



BANK OF ENGLAND

Appendix to Staff Working Paper No. 553

Some unpleasant properties of loglinearized solutions when the nominal rate is zero

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Some unpleasant properties of loglinearized solutions when the nominal rate is zero

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Appendix A Robustness Analysis

This section conducts several robustness checks. We report results for an alternative calibration strategy that holds the level of technology fixed and report results using parameterizations of our model that are based on [Christiano and Eichenbaum \(2012\)](#) and [Denes, Eggertsson, and Gilbukh \(2013\)](#). Our result that the NK model may exhibit orthodox and small fiscal multipliers at the zero bound continues to obtain when we use an alternative calibration scheme and consider other regions of the parameter space.

A.1 No-technology-shock calibration scheme

Most analyses of the zero bound that use the [Eggertsson and Woodford \(2003\)](#) Markov equilibrium also posit a single shock to demand. One advantage of that strategy is that employment is depressed in state L for the entire range of model parameters that we consider in the paper.

Our analysis focuses on parameterizations of the model that are empirically relevant in the sense that they all reproduce the GR declines in output and inflation. If we are to continue to reproduce these two facts with $z^L = z = 1$, we will need to adjust another parameter instead. We choose to adjust the Dixit-Stiglitz parameter, θ . Adjusting θ in conjunction with d^L has no effect on the AD schedule because θ is not an argument of the AD schedule. Thus, the local properties of the AD schedule are the same as before. It is downward sloping when p is sufficiently low and rotates to right as p is increased eventually turning positive. However, θ enters $\text{slope}(NKPC)$ and adjusting it in this way renders the AS schedule independent of p .¹ For our baseline parameterization of the model, $\text{slope}(AS) = 0.036$ using the no-technology-shock calibration scheme. This is about the same value of $\text{slope}(AS)$ that occurs using the baseline calibration scheme with technology shocks for $p \approx 0.415$.²

The fact that the AS curve no longer varies with p has two main consequences. The first consequence is that the locus of p 's where $\text{slope}(AD)$ and $\text{slope}(AS)$ become equal and then cross is shifted to the right. This modification enlarges the size of the Case I equilibrium region where the labor tax multiplier is unconventional. Comparing [Figure 1](#) with [Figure 4](#) in the paper we see that the Case I region starts at about the same value of p in both figures. However, using this alternative calibration scheme the size of the Case I region extends to about $p = 0.965$. This in turn shrinks the size of the two indeterminacy regions (Case II and MZB). The size of the Case III equilibrium regions is largely unaffected and it follows that there continues to be a large region where the LL solution yields the wrong slope of the AD

¹This calibration strategy implies that θ adjusts with $p\beta d^L$ in a way that keeps $\theta/(1 - p\beta d^L)$ constant.

²For this value of p , the calibrated z^L approximately equals $z = 1$.

schedule and thus produces an incorrect sign for the labor tax multiplier. This can be seen by comparing the upper panels of Figure 2 with the lower panels. In the upper panels that show the NL solution we see a large red region where employment increases when the labor tax is cut. The LL solution, in contrast, is green in this region indicating a paradox of toil.

The second consequence is that the AS curve is now flatter at higher values of p and the fiscal multipliers are correspondingly smaller. For instance, using our baseline parameterization the fiscal multipliers are smaller using the no-technology shock calibration scheme in comparison to our baseline calibration scheme when $0.415 < p \leq 0.863$. This effect can be readily discerned for the labor tax multiplier by comparing the upper panel of Figure 5 with the upper panel of Figure 2.³ It is even more pronounced for the government purchase multiplier. For instance the yellow region with government purchase multipliers between 1.05 and 1.5 begins at $p = 0.73$ in the upper panel of Figure 6 (in the paper) for our baseline parameterization. In Figure 3 the yellow region does not begin until p reaches a value of 0.86. In fact, the government purchase multiplier is less than 1.5 for all choices of $p \leq 0.95$ using the no-technology-shock calibration scheme. It is only in the immediate neighborhood of the asymptote, which occurs at $p \approx 0.965$, that the government purchase multiplier exceeds 1.5.

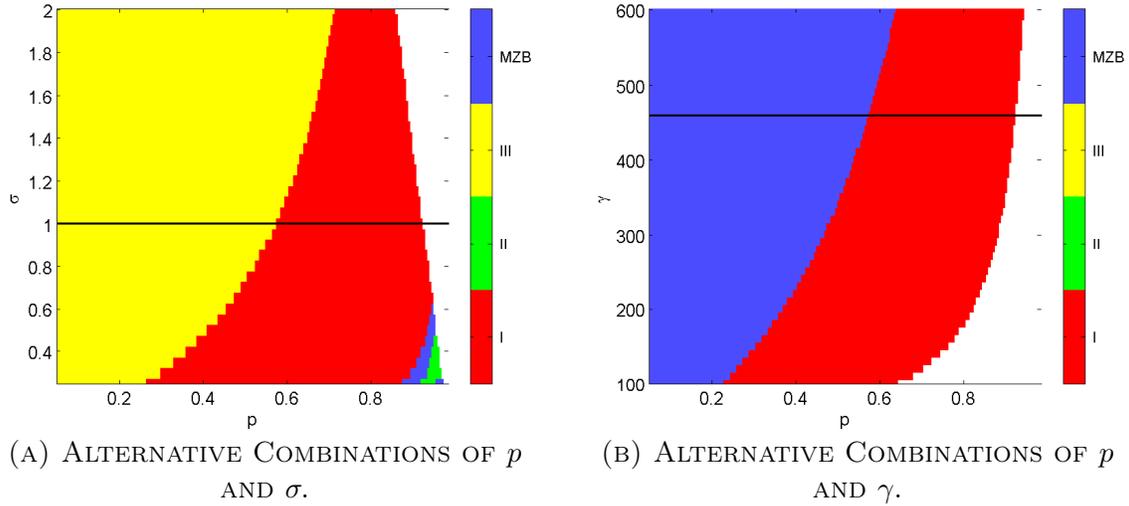
So far we have not said anything about the range of values taken on by θ . Higher values of p are associated with a smaller value of θ , and θ is declining in σ and γ . Some of the results reported in Figures 1–3 need $\theta < 1$ to hit the GR targets. These these regions are reported in white. Even though we can compute equilibria with values of $\theta < 1$ due to our subsidy scheme, θ in this range imply negative markups and are not of economic interest. To provide some indication about when this occurs suppose that γ is held fixed at its baseline value of 458.4 and $\sigma = 1$ then θ falls below 1 when p reaches 0.915. The associated values of the labor tax and government purchase multipliers are 0.56 and 1.1 respectively. But most estimates of θ are two or higher (see e.g. Broda and Weinstein (2004)). If we use our baseline parameterization of the model and limit attention to values of $\theta \geq 2$, then $p \geq 0.84$ are ruled out. Imposing this restriction implies that the government purchase GDP multiplier is 1.04 or less and that the labor tax multiplier is 0.17 or less.

In contrast to the calibration scheme with a technology shock, the no-technology-shock calibration scheme has the advantage that hours in state L are always below the steady-state. However, that calibration scheme imposes a restriction on θ instead. A common feature of both calibration schemes is that it is impossible to find empirically relevant parameter values for if $\sigma > 2$ and p is large.

Overall, the general pattern of results that emerges using this calibration scheme is consis-

³For instance, the blue region in Figure 5 (in the paper) with labor tax multipliers in excess of 1 begins at $p = 0.71$ for our baseline parameterization. The blue region in Figure 2, in contrast, begins at $p = 0.79$.

FIGURE 1: TYPES OF ZERO BOUND EQUILIBRIA FOR ALTERNATIVE VALUES OF RISK AVERSION AND PRICE ADJUSTMENT COSTS: NO TECHNOLOGY SHOCK



Notes: Red: Case I ($\text{slope}(AD) > 0 > \text{slope}(AS)$); light Green: Case II ($\text{slope}(AS) > \text{slope}(AD) > 0$); yellow: Case III ($\text{slope}(AD) < 0 < \text{slope}(AS)$); blue: Case MZB (multiple zero bound equilibria); dark Green: Case IV ($\text{slope}(AD) > 0 > \text{slope}(AS)$); white: $\theta < 1$

tent with our previous results. We find large regions where LL solution produces an incorrect sign for the labor tax multiplier and the magnitudes of the multipliers are even smaller under this calibration scheme for many choices of p . Perhaps the biggest difference is that the size of the indeterminacy regions is much smaller now. This follows from the fact that a flatter AS schedule acts to push the asymptote as indexed by p to the right.

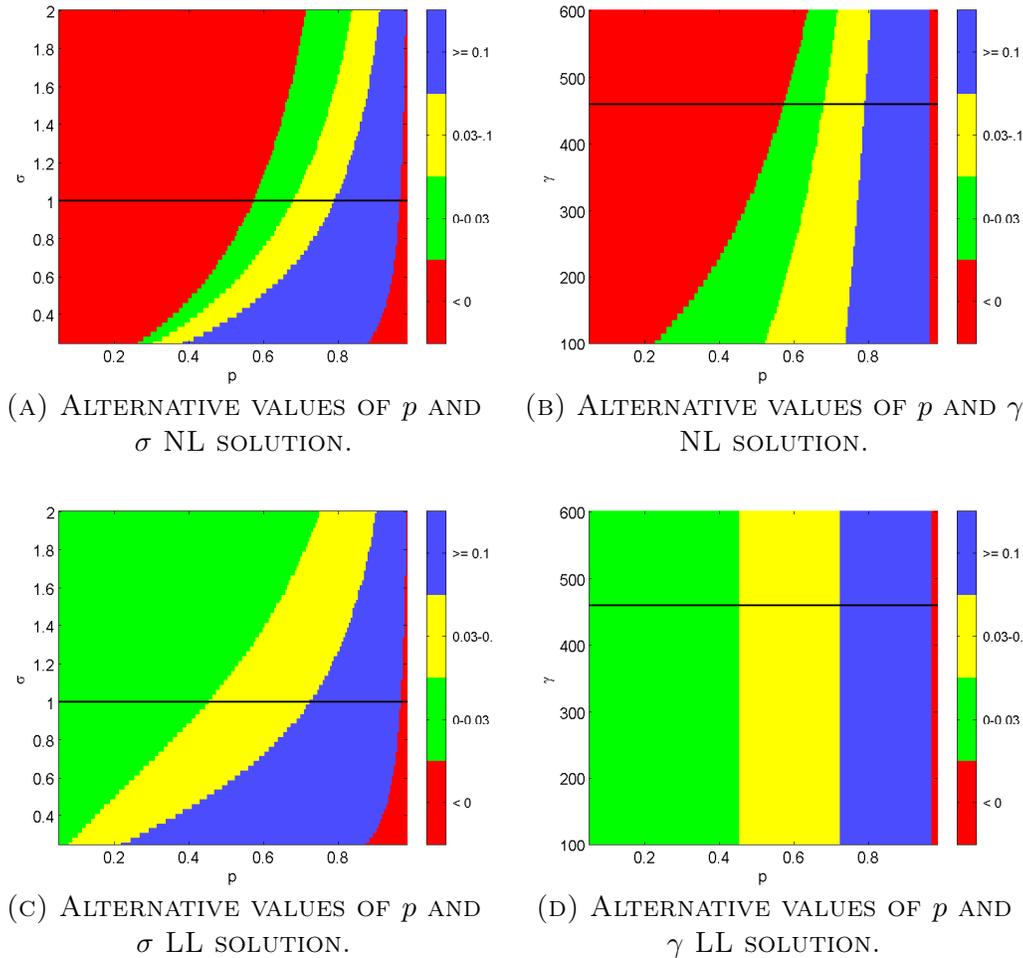
A.2 Accounting for the Great Recession with the parameterization of Christiano and Eichenbaum (2012)

Christiano and Eichenbaum (2012) find that the government purchase multiplier exceeds 2 in a nonlinear Rotemberg model that is very close to ours.⁴ In our model this can also occur but only in the neighborhood of the point where $\text{slope}(AD) = \text{slope}(AS)$. Moreover, in this neighborhood the sign and magnitudes of the fiscal multipliers are very sensitive to small perturbations of p and other structural parameters. It is thus interesting to investigate why their government purchase multipliers are so large.

Following their paper, we set the preference discount factor $\beta = 0.99$, the coefficient of relative risk aversion for consumption to $\sigma = 1$ and the curvature parameter for leisure to

⁴ Christiano, Eichenbaum, and Rebelo (2011) report similar results but it is easier for us to compare our results with the results of Christiano and Eichenbaum (2012) because they also posit Rotemberg adjustment costs.

