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## Abstract

We argue that the uncertainty over the impact of macroprudential policy need not make a policymaker more cautious. Our starting point is the classic result of Brainard (1967) which finds that uncertainty over the impact of a policy instrument will make a policymaker less active. This result is challenged in a series of richer models designed to take into account the more complex reality faced by a macroprudential policymaker. We find that the presence of unquantifiable sources of risk, potential asymmetries in policy objectives, the ability to learn from policy actions, and private sector uncertainty over policy objectives can all lead to more active policy in the face of uncertainty.

**Key words:** Macroprudential policy, robust control, Linex, uncertainty.

**JEL classification:** D81, E58.

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# 1 Introduction

The macroprudential toolkit available to policymakers across several central banks is new and largely untested. For example, in the United Kingdom, the Bank of England's Financial Policy Committee (FPC) has, since the financial crisis, received powers to alter bank capital requirements and to place restrictions on the terms of household mortgages for macroprudential purposes. Neither of these policy tools has been used previously, so their impact and the Committee's reaction function remain unclear. Moreover, in contrast to monetary policy, where price stability can be judged against the rate of inflation, the objective of macroprudential policymakers, the stability of the financial system, is inherently unobservable. Thus macroprudential policymakers face a high degree of uncertainty over the impact and effectiveness of their tools and a target variable they cannot perfectly observe. In the face of this uncertainty, a prevalent view is that a cautious approach is warranted: if a policymaker is unsure what a tool does she should use it gingerly. Indeed, this is a classic result from the literature on optimal policy under uncertainty as shown in Brainard (1967).

This paper takes the Brainard model as a starting point and asks: is the uncertainty faced by macroprudential policymakers sufficient to justify a cautious stance to macroprudential policy? The Brainard framework is stylised and static and there are multiple reasons why a policymaker may want to overlook its conclusions. In this paper, we present the results from some simple extensions to this framework to illustrate how uncertainty could alter the behaviour of policymakers. The analysis here is drawn from the existing literature; but our goal is to frame the issue of uncertainty in the macroprudential context.

As a starting point, and to fix ideas, we recast the Brainard model as a macroprudential policy problem where the policymaker attempts to stabilise the resilience of a financial system. In particular, we assume that the policymaker is trying to stabilise the level of financial stability denoted $x$ about some target $x^*$ through the use of a tool $k$ (for example a time varying capital requirement such as the countercyclical capital buffer) that controls $x$ imperfectly. The relationship between $k$ and $x$ is linear:

$$x = bk + u \tag{1}$$

and is subject to two sorts of uncertainty. First, $b$, the parameter governing how $k$ impacts $x$ is uncertain with prior mean $b^*$ and variance $\sigma_b^2$. The failure of the policymaker to observe $b$

perfectly could, for instance, reflect uncertainty over the impact of capital requirements on financial stability. Second, there is unobserved variation in the level of financial stability, $u$, which is independent of the policymaker's action and has prior mean $0$ and variance $\sigma_u^2$. To simplify the exposition, we assume the two sources of uncertainty are uncorrelated in what follows. This is a standard assumption that model and shock uncertainty are not related; however, similar results do emerge in a more general setting.

Policymakers should find an unstable financial system undesirable; however, an overly stable system may dampen economic activity and impose a burden on the financial system or consumers. To cite a cliche: policy should aim to avoid the stability of the graveyard.[1] The value of $x^*$ can therefore be thought of as the optimal level of financial stability, trading off a stable versus an active financial system. Similarly, adjusting $k$ may also impose costs; for example, by forcing banks to pay the underwriting fees associated with equity issuance. Thus we assume the policymaker has the following objective:

$$W = -\frac{1}{2}\mathbb{E}((x - x^*)^2 + \lambda(k - k^*)^2) \tag{2}$$

where $k^*$ captures a level of $k$ which will not impose any additional cost on the economy (for example, the current level of capital). The parameter $\lambda$ captures the policymaker's view over the relative cost of stabilising $k$ versus $x$ about their optimal levels.[2] Under these assumptions, as shown in the Appendix the best choice of $k$ under uncertainty (denoted $k^u$) is:

$$k^u = \frac{b^*x^* + \lambda k^*}{(b^*)^2 + \sigma_b^2 + \lambda} < \frac{b^*x^* + \lambda k^*}{(b^*)^2 + \lambda} = k^c \tag{3}$$

where $k^c$ denotes the level of $k$ the policymaker should choose if she faced no uncertainty. Policy is less active under uncertainty.[3] Further, as uncertainty over the impact of

[1] For a use of this phrase, see, for instance, the Governor of the Bank of England, Mark Carney, speech on October 24th October 2013 (pg 10): http://www.bankofengland.co.uk/publications/Documents/speeches/2013/speech690trans.pdf

[2] Note that we are assuming that the policymaker has a symmetric objective. This is potentially unrealistic both for financial stability and the costs of changing $k$: low financial stability may be more worrisome than high financial stability; and cutting capital requirements may not impose much of a burden on the financial system relative to raising them. In Section 3 we consider an asymmetric financial stability objective for the policymaker with low financial stability disproportionately costly. Further, in Section 5 we micro-found objectives for the policymaker that explicitly consider crisis prevention.

[3] This result assumes that $Cov(b, u) = 0$. In the more general case of non-zero correlation between the two forms of uncertainty, we show in the Appendix that $k^u < k^c$ so long as the uncertainty $\sigma_b^2$ over the policy instrument is sufficiently large:

$$\sigma_b > -\rho\sigma_u \frac{((b^*)^2 + \lambda)}{(b^*x^* + \lambda k^*)}$$

where $\rho$ is the correlation between $b$ and $u$. The correlation between the two forms of uncertainty needs to be sufficiently negative for the result to fail.

policy, $\sigma_b^2$, increases $k^u$ falls. This means that the policymaker should use her tool "less" as uncertainty increases and for any given level of uncertainty the policymaker should choose $k$ at a lower level than if she was certain. The intuition for this result is simply that additional uncertainty over the tool is perceived to introduce additional volatility into the economy when it is used, which is undesirable from the policymaker's perspective. A policy instrument whose impact is uncertain should be used more sparingly. Further, when there is greater uncertainty, the instrument should be used less. In this static model we associate the degree of policy activism with the level of $k$ chosen. A more natural interpretation of 'activism' might be the responsiveness of policy to shocks. The model can support such an interpretation by considering the response of $k$ to the desired level of financial stability $x^*$- greater uncertainty over the impact of policy will make the policy less responsive to the realised value of $x^*$. In Section 5 we explicitly consider a model in which policy is set after observing shocks to financial stability.

An important feature of Brainard's analysis is that the form of uncertainty matters. Brainard's results are sometimes misleadingly cited as a general rule that a policymaker should do less in the face of uncertainty. However, note that $\sigma_u^2$ does not appear in $k^u$.[4] Therefore, the second conclusion from this form of model is that being unsure over the state of the economy (for example, the inherent stability of the financial system) should not alter policymakers' behaviour.

Figure 1 illustrates the potential magnitude of these different sources of uncertainty faced by the Bank of England's FPC. The panel on the left presents the range of estimates for the impact of a change in the counter-cyclical capital buffer (CCB) on GDP, as presented by Macroeconomic Assessment Group at the Bank International Settlements (BIS), with the error bands capturing uncertainty over the impact of the policy instrument. The second chart presents a swathe of measures of banking system vulnerabilities using UK bank balance sheet data (as drawn from the FPC core indicator set for the counter-cyclical capital buffer). The indicators frequently present mixed messages and therefore the error bands in the left panel capture uncertainty over the state of the world. Taking Brainard's result at face value implies that the wider the error bands in the left panel, the less the FPC should use the CCB. However, the error bands in right panel should be less of a concern, and policy should be based on the mean value.

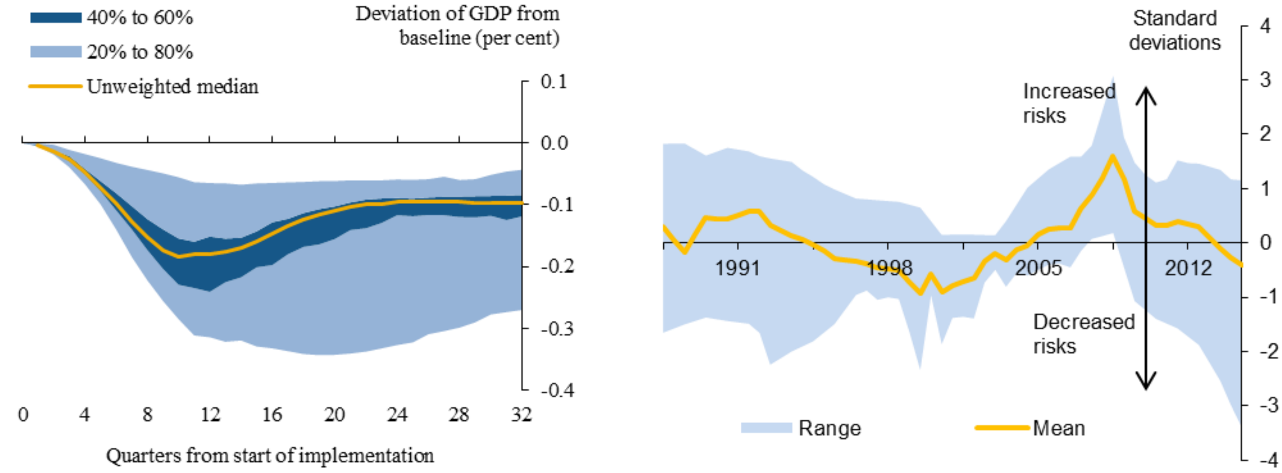
This paper argues that there are several types of uncertainty and multiple channels

[4] Again, this result assumes that $Cov(b, u) = 0$.

Figure 1: The policymaker's conundrum

Estimates of the GDP impact of a 1ppt increase in Capital Requirements

Swathe of UK Bank Balance Sheet Indicators




Notes (left panel): Source Macroeconomic Assessment Group (2010). Distributions are computed across all 89 cases contributed to the macroeconomic assessmen group that made use of standard policy forecasting and simulation models, excluding those designed to measure the impact of international spillovers. The shaded areas indicate the range between the 20th and 80th percentile.
Notes (right panel): The chart shows the UK FPC's core indicators for the UK banking system (see http://www.bankofengland.co.uk/financialstability/Pages/fpc/coreindicators.aspx), the indicators are normalised by deducting their 1987-2014 mean and dividing through by the standard deviation. We present the mean of the indicators and a shaded area denoting the min-max range.

through which uncertainty can affect policy making. The result that policy should be more cautious in the presence of uncertainty does not hold in general, particularly for specific examples that are relevant to macroprudential policy. As we shall see, if anything, the results speak to a more active policy stance in the face of uncertainty. This complements the need for policymakers to guard themselves against inaction bias. Financial stability risks are hard to measure (or unobservable) and actions to address them may have short-term costs making regulatory forbearance tempting. The lags associated with macroprudential policy instruments, both in terms of implementation and transmission, mean that there is the potential for policymakers to move too late to build resilience in the financial system ahead of crises.

To make these points, we consider several extensions to the Brainard model. In our first extension, macroprudential policy is concerned with rare events, the probability of which is difficult to quantify. In such a situation, the policymaker may wish to behave in a robust fashion, preparing for the worst case scenario. This can also lead to more active policy. A second extension considers an asymmetric objective function for financial stability in which a crisis is disproportionately costly for the policymaker. If this asymmetry is sufficiently large and the policy tool is sufficiently effective on average, policy will become more active in response to both uncertainty about the policy instrument and the state of the world, in an attempt to avoid a costly crisis. Our third extension considers a dynamic model which allows for learning. Using the tool today reduces uncertainty about its impact tomorrow, but may initially increase volatility. If the motivation to learn is sufficiently strong (i.e. the policymaker's discount rate is sufficiently low), optimal policy making can become more active with greater uncertainty. While there results are technically plausible, they ignore the political economy considerations that may make such learning strategies unpalatable. In our final extension, we consider the interaction between private sector uncertainty and uncertainty facing the policymaker. In addition to directly affecting financial institutions, macroprudential actions may have a broader impact through signalling information about risks to financial stability. We show that this signalling channel will be less powerful when there is greater private sector uncertainty about policy objectives. Consequently a private sector that is more unsure about why the policymaker is acting may require the policymaker to be more active in order to offset the diminished signalling power of the tool.

### 1.1 Related literature

Brainard (1967) lays out the canonical case for higher uncertainty leading to diminished policy activism. Beyond this classic work, this paper has several links with the academic literature on policy under uncertainty. However, most of this previous work has focused on the monetary policy context; our contribution is to recast some of the findings in the context of macroprudential policy and discuss their relevance.

In terms of policy under fundamental (or Knightian) uncertainty, the approach of using robust control and min-max dates back to at least Wald (1950); a more detailed, modern treatment is available in Hansen and Sargent (2001, 2008). Barlevy (2009) offers additional exposition over the framework. Robust control and min-max based optimal policy problems generally deliver more aggressive policy action in response. However, Onatski and Stock (2002) show that this finding does not extend to all model perturbations.

