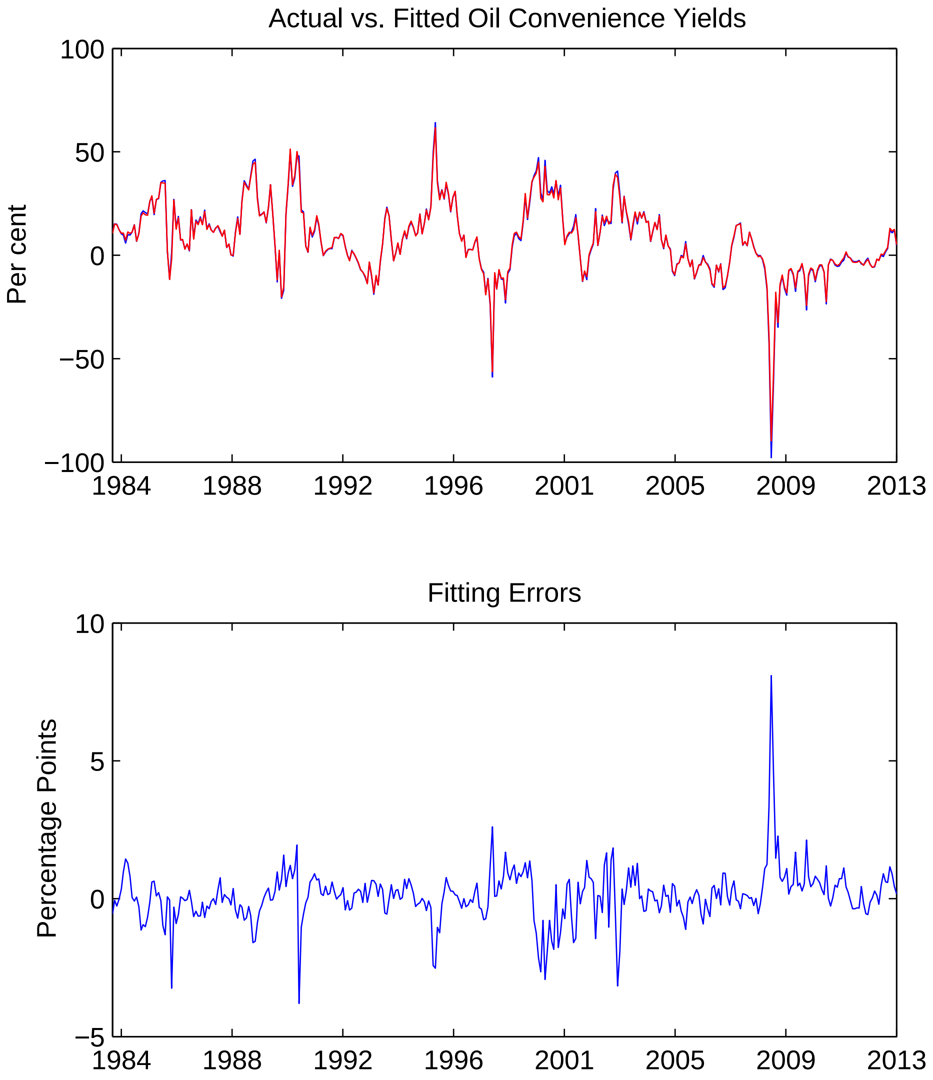


Figure 4: Actual vs. Fitted 1-Year Convenience Yields