FIGURE 2: TAX MULTIPLIER ON HOURS FOR ALTERNATIVE VALUES OF RISK AVERSION AND PRICE ADJUSTMENT COSTS: NO TECHNOLOGY SHOCK



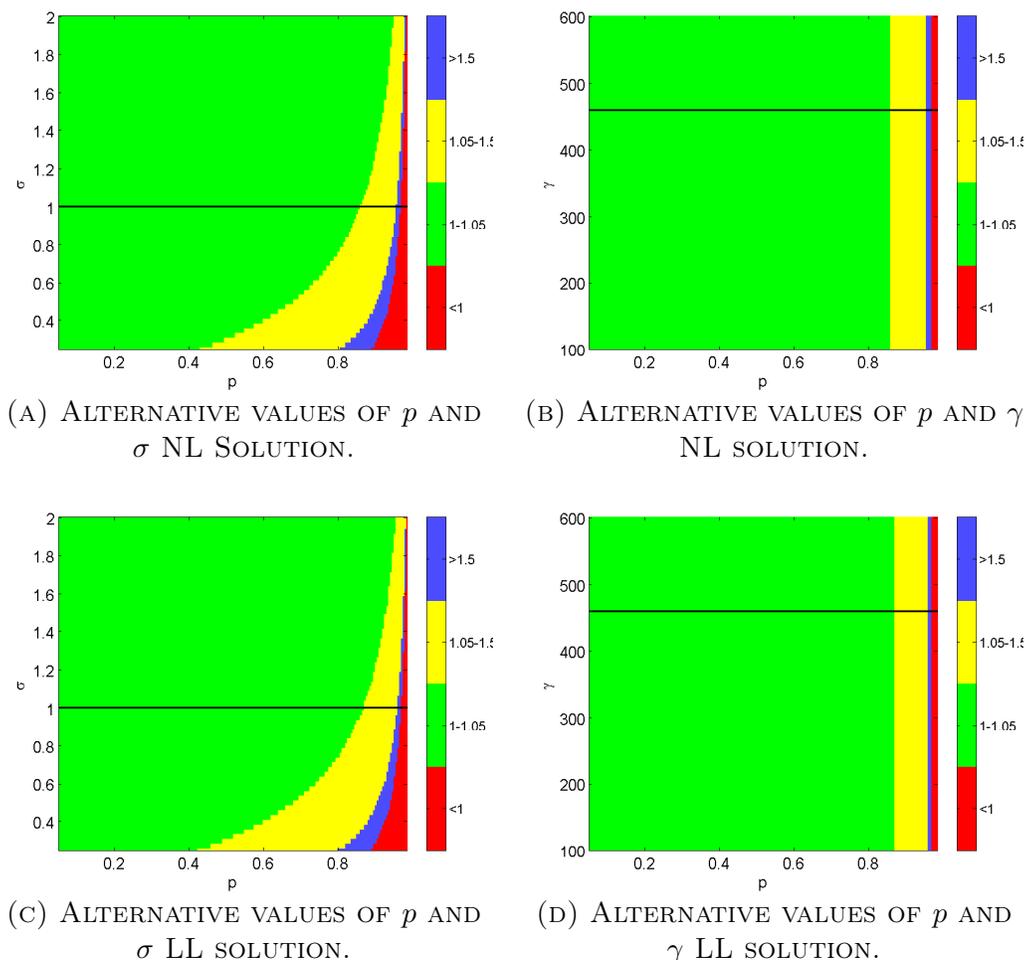
Notes: Red: labor tax multiplier is negative (employment increases when the labor tax is cut); green: labor tax multiplier is in $[0, 0.03]$; yellow: labor tax multiplier is in $(0.03, 1.0]$; blue: labor tax multiplier exceeds one; white: $\theta < 1$. The line denotes the baseline value of each parameter.

$\nu = 1$. The technology parameter θ that governs the substitutability of different types of goods is set to 3, the adjustment costs of price adjustment to $\gamma = 100$, and the conditional probability of exiting the low state to $p = 0.775$. The labor tax τ_w is set to 0.2, the government purchases share in output η to 0.2, and the subsidy to intermediate goods producers τ_s is set so that steady-state profits are zero. Finally, the coefficients on the Taylor rule are $\phi = 1.5$ and $\phi_y = 0$. With this parameterization our loglinearized system is identical to the one in [Christiano and Eichenbaum \(2012\)](#).⁵

We first examine whether some small differences in the nonlinear models are crucial for

⁵One difference between the models is that [Christiano and Eichenbaum \(2012\)](#) fix the level of government purchases as opposed to its share in output. However, the loglinearized systems are equivalent when one considers the same type of fiscal policy shock.

FIGURE 3: GOVERNMENT PURCHASE MULTIPLIER ON GDP FOR ALTERNATIVE VALUES OF RISK AVERSION AND PRICE ADJUSTMENT COSTS: NO TECHNOLOGY SHOCK



Notes: Red: government-purchase-GDP-multiplier < 1 ; green: the multiplier is in $[1, 1.05]$; yellow: the multiplier is in $[1.05, 1.5]$; blue: the multiplier exceeds 1.5; white: $\theta < 1$. The baseline parameterization is denoted with a line.

the differences in results, by solving our model using their parameter values.⁶ It turns out the differences in the two models are inconsequential and we are able to reproduce the government purchase multipliers reported in [Christiano and Eichenbaum \(2012\)](#) by setting $d^L = 1.0118$. The resulting government purchase multiplier for GDP is 2.2 using the NL solution and 2.8 using the LL solution method.

The reason their government purchase multipliers are so large is because the [Christiano and Eichenbaum \(2012\)](#) parameterization has a very steep AS schedule. Their parameterization implies that $\text{slope}(NKPC)$ is about 0.06 which is about three times larger than our

⁶We assume that the resource costs of price adjustment apply to gross production ($\gamma\pi_t^2 y_t$) whereas they assume that the resource costs of price adjustment only apply to GDP ($\gamma\pi_t^2(c_t + g_t)$).

baseline value of 0.02.⁷ From Equation (19) we know that a larger value of $\text{slope}(NKPC)$ translates directly into a steeper AS curve. A steeper AS curve also results in a much larger inflation response to a d^L shock of a given size. In particular, their parameterization associates the 7% decline in output in the LL solution with a 7% decline in the annualized inflation rate. If we solve the model using the NL equilibrium conditions instead, output and the annualized inflation rate both fall by 5%. In fact, their value of $\text{slope}(NKPC)$ is so large that the model overstates the GR inflation targets for all values of p using either the LL or the NL equilibrium conditions.⁸

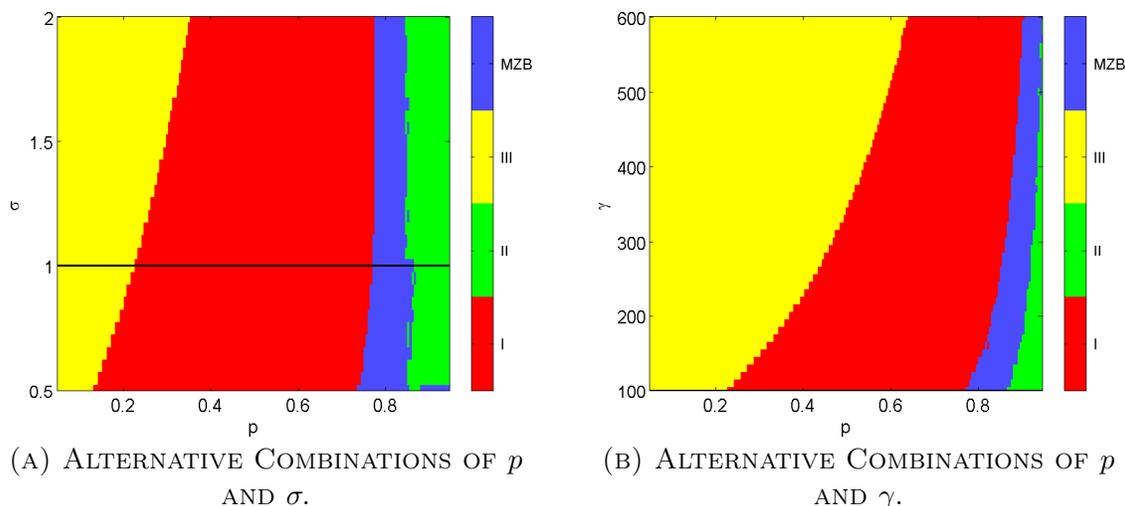
Given how different their results are from ours we would like to adjust their parameterization so that the model can hit the two GR targets using the NL solution. One way to do this is to hold fixed their choices of the structural parameters and thus $\text{slope}(NKPC)$ and to use the technology shock to hit the inflation rate. Results reported in Figures 4–7 implement this scheme. From Figure 4 we see that under this calibration strategy their model parameterization ($\sigma = 1, p = 0.775$, and $\gamma = 100$) falls in the indeterminacy region and is just to the left of the point as indexed by p where $\text{slope}(AS)$ becomes equal to $\text{slope}(AD)$ then crosses. The targeted equilibrium exhibits $\text{slope}(AD) > \text{slope}(AS) > 0$ and the resulting government purchase multiplier is 4.6. The non-targeted zero bound equilibrium exhibits $\text{slope}(AS) > \text{slope}(AD) > 0$ and there is a third equilibrium with a positive interest rate as shown in Figure 5. We have pointed out in the paper that the local properties of equilibrium and thus the sign and magnitude of the fiscal multipliers is very sensitive to the precise choice of parameters in this region of the parameter space. That result obtains here too. For instance, if p is increased from 0.775 to 0.79, the equilibrium switches to Case II and the government purchase multiplier is -10.0.

Christiano and Eichenbaum (2012) do not allow for technology shocks. So we also calibrate the model by adjusting θ and holding z fixed. This calibration scheme will also shift the asymptote to the right in the p dimension and it is possible that inferences about the size of the government purchase multiplier will be more robust. However, if we are to reproduce the GR facts using this calibration scheme we must also reduce $\text{slope}(NKPC)$ and in order to make the AS schedule flatter. We achieve this by lowering the value of θ to 1.24 and also increasing γ to 300. The reason why we have to adjust both of these parameters is because if we try to recalibrate their model by adjusting d^L and θ only, the resulting value of θ is less than 1 and thus not economically meaningful.

⁷Their parameterization implies $\text{slope}(NKPC) = 0.06$ when the share of government purchases in output is held fixed. If instead the level of government purchases is held fixed as they assume, $\text{slope}(NKPC)$ is 0.0675.

⁸This result also occurs if we set d^L to produce a 7% decline in output using the NL equilibrium conditions.

FIGURE 4: TYPES OF ZERO BOUND EQUILIBRIA FOR ALTERNATIVE VALUES OF RISK AVERSION AND PRICE ADJUSTMENT COSTS: CHRISTIANO-EICHENBAUM (2012) PARAMETERIZATION WITH TECHNOLOGY SHOCK



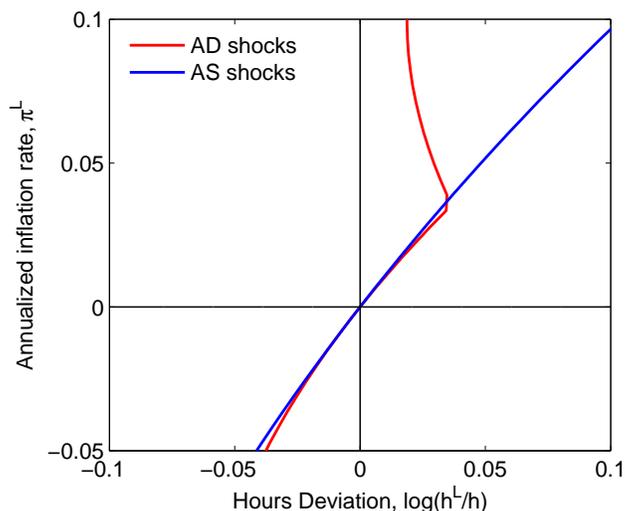
Notes: Red: Case I ($\text{slope}(AD) > 0 > \text{slope}(AS)$); light green: Case II ($\text{slope}(AS) > \text{slope}(AD) > 0$); yellow: Case III ($\text{slope}(AD) < 0 < \text{slope}(AS)$); blue; Case MZB (multiple zero bound equilibria); dark green: Case IV ($\text{slope}(AD) > 0 > \text{slope}(AS)$).