On the topic of learning through policy actions to reduce uncertainty, several papers - for example Orphanides and Williams (2007) and Wieland (2000, 2006) - show how the desire to learn can lead to policy being more aggressive in the face of uncertainty. An alternative strand of the literature (see, for example, Landes (1998), Besley (2001), Mukand and Rodrik (2002)) argue that cross-sectional variation in policy making across countries or political parties should also be encouraged to foster innovation and allow for learning about the impact of alternative policies. In contrast, other authors have challenged the gains from actively introducing policy variation in order to learn. In a quantitative assessment, Svensson and Williams (2007) show that the benefits for experimentation can often appear very modest in a generic linear quadratic forward-looking set-up allowing for model uncertainty. Cogley and Sargent (2005) also cast doubt on the benefits of setting policy with learning in mind and raise a more general point that the gains from more active policy for the purposes of learning could be limited if there are sufficient natural experiments.

The interaction of the uncertainty of the public and policymaker, alongside asymmetric information and the informational value of government policy, are discussed in the early contributions of King (1982) and Weiss (1982). More recently, Lorenzoni (2010), Angeletos and La'O (2011) and Angeletos et al. (2015)) analyze how to optimally set policy in static models with dispersed information.

## 2 Targeting the Worst Case Outcome (Robustly)

The conclusions from Brainard (1967), and indeed most modelling frameworks where policymakers maximise the expectation of an objective, rely on policymakers being able to assign (possibly subjective) probabilities to potential future scenarios. However, this is an unrealistic way of describing the uncertainty faced by macroprudential policymakers. It is not always possible to say with confidence how likely a relevant outcome will be. There is an inescapable need for policymakers to make judgements about the state of the world that cannot be backed by statistical analysis (Svensson (2002)). This sort of unquantifiable uncertainty is sometimes referred to as fundamental or Knightian uncertainty. And it is a form of uncertainty that may be particularly troublesome for macroprudential policymakers given the innovation and increasing complexity of the financial system, which can make risks difficult to quantify. Furthermore, the objective of macroprudential policymakers is defined in terms of the resilience of the financial system, suggesting a focus upon rare events whose likelihood is undetermined.

A popular method among economists for incorporating fundamental uncertainty and concerns over severe outcomes into policy making decisions is to rely upon a robust control approach. This approach favours policies that avoid large losses across scenarios regardless of how likely any given scenario is. In practice this is implemented by what is called a min-max framework:[5] a policymaker tries to set policy to minimise her losses assuming that the parts of the problem she is fundamentally uncertain about are chosen to maximise the loss. This is equivalent to making the worst case scenario as palatable as possible.

### 2.1 Modelling framework

To introduce a robust control motive into the Brainard model, we modify the relationship between $k$ and $x$ to be:

$$x = bk + u + v \tag{4}$$

where with the exception of $v$, all other variables are defined as in equation (1) in the Introduction. The term $v$ captures what we refer to as fundamental uncertainty over $x$, the level of financial stability. This is a source of uncertainty which is ambiguous and the pol-

[5] See Wald (1950).

icymaker is unable to attach a probability distribution to it. A robust approach to dealing with the uncertainty captured by $v$ is to choose policy on the assumption that the worst outcome has happened so as to avoid large losses. This is implemented in what is called a max-min framework: the policymaker chooses $k$ to maximise its objective conditional on "nature" choosing $v$ to minimise the objective:

$$max_k min_v - \frac{1}{2}\mathbb{E}[(x-x^*)^2 + \lambda(k-k^*)^2] + \frac{1}{2}\frac{\theta}{v^2}, \theta > 1 \tag{5}$$

Both nature and the policymaker choose their action conditional on each other's choice. In effect, the policymaker and nature play a phantom min-max game with each other and the pure strategy Nash equilibrium solves the model. An important element to the robust control problem is the term $\theta/v^2$. This captures how sensitive the policymaker is to the ambiguity surrounding $v$. Using the definitions of Hansen and Sargent (2008), the parameter $\theta$ is called entropy and both represents how wrong the policymaker thinks they can be about the true level of $x$ and penalises extreme realisations of $v$. An alternative interpretation, as in Hansen and Sargent (2001), is that $v$ represents model misspecification: the policymaker does not know the true model driving financial stability and thus $v$ reflects the perturbation of the true model from the model the that is being relied upon. From the perspective of the optimisation problem, $\theta/v^2$ simply serves as a constraint on nature when choosing how bad the worst case scenario can be. As shown in the Appendix, the solution for nature, taking the actions of the policymaker as given is given by

$$v = \frac{(b^*k - x^*)}{(\theta - 1)} \tag{6}$$

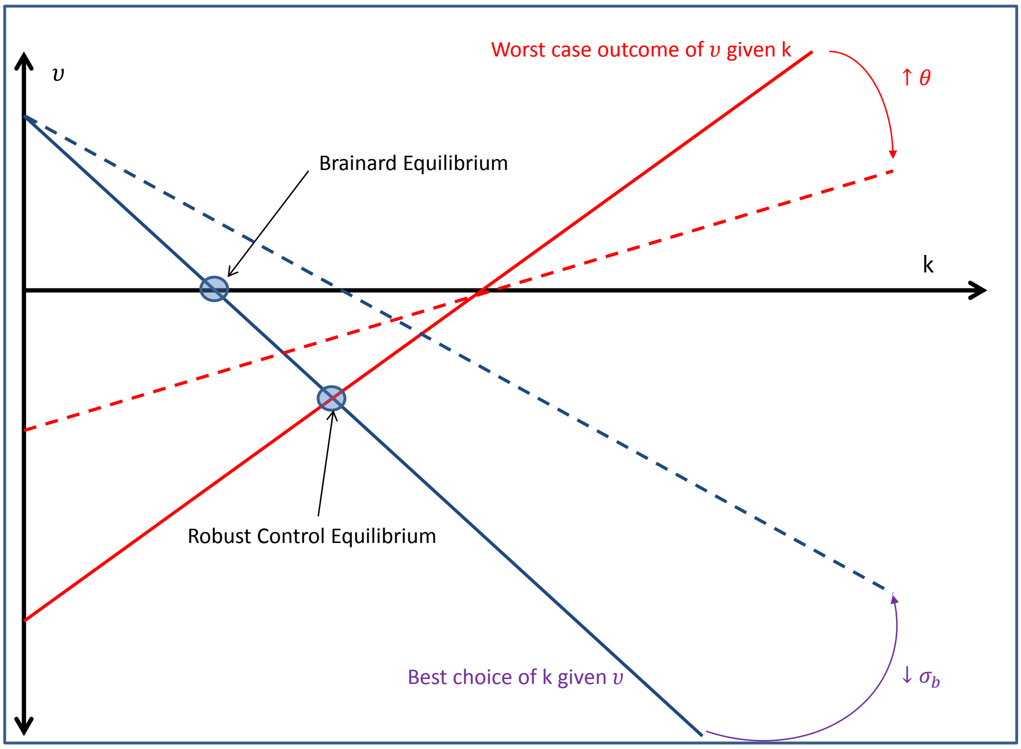
Thus as $\theta$ goes to infinity, nature is constrained to choose $v = 0$ and the policymaker ignores the fundamental uncertainty, conversely as $\theta \to 1$, the worst case scenario becomes increasingly bad and the policymaker becomes more and more sensitive to the ambiguity over $v$.[6]

In the Appendix it is also shown that the policymaker's optimal choice of $k$, taking the actions of nature as given, is given by

$$k = \frac{(b^*x^* + \lambda k^* - b^*v)}{((b^*)^2 + (\sigma_b{}^2) + \lambda)} \tag{7}$$

[6]Equation (6) can be rewritten as $k = \frac{v(\theta-1)+x^*}{b^*}$. Hence, nature's best response function intersects the $k - axis$ at the point $k = \frac{x^*}{b^*}$ which is independent of $\theta$. Therefore, any change in $\theta$ is a rotation of nature's best response line around the point $(\frac{x^*}{b^*}, 0)$ as illustrated in Figure 2.

Figure 2: Graphical Illustration of the Robust Control Model




Solving for the Nash equilibrium gives the robust control solution to $k$ as:

$$k = \frac{\frac{\theta}{(\theta-1)} b^* x^* + \lambda k^*}{(\frac{\theta}{(\theta-1)} (b^*)^2 + \sigma_b{}^2 + \lambda)} \tag{8}$$

Inspecting this solution, it is clear that as $\theta \to \infty$ and fundamental uncertainty disappears that the robust control solution for $k$ is the same as the Brainard solution (3) in the Introduction.

The main result of this section is that, so long as the uncertainty over the effect of the policy tool is sufficiently high, policy becomes more activist when there is greater fundamental uncertainty:[7]

$$\frac{dk}{d\theta} < 0 \text{ iff } \sigma_b^2 > \frac{\lambda\,(b^* k^* - x^*)}{x^*} \tag{9}$$

Thus, in particular, as the solution coincides with Brainard when $\theta \to \infty$ and fundamental uncertainty disappears, the policymaker is more activist than in the case of Brainard under this condition when there is fundamental uncertainty (i.e. for $\theta \in (1, \infty)$).

The simple explanation for this finding is that increasing fundamental uncertainty makes the worse case outcome, that the policymaker is preparing for, worse. If condition 9 holds, the worst case outcome is one where there is a large negative shock to financial

[7] Recall that fundamental uncertainty increases as $\theta$ decreases.

stability which requires a tighter policy stance in response.

Figure 2 illustrates the intuition behind the robust control model graphically. The downward sloping blue line gives the best choice of $k$ given $v$ (given by Equation 7). The intuition for the downward slope is straightforward: a negative $v$ implies a negative shock to financial stability and hence policy should be tightened to stabilise $x$. The upward sloping red line captures the "prepare for the worst" nature of the robust control problem: it is the value of $v$ that gives the worst possible outcome to the policymaker given $k$ and subject to the entropy constraint (given by Equation 6). A robust policy choice is to set $k$ where the two lines intersect: this means that the policymaker is choosing a policy which gives the best result if the very worse happens. In the standard Brainard problem in the Introduction, $v = 0$, so the Brainard solution is equivalent to where the blue line crosses the $k$ axis. In the figure the intersection between the two lines is at a higher level of $k$ than the Brainard point.[8] Hence the robust control policy is more aggressive than Brainard would suggest. The intuition for this lies with the idea that more policy action is required if the most severe scenario occurs and the higher the fundamental uncertainty, the more severe is the worst case.[9]

A second interesting feature of this model is that different forms of uncertainty have different effects. As with the Brainard case, ${\sigma_u}^2$ does not influence policy (ignoring any covariance terms). However, ${\sigma_b}^2$, which captures quantifiable uncertainty over the impact of the tool, and $\theta$, which is inversely related to the amount of fundamental uncertainty, have a meaningful and differing impact on how aggressive a policymaker should be. As illustrated by Figure 2, reducing ${\sigma_b}^2$ (subject to condition 9 holding) effectively rotates the blue line outwards meaning that a higher $k$ is optimal. This is exactly in line with the intuition expressed in the Introduction where reducing the uncertainty over the impact of the tool means it should be used more aggressively. In contrast, increasing $\theta$ reduces fundamental uncertainty; this flattens the red line and means that the worst case outcome is not as bad implying that policy does not need to be so activist.

## 2.2 Discussion

While a robust approach to policymaking usually suggests policy should be more active, it may say little over the direction in which policy should be more aggressive. To see this, con-

[8] Condition 9 is required for this to hold.

[9] This is a modelling assumption. It is not the case that all robust control problems lead to a more aggressive policy, but it is a typical feature of this sort of model. See Onatski and Stock (2002) for a discussion in the context of monetary policy. Barlevy (2009) offers some counter examples.

sider the response of fiscal policy in the event of a severe recession. One extreme outcome may be high unemployment, negative growth and deflation. Therefore, the robust response may be to embark upon aggressive fiscal expansion. However, an alternative scenario is that the recession puts sufficient strain on public finances to bring the government's solvency into question, leading to rising sovereign risk premia and additional stress on the financial system and the economy. So an alternative robust policy may be an aggressive tightening of the fiscal stance. Therefore, a policy designed to avoid either extreme outcome requires an aggressive shift in the policy stance but the exact direction depends on which scenario policymakers are trying to avoid.

At face value, for the macroprudential context this lack of direction may seem less of an issue. Macroprudential policymakers are chiefly concerned by the extreme tail risks posed by financial crises. This asymmetry means that, at most points in time, a policymaker behaving in a robust way would be hawkish in setting her macroprudential tools. However, there is always a risk that under certain conditions severe negative outcomes can emerge if policy is set too tightly. In the spirit of Lucas (1987), one can imagine that the worst outcome of macroprudential policy is to lower the long-term trend growth rate. The question would then be whether an overly burdensome regulatory regime had a greater impact on long term growth than financial crises.

Despite its popularity in the academic literature, a robust control approach can appear abstract when applied to practical policymaking. It has real world applications nonetheless. For example, in right hand chart in Figure 1, there is a swathe of potential indicators of risks to bank balance sheets: a robust policymaking strategy would be to focus on the upper end of the range, as that is the worst case model, rather than the mid-point of the swathe. Second, the principle of calibrating policy to prepare for severe outcomes is already embodied within the UK macroprudential framework via stress testing. Stress tests by definition provide a sense of economic outcomes if an extreme scenario emerges.

# 3 Asymmetric Loss Function

The Brainard model in the Introduction and the extension of the previous section share a common *symmetric* financial stability loss function $(x - x^*)^2$, in which financial stability being above the target level $x^*$ is equally as costly to the policymaker as financial stability being below target. Such objective functions have been frequently used for modeling inflation targeting, as high inflation and deflation can both be costly. However, it is not clear that this functional form is appropriate for financial stability: the losses associated with a financial crisis may be significantly greater than those imposed by having excessive stability. In this section we consider an extension to the Brainard model with an asymmetric loss function that captures this property.