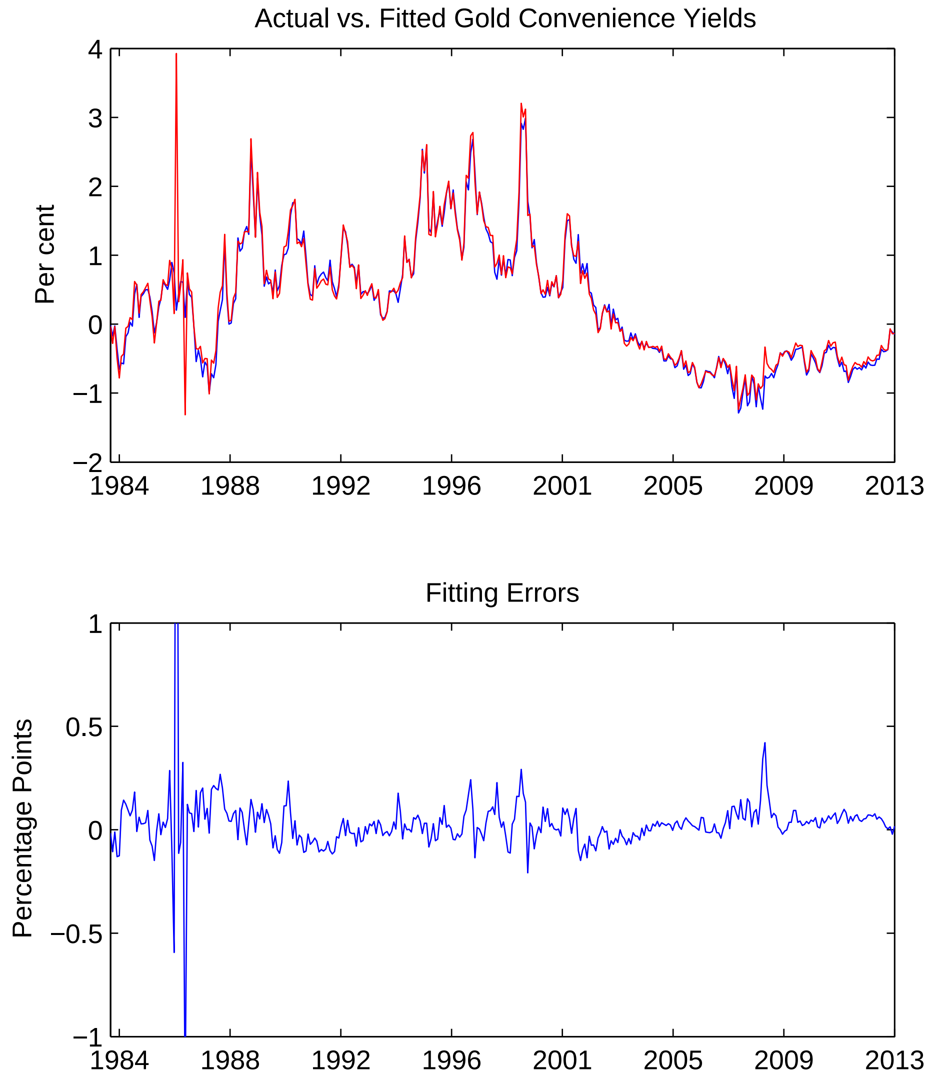
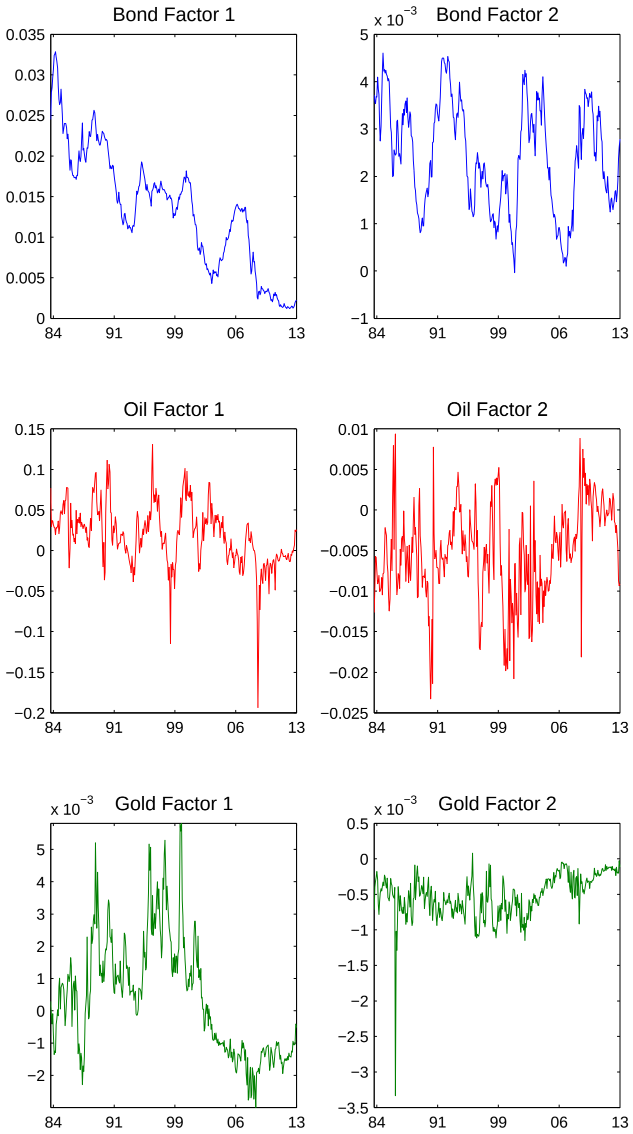