Figures 8-10 report results using the no-technology-shock calibration scheme. We saw above that the no-technology shock calibration scheme resulted in lower fiscal multipliers for larger values of p because $\text{slope}(AS)$ is independent of p .⁹ This is also true here. For instance, the equilibrium is now determinate and of type Case II and the government purchase multiplier is 1.06 when ($\sigma = 1, \theta = 1.24, \gamma = 300, p = 0.775$). In fact, the government purchase multiplier only exceeds 1.5 when $p \in [0.94, 0.965]$ but in that area, $\theta < 1$. (8).

The no-technology-shock results have several other noteworthy features. Now Case MZB equilibria occur at higher values of σ even when p is small and far away from the asymptote. The targeted zero bound equilibrium in this region continues to have $\text{slope}(AD) < 0$ and $\text{slope}(AS) > 0$ and it follows that the labor tax multiplier has an orthodox sign in this entire region (Figure 9). In the non-target equilibrium inflation and output exceed their steady-state levels but it is still a ZLB equilibrium because d^L is large. Note also that the overall size of the region with a downward-sloping AD schedule is smaller in Figure 8 as compared to our baseline calibration without technology shocks reported in the left panel of Figure 1. This is because the value of $\gamma = 300$ is lower than our baseline value of 458.4. The pattern of equilibria when γ is varied is qualitatively similar in the right panels of Figures 1 and 8. Most of the results with $\gamma < 300$ are not economically meaningful because the associated

⁹The value of $\text{slope}(AS) = 0.0342$ using our Christiano and Eichenbaum (2012) reference parameterization of the model: ($\sigma = 1, \theta = 1.24, \gamma = 300, p = 0.775$).

FIGURE 5: ZERO-LOWER BOUND EQUILIBRIUM AT $p = 0.775$: CHRISTIANO-EICHENBAUM (2012) PARAMETERIZATION WITH TECHNOLOGY SHOCK



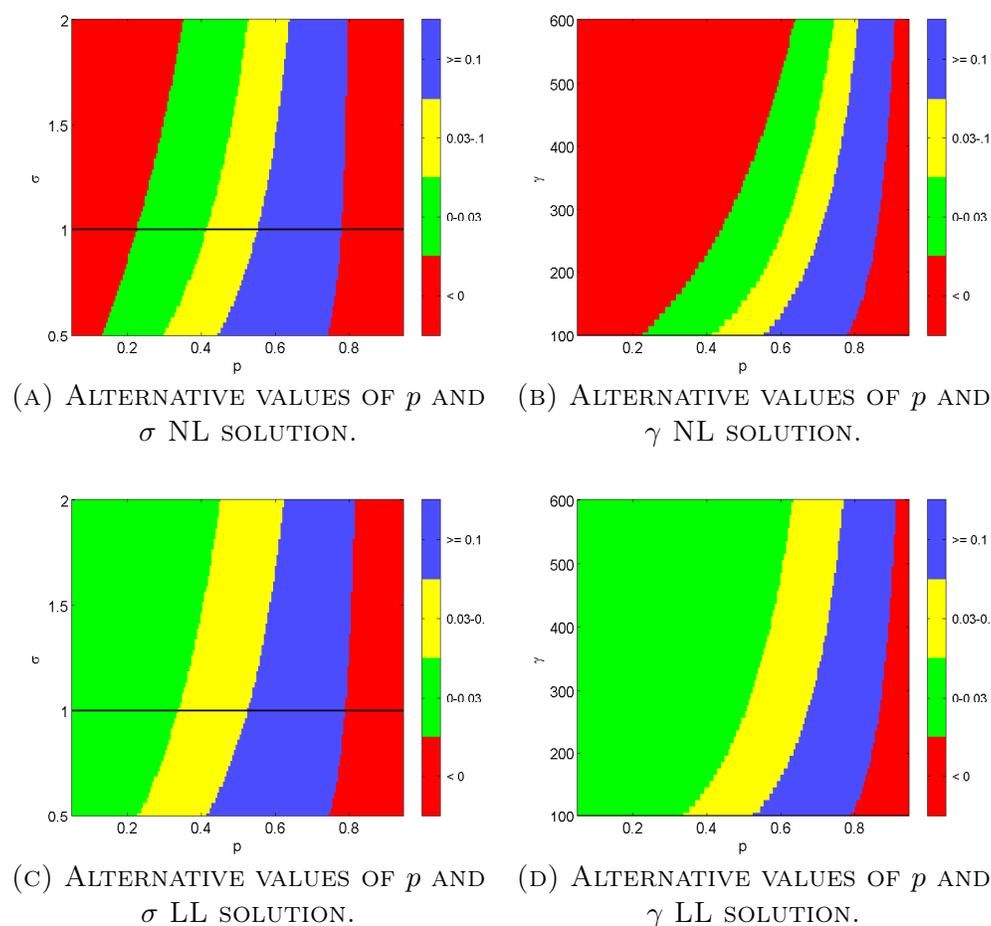
value of $\theta < 1$. Still, θ is positive and we can compute an equilibrium due to the tax subsidy. Thus, to facilitate comparison with our other results, the right panel of figures 8 - 10 also report values of γ that are less than 300. At higher levels of γ the LL solution significantly overstates the size of the region where the sign of the employment response to a tax cut is unorthodox. In regions where the NL solution indicates that the equilibrium is of Case I and thus unorthodox, the LL solution overstates the size of the labor tax.

A.3 Accounting for the Great Recession with the parameterization of Denes, Eggertsson and Gilbukh (2013)

We now consider the parameterization of Denes, Eggertsson, and Gilbukh (2013). Their estimated parameterization is interesting because their estimates imply a much smaller value of $\text{slope}(NKPC) = 0.0075$ than we have considered up to this point.

Denes, Eggertsson, and Gilbukh (2013) consider a NK framework with Calvo price adjustment and firm specific labor markets and a single shock to d^L . This is a different model from ours and the results that follow should not be interpreted as saying anything about their structural model. The common link between their model and ours is that they solve their model using the LL solution method we described in the paper and the loglinearized reduced form of their model and is identical to ours. They estimate their model parameters using an overidentified Quasi-Bayesian method of moments procedure with two moments that they associate with the GR: an output decline of 10% and an (annualized) decline in the inflation rate of -2%. The resulting estimates are: $(p, \theta, \beta, \sigma, \nu) = (0.857, 13.23, 0.997, 1.22, 1.69)$.

FIGURE 6: LABOR TAX MULTIPLIER ON EMPLOYMENT FOR ALTERNATIVE VALUES OF RISK AVERSION AND PRICE ADJUSTMENT COSTS: CHRISTIANO-EICHENBAUM (2012) PARAMETERIZATION WITH TECHNOLOGY SHOCK



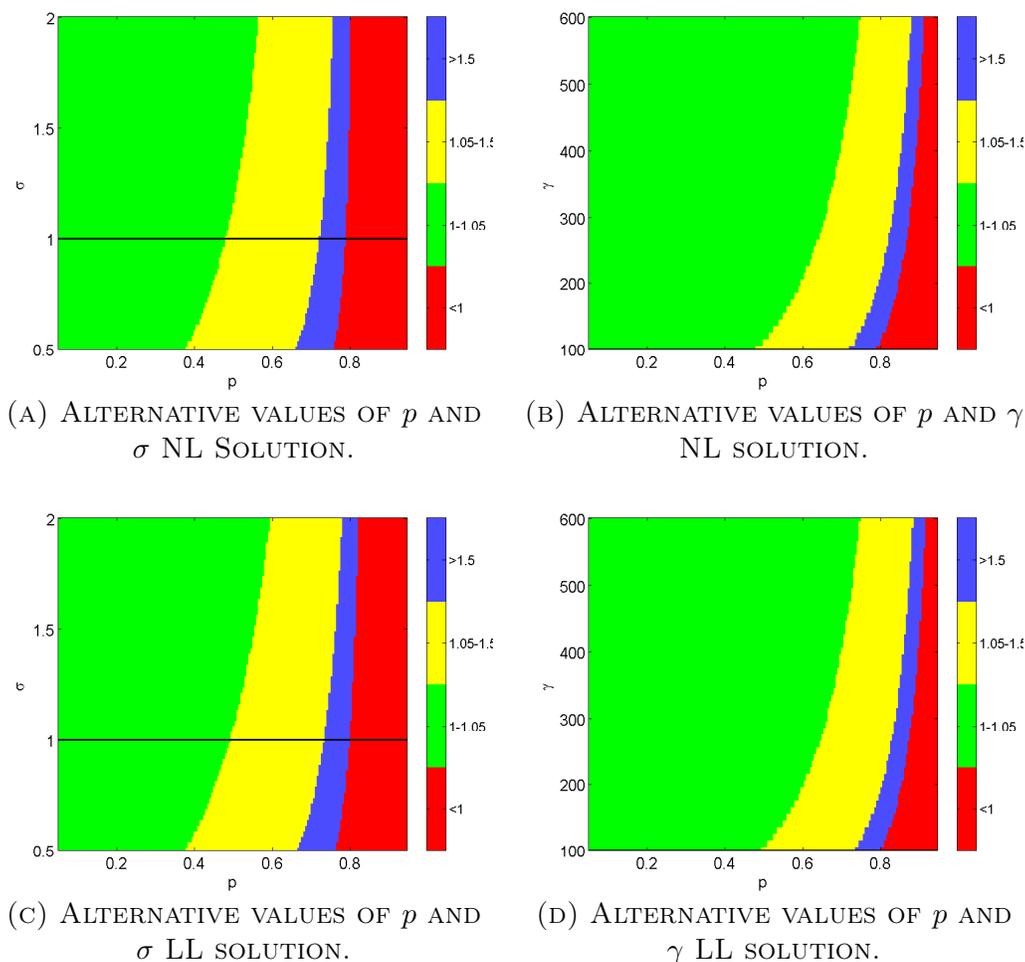
Notes: Red: labor tax multiplier is negative (employment increases when the labor tax is cut); green: labor tax multiplier is in $[0, 0.03]$; yellow: labor tax multiplier is in $(0.03, 1.0]$; blue: labor tax multiplier exceeds one. The baseline values of each parameter are denoted with a black line.

Finally, their fiscal parameters are set in the same way that we have assumed up to now.

We are interested in understanding the properties of our model in this region of the parameter space. However, in order to do that we must first make some small adjustments to their estimates. Our practice has been to calibrate the model using our *nonlinear* equilibrium conditions. Here we adjust d^L and γ to reproduce our GR inflation and output targets using our NL equilibrium conditions.¹⁰ The resulting value of $\gamma = 6341$. These adjustments have only a very small effect on $\text{slope}(NKPC)$. It rises from 0.0075 using the estimated parameterization of [Denes, Eggertsson, and Gilbukh \(2013\)](#) to 0.0086 using our calibrated

¹⁰The value of γ implied by their estimated reduced form is very large and calibrating our model in this way brings the value of γ down a bit and allows us to use their estimated value of θ as a reference point.

FIGURE 7: GOVERNMENT PURCHASE MULTIPLIER ON GDP FOR ALTERNATIVE VALUES OF RISK AVERSION AND PRICE ADJUSTMENT COSTS: CHRISTIANO-EICHENBAUM (2012) PARAMETERIZATION WITH TECHNOLOGY SHOCK

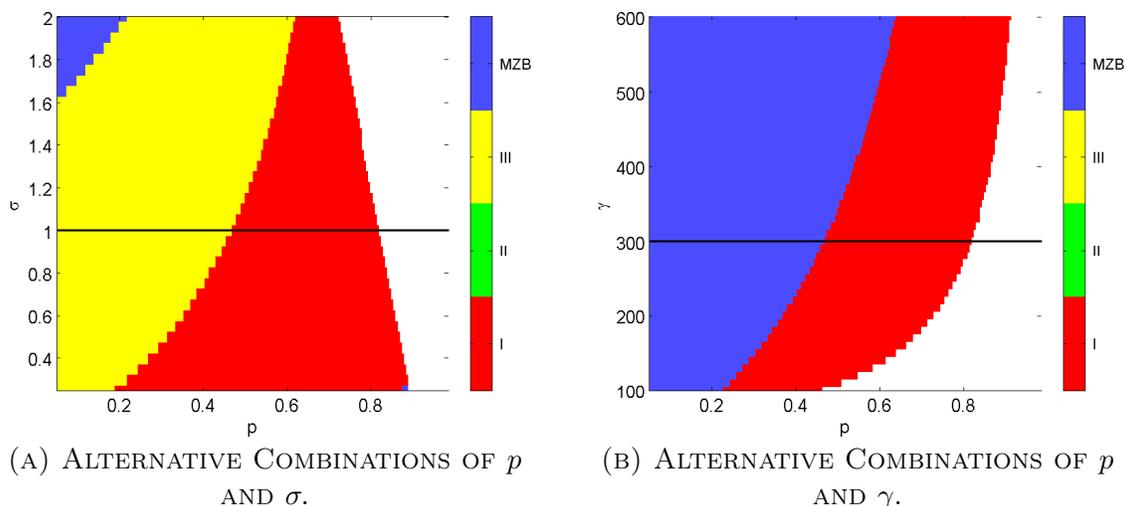


Notes: Red: government-purchase-GDP-multiplier < 1 ; green: the multiplier is in $[1, 1.05]$; yellow: the multiplier is in $[1.05, 1.5]$; blue: the multiplier exceeds 1.5. The baseline parameterization is denoted with a line.

values of d^L and γ .