## 3.1 Modeling framework

A tractable way to model an asymmetric loss function is to use the Linex function (Varian (1975)).[10] We consider the following objective function for the macroprudential policymaker:

$$W = -\mathbb{E}\left\{\frac{e^{a(x^*-x)} - a\left(x^* - x\right) - 1}{a^2} + \frac{\lambda}{2}\left(k - k^*\right)^2\right\} \tag{10}$$

where $a > 0$ and the remaining variables are defined as in the Introduction. An attractive property of (10) is that it nests the benchmark quadratic loss function used in the basic Brainard model, collapsing to it for $a \to 0$.[11] When $a > 0$, equation (10) is asymmetric, with a greater loss when financial stability is low $(x < x^*)$ than when it is too high $(x > x^*)$. Further, the greater $a$ is, the greater this differential, and the more costly low financial stability is relative to high financial stability.[12] Figure 3 plots the financial stability loss

[10] We would like to thank David Aikman for suggesting this functional form.

[11] Formally, using L'Hôpital's rule, it can be shown that

$$\lim_{a \to 0} - \mathbb{E}\left\{\frac{e^{a\left(x^*-x\right)} - a\left(x^* - x\right) - 1}{a^2} + \frac{\lambda}{2}\left(k - k^*\right)^2\right\} = -\frac{1}{2}\mathbb{E}\left\{\left(x - x^*\right)^2 + \lambda\left(k - k^*\right)^2\right\}$$

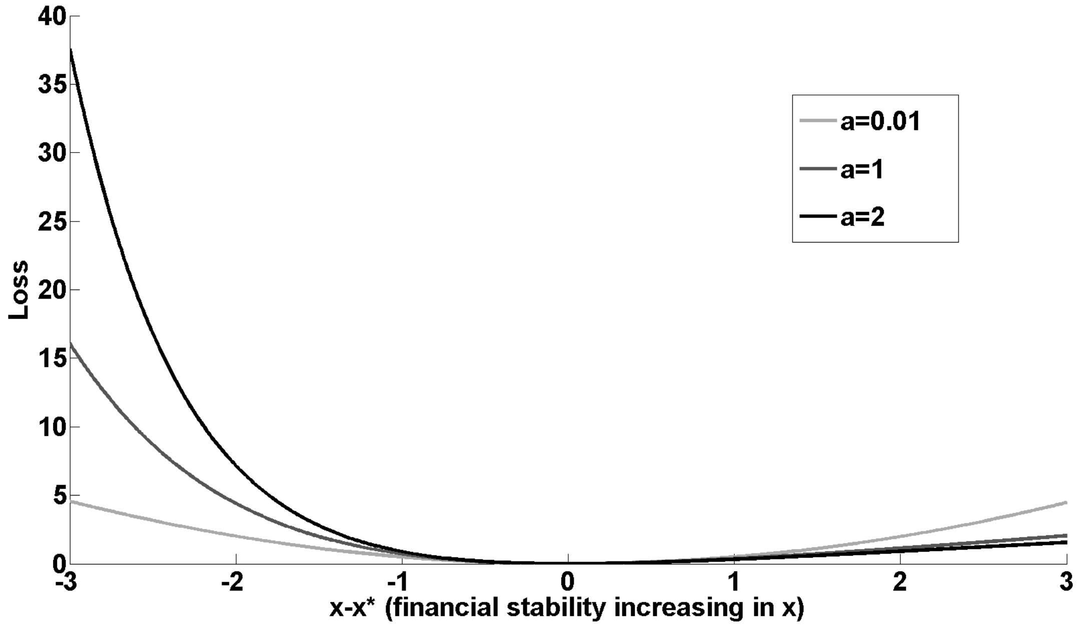
[12] Formally, define the financial stability loss function as

$$L\left(x\right) := \frac{e^{a\left(x^*-x\right)} - a\left(x^* - x\right) - 1}{a^2}$$

and the loss when financial stability is low relative to when it's high (for a given deviation from target $\epsilon$) as $\Gamma\left(\epsilon, a\right) := \frac{L\left(x^*-\epsilon\right)}{L\left(x^*+\epsilon\right)}$. In the Appendix we show that for $a > 0$, $\Gamma\left(\epsilon, a\right) > 1$ and this asymmetry increases with $a$:

$$\frac{\partial \Gamma\left(\epsilon, a\right)}{\partial a} > 0$$

**Figure 3: Graphical Illustration of the Linex Function**




function for a range of values of $a$ (holding $k = k^*$). As can be seen, for larger values of $a$, the costs of very low financial stability (to the left of $0$) can be significantly higher than the costs of very high financial stability (to the right of $0$).

As in the basic Brainard model the policy tool $k$ influences financial stability $x$ in a linear way

$$x = bk + u \tag{11}$$

with uncertainty over the impact of the tool, $b$, and shocks to financial stability, $u$. For tractability we assume that $b \sim N\left(b^*, \sigma_b^2\right),\ u \sim N\left(0, \sigma_u^2\right)$ and $Cov\left(b, u\right) = 0$. Using (11) and the properties of the log-normal distribution we show in the Appendix that the objective function can then be written as:

$$W = -\left\{ \frac{e^{\left(ax^* - ab^*k + \frac{a^2\left(\sigma_b^2 k^2 + \sigma_u^2\right)}{2}\right)} - a\left(x^* - b^*k\right) - 1}{a^2} + \frac{\lambda}{2}\left(k - k^*\right)^2 \right\} \tag{12}$$

This function is strictly concave and has a unique global maximum $\tilde{k} > 0$ (See the Appendix for the proof).

In the Brainard model of the Introduction, greater uncertainty over the effectiveness of the policy tool leads to less active policy. However, if the financial stability objective is sufficiently asymmetric and the tool is sufficiently effective on average, this result is reversed:[13]

$$\frac{d\tilde{k}}{d\sigma_b^2} > 0 \tag{13}$$

With a symmetric financial stability objective function, greater uncertainty over the impact of the tool makes the policymaker more cautious as using the tool introduces greater variance to financial stability, which is costly both when too high and too low. However, when the losses from a financial crisis are sufficiently greater than those from too much stability and the policy tool is reasonably effective, greater uncertainty over the impact of the tool will make policy more active in order to reduce the chance of a low level of financial stability. Such activism brings large benefits in reducing the cost of a financial crisis with relatively small costs if financial stability ends up being too high.

A second key result of Brainard is that uncertainty over the level of financial stability has no impact on policy. With a symmetric objective function, the costs of financial stability being above or below target are equal and do not necessitate a policy response. However if the financial stability objective is asymmetric and $b^*$ is sufficiently large, greater uncertainty over the state of the world will result in more active policy in a bid to avert a costly crisis:[14]

$$\frac{d\tilde{k}}{d\sigma_u^2} > 0 \tag{14}$$

The greater the uncertainty over the state of the world, the greater the chance of a costly financial crisis for a given policy stance. When policy is reasonably effective on average, with more active policy the policymaker has a relatively large potential benefit in averting a costly crisis and a relatively small potential cost if circumstances turn out to be more benign. Under these circumstances the benefits outweigh the costs resulting in more active policy.

### 3.2 Discussion

There is good reason to think macroprudential policymakers should have an asymmetric preference when it comes to financial stability. The substantial skew with regards to negative economic outcomes when financial stability risks materialise mean that the costs of missed downside risks may be much larger than benefits of erring towards loser pol-

[13]Formally, in the Appendix we show that if $a > \frac{-x^* + \sqrt{(x^*)^2 + 4\sigma_u^2}}{\sigma_u^2}$ then $\frac{d\tilde{k}}{d\sigma_b^2} > 0$ for sufficiently large $b^*$.

[14]Specifically, in the Appendix we show that if $a > 0$ and $b^* \geq \frac{\lambda k^* a \sigma_b^2}{(\lambda + \sigma_b^2)}$ then $\frac{d\tilde{k}}{d\sigma_u^2} > 0$.

icy. Theoretical models that deliver endogenous crises via occasionally binding constraints (see for example, Bianchi and Mendoza (2010), Jeanne and Korinek (2013), Korinek (2011) and Korinek and Simsek (2014)) illustrate how severe asymmetries can emerge in stylised macroeconomic models and show that this can provide a normative justification for macroprudential policies that are contractionary in normal circumstances. The intuition for the findings above are similar: as the policymaker is much more adverse to low financial stability outcomes, when uncertainty over the state of the world increases she buys insurance against low financial stability by tightening policy. The cost of this is an overly tight policy choice in situations when the financial system is stable. A similar intuition emerges for increased uncertainty over tool effectiveness: the policymaker insures herself against outcomes where her a tool is weaker than she expects (which implies low financial stability) with tighter policy.

# 4 Learning About the Effects of Policy

A natural way to respond to uncertainty over a policy instrument is to attempt to learn about it. Research either using evidence from other countries, from natural experiments or by using calibrated theoretical models can fill this gap. However, there is rarely a perfect substitute for using the instrument itself. Furthermore, in the macroprudential context, the framework for making and communicating changes to the tools is also new. Equivalent instruments have been used for microprudential purposes (such as bank capital requirements) but the signalling value (see Section 5) and system-wide consequences of a macroprudential action may lead to a different impact, limiting what can be learned from previous uses of the tools.

In this section, we adapt the Brainard model to allow the policymaker to learn about the impact of her tool by observing what happens when she moves it. There is a trade-off: being active with a policy tool today leads to additional volatility, but by observing the tool's effect the policymaker will be less uncertain in future. This means that the policymaker has an incentive to be more active initially. Whilst it may not be possible to directly measure or observe financial stability, so long as the policymaker can observe signals which convey some information about risks to the stability of the financial system (e.g. prices in financial markets), they will be able to learn about the effectiveness of their tool.

In practise, communicating a policy based on experimentation to the public is challenging. And an initial period of volatility and missed objectives during a period of learning may be politically costly for a central bank. Our model does not capture these effects. Also, policymakers will be able to learn when they use their tools even if the tools are used cautiously without learning in mind.

## 4.1 Modelling framework

Here we lay out a two period $(t = 1, 2)$ version of the model presented in the Introduction.

Let

$$W_t := -\frac{1}{2}\mathbb{E}[(x_t - x^*)^2 + \lambda\,(k_t - k^*)^2] \tag{15}$$

And suppose the policymaker's objective in period 1 is now to maximise

$$W_1 + \delta W_2 : \delta > 0 \tag{16}$$

As before, the policy instrument $k$ has an uncertain impact on $x_t$ described by

$$x_t = bk_t + u_t \tag{17}$$

The parameter $\delta$ determines how much the policymaker values future periods; it has multiple interpretations in this context as discussed below. The other parameters share the interpretation they have in the Introduction. For tractability we now suppose that $b$ and $u_t$ are normally and independently distributed with the following initial priors: $b \sim N\left(b^*, \sigma_b^2\right)$ and $u_t \sim N\left(0, \sigma_u^2\right)$.

The crucial assumption underpinning this two period model is that by setting policy in period 1 the policymaker learns something about $b$ in period 2. To embed this feature in the model we have to specify the information available to the policymaker in each period. We assume that in period 1, when $k_1$ is chosen, the policymaker has the same information as in the Brainard model. At the end of period 1 the policymaker observes $x_1$, which given $k_1$, provides an additional signal over $b$, reducing the uncertainty around the impact of the tool. However, since $u_1$ is not observed, the policymaker cannot perfectly distinguish between movements in $x_1$ that are due to $k_1$ or due to $u_1$. This results in a simple signal extraction problem where the policymaker updates the expectation she has of $b$ and uncertainty that surrounds that expectation (as shown in the Appendix):

$$\mathbb{E}(b|x_1) = b^* + \frac{b^* k_1}{((k_1)^2 {\sigma_b}^2 + {\sigma_u}^2)}(x_1 - b^* k_1) \tag{18}$$

$$Var(b|x_1) = {\sigma_b}^2 - \frac{({\sigma_b}^2)^2}{({\sigma_b}^2 + {\sigma_u}^2/(k_1)^2)} \tag{19}$$

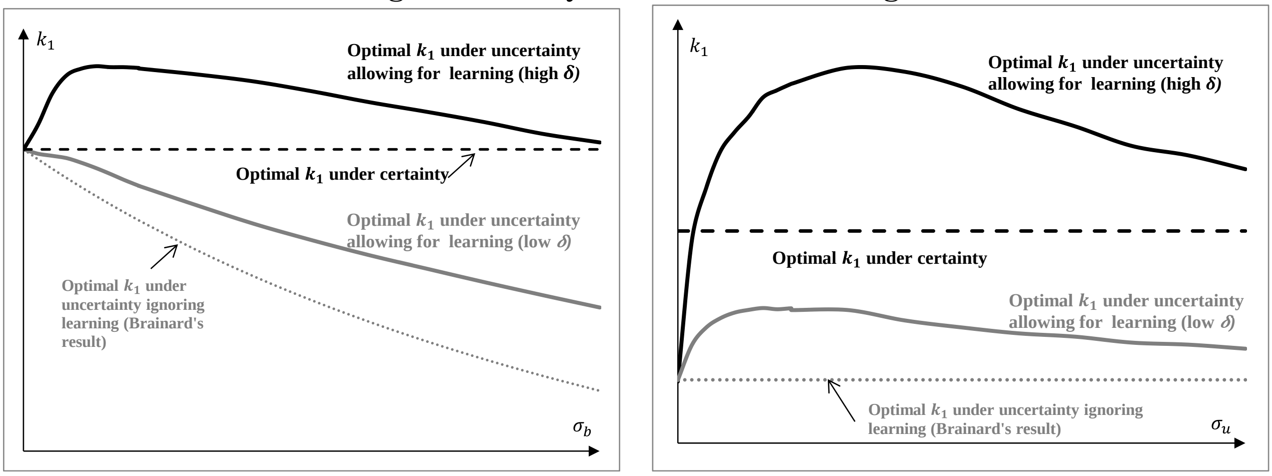
Equation (19) implies that the updated variance in period 2 is strictly less than in period 1 when $k_1 > 0 : (Var(b|x_1) < {\sigma_b}^2)$; thus uncertainty is reduced between the two periods.[15] Moreover, increasing $k_1$ reduces $Var(b|x_1)$: the policymaker becomes more certain in the second period the more she acts in the first. The problem in period 2 is almost identical to the static Brainard model of the Introduction and the policymaker tries to maximise $W_2$ conditional on a prior mean and variance about $b$. The only difference is that now those

[15] Note that this result that $Var(b|x_1) < {\sigma_b}^2$ is not dependent on the assumption that $Cov(b,u) = 0$. If we allow $Cov(b,u) = \rho\sigma_u\sigma_b$, then:

$$Var(b|x_1) = {\sigma_b}^2 - \frac{(k_1{\sigma_b}^2 + \rho\sigma_u\sigma_b)^2}{(k_1^2{\sigma_b}^2 + {\sigma_u}^2 + 2k_1\rho\sigma_u\sigma_b)}$$

The second term captures how much can be learned about $b$ by acting in period 1. This term is strictly positive, hence $Var(b|x_1) < {\sigma_b}^2$.