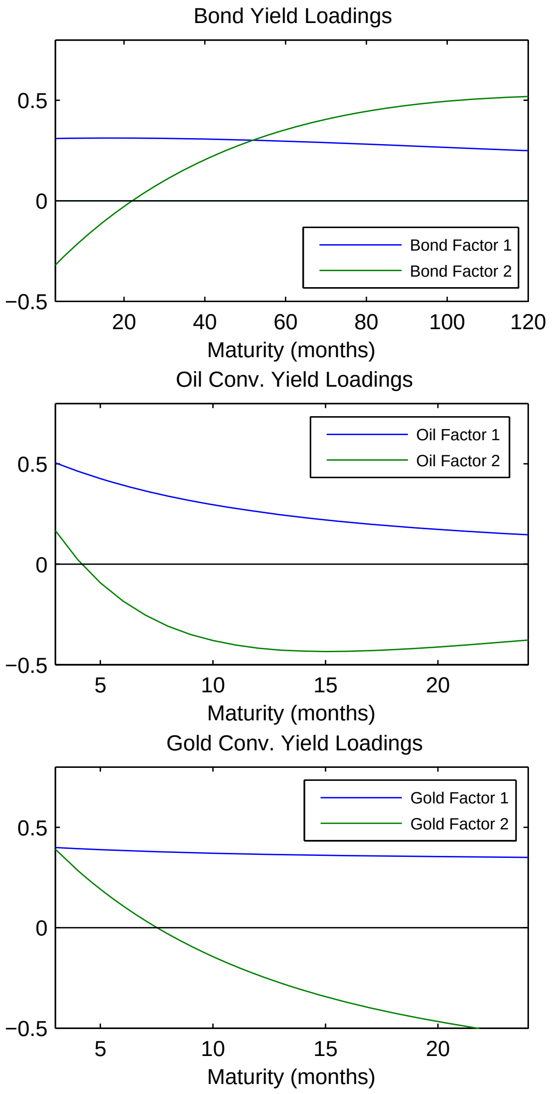
Figure 5: Estimated Bond and Oil Factors[10]



---

[10] Bond factors come from the bond-oil model

Figure 6: Bond and Convenience Yield Factor loadings[11]



---

[11] Bond yield loadings come from the bond-oil model

Figure 7: Variance Decomposition Oil (Left Hand Side) and Gold (Right Hand Side)