Why is γ so large for this parameterization? Our discussion in Section 4.1 and 4.2 of the paper implies that $\text{slope}(NKPC)$ must be small to produce a Case 1 equilibrium when p is large. Indeed, the value of $\text{slope}(NKPC)$ here is less than half the size implied by our baseline calibration. To understand why γ is large observe that the [Denes, Eggertsson, and Gilbukh \(2013\)](#) estimates of the other parameters in $\text{slope}(NKPC)$ including θ , σ , and ν are all much higher than our estimates. The fact that these other parameters are so large implies that γ must also be very large if $\text{slope}(NKPC)$ is to be small enough to reproduce the GR targets at high values of p , see also Appendix .

FIGURE 8: TYPES OF ZERO BOUND EQUILIBRIA FOR ALTERNATIVE VALUES OF PRICE ADJUSTMENT COSTS AND RISK AVERSION: CHRISTIANO-EICHENBAUM (2012) PARAMETERIZATION WITHOUT TECHNOLOGY SHOCK



Notes: Red: Case I ($\text{slope}(AD) > 0 > \text{slope}(AS)$); light green: Case II ($\text{slope}(AS) > \text{slope}(AD) > 0$); yellow: Case III ($\text{slope}(AD) < 0 < \text{slope}(AS)$); blue: Case MZB (multiple zero bound equilibria); dark green: Case IV ($\text{slope}(AD) > 0 > \text{slope}(AS)$); white: $\theta < 1$.

The most noteworthy new property of the model is shown in Figure 11.¹¹ The Case III region is now much smaller and instead there is a very large Case MZB region at low values of p . The size of this region increases with σ and γ . Throughout this region there are two zero bound equilibria, the targeted equilibrium has $\text{slope}(AD) < 0 < \text{slope}(AS)$ and the second has $0 < \text{slope}(AD) < \text{slope}(AS)$.

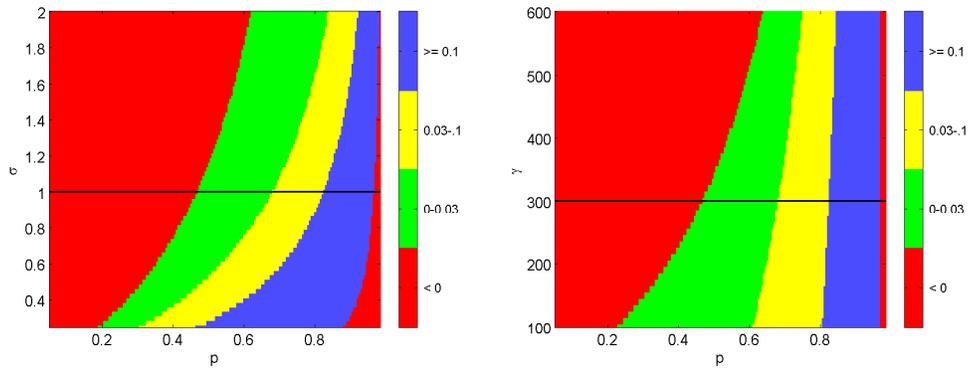
The model has a unique Case III equilibrium at the reference value of $p = 0.857$, estimated by Denes, Eggertsson, and Gilbukh (2013). This implies that employment increases in response to a cut in the labor tax (Figure 12). Using the LL solution one would conclude instead that there is a paradox of toil and that it is large (0.11).

The size of the government purchase multiplier using the NL equilibrium conditions is 1.08 at the reference parameterization and with marginally higher σ it would fall below 1.05 (Figure 13). There are some larger differences between the NL and LL government purchase multipliers here. The LL solution produces larger government purchase multipliers at lower values of p . However, once again we see that a government purchase multiplier in excess of 1.5 only occurs in a very small region of the parameter space as indexed by p , σ and γ .

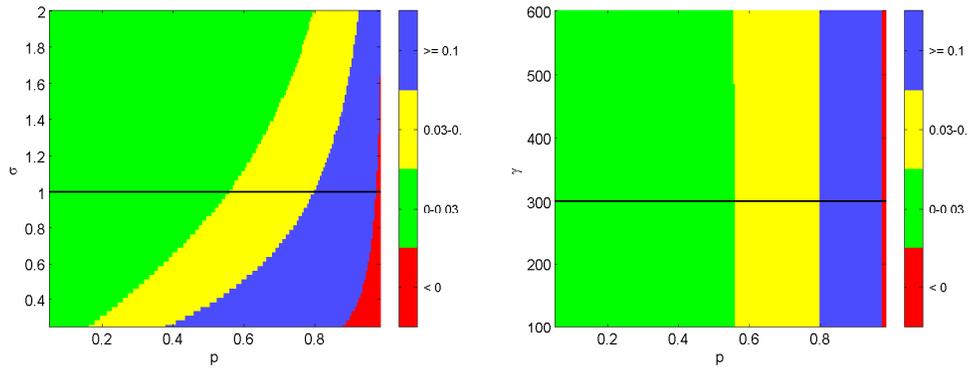
To summarize, the results we have reported here are consistent with the message of our paper. The NK model may also exhibit orthodox and small fiscal multipliers at the zero

¹¹To conserve on space we only report figures using the no-technology-shock calibration scheme for this parameterization of our model.

FIGURE 9: TAX MULTIPLIER ON EMPLOYMENT FOR ALTERNATIVE VALUES OF RISK AVERSION: CHRISTIANO-EICHENBAUM (2012) PARAMETERIZATION WITHOUT TECHNOLOGY SHOCK



(A) ALTERNATIVE VALUES OF p AND σ NL SOLUTION. (B) ALTERNATIVE VALUES OF p AND γ NL SOLUTION.

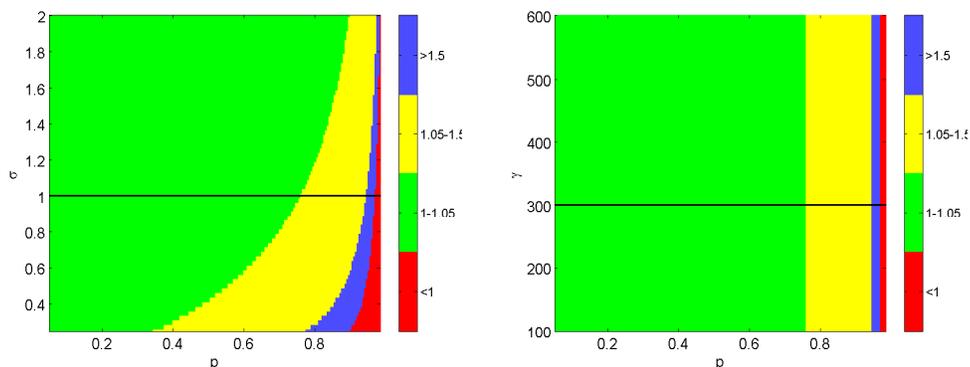


(C) ALTERNATIVE VALUES OF p AND σ LL SOLUTION. (D) ALTERNATIVE VALUES OF p AND γ LL SOLUTION.

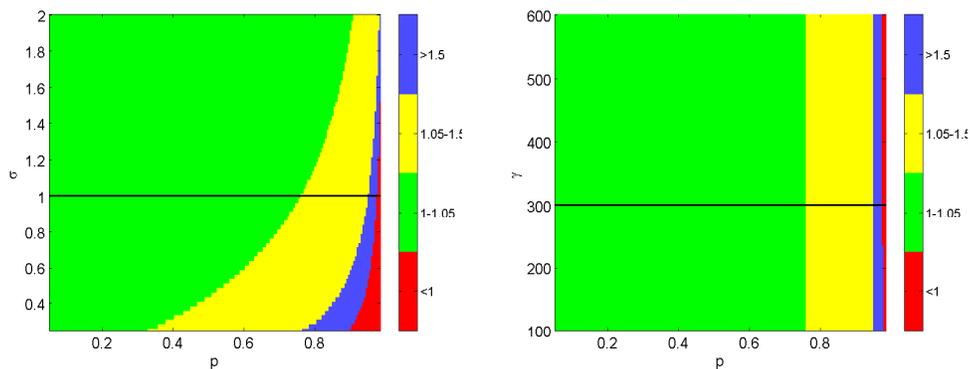
Notes: Red: labor tax multiplier is negative (employment increases when the labor tax is cut); green: labor tax multiplier is in $[0, 0.03]$; yellow: labor tax multiplier is in $(0.03, 1.0]$; blue: labor tax multiplier exceeds one; white: $\theta < 1$. The line denotes the baseline value of each parameter.

bound in these other regions of the parameter space.

FIGURE 10: GOVERNMENT PURCHASE MULTIPLIER ON GDP FOR ALTERNATIVE VALUES OF RISK AVERSION: CHRISTIANO-EICHENBAUM (2012) PARAMETERIZATION WITHOUT TECHNOLOGY SHOCK



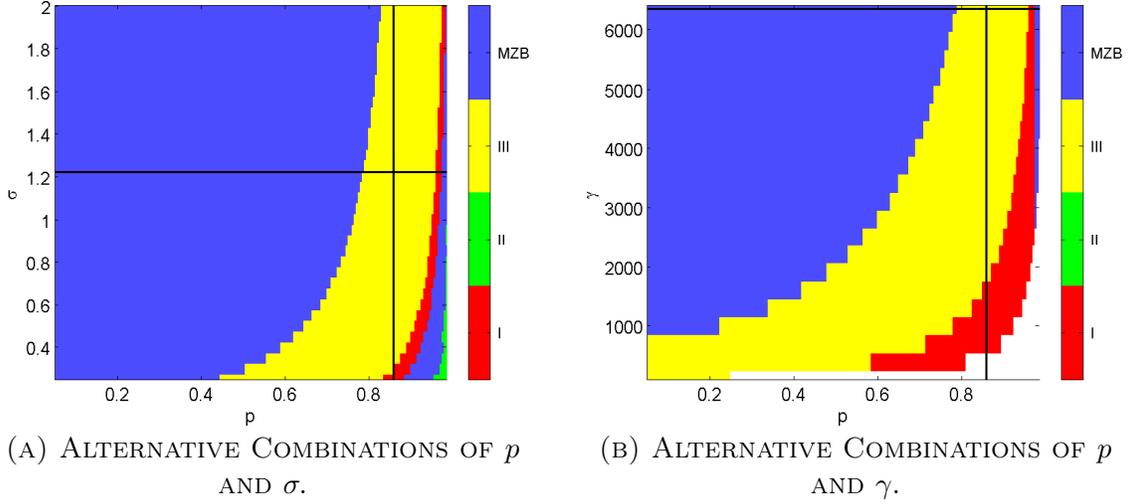
(A) ALTERNATIVE VALUES OF p AND σ NL SOLUTION. (B) ALTERNATIVE VALUES OF p AND γ NL SOLUTION.



(C) ALTERNATIVE VALUES OF p AND σ LL SOLUTION. (D) ALTERNATIVE VALUES OF p AND γ LL SOLUTION.

Notes: Red: government-purchase-GDP-multiplier < 1 ; green: the multiplier is in $[1, 1.05]$; yellow: the multiplier is in $[1.05, 1.5]$, blue: the multiplier exceeds 1.5; white: $\theta < 1$. The baseline parameterization is denoted with a line.

FIGURE 11: TYPES OF ZERO BOUND EQUILIBRIA FOR ALTERNATIVE VALUES OF RISK AVERSION AND PRICE ADJUSTMENT COSTS: DENES ET AL. (2013).
PARAMETERIZATION WITH NO TECHNOLOGY SHOCKS.