Figure 4: Policy Choice With Learning




priors are given by $\mathbb{E}(b|x_1)$ and $Var(b|x_1)$, which depend on the choice of $k_1$ in the previous period:

$$k_2 = \frac{(\mathbb{E}(b|x_1)x^* + \lambda k^*)}{((\mathbb{E}(b|x_1))^2 + Var(b|x_1) + \lambda)} \tag{20}$$

As a result, the policymaker faces a trade-off: actively using a tool in period 1 may not be immediately desirable due to the Brainard result from the Introduction but being active will make the policymaker better off in future via lower uncertainty. To illustrate how this trade-off manifests and interacts with other parameters in the model in Figure 4 we present the optimal choice of $k_1$ across different levels of ${\sigma_u}^2$ and ${\sigma_b}^2$ and for two different levels of $\delta$.[16] These values of $k_1$ are presented in comparison to the optimal choice if the policymaker faced no uncertainty and if policy was set using the Brainard model.

Consider the left-hand panel of Figure 4 first. If ${\sigma_b}^2 = 0$ (i.e. the policymaker has no initial uncertainty over the effect of $k$): the best choice for $k_1$ is the same across all cases. In line with Brainard, uncertainty over $u_t$ is not relevant. As $\sigma_b^2$ increases we see that policy under the Brainard model (which ignores learning) diminishes in line with the solution in the Introduction. Once learning is accounted for, $k_1$ is set at a higher level than under Brainard. The parameter $\delta$ has an important role to play in determining how the possibility to learn affects activism. As $\delta$ increases, the policymaker puts more weight on future outcomes. Hence, learning about how policy works in period 2 becomes a greater priority leading to a more activist policy. In reality policymakers will need to make decisions over many periods, thus the loss at period 2 could be thought of as a reduced form way

[16] In the Appendix we show how, using these formulas, the policymaker's maximisation problem can be set-up in terms of $k_1$ only. This results in a highly non-linear function which is solved numerically using Monte Carlo methods.

of capturing the continuation value of the problem.[17] This would suggest that a large $\delta$, potentially greater than 1, is appropriate.[18] With $\delta$ high and when $k_1$ is chosen internalising learning, we see that policy is actually more activist than it would be under certainty. Learning more than trumps the costs of being overly activist today. There is a hump-shaped pattern with respect to $\sigma_b^2$: for low levels of uncertainty increasing ${\sigma_b}^2$ increases the gain from learning more than the cost of increased activism. However, for higher levels of ${\sigma_b}^2$ this trade off switches around, to the point where as ${\sigma_b}^2 \to \infty$ the costs from using policy are so great initially and policy tends towards Brainard and complete inaction. For relatively low δ, a similar mechanism exists but the value of learning is insufficient to drive the policy choice beyond the certainty equivalent case.

The irrelevance of uncertainty over the state of the economy for the policy choice does not extend to the learning case: the ability of a policymaker to learn about her instrument depends crucially on how easily she can disentangle its effects from other shocks that are hitting the economy at the same time. The right-hand panel of Figure 4 considers how policy varies with ${\sigma_u}^2$. One obvious point is that the Brainard policy response is insensitive to ${\sigma_u}^2$ (the uncertainty around financial stability that is unrelated to policy) as described in the Introduction. In the learning model, increasing δ increases activism relative to the Brainard level. We also see a similar hump-shaped pattern with respect to ${\sigma_u}^2$. This is because how much can be learned about $b$ by altering $k_1$ has a non-linear relationship with the uncertainty over $u_1$. If ${\sigma_u}^2$ is zero, there is nothing more that can be learned from altering $k_1$ as $b$ is observed perfectly at the end of period 1 and thus optimal $k_1$ is the same as Brainard. As ${\sigma_u}^2$ rises from zero, the uncertainty over b in period 2 increases, hence the policymaker chooses to be more active to learn more about the coefficient. However, as ${\sigma_u}^2$ becomes increasingly large the ability of $x_1$ to be informative about $b$ diminishes and the policymaker becomes less able to learn and is therefore less active. At the limit ${\sigma_u}^2 \to \infty$, there is nothing that can be learned from $x_1$ and policy reverts to the Brainard solution.[19] This implies that, in contrast to the result in the Introduction, the value of ${\sigma_u}^2$ can affect the best policy choice.

[17] Prescott (1972) finds that sequentially solving a two period learning problem, i.e. the policymaker only optimises over the current period and the next, is a good approximation to a policy problem with a much longer or even infinite planning horizon.

[18] An alternative reduced form interpretation of $\delta$ is that the policymaker will receive more information about how policy works, separate from the observation of $k_1$, in future periods. This could be from research into how the tool works, by exploiting natural experiments for example. A simple way of accounting for increased future information is reducing the expected future loss; a lower $\delta$ in other words and a less activist policy.

[19] These results for $\sigma_u^2$ are shown formally in the Appendix.

### 4.2 Discussion

The desire to learn is intimately related to how uncertain the policymaker thinks they will be in future. From today's perspective the macroprudential toolkit is largely untested. However, new research may eventually come on stream and contribute to developing a better understanding of how the tools operate. Policymakers today have to decide on the extent to which they think research can substitute for seeing the tool in action. Given that both the empirical and theoretical literature are still in their relative infancy, policymakers may need to have a bias towards action.

Wieland (2000) provides an example that illustrates the limits to using research as a means of reducing uncertainty (although his paper is not written in such a light). Wieland argues forcefully that the Bundesbank should have responded more aggressively to high inflation in the wake of Germany's reunification in the 1990s. He argues that reunification shifted parameters in the economy's Phillips Curve which biased the central bank's own estimates of the structure of the economy. The resultant loose policy led to a period of rapid money growth and inflation. Wieland shows that the Bundesbank was sub-optimally passive in response to the change in circumstances: it would have been better off acting aggressively at the cost of additional volatility initially in order to learn how the economy had changed and reduce the inflationary bias in the policy stance. Of course, if there were other cases of reunification in economies similar to East and West Germany, research could have been conducted on these to obtain better estimates. But the reunification was unprecedented; the Bundesbank had to learn by doing. To some extent this is the position of macroprudential policymakers today as well.

Learning about the effectiveness of macroprudential policy may take much longer than learning about the effectiveness of monetary policy due to the relatively infrequent occurrence of financial crises. Policymakers will not be aware that the financial system is less stable than it appeared until after a crisis occurred. Therefore, over time, there are relatively few events to learn from. As such, the case for gradualism in the face of uncertainty from that comes from certain sections of the monetary policy literature[20] which advocate a gradual adjustment of interest rates in order to learn about the impact of each interest rate adjustment on price stability may be less applicable to financial stability, where such learning would take too long. On the other hand, the length of financial cycles means that, if policymakers believe that they can readily observe financial stability, there is more

[20] See Sack (1998).

opportunity to learn about the impact of policy tools ahead of crises.

The immediate practical issue with using a tool to learn is one of communication and political sensitivity. A macroprudential policymaker would probably struggle to articulate to banks that it was forcing them to raise more capital in order to determine the economic consequences. Deliberately experimenting with the economy with no other goal in mind is highly inadvisable and should run counter to macroprudential policy objectives. However, it is unnecessary to do policy that is harmful in order to learn. The learning mechanism provides an argument to suggest that there are benefits from not being too cautious, not that policy should be set with wild abandon.

## 5 Policy Transparency and Private Sector Uncertainty

It is not only policymakers that face uncertainty with regards to macroprudential policy-making. The public are also uncertain about risks to financial stability, the impact of macroprudential policy, and how the policymaker herself will behave. Furthermore, the policymaker may have informational advantages over the public through access to confidential regulatory data, supervisory intelligence, and the results of stress tests. Thus a macroprudential policymaker needs to consider how her policy actions interact with the information available to the public.

In the models used in the Introduction and the previous three sections, the private sector has responded to policy actions in a mechanical fashion. In this section, we allow for strategic interaction between the policymaker and the private sector. The setup moves beyond a simple stabilisation problem into a framework where (private) risk taking is an endogenous choice (albeit modeled in a stylised fashion). The transmission mechanism of policy is similar, as is the eventual objective of the policymaker, to that discussed in the previous sections. However, this framework allows us to explore how private sector uncertainty affects the transmission mechanism and, hence, how policymakers should behave in response.

In the model, there are two states of the world, a good state and a crisis state. The policymaker can use her tool to reduce the probability of crises and make them less harmful. However, there is an offsetting effect: when the world is safer, the private sector would like to take more risk.[21] The risk taken by the private sector is excessive; we assume that the social cost of a crisis is greater than the private cost.

We also assume that the policymaker is better informed about financial stability (and hence the risk of a crisis) than the private sector. As such, the policy action is an additional signal for the private sector about crisis risk. Crucially this signal is imperfect: we also assume that the private sector is not fully aware of the objectives of the policymaker and hence cannot perfectly distinguish actions caused by risks to financial stability from those caused by an aversion to crises. An example of this would be if banks did not know if the policymaker set a high CCB rate because it was aware of a specific financial stability risk or simply had a high preference for avoiding a crisis.

[21] This is analogous to a Minsky cycle: periods of low volatility lead to greater risk taking. The difference in our model is that it is policy, which by making the world more resilient, encourages greater risk taking.

## 5.1 Modelling framework

Formally, the model takes the form of a Stackelberg game. The policymaker first chooses the level of her policy instrument, $k$; then, having observed $k$, the private sector chooses their desired level of risk-taking $x$. The policymaker internalises the impact of her policy choice on the private sectors response when choosing her desired level of $k$. The two states of the world are denoted $H$ and $L$, where $L$ is the crisis state. The probability, $p\left(x,k\right)$, of reaching the crisis state $L$ is increasing in private sector risk taking $x$, but decreasing in the level of capital set by the public sector, $k$:

$$p(x,k) = p^{*} + x - \gamma k + \omega \tag{21}$$

There is uncertainty over the effectiveness of this policy tool in reducing the probability of crisis with both the policymaker and private sector sharing a common prior $\gamma \sim \left(\gamma^{*}, \sigma_{\gamma}^{2}\right)$.[22] There is also uncertainty over the state of the world, with the probability of a crisis affected by a shock $\omega$. The private sector do not observe this shock but believe it to have a prior distribution $\omega \sim N(0, \sigma_{\omega}^{2})$. However, crucially, in contrast to the private sector, the policymaker observes the shock to the likelihood of a crisis, $\omega$, before choosing $k$.[23]

The model is solved backwards, beginning with the private sector's problem. Formally, this is given as:

$$max_{x}\left(\mathbb{E}[(1-p\left(x,k\right))\,\pi^{H}\left(x,k\right)+p(x,k)\,\pi^{L}\left(x,k\right)]\right) \tag{22}$$

subject to equation (21) and:

$$\pi^{H}\left(x,k\right) = \beta_{H}^{pr}x - a_{H}k,\ \beta_{H}^{pr} > 1 \tag{23}$$

$$\pi^{L}\left(x,k\right) = -x + a_{L}k \tag{24}$$

The functions $\pi^{H}\left(x,k\right)$ and $\pi^{L}\left(x,k\right)$ denote the payout (profits) of the private sector in states $H$ and $L$ respectively. Greater private sector risk taking results in higher profits, $\pi^{H}$, outside of a crisis, but lower profits, $\pi^{L}$, if the crisis state arises. The policy tool has the opposite effect, having a net negative impact on profits in the good state, but, through

[22] As policymaker has the same information as the private sector regarding $\gamma$, her choice of $k$ does not convey any information about this parameter.

[23] This modeling assumption is made for ease of exposition. All that is required for this signalling channel is that the policymaker has more information about $\omega$ than the private sector.

providing resilience to the financial system, a net positive benefit in the crisis.