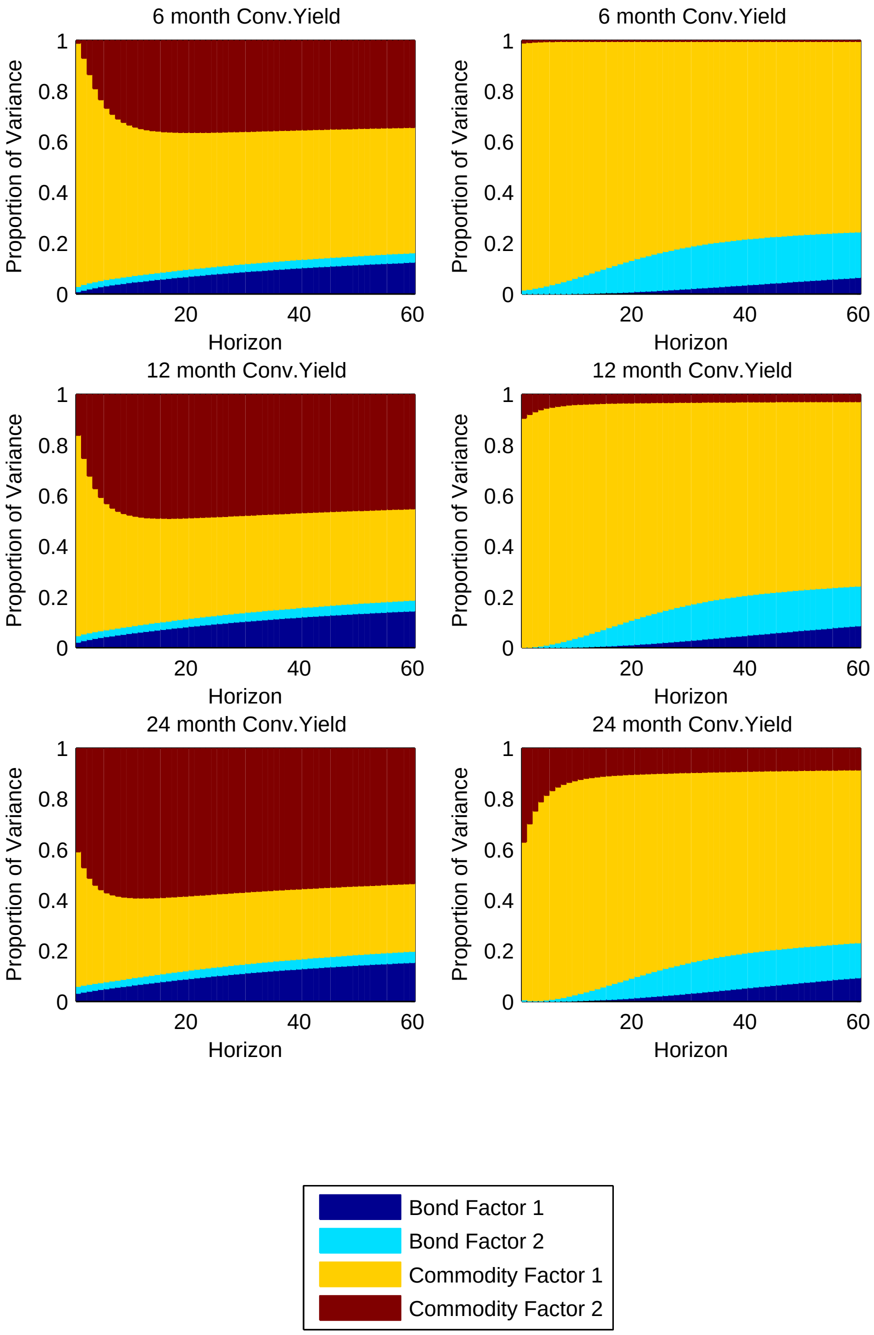


Figure 8: Oil and Gold Risk Premium Estimates
(shaded areas NBER recessions)