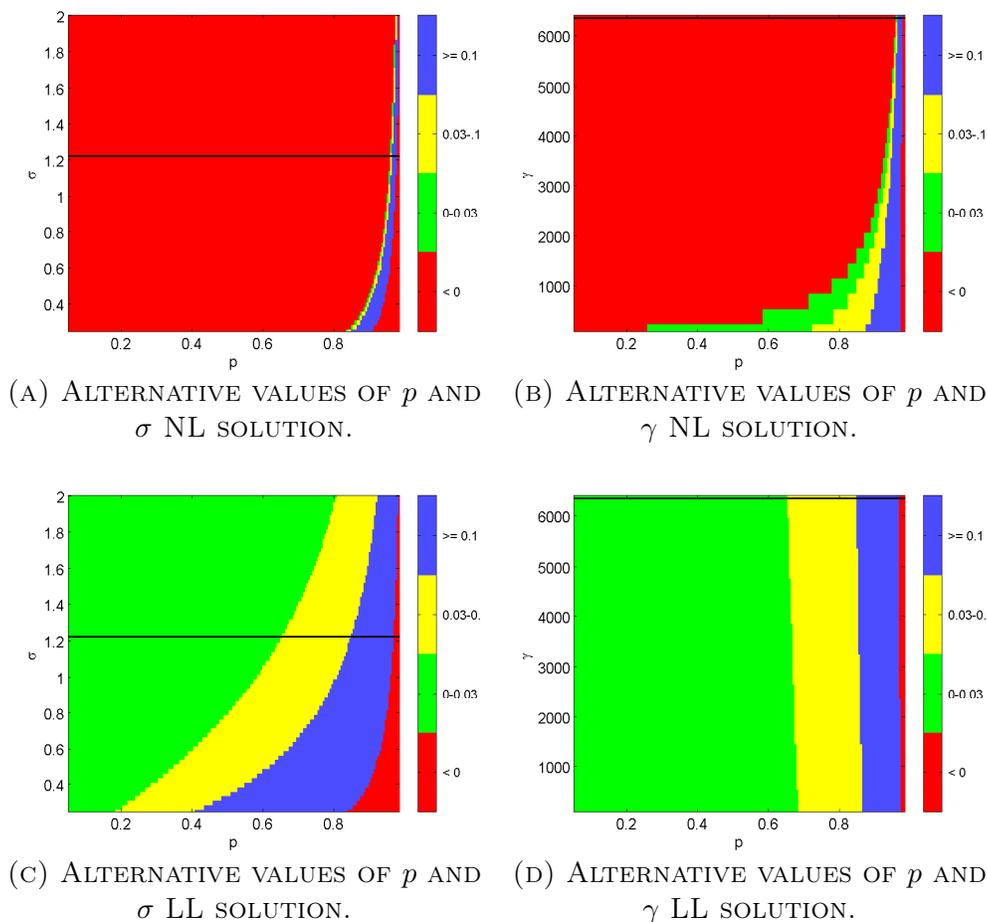
Notes: Red: Case I ($\text{slope(AD)} > 0 > \text{slope(AS)}$); light green: Case II ($\text{slope(AS)} > \text{slope(AD)} > 0$); yellow: Case III ($\text{slope(AD)} < 0 < \text{slope(AS)}$); blue: Case MZB (multiple zero bound equilibria); dark green: Case IV ($\text{slope(AD)} > 0 > \text{slope(AS)}$); white: $\theta < 1$.

Appendix B Calvo model with a single labor market

This section presents the equilibrium conditions in the Calvo model with a single labor market. They are given by

$$\begin{aligned}
 c_t^\sigma h_t^\nu &= w_t(1 - \tau_{w,t}), \\
 1 &= \beta d_{t+1} E_t \left\{ \frac{1 + R_t}{1 + \pi_{t+1}} \left(\frac{c_t}{c_{t+1}} \right)^\sigma \right\} \\
 gdp_t &= \frac{1}{x_t} z_t h_t, \\
 c_t &= \left(\frac{1}{x_t} - \eta_t \right) z_t h_t, \\
 as_{1,t} &= gdp_t + \beta \alpha d_{t+1} E_t \left[\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} (1 + \pi_{t+1})^{\theta-1} as_{1,t+1} \right], \\
 as_{2,t} &= \frac{c_t^\sigma h_t^\nu}{(1 - \tau_{w,t})} \frac{gdp_t}{z_t} + \beta \alpha d_{t+1} E_t \left[\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} (1 + \pi_{t+1})^\theta as_{2,t+1} \right], \\
 \tilde{P}_t &= \frac{as_{2,t}}{as_{1,t}}, \\
 x_t &= (1 - \alpha) \tilde{P}_t^{-\theta} + \alpha (1 + \pi_t)^\theta x_{t-1}, \\
 1 &= (1 - \alpha) \tilde{P}_t^{1-\theta} + \alpha (1 + \pi_t)^{\theta-1} \\
 R_t &= \max(0, r_t^e + \phi_\pi \pi_t + \phi_y \widehat{gdp}_t)
 \end{aligned}$$

FIGURE 12: TAX MULTIPLIER ON EMPLOYMENT FOR ALTERNATIVE VALUES OF RISK AVERSION AND PRICE ADJUSTMENT COSTS: DENES ET AL. (2013).
PARAMETERIZATION WITH NO TECHNOLOGY SHOCKS

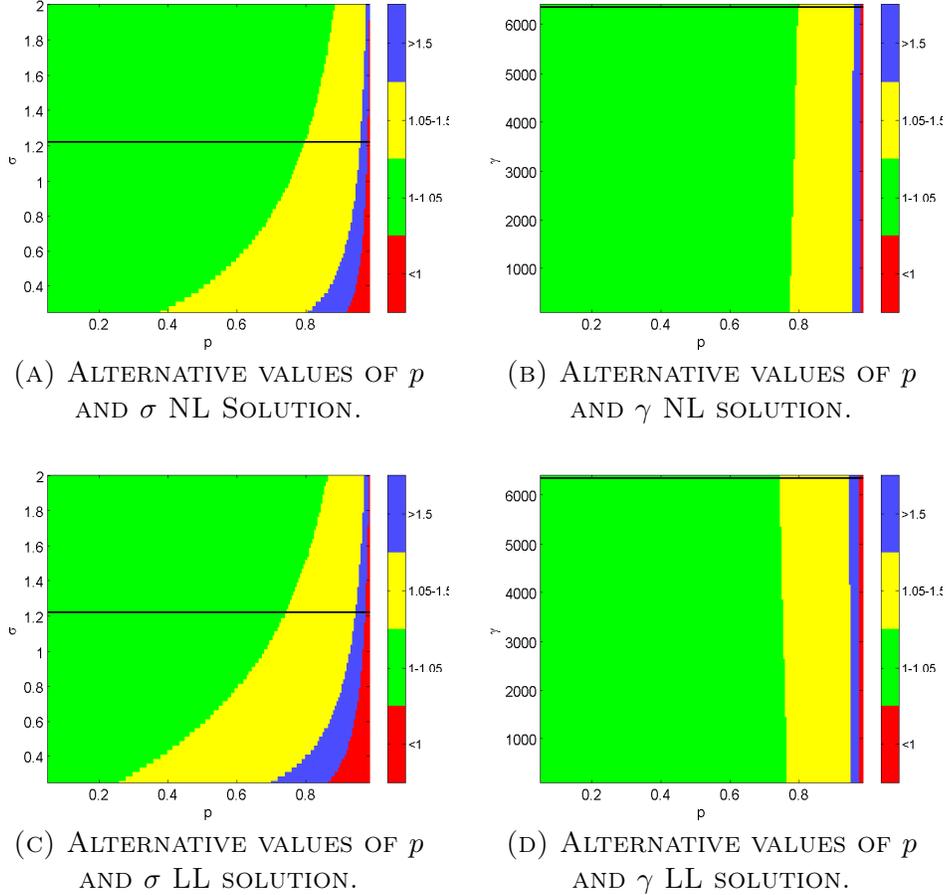


Notes: Red: labor tax multiplier is negative (employment increases when the labor tax is cut); green: labor tax multiplier is in $[0, 0.03]$; yellow: labor tax multiplier is in $(0.03, 1.0]$; blue: labor tax multiplier exceeds one; white: $\theta < 1$. The line denotes the baseline value of each parameter.

where \tilde{P}_t is the real price which is chosen by firms that can change their nominal prices at time t , x_t summarizes the relative price dispersion, and α is the probability that a firm is unable to change its price. The term $1/x_t$ introduces the wedge between GDP (y_t) and gross output ($z_t h_t$), and $1 - 1/x_t$ acts as κ_t in our baseline Rotemberg model. In NK models with Calvo price setting, there is a non-degenerate relative price distribution and as a result the allocation of the factors of production (labor in this model) is inefficient, i.e. higher price dispersion reduces GDP compared to the maximal production level that is possible with the same level of factor input.

Because x_t is a state variable, the equilibrium conditions cannot be summarized by the AD and the AS schedules without any additional simplifying assumptions. We make the following

FIGURE 13: GOVERNMENT PURCHASE MULTIPLIER ON GDP FOR ALTERNATIVE VALUES OF RISK AVERSION AND PRICE ADJUSTMENT COSTS: DENES ET AL. (2013) PARAMETERIZATION WITH NO TECHNOLOGY SHOCKS, NONLINEAR (TOP) VS. LOGLINEAR (BOTTOM)



Notes: Red: government-purchase-GDP-multiplier < 1 ; green: the multiplier is in $[1, 1.05]$; yellow: the multiplier is in $[1.05, 1.5]$; blue: the multiplier exceeds 1.5; white: $\theta < 1$. The baseline parameterization is denoted with a line.

assumption: x is constant at x^L when the shocks are (d^L, z^L) and becomes 1 immediately after the shocks dissipate. This allows us to use the AD and the AS schedules to characterize a zero bound equilibrium.

Once the shocks dissipate, all variables jump to the zero inflation steady-state, where $x = 1$, $gdp = h = \{(1 - \tau_w)/(1 - \eta)^\sigma\}^{1/(\sigma+\nu)}$, $as_1 = h/(1 - \beta\alpha)$, $as_2 = (1 - \eta)^\sigma h^{1+\sigma+\nu}/\{(1 - \beta\alpha)(1 - \tau_w)\}$, and $\tilde{P} = 1$. In the L state, by assumption, x^L can be written as

$$x^L = \frac{(1 - \alpha)(\tilde{P}^L)^{-\theta}}{1 - \alpha(1 + \pi^L)^\theta} = \frac{1 - \alpha}{1 - \alpha(1 + \pi^L)^\theta} \left\{ \frac{1 - \alpha(1 + \pi^L)^{\theta-1}}{1 - \alpha} \right\}^{\frac{\theta}{\theta-1}}.$$

The AD schedule is identical to the AD schedule in the Rotemberg model, except that the

term κ^L is now equal to $(x^L - 1)/x^L$ (equation (13) in the main paper).

The AS schedule is a little bit more complicated. First observe

$$\tilde{P}^L = \frac{(c^L)^{-\sigma} as_2^L}{(c^L)^{-\sigma} as_1^L}.$$

Then using

$$(c^L)^{-\sigma} as_1^L = (c^L)^{-\sigma} gdp^L + \beta\alpha d^L \{p(c^L)^{-\sigma}(1 + \pi^L)^{\theta-1} as_1^L + (1-p)c^{-\sigma} as_1\}$$

and

$$(c^L)^{-\sigma} as_2^L = \frac{(c^L)^\sigma (h^L)^\nu}{(1 - \tau_w^L)} \frac{gdp^L}{z^L} + \beta\alpha d^L \{p(c^L)^{-\sigma}(1 + \pi^L)^\theta as_2^L + (1-p)c^{-\sigma} as_2\},$$

we obtain ¹²

$$\tilde{P}^L = G(h^L, \pi^L).$$

Because

$$\tilde{P}^L = \left\{ \frac{1 - \alpha(1 + \pi^L)^{\theta-1}}{1 - \alpha} \right\}^{\frac{1}{1-\theta}},$$

the AS schedule is characterized by

$$\left\{ \frac{1 - \alpha(1 + \pi^L)^{\theta-1}}{1 - \alpha} \right\}^{\frac{1}{1-\theta}} = G(h^L, \pi^L).$$

Figure 14 shows the AD-AS diagram for $p = 0.8$ and $p = 0.9$. For both cases the AD and the AS schedules are upward sloping, but as in the Rotemberg model the slope configurations switch: the AD schedule is steeper in the former case but is flatter in the latter case.