The objective of the public sector is similar to that of the private sector but differs in a few crucial respects. First, the social payout in the crisis state is given by $\pi^L(x,k) - C$, where $C > 0$ represents the policymaker's additional preference for avoiding a crisis. This is not known perfectly by the private sector, rather the private sector has a prior belief that $C$ is distributed as $C \sim N(C^*, \sigma_C^2)$, and is independent of $\omega$. The greater the variance term $\sigma_C^2$, the greater the private sector's uncertainty over the policymaker's disutility from a crisis. The policymaker's choice of $k$ thus conveys some information to the private sector about the state of the world. However, $k$ is a noisy signal because there is uncertainty over $C$; a higher $k$ than the private sector anticipates could be either due to the policymaker having a greater aversion to a crisis than thought, or because of a shock, $\omega$, that makes a crisis more likely. The policymaker internalises that the private sector responds to $k$ both as a variable that directly influences the objective and as an additional signal of risk, when setting their policy tool $k$. This mechanism allows us to explore the signalling impact of macroprudential policy.[24]

The model is solved via the method of undetermined coefficients and we postulate (and then verify) that the private sectors' belief about the shock to the probability of a crisis, $\omega$, having observed $k$, is linear in the choice of policy tool:

$$\mathbb{E}[\omega|k] = \Gamma_0 + \Gamma_1 k \tag{25}$$

Given this, as shown in the Appendix, the optimal choice of $x$ for the private sector is given by:

$$x = \delta_0 + k\left(\delta_1 - \frac{\Gamma_1}{2}\right) \tag{26}$$

where $\delta_0, \delta_1$ are functions of underlying parameters. We show in the appendix that $\delta_1 > 0$, and thus, absent any signalling channel, $k$ and $x$ are strategic complements: the greater the choice of $k$ by the policymaker results the greater the risk taking by the private sector. This arises because a higher choice of $k$ by the policymaker reduces the probability of a crisis and also private sector losses in the crisis state. However, the signalling channel can partially offset this relationship: when $\Gamma_1 > 0$ a greater choice of $k$ is inferred to be due to a greater probability of crisis, which results in the private sector reducing $x$, taking less risk. It is shown in the Appendix that the policymaker's solution for $k$ is linear in the policy

[24] The final difference is that we assume that the public benefit of private sector risk taking in the good state is lower than the private sector benefit: $\beta_H^{pr} > \beta_H^{pub}$.

preference $C$ and shock to the probability of a crisis $\omega$:

$$k = \phi_0 + \phi_1 C + \phi_2 \omega \tag{27}$$

Thus, from the perspective of the private sector, with their priors over $\omega$,$C$ distributed normally and independently, $k$ will also be normally distributed, and, using a standard mathematical result, so too is the distribution of $\omega$ conditional upon observing k. It follows that

$$\mathbb{E}[\omega|k] = -(\phi_0 + \phi_1 C^*)\frac{(\phi_2 \sigma_\omega^2)}{(\sigma_C^2 \phi_1^2 + (\phi_2^2 \sigma_\omega^2))} + \frac{(\phi_2 \sigma_\omega^2)}{(\sigma_C^2 \phi_1^2 + (\phi_2^2 \sigma_\omega^2))} k \tag{28}$$

This verifies the functional form assumption and the model is solved by equating coefficients and solving for $\Gamma_0, \Gamma_1$. There is no closed-from solution for $\Gamma_0, \Gamma_1$, however, we establish in the Appendix that there a unique solution to the model under some restrictions on the parameters.[25] In this solution $\Gamma_1 > 0$, thus the policy tool carries a signalling value about the probability of a crisis. Further, as is intuitive, $\phi_1 > 0, \phi_2 > 0$ and the policymaker sets $k$ higher both when the social cost of a crisis is greater and when there is shock making a crisis more likely. We use the model to examine how the private sector uncertainty affects policymaker activism. In particular, we examine how the policymaker sets $k$ in response to $\omega$, as captured by the $\phi_2$ parameter. The key concept in analysing this is the signal to noise ratio, defined as:

$$SNR = \frac{\sigma_\omega^2}{\sigma_C^2} \tag{29}$$

The private sector care about $\omega$, but do not care about the social cost of a crisis $C$. Hence, $k$ is a useful signal to them to the extent that it provides information about $\omega$, whilst $C$ adds noise to this signal. The greater the signal to noise ratio, the more informative $k$ is as a signal about $\omega$. In particular, the greater the public sector transparency about $C$, the lower $\sigma_C^2$ will be, and the greater SNR.

The main result of the model, as established in the Appendix, is that the greater the signal to noise ratio, the less active the policymaker is in responding to $\omega$:

[25]Specifically we assume that $\frac{(a_H + a_L)}{(\beta_H^{pr}+1)} < \gamma^* < \frac{(a_H + a_L)}{(\beta_H^{pr}+1)}\left[\frac{3(\beta_H^{pr} - \beta_H^{pub})}{2(\beta_H^{pub}+1)} + 1\right]$ The first inequality ensures that higher k is expected to reduce the probability of a crisis (taking into account the private sector response). The second condition ensures that the net benefit of policy is sufficiently greater in the crisis than non-crisis state.

$$\frac{d\phi_2}{d\left(\frac{\sigma_\omega^2}{\sigma_C^2}\right)} < 0$$

As the signal to noise ratio increases, $k$ becomes a more informative signal about the realisation of $\omega$ and consequently more weight is placed on the signal, resulting in $\Gamma_1$ being larger. This in turn reduces the responsiveness of $x$ to $k$ as the private sector view a larger part of the rise in $k$ being due to a greater probability of crisis. Consequently, with the private sector taking less risk in response to a given policy choice, $k$ does not need to be set as high to offset the increased probability of a crisis, resulting in a lower $\phi_2$. This implies that when the public sector is more transparent about $C$, they will not need to be as active in response to information about the probability of a crisis, as the private sector will view $k$ as a more informative signal about the probability of a crisis, reducing the amount of risk they take for a given choice of $k$. Thus, in the presence of greater transparency over policy objectives, policy will not need to be as aggressive due to its increased signalling power.

### 5.2 Discussion

This result highlights the importance of a policymaker ensuring that private agents understand why a policy action is taken, in order to maximise the signalling value of a policy decision. In practice, a macroprudential policymaker has a variety of ways of communicating to the private sector beyond her actions. The explanation of macroprudential policy decisions given alongside the action can help the public learn and extract an informative signal about the policymaker's views on financial stability risks. Alternatively, the macroprudential policymaker may signal their likely future actions in response to a build-up in risks by clarifying their 'reaction function'. The publication of core financial stability indicators may provide some clarity to the policymaker's reaction function. Communicating likely future actions may affect expectations about the evolution of the economy, and hence influence behaviour today.

Changes in private sector uncertainty have affected the transmission mechanism of other policy tools. As noted by Beechey (2008), the news on 6 May 1997 that a Monetary Policy Committee would be created and the Bank of England would be granted operational independence with an explicit inflation target was followed by a substantial reduction in market-implied short and long term inflation expectations, arguably because this reduced uncertainty about the objectives of monetary policy. The forward rates of inflation compensation 5-10 years ahead fell around 35 basis points on the day of the announcement, with a further decline over the following days. This in turn affected the transmission mechanism of monetary policy.

# 6 Conclusion

As this paper has highlighted, there are several types of uncertainty and multiple channels through which uncertainty can affect policy making. The well-known result of Brainard (1967) that policy should be more cautious in the presence of uncertainty does not hold in general, particularly for specific examples that are relevant to macroprudential policymakers. If anything, the need to learn about these relatively untested tools and the focus on avoiding tail risks speak to more active policy making. Moreover, private sector uncertainty over the financial stability objectives, preferences and reaction function may diminish the potency of the signalling impact of macroprudential policy, requiring more active policy or communication. One limitation of the analysis here is that it takes place in context of static or two period models. Hence it is a challenge to disentangle tighter steady state macroprudential policy in response to increased risks from a more active response to economic shocks as they emerge. This would be an interesting area for future analysis.

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# A Simple Set Up A La Brainard

**Proposition 1** *The solution to the Brainard problem is given by*

$$k^u = \frac{b^* x^* + \lambda k^* - \rho \sigma_b \sigma_u}{(b^*)^2 + \sigma_b^2 + \lambda}$$

**Proof.** The policymaker's problem can be written as

$$\max_k -\frac{1}{2}\mathbb{E}[(x - x^*)^2 + \lambda (k - k^*)^2] : x = bk + u$$

Substituting in the constraint gives

$$\max_k -\frac{1}{2}\mathbb{E}[(bk + u - x^*)^2 + \lambda (k - k^*)^2]$$

Which has a FOC

$$-\frac{1}{2}\mathbb{E}[2(bk + u - x^*)b + 2\lambda (k - k^*)] = 0$$

Rearranging gives

$$\mathbb{E}[b^2 k + bu - bx^* + \lambda (k - k^*)] = 0$$
$$k \left(\mathbb{E}[b^2] + \lambda\right) = \mathbb{E}[bx^*] + \lambda k^* - \mathbb{E}[bu]$$

And finally as, noting that mean $u = 0$, the solution is given by:

$$k^u = \frac{b^* x^* + \lambda k^* - \rho \sigma_b \sigma_u}{(b^*)^2 + \sigma_b^2 + \lambda} \tag{30}$$

When $\rho = 0$ this gives equation (3) in the text.

■

**Lemma 2** *Let $k^c$ be the optimal solution under certainty. Then $k^u < k^c$ and policy is expected to be less activist under uncertainty iff*

$$\sigma_b > -\rho \sigma_u \frac{\left((b^*)^2 + \lambda\right)}{(b^* x^* + \lambda k^*)}$$

**Proof.** When there is no uncertainty, $\rho = 0$ and $\sigma_b^2 = 0$ and so from (30) the solution reduces to (evaluated at average $b$)

$$k^c = \frac{b^* x^* + \lambda k^*}{(b^*)^2 + \lambda}$$

Thus $k^u < k^c$ iff

$$\frac{b^* x^* + \lambda k^* - \rho\sigma_b\sigma_u}{(b^*)^2 + \sigma_b^2 + \lambda} < \frac{b^* x^* + \lambda k^*}{(b^*)^2 + \lambda}$$

$$(b^* x^* + \lambda k^* - \rho\sigma_b\sigma_u)\left((b^*)^2 + \lambda\right) < \left((b^*)^2 + \sigma_b^2 + \lambda\right)(b^* x^* + \lambda k^*)$$

$$(b^* x^* + \lambda k^*)\left((b^*)^2 + \lambda\right) - \rho\sigma_b\sigma_u((b^*)^2 + \lambda) < \left((b^*)^2 + \lambda\right)(b^* x^* + \lambda k^*) + \sigma_b^2 (b^* x^* + \lambda k^*)$$

$$-\rho\sigma_b\sigma_u\left((b^*)^2 + \lambda\right) < \sigma_b^2 (b^* x^* + \lambda k^*)$$

$$-\frac{\rho\sigma_u\left((b^*)^2 + \lambda\right)}{(b^* x^* + \lambda k^*)} < \sigma_b$$

■

# B Robust Control

**Proposition 3** *The Nash solution to the policymaker's problem is given by*

$$k = \frac{\theta b^* x^* - (\theta - 1)\rho\sigma_b\sigma_u + (\theta - 1)\lambda k^*}{\theta (b^*)^2 + \sigma_b^2(\theta - 1) + (\theta - 1)\lambda}$$

*Further, the best responses of nature and the policymaker are given respectively by*

$$v = \frac{b^* k - x^*}{\theta - 1}$$

$$k = \frac{x^* b^* - \mathbb{E}(bu) + \lambda k^* - b^* v}{(b^*)^2 + \sigma_b^2 + \lambda}$$

*The results in the text follow for the special case of* $\mathbb{E}(bu) = 0$.