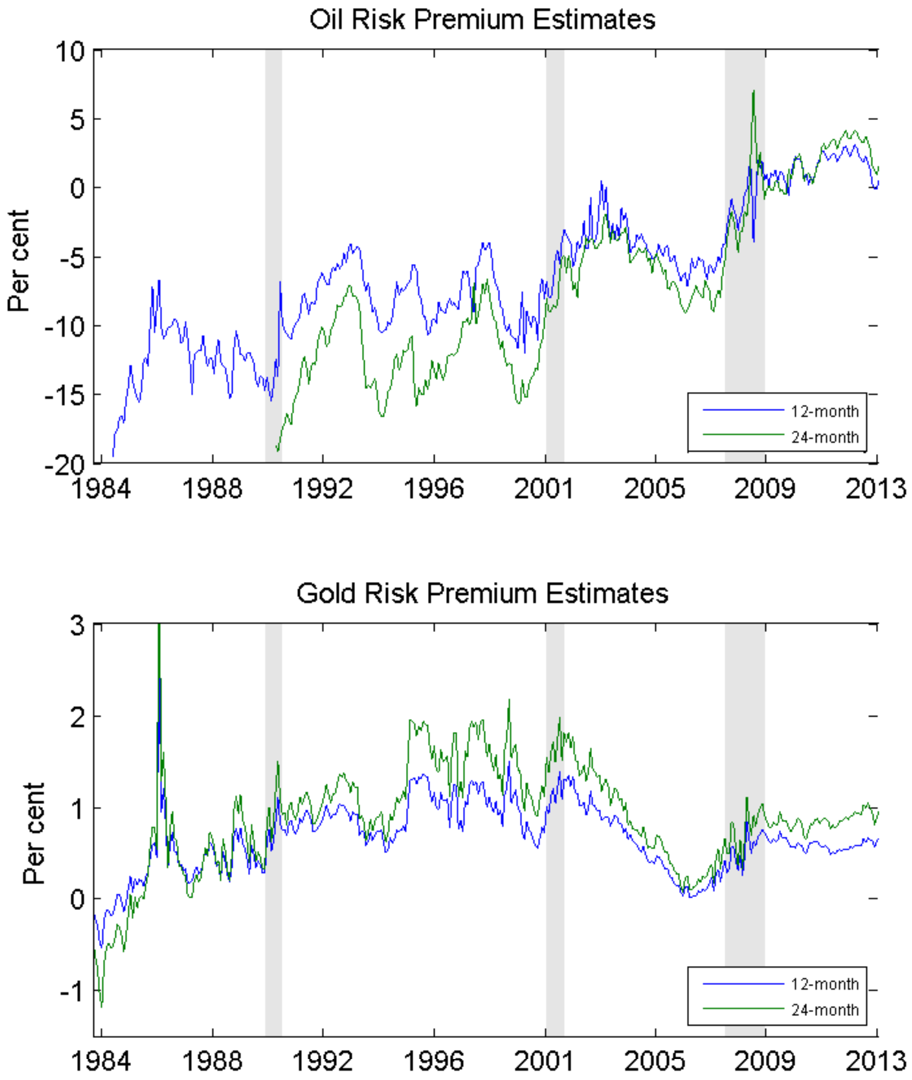
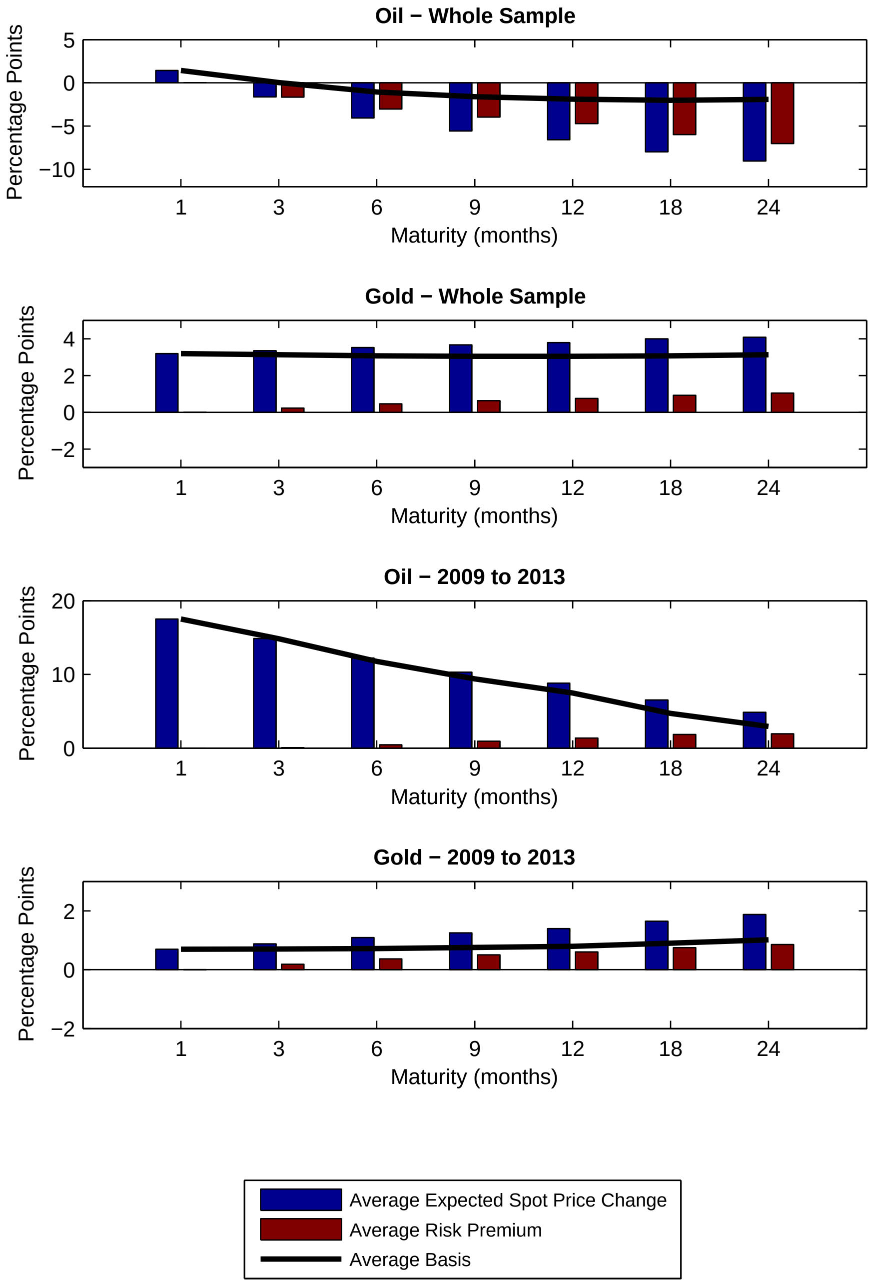


Figure 9: Futures-Spot Basis Decomposition



Oil − Whole Sample

Gold − Whole Sample

Oil − 2009 to 2013

Gold − 2009 to 2013