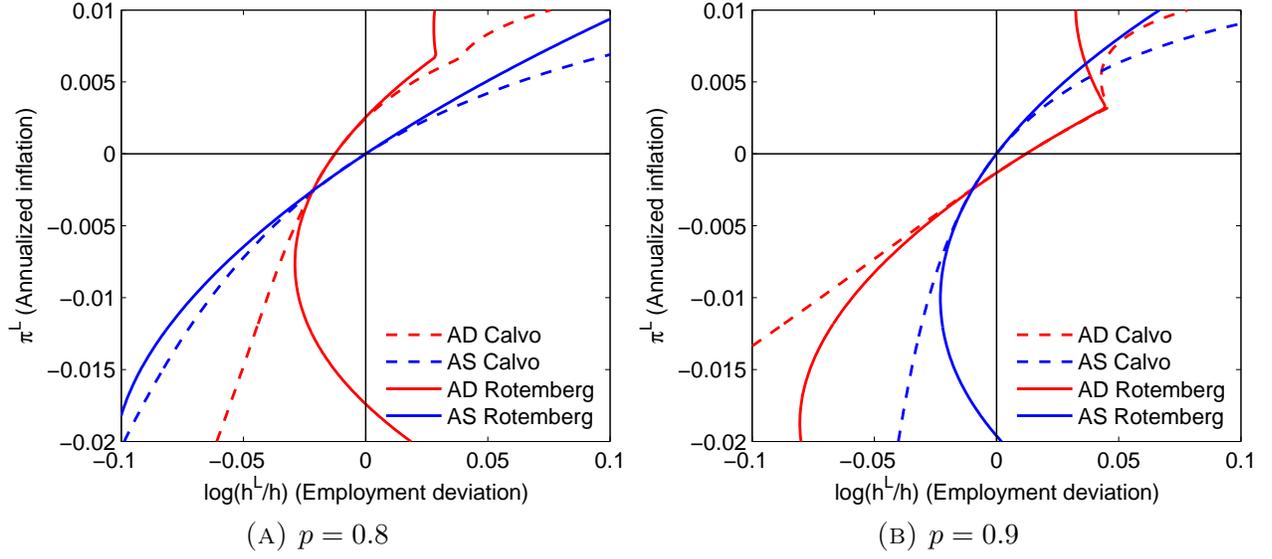
Appendix C Existence of a zero bound equilibrium in the LL model

To make the argument more transparent we assume that $\hat{\eta}^L = \hat{\tau}_w^L = 0$.

Proposition 1 Existence of a zero bound equilibrium in the LL model. *Suppose $\hat{\eta}^L = \hat{\tau}_w^L = 0$, $(\phi_\pi, \phi_y) \geq (p, 0)$, $\hat{d}^L \geq 0$, $\hat{z}^L \leq 0$, $0 < p \leq 1$, $\sigma \geq 1$, and that AD^{LL} and AS^{LL}*

¹²This is because both c^L and gdp^L can be expressed by x^L , h^L , and exogenous variables and parameters, and because x^L is a function of π^L .

FIGURE 14: ZERO-BOUND EQUILIBRIA IN THE CALVO VS. ROTEMBERG MODEL



do not coincide in state L . Then there exists a unique zero bound equilibrium with deflation and depressed labor input, $(\pi^L, \hat{h}^L) < (0, 0)$, if

Case I

1a) $\text{slope}(AD^{LL}) > \text{slope}(AS^{LL})$ and

1b) $(\text{slope}(AD^{LL}) - \text{slope}(AS^{LL}) \frac{\sigma-1}{\sigma+\nu}) \hat{z}^L - \frac{\hat{r}_L^e}{p} > 0$,

or

Case II

2a) $\text{slope}(AD^{LL}) < \text{slope}(AS^{LL})$ and

2b) $(\text{slope}(AD^{LL}) - \text{slope}(AS^{LL}) \frac{\sigma-1}{\sigma+\nu}) \hat{z}^L - \frac{\hat{r}_L^e}{p} < 0$.

If the parameters do not satisfy either both 1a) and 1b) or alternatively both 2a) and 2b), then there is no zero bound equilibrium with depressed labor input $\hat{h}^L < 0$. ¹³

¹³The final statement leaves open the possibility that a zero bound equilibrium with $\hat{h}^L \geq 0$ exists for parameterizations that satisfy 1a) and 2b) (or 1b) and 2a)). This is possible when z^L is sufficiently low. If it is assumed that $\hat{z}^L = 0$, then the final clause can be removed and any ZLB equilibrium must satisfy both (1a) and (1b) or both (2a) and (2b). For further details see [Braun, Körber, and Waki \(2012\)](#).

Proof The AD and the AS schedules are

$$\pi^L = [\text{slope}(AD)\hat{z}^L - \frac{\hat{r}_L^e}{p}] + \text{slope}(AD)\hat{h}^L, \text{ and } \pi^L = \text{slope}(AS)\frac{\sigma-1}{\sigma+\nu}\hat{z}^L + \text{slope}(AS)\hat{h}^L.$$

They are upward-sloping, for both $\text{slope}(AD)$ and $\text{slope}(AS)$ are positive.

First, assume (1a) and (1b). They imply that the AD schedule is strictly steeper than the AS schedule, and that the intercept term is strictly higher for the AD schedule than for the AS schedule. It follows that at the intersection $\hat{h}^L < 0$. Solving for π^L , we obtain

$$\begin{aligned} \pi^L &= \frac{1}{\text{slope}(AS) - \text{slope}(AD)} \left[\text{slope}(AS)\left\{\text{slope}(AD)\hat{z}^L - \frac{\hat{r}_L^e}{p}\right\} - \text{slope}(AD)\text{slope}(AS)\frac{\sigma-1}{\sigma+\nu}\hat{z}^L \right] \\ &= \frac{\text{slope}(AS)}{\text{slope}(AS) - \text{slope}(AD)} \left[\text{slope}(AD)\hat{z}^L + \text{slope}(AD)\frac{\sigma-1}{\sigma+\nu}(-\hat{z}^L) - \frac{\hat{r}_L^e}{p} \right]. \end{aligned}$$

Since $\text{slope}(AS) - \text{slope}(AD) < 0$, π^L is negative at the intersection if and only if the terms in the square brackets are positive.

$$\begin{aligned} &\text{slope}(AD)\hat{z}^L + \text{slope}(AD)\frac{\sigma-1}{\sigma+\nu}(-\hat{z}^L) - \frac{\hat{r}_L^e}{p} \\ &\geq \text{slope}(AD)\hat{z}^L + \text{slope}(AS)\frac{\sigma-1}{\sigma+\nu}(-\hat{z}^L) - \frac{\hat{r}_L^e}{p} \\ &\quad (\text{By assumption, } (\sigma-1)(-\hat{z}^L) \geq 0 \text{ and } \text{slope}(AS) - \text{slope}(AD) < 0.) \\ &> 0. \quad (\text{By condition 1a).} \end{aligned}$$

Thus, at the intersection of the AD and the AS schedules, $(\pi^L, \hat{h}^L) < (0, 0)$.

What remains to show is that at the intersection, the Taylor rule implies zero nominal interest rate. The linear part of the Taylor rule prescribes

$$\begin{aligned} &\hat{r}^e + \phi_\pi\pi^L + \phi_y\hat{y}^L \\ &< p \left(\text{slope}(AD) - \text{slope}(AS)\frac{\sigma-1}{\sigma+\nu} \right) \hat{z}^L + \phi_\pi\pi^L + \phi_y\hat{y}^L. \quad (\text{Condition 1a).} \end{aligned}$$

Since $\hat{y}^L = \hat{z}^L + \hat{h}^L$, we know that $(\hat{z}^L, \pi^L, \hat{y}^L)$ are all negative. Condition 1b) implies that the coefficient on \hat{z}^L is strictly positive. Together with the assumption $(\phi_\pi, \phi_y) \geq 0$, it follows that the RHS of the above inequality is strictly negative, and the nominal interest

rate at the intersection is zero.

Next, assume (2a) and (2b). Proof is almost the same as that in the case with (1a) and (1b). The only difference is that ϕ_π needs to be sufficiently large to have $\hat{r}_L^e + \phi_\pi \pi^L + \phi_y \hat{y}^L < 0$. Since the AD schedule is upward sloping and $\hat{h}^L < 0$, π^L is smaller than the intercept of the AD schedule, $\text{slope}(AD)\hat{z}^L - \hat{r}_L^e/p$. Thus, the assumption $\phi_\pi \geq p$ implies

$$\hat{r}_L^e + \phi_\pi \pi^L + \phi_y \hat{y}^L \leq \hat{r}_L^e + p\pi^L \leq \hat{r}_L^e + p\{\text{slope}(AD)\hat{z}^L - \hat{r}_L^e/p\} \leq p \times \text{slope}(AD)\hat{z}^L \leq 0.$$

The nominal interest rate is thus zero.

Finally, suppose 1a) holds but 1b) doesn't. Then the AD is no steeper than the AS, and the intercept of the AD is larger than the intercept of the AS. When the AD and the AS are parallel but their intercepts differ, then there is no intersection and thus no equilibrium with $R = 0$. When the AS is strictly steeper than the AD, then their intersection satisfies $\hat{h}^L > 0$, and there is no ZLB equilibrium with $\hat{h}^L \leq 0$.

The same argument goes through for the case where 2a) holds but 2b) doesn't. \square

Appendix D Loglinearization of the AD and the AS schedules using the L state as a reference point and formulas for multipliers

D.1 Loglinearization of the AD and the AS schedules

When computing multipliers it is sometimes convenient to loglinearize the AD and AS schedules about state L instead. Let $\Delta\pi = \pi - \pi^L$, $\Delta h = \ln(h/h^L)$, $\Delta z = \ln(z/z^L)$, $\Delta\eta = \eta - \eta^L$, and $\Delta\tau_w = \tau_w - \tau_w^L$, then the loglinearized AD equation is

$$\begin{aligned} 1 &= (1-p)\beta d^L \frac{(1-\kappa^L - \eta^L)^\sigma (z^L)^\sigma (h^L)^\sigma}{(1-\eta)^\sigma z^\sigma h^\sigma} \left(1 + \sigma(\Delta h + \Delta z) - \frac{\sigma\Delta\eta}{1-\kappa^L - \eta^L} - \frac{\sigma\gamma\pi^L\Delta\pi}{1-\kappa^L - \eta^L}\right) \\ &\quad + \frac{p\beta d^L}{1+\pi^L} \left(1 - \frac{\Delta\pi}{1+\pi^L}\right) \\ &= \frac{p\beta d^L}{1+\pi^L} \left(1 - \frac{\Delta\pi}{1+\pi^L}\right) + \left(1 - \frac{p\beta d^L}{1+\pi^L}\right) \left(1 + \sigma(\Delta h + \Delta z) - \frac{\sigma\Delta\eta}{1-\kappa^L - \eta^L} - \frac{\sigma\gamma\pi^L\Delta\pi}{1-\kappa^L - \eta^L}\right). \end{aligned}$$

Thus

$$\begin{aligned}
slope(AD) &= \frac{(1 - \frac{p\beta d^L}{1+\pi^L})\sigma}{\frac{p\beta d^L}{(1+\pi^L)^2} + (1 - \frac{p\beta d^L}{1+\pi^L})\frac{\sigma\gamma\pi^L}{1-\kappa^L-\eta^L}} \\
icept(AD) &= -\frac{(1 - \frac{p\beta d^L}{1+\pi^L})\frac{\sigma}{1-\kappa^L-\eta^L}}{\frac{p\beta d^L}{(1+\pi^L)^2} + (1 - \frac{p\beta d^L}{1+\pi^L})\frac{\sigma\gamma\pi^L}{1-\kappa^L-\eta^L}} [\Delta\eta - (1 - \kappa^L - \eta^L)\Delta z] \\
&= slope(AD)\Delta z - slope(AD)\frac{\Delta\eta}{1 - \kappa^L - \eta^L}.
\end{aligned}$$