**Proof.** Nature's problem is to solve

$$min_v - \frac{1}{2}\mathbb{E}[(bk + u + v - x^*)^2 + \lambda(k - k^*)^2] + \frac{\theta}{2}v^2$$

Given $\theta > 1$ nature's objective is convex with linear constraints such that an interior optimum exists. Nature's first order condition is, taking $k$ as given:

$$-\frac{1}{2}\mathbb{E}[2(bk + u + v - x^*)] + \theta v = 0$$

Giving (noting that $v$ is known as it's nature's choice variable)

$$\theta v = (b^* k + v - x^*)$$

$$v = \frac{b^* k - x^*}{\theta - 1} \tag{31}$$

Taking $v$ as given, the first order condition of the policy problem, wrt its choice variable $k$ is given by

$$-\frac{1}{2}\mathbb{E}[2(bk + u + v - x^*)b + 2\lambda(k - k^*)] = 0$$
$$\mathbb{E}[(b^2 k + bu + bv - bx^*) + \lambda(k - k^*)] = 0$$
$$k\left[\mathbb{E}(b^2) + \lambda\right] = x^* b^* - \mathbb{E}(bu) + \lambda k^* - \mathbb{E}(bv)$$

Thus, we have (noting that $v$ is not stochastic, given that nature has the same information set as the policy maker)

$$k = \frac{x^* b^* - \mathbb{E}(bu) + \lambda k^* - \mathbb{E}(bv)}{\mathbb{E}(b^2) + \lambda}$$
$$k = \frac{x^* b^* - \mathbb{E}(bu) + \lambda k^* - b^* v}{\mathbb{E}(b^2) + \lambda}$$

Substituting in (31) for $v$ and rearranging gives:

$$\mathbb{E}(b^2)k + \mathbb{E}(bu) - b^* x^* + \mathbb{E}\left(\frac{bb^* k - bx^*}{(\theta - 1)}\right) + \lambda(k - k^*) = 0$$
$$\mathbb{E}(b^2)k + \mathbb{E}(bu) - b^* x^* + \left(\frac{(b^*)^2 k - b^* x^*}{(\theta - 1)}\right) + \lambda(k - k^*) = 0$$

$$\left[\mathbb{E}(b^2) + \frac{(b^*)^2}{\theta - 1} + \lambda\right] k = b^* x^* \left[1 + \frac{1}{(\theta - 1)}\right] + \lambda k^* - \mathbb{E}(bu)$$
$$\left[\mathbb{E}(b^2)(\theta - 1) + (b^*)^2 + (\theta - 1)\lambda\right] k = b^* x^* \theta + (\lambda k^* - \mathbb{E}(bu))(\theta - 1)$$
$$\left[\left((b^*)^2 + \sigma_b^2\right)(\theta - 1) + (b^*)^2 + (\theta - 1)\lambda\right] k = b^* x^* \theta + (\lambda k^* - \mathbb{E}(bu))(\theta - 1)$$

Thus

$$k = \frac{\theta b^* x^* - (\theta - 1)\rho\sigma_b\sigma_u + (\theta - 1)\lambda k^*}{\theta (b^*)^2 + \sigma_b^2(\theta - 1) + (\theta - 1)\lambda}$$

This completes the proof. ∎

**Lemma 4** *When fundamental uncertainty disappears ($\theta \to \infty$) the optimal solution reduces to (30), the solution in the Brainard set-up. The Robust control set-up is thus more general nesting the latter.*

**Proof.** From above, we have

$$k = \frac{\theta b^* x^* - (\theta - 1)\rho\sigma_b\sigma_u + (\theta - 1)\lambda k^*}{\theta\left(b^*\right)^2 + \sigma_b^2(\theta - 1) + (\theta - 1)\lambda}$$

We can write this as

$$k = \frac{\frac{\theta}{(\theta-1)}b^* x^* - \rho\sigma_b\sigma_u + \lambda k^*}{\frac{\theta}{(\theta-1)}\left(b^*\right)^2 + \sigma_b^2 + \lambda}$$

Now

$$\lim_{\theta\to\infty}\frac{\theta}{(\theta-1)} = \lim_{\theta\to\infty}\frac{1}{(1-\frac{1}{\theta})} = \frac{1}{1-0} = 1$$

Hence, using the Algebra of Limits

$$\lim_{\theta\to\infty} k = \frac{b^* x^* - \rho\sigma_b\sigma_u + \lambda k^*}{\left(b^*\right)^2 + \sigma_b^2 + \lambda}$$

Which is identical to (30). ■

**Lemma 5** *With the robust control model*

$$\frac{dk}{d\theta} < 0$$

*iff*

$$\sigma_b^2 > \frac{\lambda\left(b^* k^* - x^*\right) - b^*\rho\sigma_b\sigma_u}{x^*} \tag{32}$$

*Thus, given that fundamental uncertainty* ***increases*** *as $\theta$ decreases, under (32), the policymaker is more activist when fundamental uncertainty is greater. Thus, in particular, the policymaker is more activist than in the Brainard model when applying Robust Control when (32) holds.*

**Proof.** Let

$$f(\theta) := \theta b^* x^* - (\theta - 1)\rho\sigma_b\sigma_u + (\theta - 1)\lambda k^*$$

$$g(\theta) := \theta\left(b^*\right)^2 + \sigma_b^2(\theta - 1) + (\theta - 1)\lambda$$

Then we can write

$$k = \frac{f(\theta)}{g(\theta)}$$

And hence

$$\frac{dk}{d\theta} = \frac{f'(\theta)\, g(\theta) - f(\theta)\, g'(\theta)}{[g(\theta)]^2}$$

Thus $\frac{dk}{d\theta} < 0$ iff $f'(\theta)\, g(\theta) < f(\theta)\, g'(\theta)$. This condition can be written as

$$[b^* x^* - \rho\sigma_b\sigma_u + \lambda k^*]\left[\theta\,(b^*)^2 + \sigma_b^2(\theta-1) + (\theta-1)\lambda\right] < [\theta b^* x^* - (\theta-1)\,\rho\sigma_b\sigma_u + (\theta-1)\lambda k^*]\left[(b^*)^2 + \sigma_b^2 + \lambda\right]$$

This can be further written as

$$\begin{aligned}&(\theta-1)\,[b^* x^* - \rho\sigma_b\sigma_u + \lambda k^*]\left[(b^*)^2 + \sigma_b^2 + \lambda\right] + [b^* x^* - \rho\sigma_b\sigma_u + \lambda k^*]\,(b^*)^2 \\ &< (\theta-1)\,[b^* x^* - \rho\sigma_b\sigma_u + \lambda k^*]\left[(b^*)^2 + \sigma_b^2 + \lambda\right] + b^* x^*\left[(b^*)^2 + \sigma_b^2 + \lambda\right]\end{aligned}$$

Canceling common terms this reduces to

$$\begin{aligned}[b^* x^* - \rho\sigma_b\sigma_u + \lambda k^*]\,(b^*)^2 &< b^* x^*\left[(b^*)^2 + \sigma_b^2 + \lambda\right] \\ (b^*)^2 x^* - b^*\rho\sigma_b\sigma_u + b^*\lambda k^* &< x^*(b^*)^2 + x^*\sigma_b^2 + x^*\lambda \\ -b^*\rho\sigma_b\sigma_u + b^*\lambda k^* &< x^*\sigma_b^2 + x^*\lambda\end{aligned}$$

$$\sigma_b^2 > \frac{\lambda\,(b^* k^* - x^*) - b^*\rho\sigma_b\sigma_u}{x^*}$$

This completes the proof. ■

# C Asymmetric Objective Function

**Lemma 6** *Let the policy objective $W$ be given by (10) in Section 3. Then*

$$\underset{a\to 0}{lim} W = -\frac{1}{2}\mathbb{E}\left\{(x - x^*)^2 + \frac{\lambda}{2}(k - k^*)^2\right\}$$

*which is the objective function in the baseline Brainard case.*

**Proof.** We show this using L'Hôpital's rule. Let

$$f(a) := e^{a(x^* - x)} - a\,(x^* - x) - 1$$

$$g(a) := a^2$$

Then $f''(a) = e^{a(x^*-x)}(x^*-x) - (x^*-x)$ and $f'''(a) = e^{a(x^*-x)}(x^*-x)^2$. Thus $\underset{a\to 0}{lim} f(a) = 1-1=0$, $\underset{a\to 0}{lim} f'(a) = (x^*-x)-(x^*-x) = 0$ and $\underset{a\to 0}{lim} f''(a) = (x^*-x)^2$.

Further, $g'(a) = 2a$ and $g''(a) = 2$ thus $\underset{a\to 0}{lim} g(a) = 0$ ,$\underset{a\to 0}{lim} g'(a) = 0$ and $\underset{a\to 0}{lim} g''(a) = 2$. Thus, by L'Hôpital's rule we have

$$\underset{a\to 0}{lim}\frac{e^{a(x^*-x)} - a(x^*-x) - 1}{a^2} = \frac{(x^*-x)^2}{2} = \frac{(x-x^*)^2}{2}$$

The result then follows from the Algebra of Limits.

■

The next lemma demonstrates the asymmetric nature of the loss function, with a greater loss when financial stability, as measure by$x$, being below than above it's target value $x^*$.

**Lemma 7** *Define the financial stability loss function* $L(x)$ *as*

$$L(x) := \frac{e^{a(x^*-x)} - a(x^*-x) - 1}{a^2}$$

*The loss when financial stability is too low, relative to when it's too high is given by (for* $\epsilon > 0$)

$$\Gamma(\epsilon, a) := \frac{L(x^*-\epsilon)}{L(x^*+\epsilon)} = \frac{e^{a\epsilon} - a\epsilon - 1}{e^{-a\epsilon} + a\epsilon - 1}$$

*Then we have no asymmetry when* $a \to 0$*:*

$$\underset{a\to 0}{lim}\Gamma(\epsilon, a) = 1$$

*And greater asymmetry the greater* $a$ *is:*

$$\frac{\partial\Gamma(\epsilon, a)}{\partial a} > 0$$

*Hence, the greater* $a$ *is, the greater the loss when financial stability is too low relative to being too high.*

**Proof.** Beginning with the first property

$$\Gamma(\epsilon, a) = \frac{L(x^*-\epsilon)}{L(x^*+\epsilon)}$$

Hence, using the Algebra of Limits, by the above lemma,

$$\lim_{a \to 0} \Gamma(\epsilon, a) = \frac{\frac{(\epsilon)^2}{2}}{\frac{(-\epsilon)^2}{2}} = \frac{\epsilon^2}{\epsilon^2} = 1$$

Turning to the second property, we have that

$$\frac{\partial \Gamma(\epsilon, a)}{\partial a} = \frac{(\epsilon e^{a\epsilon} - \epsilon)(e^{-a\epsilon} + a\epsilon - 1) - (e^{a\epsilon} - a\epsilon - 1)(-\epsilon e^{-a\epsilon} + \epsilon)}{(e^{-a\epsilon} + a\epsilon - 1)^2}$$

This can be written as

$$\frac{\partial \Gamma(\epsilon, a)}{\partial a} = \frac{(e^{a\epsilon} - 1)(e^{-a\epsilon} + a\epsilon - 1) + (e^{a\epsilon} - a\epsilon - 1)(e^{-a\epsilon} - 1)}{\frac{(e^{-a\epsilon} + a\epsilon - 1)^2}{\epsilon}}$$

Collecting terms on the numerator we have

$$1 + a\epsilon e^{a\epsilon} - e^{a\epsilon} - e^{-a\epsilon} - \epsilon a + 1 + 1 - e^{a\epsilon} - a\epsilon e^{-a\epsilon} + a\epsilon - e^{-a\epsilon} + 1$$

Collecting terms this can be simplified to

$$4 + e^{a\epsilon}(a\epsilon - 2) - e^{-a\epsilon}(a\epsilon + 2) \tag{33}$$

It remains to show (33) is positive for $a, \epsilon > 0$. To simplify, we define the following function:

$$h(z) := 4 + e^{z}(z - 2) - e^{-z}(z + 2)$$

It's then enough to show that $h(z) > 0$ for $z > 0$. We have $h(0) = 4 - 2 - 2 = 0$. Further,

$$h'(z) = e^{z}(z - 2) + e^{z} + e^{-z}(z + 2) - e^{-z} = e^{z}(z - 1) + e^{-z}(z + 1)$$

Now $h'(0) = -1 + 1 = 0$. Further

$$h''(z) = e^{z}(z - 1) + e^{z} - e^{-z}(z + 1) + e^{-z} = \left(e^{z} - e^{-z}\right) z$$

Now, for $z > 0$ $e^{z} > 1 > e^{-z}$ and hence $h''(z) > 0$ for $z > 0$. Thus as $h'(0) = 0$ we have $h'(z) > 0$ for $z > 0$. Given that $h(0) = 0$ we thus also have $h(z) > 0$ for $z > 0$. Thus we have $\frac{\partial \Gamma(\epsilon, a)}{\partial a} > 0$. This completes the proof of the lemma.

■

We now consider the solution to the policymaker's problem.

**Lemma 8** *Suppose the policymaker has objective function given by (10) with* $x = bk + u$ *and* $b \sim N\left(b^*, \sigma_b^2\right), \ u \sim N\left(0, \sigma_u^2\right)$ *and* $Cov\left(b, u\right) = 0.$

*Then we can write the policymaker's objective function as*

$$W = -\left\{ \frac{e^{\left(ax^* - ab^*k + \frac{a^2\left(\sigma_b^2 k^2 + \sigma_u^2\right)}{2}\right)} - a\left(x^* - b^*k\right) - 1}{a^2} + \frac{\lambda}{2}\left(k - k^*\right)^2 \right\} \tag{34}$$

*This function is maximised with respect to* $k$*.*

**Proof.** Substituting $x = bk + u$ into (10) gives:

$$W = -\mathbb{E}\left\{ \frac{e^{a(x^* - bk - u)} - a\left(x^* - bk - u\right) - 1}{a^2} + \frac{\lambda}{2}\left(k - k^*\right)^2 \right\} \tag{35}$$

Given the distributional assumptions on $b, u$ we have

$$a\left(x^* - bk - u\right) \sim N\left(ax^* - ab^*k, a^2\left(k^2\sigma_b^2 + \sigma_u^2\right)\right)$$

Now, it's a standard result about the log-normal distribution that if $Z \sim N(\mu, \sigma^2)$then $\mathbb{E}\left(e^Z\right) = e^{\mu + \frac{\sigma^2}{2}}$. Thus we have

$$\mathbb{E}\left\{e^{a(x^* - bk - u)}\right\} = e^{\left\{ax^* - ab^*k + \frac{a^2\left(k^2\sigma_b^2 + \sigma_u^2\right)}{2}\right\}}$$

Hence we can write (35) as

$$W = -\left\{ \frac{e^{\left(ax^* - ab^*k + \frac{a^2\left(k^2\sigma_b^2 + \sigma_u^2\right)}{2}\right)} - a\left(x^* - b^*k\right) - 1}{a^2} + \frac{\lambda}{2}\left(k - k^*\right)^2 \right\}$$

This completes the proof of the lemma. ■

We now show that this function has a unique global maximum.