Loglinearizing the AS equation yields:

$$\begin{aligned}
0 &= \frac{(1 - \kappa^L - \eta^L)^\sigma h_L^{\sigma+\nu}}{(1 - \tau_w^L)(z^L)^{1-\sigma}} \left[1 + (\sigma + \nu)\Delta h - (1 - \sigma)\Delta z - \frac{\sigma\Delta\eta}{1 - \kappa^L - \eta^L} + \frac{\Delta\tau_w}{1 - \tau_w^L} - \frac{\sigma\gamma\pi^L\Delta\pi}{1 - \kappa^L - \eta^L} \right] \\
&\quad - (1 - p\beta d^L)\frac{\gamma}{\theta} [\pi^L(1 + \pi^L) + (1 + 2\pi^L)\Delta\pi] \\
&= -(1 - p\beta d^L)\frac{\gamma}{\theta}(1 + 2\pi^L)\Delta\pi + \left[(1 - p\beta d^L)\frac{\gamma}{\theta}\pi^L(1 + \pi^L) + 1 \right] \\
&\quad \times \left[(\sigma + \nu)\Delta h - (1 - \sigma)\Delta z - \frac{\sigma\Delta\eta}{1 - \kappa^L - \eta^L} + \frac{\Delta\tau}{1 - \tau_w^L} - \frac{\sigma\gamma\pi^L\Delta\pi}{1 - \kappa^L - \eta^L} \right].
\end{aligned}$$

Thus

$$\begin{aligned}
slope(AS) &= \frac{[(1 - p\beta d^L)\frac{\gamma}{\theta}\pi^L(1 + \pi^L) + 1](\sigma + \nu)}{(1 - p\beta d^L)\frac{\gamma}{\theta}(1 + 2\pi^L) + [(1 - p\beta d^L)\frac{\gamma}{\theta}\pi^L(1 + \pi^L) + 1]\frac{\sigma\gamma\pi^L}{1-\kappa^L-\eta^L}} \\
icept(AS) &= \frac{[(1 - p\beta d^L)\frac{\gamma}{\theta}\pi^L(1 + \pi^L) + 1]}{(1 - p\beta d^L)\frac{\gamma}{\theta}(1 + 2\pi^L) + [(1 - p\beta d^L)\frac{\gamma}{\theta}\pi^L(1 + \pi^L) + 1]\frac{\sigma\gamma\pi^L}{1-\kappa^L-\eta^L}} \\
&\quad \times \left[-(1 - \sigma)\Delta z - \frac{\sigma\Delta\eta}{1 - \kappa^L - \eta^L} + \frac{\Delta\tau}{1 - \tau_w^L} \right] \\
&= slope(AS)\frac{1}{\sigma + \nu} \left[-(1 - \sigma)\Delta z - \frac{\sigma\Delta\eta}{1 - \kappa^L - \eta^L} + \frac{\Delta\tau}{1 - \tau_w^L} \right].
\end{aligned}$$

Loglinearizing the aggregate resource constraint $c_L = (1 - \eta^L - \kappa^L)z^L h^L$ yields

$$\Delta c = \Delta h + \Delta z - \frac{\Delta\eta}{1 - \kappa^L - \eta^L} - \frac{\gamma\pi^L\Delta\pi}{1 - \kappa^L - \eta^L}.$$

Loglinearizing GDP $gdp^L = (1 - \kappa^L)z^L h^L$ yields

$$\Delta gdp = \Delta h + \Delta z - \frac{\gamma\pi^L\Delta\pi}{1 - \kappa^L}.$$



D.2 Multiplier formulas

Labor tax multiplier

Our multiplier measures are based on the above system that is loglinearized around a zero bound equilibrium. Note that

$$\Delta h = \frac{icept(AS) - icept(AD)}{slope(AD) - slope(AS)},$$

and

$$\Delta\pi = slope(AD)\Delta h + icept(AD) = slope(AS)\Delta h + icept(AS).$$

The labor tax multiplier on hours is thus

$$\frac{\partial\Delta h}{\partial\Delta\tau_w} = \frac{1}{slope(AD) - slope(AS)} \frac{\partial icept(AS)}{\partial\Delta\tau_w} = \frac{1}{\frac{slope(AD)}{slope(AS)} - 1} \frac{1}{\sigma + \nu} \frac{1}{1 - \tau_w}.$$

And the multiplier on inflation is:

$$\frac{\partial\Delta\pi}{\partial\Delta\tau_w} = slope(AD) \frac{\partial\Delta h}{\partial\Delta\tau_w} = \frac{slope(AD)}{\frac{slope(AD)}{slope(AS)} - 1} \frac{1}{\sigma + \nu} \frac{1}{1 - \tau_w}.$$

These multipliers are the *marginal* ones, for they are derived from elasticities.

Clearly, the slopes of the AD and the AS schedules are crucial for the multipliers. The sign of the multiplier on hours is positive when the relative slope, $slope(AD)/slope(AS)$, is bigger than one, and negative when it is less than one. Therefore, whenever the AD and the AS schedules have different signs, the multiplier on hours is negative. The multiplier is positive only when both schedules have the same signed slopes and the AD schedule is steeper. The absolute size of the multiplier explodes as the two schedules' slopes become closer.

Government purchase multiplier

To calculate the government purchases multiplier, it is convenient to start by deriving the multipliers associated with perturbations in the share of government purchases in output:

$$\frac{\partial\Delta h}{\partial\Delta\eta} = \frac{\frac{slope(AD)}{slope(AS)} - \frac{\sigma}{\sigma + \nu}}{\frac{slope(AD)}{slope(AS)} - 1} \frac{1}{1 - \kappa^L - \eta^L}$$

and

$$\begin{aligned}\frac{\partial \Delta \pi}{\partial \Delta \eta} &= \text{slope}(AD) \frac{\frac{\text{slope}(AD)}{\text{slope}(AS)} - \frac{\sigma}{\sigma + \nu}}{\frac{\text{slope}(AD)}{\text{slope}(AS)} - 1} \frac{1}{1 - \kappa^L - \eta^L} - \text{slope}(AD) \frac{1}{1 - \kappa^L - \eta^L} \\ &= \frac{\text{slope}(AD)}{\frac{\text{slope}(AD)}{\text{slope}(AS)} - 1} \frac{\nu}{\sigma + \nu} \frac{1}{1 - \kappa^L - \eta^L} \\ \frac{\partial \Delta gdp}{\partial \Delta \eta} &= \frac{\partial \Delta h}{\partial \Delta \eta} - \frac{\gamma \pi^L}{1 - \kappa^L} \frac{\partial \Delta \pi}{\partial \Delta \eta}.\end{aligned}$$

Since $\Delta g = \Delta \eta / \eta^L + \Delta h$,

$$\frac{\partial \Delta g}{\partial \Delta \eta} = \frac{1}{\eta^L} + \frac{\partial \Delta h}{\partial \Delta \eta}.$$

We can then calculate the multipliers associated with perturbations in the level of government purchases as follows

$$\begin{aligned}\text{Government purchases hours multiplier} &: \left(h^L \times \frac{\partial \Delta h}{\partial \Delta \eta} \right) / \left(g^L \times \frac{\partial \Delta g}{\partial \Delta \eta} \right) \\ \text{Government purchases GDP multiplier} &: \left(gdp^L \times \frac{\partial \Delta gdp}{\partial \Delta \eta} \right) / \left(g^L \times \frac{\partial \Delta g}{\partial \Delta \eta} \right) \\ \text{Government purchases inflation multiplier} &: \left(\frac{\partial \Delta \pi}{\partial \Delta \eta} \right) / \left(g^L \times \frac{\partial \Delta g}{\partial \Delta \eta} \right).\end{aligned}$$

Note that the government purchases increase with η when $\partial \Delta g / \partial \Delta \eta$ is positive. In such a case, the sign of the consumption response determines whether the government purchase multiplier on GDP is bigger than or less than one. Because the Euler equation implies that consumption and inflation are positively related when the nominal rate is constant, it suffices to know whether the inflation response is positive or not. What determines its sign and size is

$$\text{slope}(AD) / \left\{ \frac{\text{slope}(AD)}{\text{slope}(AS)} - 1 \right\}.$$

If the AD schedule is upward-sloping, the inflation response is positive when the AS schedule is also upward-sloping but flatter than the AD schedule. If both schedules are upward-sloping and the AS schedule is steeper, then the inflation response is negative. If the AD schedule is instead downward-sloping, the inflation response is positive either (i) when the AS schedule is upward-sloping, or (ii) when the AS schedule is downward-sloping and steeper than the AD schedule.

Effects of a technology shock

The responses of employment, output and inflation to a change in technology can be derived in the following way

$$\frac{\partial \Delta h}{\partial \Delta z} = \frac{\frac{\sigma-1}{\sigma+\nu} - \frac{\text{slope}(AD)}{\text{slope}(AS)}}{\frac{\text{slope}(AD)}{\text{slope}(AS)} - 1},$$

and it follows that

$$\frac{\partial \Delta y}{\partial \Delta z} = \frac{\partial \Delta h}{\partial \Delta z} + 1$$

and

$$\frac{\partial \Delta \pi}{\partial \Delta z} = \text{slope}(AS) \left[\frac{\partial \Delta h}{\partial \Delta z} + \frac{\sigma - 1}{\sigma + \nu} \right].$$

Observe, that output increases and hours and inflation fall in response to an improvement in technology when $\sigma = 1$ in a Case III equilibrium since $\text{slope}(AD) < 0 < \text{slope}(AS)$. In a Case II equilibrium we have ($0 < \text{slope}(AD) < \text{slope}(AS)$) and it follows that an improvement in technology increases employment, output and the inflation rate when $\sigma = 1$.

Appendix E Loglinear Slope of New Keynesian Phillips Curve

This section provides a more detailed analysis on the restrictions imposed by the calibration target.

Denote the slope coefficient (on output) in the loglinear New Keynesian Phillips curve by

$$\text{slope}(NKPC) := \frac{\theta(\sigma + \nu)}{\gamma},$$

. Then the AS schedule can be written as

$$\pi^L = \frac{\text{slope}(NKPC)}{(1 - p\beta)} \hat{h}^L + \frac{\text{slope}(NKPC)}{(1 - p\beta)(\sigma + \nu)} \left[-\sigma \frac{\hat{\eta}^L}{1 - \eta} + \frac{\hat{\tau}_w^L}{1 - \tau_w} - (1 - \sigma) \hat{z}^L \right]. \quad (1)$$

This relationship holds not only for our model but also for a wide range of NK models including those with Calvo price setting. Importantly, $\text{slope}(NKPC)$ is independent of

TABLE 1: VALUES OF p AND $slope(NKPC)$ THAT REPRODUCE THE GREAT RECESSION TARGETS

p	$slope(NKPC)$
0.16	0.03
0.44	0.02
0.58	0.015
0.66	0.012
0.72	0.01
0.86	0.005

(p, \hat{d}^L) .

First, we argue that **if $\hat{z}^L = \hat{\eta}^L = \hat{\tau}^L = 0$, the loglinearized model is unable to reproduce the Great Recession target with high p unless $slope(NKPC)$ is sufficiently low.**

Under the stated assumptions,

$$1 - p\beta = slope(NKPC) \frac{\hat{h}^L}{\pi^L} \Leftrightarrow p = \frac{1}{\beta} \left[1 - slope(NKPC) \frac{\hat{h}^L}{\pi^L} \right].$$

Our Great Recession target is $(\hat{h}^L, \pi^L) \approx (-0.07, -0.01/4)$, hence $\hat{h}^L/\pi^L \approx 28$. This implies the following:

- (A) For the model to reproduce the Great Recession target, it is necessary that $slope(NKPC) \leq 1/28 \approx 0.036$ (this is implied by the non-negativity of p)
- (B) When $\beta \approx 1$, the value of p with which the model reproduces the Great Recession targets is reported in Table 1:

To put these numbers in perspective, consider values of the Calvo parameter implied by these values of $slope(NKPC)$. In the NK model with Calvo price setting and a homogeneous labor market, $slope(NKPC)$ is given by the formula

$$slope(NKPC) = \frac{(1 - \alpha)(1 - \beta\alpha)}{\alpha}(\sigma + \nu),$$

where α is the Calvo parameter. If $(\sigma, \nu) = (1, 1)$ and $\beta \approx 1$, the right hand side equals 0.01 when the Calvo parameter α is as high as 0.93, and equals 0.02 when α is around 0.905. If $(\sigma, \nu) = (1, 0.28)$ and $\beta \approx 1$ as in our baseline specification, the right hand side equals 0.01 when α is around 0.916, and equals 0.02 when α is around 0.883.¹⁴ Intuitively, the GR

¹⁴In our model, $slope(NKPC)$ is about 0.021 and thus it corresponds to a Calvo parameter of $\alpha \approx 0.883$.

target inflation rate is so much lower than the target output decline that the New Keynesian Phillips Curve has to be very flat in order to be consistent with the target.