**Lemma 9** *The first derivative of* $W$ *is given by:*

$$\frac{dW}{dk} = -\left\{ \frac{1}{a}\left[ e^{\left(ax^* - ab^*k + \frac{a^2\left(k^2\sigma_b^2 + \sigma_u^2\right)}{2}\right)}\left\{-b^* + ak\sigma_b^2\right\} + b^* \right] + \lambda\left(k - k^*\right) \right\}$$

*The second derivative of $W$ is given by:*

$$\frac{d^2W}{dk^2} = -\left\{\frac{1}{a^2}\left[e^{\left(ax^*-ab^*k+\frac{a^2\left(k^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}\left\{-ab^*+a^2k\sigma_b^2\right\}^2+e^{\left(ax^*-ab^*k+\frac{a^2\left(k^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}a^2\sigma_b^2\right]+\lambda\right\}$$

*Further the function is strictly concave:*

$$\frac{d^2W}{dk^2} < 0$$

**Proof.** Using equation 34 we have

$$\frac{dW}{dk} = -\left\{\frac{1}{a^2}\left[e^{\left(ax^*-ab^*k+\frac{a^2\left(k^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}\left\{-ab^*+a^2k\sigma_b^2\right\}+ab^*\right]+\lambda\left(k-k^*\right)\right\}$$

Thus, we have that

$$\frac{d^2W}{dk^2} = -\left\{\frac{1}{a^2}\left[e^{\left(ax^*-ab^*k+\frac{a^2\left(k^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}\left\{-ab^*+a^2k\sigma_b^2\right\}^2+e^{\left(ax^*-ab^*k+\frac{a^2\left(k^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}a^2\sigma_b^2\right]+\lambda\right\}$$

It's clear from this expression that $\frac{d^2W}{dK^2} < 0$. This completes the proof of the lemma.

■

We thus have the following proposition:

**Proposition 10** *The unique global maximum solution for $W$ is given by the $\tilde{k}$ that satisfies-*

$$\frac{-1}{a}\left(e^{\left(ax^*-ab^*\tilde{k}+\frac{a^2\left(\tilde{k}^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}\left\{-b^*+a\tilde{k}\sigma_b^2\right\}+b^*\right)-\lambda\left(\tilde{k}-k^*\right)=0 \quad (36)$$

**Corollary 11** *If $a > 0, b^* > 0, x^* > 0$,$\lambda > 0$ and $k^* \geq 0$ then the unique global maximum solution for $W$ is contained in the following range:*

$$\tilde{k}\epsilon\left(0, max\left\{\frac{b^*}{a\sigma_b^2}, k^*-\frac{b^*}{\lambda a}\right\}\right)$$

**Proof.** From above we have

$$\frac{dW}{dk}|_{k=0} = -\left\{\frac{1}{a}\left[e^{\left(ax^*+\frac{a^2\sigma_u^2}{2}\right)}\left\{-b^*\right\}+b^*\right]+\lambda\left(-k^*\right)\right\}$$

This can then be written as

$$\frac{b^*}{a}\left[e^{\left(ax^*+\frac{a^2\sigma_u^2}{2}\right)}-1\right]+\lambda k^*$$

Under the given conditions, this is positive. Hence, as $W$ is strictly concave, the derivative is positive iff $k<\tilde{k}$ and so we must have $0<\tilde{k}$.

We now consider the upper bound. Recall we have

$$\frac{dW}{dk}=-\left\{\frac{1}{a}\left[e^{\left(ax^*-ab^*k+\frac{a^2\left(k^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}\left\{-b^*+ak\sigma_b^2\right\}+b^*\right]+\lambda\left(k-k^*\right)\right\}$$

Thus, if $k\geq max\left\{\frac{b^*}{a\sigma_b^2},k^*-\frac{b^*}{\lambda a}\right\}$ then $-b^*+ak\sigma_b^2\geq 0$ and $\frac{b^*}{a}+\lambda\left(k-k^*\right)\geq 0$ then $\frac{dW}{dk}<0$. However, as $W$ is strictly concave, the derivative is negative iff $k>\tilde{k}$ and so we must have $\tilde{k}<max\left\{\frac{b^*}{a\sigma_b^2},k^*-\frac{b^*}{\lambda a}\right\}$. This completes the proof.

■

## C.1 Response of Policy to $\sigma_b^2$

The following lemma establishes the formula for $\frac{d\tilde{k}}{d\sigma_b^2}$

**Lemma 12** *The policymaker's optimal choice of their tool responds to uncertainty over the impact of the tool as follows*

$$\frac{d\tilde{k}}{d\sigma_b^2}=\frac{-e^{\left(ax^*-ab^*\tilde{k}+\frac{a^2\left(\tilde{k}^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}\tilde{k}\left(\frac{a\tilde{k}\left\{-b^*+a\tilde{k}\sigma_b^2\right\}}{2}+1\right)}{e^{\left(ax^*-ab^*k+\frac{a^2\left(k^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}\left\{-b^*+ak\sigma_b^2\right\}^2+e^{\left(ax^*-ab^*k+\frac{a^2\left(k^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}\sigma_b^2+\lambda} \tag{37}$$

**Proof.** From (36) we can write the optimality condition that implicitly defines $\tilde{k}$ as

$$g\left(\tilde{k}\left(\sigma_b^2\right),\sigma_b^2\right)\equiv 0$$

Then, totally differentiating this with respect to $\sigma_b^2$ we have

$$\frac{\partial g}{\partial\tilde{k}}\left(\frac{d\tilde{k}}{d\sigma_b^2}\right)+\frac{\partial g}{\partial\sigma_b^2}=0$$

Now, from the prior lemma, $\frac{\partial g}{\partial \tilde{k}} < 0$ and so the following is w.d. (i.e. the Implicit Function Theorem holds):

$$\frac{d\tilde{k}}{d\sigma_b^2} = \frac{-\frac{\partial g}{\partial \sigma_b^2}}{\frac{\partial g}{\partial \tilde{k}}}$$

Given that $\frac{\partial g}{\partial \tilde{k}} < 0$ we have that $\frac{d\tilde{k}}{d\sigma_b^2} > 0$ iff $\frac{\partial g}{\partial \sigma_b^2} > 0$. We now turn to computing this partial derivative.

We can write

$$g\left(\tilde{k}\left(\sigma_b^2\right), \sigma_b^2\right) = -\frac{e^{\left(ax^* - ab^*\tilde{k} + \frac{a^2\left(\tilde{k}^2\sigma_b^2 + \sigma_u^2\right)}{2}\right)}\left\{-b^* + a\tilde{k}\sigma_b^2\right\}}{a} + C$$

where $C$ is independent of $\sigma_b^2$. Thus, taking the derivative we have

$$\frac{\partial g}{\partial \sigma_b^2} = -\left\{\frac{e^{\left(ax^* - ab^*\tilde{k} + \frac{a^2\left(\tilde{k}^2\sigma_b^2 + \sigma_u^2\right)}{2}\right)}\left\{-b^* + a\tilde{k}\sigma_b^2\right\}\left\{\frac{a^2\tilde{k}^2}{2}\right\}}{a} + \frac{e^{\left(ax^* - ab^*\tilde{k} + \frac{a^2\left(\tilde{k}^2\sigma_b^2 + \sigma_u^2\right)}{2}\right)}a\tilde{k}}{a}\right\}$$

Gathering terms this can be written as

$$\frac{\partial g}{\partial \sigma_b^2} = -e^{\left(ax^* - ab^*\tilde{k} + \frac{a^2\left(\tilde{k}^2\sigma_b^2 + \sigma_u^2\right)}{2}\right)}\tilde{k}\left(\frac{a\tilde{k}\left\{-b^* + a\tilde{k}\sigma_b^2\right\}}{2} + 1\right)$$

■

**Remark 13** *In the case of $a \to 0$ this reduces to*

$$\frac{d\tilde{k}}{d\sigma_b^2} = \frac{-\tilde{k}}{\{b^*\}^2 + \sigma_b^2 + \lambda} < 0$$

*Which is precisely the derivative in the Brainard case.*[26]

**Corollary 14** *Given that $\tilde{k} > 0$, the policymaker becomes more active when there is greater uncertainty over the impact of their tool so long as*

$$\left(\frac{a\tilde{k}\left\{-b^* + a\tilde{k}\sigma_b^2\right\}}{2} + 1\right) < 0 \tag{38}$$

[26] To see this note that in the Brainard case we have $\tilde{k} = \frac{b^*x^* + \lambda k^*}{(b^*)^2 + \sigma_b^2 + \lambda}$. Hence we have

$$\frac{d\tilde{k}}{d\sigma_b^2} = -\frac{(b^*x^* + \lambda k^*)}{\left((b^*)^2 + \sigma_b^2 + \lambda\right)^2} = -\frac{\tilde{k}}{(b^*)^2 + \sigma_b^2 + \lambda}$$

To build up to the main proposition in this section we make use of the following lemma.

**Lemma 15** *The following relationship holds as the mean effectiveness of the policy tool* $(b^*)$ *becomes arbitrarily large:*

$$\lim_{b^*\to\infty} b^*\tilde{k} = x^* + \frac{a\sigma_u^2}{2}$$

**Proof.** We first establish that $b^*\tilde{k}$ is bounded. We clearly have that $b^*\tilde{k}\geq 0$ so it's sufficient to show that $\exists S \geq 0 : b^*\tilde{k} \leq S \;\; \forall b^* \geq 0$. To show this, we suppose for a contradiction that it's not true. Then $b^*\tilde{k} \to \infty$ as $b^* \to \infty$. Then we can write the FOC as

$$\frac{-1}{a}\left(e^{\left(ax^*-ab^*\tilde{k}+\frac{a^2\left(\tilde{k}^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}a\tilde{k}\sigma_b^2 + b^*\left\{1-e^{\left(ax^*-ab^*\tilde{k}+\frac{a^2\left(\tilde{k}^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}\right\}\right) - \lambda\left(\tilde{k}-k^*\right) = 0 \quad (39)$$

We first establish that

$$e^{\left(ax^*-ab^*\tilde{k}+\frac{a^2\left(\tilde{k}^2\sigma_b^2+\sigma_u^2\right)}{2}\right)} \to 0 \text{ as } b^* \to\infty \quad (40)$$

We have $\tilde{k} \leq max\left\{\frac{b^*}{a\sigma_b^2}, k^* - \frac{b^*}{\lambda a}\right\}$ , hence for large $b^*$ we have $\tilde{k} \leq \frac{b^*}{a\sigma_b^2}$. Thus we have

$$ax^* - ab^*\tilde{k} + \frac{a^2\left(\tilde{k}^2\sigma_b^2+\sigma_u^2\right)}{2} \leq ax^* + \frac{a^2\sigma_u^2}{2} - ab^*\tilde{k} + \frac{a^2\sigma_b^2}{2}\tilde{k}\left(\frac{b^*}{a\sigma_b^2}\right)$$

Thus we have

$$ax^* - ab^*\tilde{k} + \frac{a^2\left(\tilde{k}^2\sigma_b^2+\sigma_u^2\right)}{2} \leq ax^* + \frac{a^2\sigma_u^2}{2} - ab^*\tilde{k} + \frac{a}{2}\tilde{k}b^* = ax^* + \frac{a^2\sigma_u^2}{2} - \frac{ab^*\tilde{k}}{2}$$

As $b^*\tilde{k}$ is unbounded, the RHS tends to $-\infty$ as $b^* \to \infty$ and hence so too does the LHS and (40) holds.

Hence, using the Algebra of Limits $b^*\left\{1-e^{\left(ax^*-ab^*\tilde{k}+\frac{a^2\left(\tilde{k}^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}\right\} \to \infty$ .

Further $e^{\left(ax^*-ab^*\tilde{k}+\frac{a^2\left(\tilde{k}^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}a\tilde{k}\sigma_b^2 \geq 0$ and $k \geqslant 0$. Thus, in sum, the LHS of the (39) tends to $-\infty$ as $b^* \to \infty$, a contradiction, as the FOC no longer holds. Hence we must have $b^*\tilde{k}$ bounded.

Now with $b^*\tilde{k}$ bounded and non-negative $\exists S \geq 0 : 0 \leq b^*\tilde{k} \leq S \; \forall b^*$. Thus $0 \leq \tilde{k} \leq \frac{S}{b^*}$ and hence as $\lim_{b^*\to\infty} \frac{S}{b^*} = 0$ by the Sandwich Theorem we have $\lim_{b^*\to\infty} \tilde{k} = 0$. Thus we must have $e^{\left(ax^*-ab^*\tilde{k}+\frac{a^2\left(\tilde{k}^2\sigma_b^2+\sigma_u^2\right)}{2}\right)}a\tilde{k}\sigma_b^2 \to 0$ as the exponential term is bounded and $\lim_{b^*\to\infty} \tilde{k} = 0$.

Thus for (39) to hold we must have $e^{\left(ax^*-ab^*\tilde{k}+\frac{a^2\left(\tilde{k}^2\sigma_b^2+\sigma_u^2\right)}{2}\right)} \to 1$ otherwise the FOC will tend to $-\infty$ or $\infty$.

Thus as $log()$ is strictly monotonic increasing, we must have

$$\lim_{b^*\to\infty}\left\{ax^* - ab^*\tilde{k} + \frac{a^2\left(\tilde{k}^2\sigma_b^2+\sigma_u^2\right)}{2}\right\} = 0$$

Given $\lim_{b^*\to\infty}\tilde{k} = 0$ it follows that

$$\lim_{b^*\to\infty}\left\{ax^* - ab^*\tilde{k} + \frac{a^2\sigma_u^2}{2}\right\} = 0$$

and hence

$$\lim_{b^*\to\infty} b^*\tilde{k} = x^* + \frac{a\sigma_u^2}{2}$$

This completes the proof of the lemma.

■

Using the lemma we establish the following proposition.