Note that the arguments so far are conditional on $\hat{z}^L = 0$. Deep recessions may bring about some production efficiency loss through e.g. resource misallocation, and/or lower utilization rates for production factors. When $\hat{z}^L < 0$ is allowed, the AS schedule is

$$\pi^L = \frac{\text{slope}(NKPC)}{(1-p\beta)} \hat{y}^L - \frac{\text{slope}(NKPC)}{(1-p\beta)} \frac{1+\nu}{\sigma+\nu} \hat{z}^L. \quad (2)$$

Note that we rewrite it with $\hat{y}^L = \hat{h}^L + \hat{z}^L$. This is because we are fixing the target values for inflation rate and GDP, and with a technology shock GDP and labor input are different. Restrictions imposed by the calibration targets are most transparently seen when labor input \hat{h}^L is replaced by $\hat{y}^L - \hat{z}^L$. This leads to the following expression

$$p = \frac{1}{\beta} \left[1 - \text{slope}(NKPC) \left\{ \frac{\hat{y}^L}{\pi^L} - \frac{1+\nu}{\sigma+\nu} \frac{\hat{z}^L}{\pi^L} \right\} \right]. \quad (3)$$

For pre-specified targets $(\hat{y}^L, \pi^L) < (0, 0)$, lowering $\hat{z}^L < 0$ increases the implied value for p for given preference parameters and $\text{slope}(NKPC)$. For instance, If $\sigma = 1$ and $\beta \approx 1$, then a value of p of 0.76 can be produced by $\text{slope}(NKPC)$ of about 0.015 together with $\hat{z}^L = -0.03$, or $\text{slope}(NKPC)$ of about 0.02 in conjunction with $\hat{z}^L = -0.04$. This is intuitive: for a given \hat{y}^L , negative technology shocks add inflationary pressure, and the NKPC does not need be so flat to produce a small amount of disinflation together with a relatively large decline in output.

It is worth mentioning that the above discussion does not use any information about the AD schedule, and hence these results also obtain in the true nonlinear model as long as loglinearization methods approximates the AS schedule well.

Next, we point out that **when p is close to one an asymptote or a Case 1 equilibrium is only possible if $\text{slope}(NKPC)$ is very small.** To understand this, observe that $\text{slope}(AD) \gtrless \text{slope}(AS)$ can be written as

$$\frac{(1-p)(1-p\beta)}{p} \gtrless \frac{\text{slope}(NKPC)}{\sigma}.$$

When the left hand side is larger (smaller) than the right hand side, the AD schedule is steeper (flatter) than the AS schedule.

This relationship has several implications. First, let \bar{p} be the value of p which satisfies the above with equality. When $p \rightarrow \bar{p}$, $\text{slope}(AD)/\text{slope}(AS) \rightarrow 1$ and the denominators in the

multiplier formulas go to zero as well. This results in an asymptote with very large positive or negative fiscal multipliers on each side of it. Second, since the left hand side of this inequality is decreasing in p , if we want to entertain very high p and $\text{slope}(AD) > \text{slope}(AS)$, then the right hand side $\text{slope}(NKPC)/\sigma$ must be sufficiently low. For example, when $\beta \approx 1$ and $p = 0.9$, the left hand side is approximately 0.0111. When $\sigma = 1$, then $\text{slope}(NKPC) < 0.0111$ must hold. This restriction is not very tight for modestly large p : e.g. for $p = 0.8$ and $\beta \approx 1$, the left hand side is around 0.05, and when $\sigma = 1$, the requirement is $\text{slope}(NKPC) < 0.05$.

Appendix F Estimation

Our Bayesian estimation procedure uses the log-linearized equilibrium conditions to solve the model and derive the likelihood function and assumes that agents assigned zero ex-ante probability to the possibility of $R = 0$. The estimated model has a more general shock structure than the two-state Markov model presented in Section 2 of the paper. In addition, to shocks to demand d_t and technology z_t , we allow for a shock to monetary policy ϵ_t . This makes it possible to estimate the model using three observables, the output gap, inflation and the nominal interest rate.¹⁵ The specification of the model that is estimated is given by the following equations. The nonlinear aggregate demand schedule is:

$$1 = \beta d_t E_t \frac{(1 + R_t)(gdp_{t+1}(1 - \eta))^{-\sigma}}{(1 + \pi_{t+1})(gdp_t(1 - \eta))^{-\sigma}} \quad (4)$$

and the aggregate supply schedule is:

$$\begin{aligned} \gamma \pi_t(1 + \pi_t) + (1 + \tau_s)(\theta - 1) &= \theta \frac{(gdp_t(1 - \eta))^\sigma gdp_t^\nu}{(1 - \tau_w)z_t^{1+\nu}(1 - \kappa_t)^\nu} \\ &+ \beta d_t E_t \frac{(gdp_t(1 - \eta))^\sigma gdp_{t+1}(1 - \kappa_t)}{(gdp_{t+1}(1 - \eta))^\sigma gdp_t(1 - \kappa_{t+1})} \gamma \pi_{t+1}(1 + \pi_{t+1}) \end{aligned} \quad (5)$$

The Taylor Rule is given by:

$$R_t = \rho R_{t-1} + (1 - \rho)(\phi_\pi \pi_t + \phi_y \hat{y}_t) + \epsilon_t \quad (6)$$

where \hat{y}_t , the log GDP gap, is given by:

$$\hat{y}_t = \ln((\exp(gdp_t)(1 - \eta))^{\sigma/(\sigma+\nu)} / (1 - \tau_w)^{1/(\sigma+\nu)}). \quad (7)$$

¹⁵Our measure of the output gap uses the Congressional Budget Office measure of potential GDP.

The shocks to demand and technology are assumed to follow AR 1 rules:

$$\log d_t = \rho_d \log d_{t-1} + u_{d,t} \quad (8)$$

$$\log z_t = \rho_z \log z_{t-1} + u_{z,t} \quad (9)$$

$$\epsilon_t = \rho_\epsilon \epsilon_{t-1} + u_{\epsilon,t} \quad (10)$$

where the shocks are assumed to be Gaussian with zero means and variance-covariance matrix

$$\Omega \equiv \begin{bmatrix} \sigma_d & 0 & 0 \\ 0 & \sigma_z & 0 \\ 0 & 0 & \sigma_\epsilon \end{bmatrix} \quad (11)$$

We estimated the model using version 4.3.3 of Dynare. When computing the posteriors, we specified Metropolis Hastings chains of length 60,000 and used 10 parallel chains. After some experimentation we set the scale of the jumping distribution for the Metropolis-Hastings algorithm to 0.68 which produced an acceptance ratio that ranged from 0.2-0.3. The other DYNARE options for Metropolis Hastings were set at their default values.

Priors, posterior modes, posterior means and 5 and 95 percentiles for all estimated parameters are reported in Table 2.

Appendix G Calibration

For our baseline exercises, we fix $(\beta, \sigma, \nu, \theta, \gamma)$ at their estimated/calibrated values. For given p , we adjust (d^L, z^L) to hit the inflation and output targets (π^L, gdp^L) . The level of technology in the high state (H) normalized to 1, and thus the steady-state values of all prices and allocations are known. We also know the value of consumption in the L state $c^L = (1 - \kappa^L - \eta^L)gdp^L / (1 - \kappa^L)$, because $z^L h^L = gdp^L / (1 - \kappa^L)$.

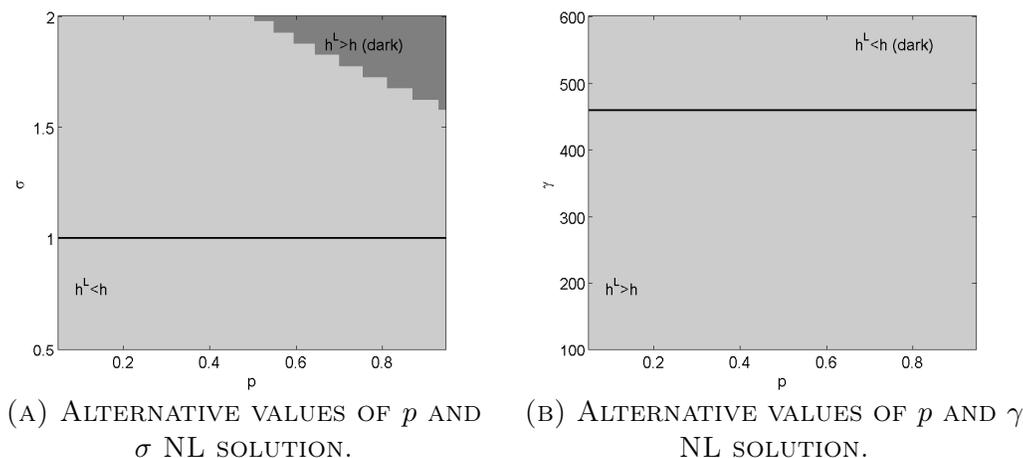
For a given p , we can solve the AD equation for d^L :

$$d^L = \left[\frac{p\beta}{1 + \pi^L} + (1 - p)\beta \left(\frac{c^L}{c} \right)^\sigma \right]^{-1}.$$

We then solve the AS equation for z^L :

$$\begin{aligned} \pi^L(1 + \pi^L) &= \frac{1}{1 - p\beta d^L} \frac{\theta}{\gamma} \left[\frac{(1 - \kappa^L - \eta^L)^\sigma (h^L)^{\sigma+\nu}}{(1 - \tau_w^L)(z^L)^{1-\sigma}} - 1 \right] \\ &= \frac{1}{1 - p\beta d^L} \frac{\theta}{\gamma} \left[\frac{(c^L)^{\sigma+\nu}}{(1 - \tau_w^L)(z^L)^{1+\nu}(1 - \kappa^L - \eta^L)^\nu} - 1 \right]. \end{aligned}$$

FIGURE 15: REGIONS WHERE EMPLOYMENT IS DEPRESSED AT THE ZERO LOWER BOUND FOR ALTERNATIVE VALUES OF p , σ AND γ .



Notes: Light gray: employment is below its steady-state value; dark gray: employment exceeds its steady-state value. The line denotes the baseline value of each parameter.

Note that all variables in this second equation are known except for z^L .

When considering the specification with constant technology, we restrict $z^L = 1$, we vary θ to hit the target. This proceeds in the following way. The preference shock d^L is calibrated first in the same way as before. This step does not require knowledge of θ . Then we use the second equation which is derived from the AS equation, to solve for θ . When calibrating our model to the parameterization of [Denes, Eggertsson, and Gilbukh \(2013\)](#), we restrict $z^L = 1$ and fix θ at their estimated value of this parameter, and then adjust d^L and γ instead to satisfy the above two equations.

Appendix H Employment at the zero bound

We have calibrated the model to produce a 7% decline in GDP. The resource costs of price adjustment, however, drive a wedge between GDP and employment and it is possible in some situations for employment in state L to exceed its steady-state level even though GDP is below its steady-state level. Figure 15 displays when this occurs. The reason this occurs is that technology in this situations is very depressed. If instead technology is held fixed as described in Section A, employment is always depressed at the zero bound.

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TABLE 2: PARAMETER ESTIMATES.

Parameter	Prior			Posterior			
	distribution	mean	std. dev.	mode	mean	5%	95%
ν	gamma	0.5	0.25	0.28	0.37	0.08	0.63
γ	normal	150	200	458.4	510.6	315.4	703.8
ϕ_y	normal	0	1	1.63	1.72	1.06	2.33
ϕ_π	normal	3	1	3.46	3.58	2.38	4.77
ρ_r	beta	0.75	0.1	0.86	0.85	0.81	0.90
ρ_ϵ	beta	0.75	0.1	0.86	0.86	0.81	0.91
ρ_z	beta	0.75	0.12	0.96	0.95	0.93	0.98
ρ_d	beta	0.75	0.1	0.88	0.88	0.83	0.92
σ_ϵ	inverse gamma	0.007	0.008	0.0014	0.0015	0.0012	0.0018
σ_z	inverse gamma	0.01	0.008	0.0052	0.0058	0.0038	0.0078
σ_d	inverse gamma	0.007	0.008	0.0027	0.0027	0.0023	0.0031

These estimates use U.S. quarterly data on the output gap, inflation rate and Federal Funds rate with a sample period of 1985:I-2007:IV.