**Proposition 16** *Suppose*

$$a > \frac{-x^* + \sqrt{(x^*)^2 + 4\sigma_u^2}}{\sigma_u^2}$$

*Then* $\frac{d\tilde{k}}{d\sigma_b^2} > 0$ *for all sufficiently large* $b^*$.

*In other words, if the asymmetric losses from low financial stability are sufficiently great, when the expected impact of the policy tool is sufficiently large, then greater uncertainty over the impact of the tool leads to greater activism.*

**Proof.** From above, $\frac{d\tilde{k}}{d\sigma_b^2} > 0$ when $\left(\frac{a\tilde{k}\left\{-b^*+a\tilde{k}\sigma_b^2\right\}}{2} + 1\right) < 0$. We can write the LHS as

$$\frac{a\tilde{k}b^*\left\{-1 + a\frac{\tilde{k}}{b^*}\sigma_b^2\right\}}{2} + 1$$

By the lemma,

$$\lim_{b^*\to\infty}\frac{\tilde{k}}{b^*} = \frac{x^*}{(b^*)^2} + \frac{a\sigma_u^2}{2\,(b^*)^2} = 0$$

Hence

$$\lim_{b^*\to\infty}\left\{\frac{a\tilde{k}b^*\left\{-1 + a\frac{\tilde{k}}{b^*}\sigma_b^2\right\}}{2} + 1\right\} = \frac{-a\left\{x^* + \frac{a\sigma_u^2}{2}\right\}}{2} + 1$$

Now, the RHS is negative so long as

$$-a\left\{x^{*}+\frac{a\sigma_{u}^{2}}{2}\right\}<-2$$

or

$$0<a^{2}\frac{\sigma_{u}^{2}}{2}+ax^{*}-2$$

Let $f(a):=a^{2}\frac{\sigma_{u}^{2}}{2}+ax^{*}-2$. Then $f\left(0\right)<0$ and $f\left(a\right)\rightarrow\infty$ as $a\rightarrow\pm\infty$ hence it has two real roots (one positive, one negative) and will be positive for all $a$ greater than the positive root. This positive root is given by

$$\frac{-x^{*}+\sqrt{\left(x^{*}\right)^{2}+4\sigma_{u}^{2}}}{\sigma_{u}^{2}}$$

Hence, when $a>\frac{-x^{*}+\sqrt{\left(x^{*}\right)^{2}+4\sigma_{u}^{2}}}{\sigma_{u}^{2}}$ we'll have $\underset{b^{*}\rightarrow\infty}{lim}\left(\frac{a\tilde{k}\left\{-b^{*}+a\tilde{k}\sigma_{b}^{2}\right\}}{2}+1\right)<0$ and $\frac{d\tilde{k}}{d\sigma_{b}^{2}}>0$. This completes the proof.

■

## C.2 Response of Policy to $\sigma_{u}^{2}$

The following lemma establishes the formula for $\frac{d\tilde{k}}{d\sigma_{u}^{2}}$.

**Lemma 17** *The policymaker's optimal choice of their tool responds to uncertainty over the impact of the tool as follows*

$$\frac{d\tilde{k}}{d\sigma_{u}^{2}}=\frac{-e^{\left(ax^{*}-ab^{*}\tilde{k}+\frac{a^{2}\left(\tilde{k}^{2}\sigma_{b}^{2}+\sigma_{u}^{2}\right)}{2}\right)}\left\{-b^{*}+a\tilde{k}\sigma_{b}^{2}\right\}\frac{\sigma_{u}^{2}}{2}a}{e^{\left(ax^{*}-ab^{*}k+\frac{a^{2}\left(k^{2}\sigma_{b}^{2}+\sigma_{u}^{2}\right)}{2}\right)}\left\{-b^{*}+ak\sigma_{b}^{2}\right\}^{2}+e^{\left(ax^{*}-ab^{*}k+\frac{a^{2}\left(k^{2}\sigma_{b}^{2}+\sigma_{u}^{2}\right)}{2}\right)}\sigma_{b}^{2}+\lambda} \tag{41}$$

**Proof.** As with the case of a change in $\sigma_{b}^{2}$ the derivative can be expressed via the implicit function theorem (with $g()$ defined as before):

$$\frac{d\tilde{k}}{d\sigma_{u}^{2}}=\frac{\frac{\partial g}{\partial\sigma_{u}^{2}}}{-\frac{\partial g}{\partial\tilde{k}}}$$

The expression for $-\frac{\partial g}{\partial\tilde{k}}$ is as above, given by the second derivative of $W$ w.r.t. $k$.

Turning to $\frac{\partial g}{\partial\sigma_{u}^{2}}$ as above we have

$$g\left(\tilde{k}\left(\sigma_{u}^{2}\right),\sigma_{u}^{2}\right)=-\frac{e^{\left(ax^{*}-ab^{*}\tilde{k}+\frac{a^{2}\left(\tilde{k}^{2}\sigma_{b}^{2}+\sigma_{u}^{2}\right)}{2}\right)}\left\{-b^{*}+a\tilde{k}\sigma_{b}^{2}\right\}}{a}+C$$

where $C$ is independent of $\sigma_u^2$. Thus

$$\frac{\partial g}{\partial \sigma_u^2} = -\frac{e^{\left(ax^* - ab^*\tilde{k} + \frac{a^2\left(\tilde{k}^2\sigma_b^2 + \sigma_u^2\right)}{2}\right)}\left\{-b^* + a\tilde{k}\sigma_b^2\right\}}{a}\frac{a^2}{2}$$

Combining the expressions for the numerator and denominator gives the result.

■

**Corollary 18** *When* $a \to 0$ $\frac{d\tilde{k}}{d\sigma_u^2} \to 0$ *as in the case of Brainard.*[27] *More generally when* $a > 0$

$$\frac{d\tilde{k}}{d\sigma_u^2} > 0 \text{ if } a\tilde{k}\sigma_b^2 < b^* \tag{42}$$

*Further, it's clear that, upon comparing (42) with (38) that*

$$\frac{d\tilde{k}}{d\sigma_b^2} > 0 \Rightarrow \frac{d\tilde{k}}{d\sigma_u^2} > 0$$

*Thus, when* $a$, $b^*$ *are sufficiently large (as defined above) we'll have both* $\frac{d\tilde{k}}{d\sigma_b^2}, \frac{d\tilde{k}}{d\sigma_u^2} > 0$.

We now state the main result for how uncertainty over the state of the world affects optimal policy.

**Proposition 19** *Suppose* $a > 0$ *and* $\frac{\lambda k^* a\sigma_b^2}{\left(\lambda + \sigma_b^2\right)} \leq b^*$ *then policy becomes more active as uncertainty over the state of the world increases:*

$$\frac{d\tilde{k}}{d\sigma_u^2} > 0$$

**Proof.** From (42) when $a > 0$ $\frac{d\tilde{k}}{d\sigma_u^2} > 0$ if $a\tilde{k}\sigma_b^2 < b^*$. However, we have that

$$\tilde{k}\epsilon\left(0, max\left\{\frac{b^*}{a\sigma_b^2}, k^* - \frac{b^*}{\lambda a}\right\}\right)$$

Thus when $\frac{b^*}{a\sigma_b^2} \geq k^* - \frac{b^*}{\lambda a}$ or equivalently $b^*\left\{\frac{1}{\sigma_b^2} + \frac{1}{\lambda}\right\} \geq ak^*$, we have $\tilde{k} < \frac{b^*}{a\sigma_b^2}$.

■

[27] Note that

$$a\tilde{k}\epsilon\left(0, max\left\{\frac{b^*}{\sigma_b^2}, k^* - \frac{b^*}{\lambda}\right\}\right)$$

so that $a\tilde{k}$ is bounded as $a \to 0$.

## D Learning

**Lemma 20** *Suppose* $Cov(b,u) = 0$. *Then for a given choice of* $k_1$, *the posterior mean and variance of* $b$ *(having observed* $x_1$*) are given by*

$$\mathbb{E}(b|x_1) = b^* + \frac{k_1{\sigma_b}^2}{((k_1)^2{\sigma_b}^2 + {\sigma_u}^2)}(x_1 - b^* k_1)$$

$$Var(b|x_1) = {\sigma_b}^2 - \frac{({\sigma_b}^2)^2}{({\sigma_b}^2 + {\sigma_u}^2/(k_1)^2)}$$

**Proof.** We note that $x_1$ will be observed before $k_2$ is chosen. Thus, following the Brainard solution in the static case, the optimal solution for $k_2$ given $Cov(b,u) = 0$ will be

$$k_2 = \frac{x^*\mathbb{E}[b|\mathcal{I}_2] + \lambda k^*}{(\mathbb{E}[b^2|\mathcal{I}_2] + \lambda)} \tag{43}$$

where $\mathcal{I}_2$ is the policymaker's information set at $t = 2$ and we have $x_1, k_1 \in \{\mathcal{I}_2\}$. Given the distributional assumptions on $b, u$ and given that $k_1$ is non-stochastic as chosen by the policymaker, we have that

$$x_1 \sim N\left(b^* k_1, (k_1)^2 \sigma_b^2 + \sigma_u^2\right)$$

Then, as $b, x_1$ are both normally distributed we have that

$$\mathbb{E}[b|x_1] = b^* + \frac{Cov(b,x_1)}{Var(x_1)}(x_1 - \mathbb{E}(x_1)) \tag{44}$$

$$Var(b|x_1) = \sigma_b^2 - \frac{(Cov(b,x_1))^2}{Var(x_1)} \tag{45}$$

Now,

$$Cov(b,x_1) = Cov(b, bk_1 + u) = k_1\sigma_b^2$$

Applying this to (44) and (45) completes the proof.

■

**Proposition 21** *Suppose* $Cov(b,u) = 0$, *then maximisation problem of the policymaker can be written as*

$$\max_{k_1} -\frac{1}{2}\mathbb{E}_1\left\{(bk_1 + u_1 - x^*)^2 + \lambda(k_1 - k^*)^2\right\} - \frac{\delta}{2}\mathbb{E}_1\left\{(bk_2 + u_2 - x^*)^2 + \lambda(k_2 - k^*)^2\right\}$$

$$: k_2 = \frac{x^* \left\{ b^* + \frac{k_1 \sigma_b^2}{(k_1)^2 \sigma_b^2 + \sigma_u^2} \left( (b - b^*) k_1 + u_1 \right) \right\} + \lambda k^*}{\left\{ b^* + \frac{k_1 \sigma_b^2}{(k_1)^2 \sigma_b^2 + \sigma_u^2} \left( (b - b^*) k_1 + u_1 \right) \right\}^2 + \frac{\sigma_b^2}{1 + \frac{(k_1)^2 \sigma_b^2}{\sigma_u^2}} + \lambda}$$

*This problem is solved numerically.*

**Proof.** The nature of the objective function is immediate. The form of the constraint follows from the formula for (43) combined with the prior lemma, noting that $\mathbb{E}[b^2|\mathcal{I}_2] = \{\mathbb{E}[b|\mathcal{I}_2]\}^2 + Var(b|x_1)$. ■

**Lemma 22** *Suppose $\sigma_u^2 = 0$ or $\sigma_u^2 \to \infty$ then $k_2$ is independent of $k_1$ and the solution for $k_1$ is as in the static case*

**Proof.** From the prior proposition, it's clear that when $\sigma_u^2 \to \infty$ $k_2$ is independent of $k_1$ and hence the solution for $k_1$ is as in the static Brainard case.

Now suppose $\sigma_u^2 = 0$. Then then we have (noting that as$\mathbb{E}_1(u_1)$ is zero, $\sigma_u^2 = 0$ implies $u_1 = 0$):

$$k_2 = \frac{x^* \left\{ b^* + \frac{k_1 \sigma_b^2}{(k_1)^2 \sigma_b^2} \left( (b - b^*) k_1 \right) \right\} + \lambda k^*}{\left\{ b^* + \frac{k_1 \sigma_b^2}{(k_1)^2 \sigma_b^2} \left( (b - b^*) k_1 \right) \right\}^2 + \lambda} = \frac{x^* \left\{ b^* + \frac{(b-b^*)k_1}{k_1} \right\} + \lambda k^*}{\left\{ b^* + \frac{(b-b^*)k_1}{k_1} \right\}^2 + \lambda} = \frac{x^* b + \lambda k^*}{b^2 + \lambda}$$

Hence $k_2$ is independent of $k_1$ and thus the optimal solution for $k_1$ will be as in the static Brainard case. Here the information regarding $b$ can be inferred perfectly for all levels of $k_1$, so the choice of $k_1$ is irrelevant for the amount that can be learned. ■

# E Private Sector Uncertainty

In this section we first lay out the private sector's problem and their solution to it, before turning to the policymaker's problem, which -as the Stackelberg leader- takes the private sector response into account.

**Private Sector Problem**

The private sector's problem has the following form (with specific parameter $\beta_H^{pr}$) :

$$max_x \mathbb{E} \left[ (1 - p(x, k)) \pi^H (x, k) + p(x, k) \pi^L (x, k) | \mathcal{I}^a \right]$$

$$p(x,k) = p^* + x - \gamma k + \omega,$$

$$\pi^H(x,k) = \beta_H^{pr} x - a_H k, \ \beta_H > 1$$

$$\pi^L(x,k) = -x + a_L k$$

$$\omega \sim iid(0, \sigma_\omega^2)\gamma \sim iid(\gamma^*, \sigma_\gamma^2)$$

Where all parameters are strictly positive and share the same interpretation as above. Note that the social cost of a crisis, $C$, is omitted from the private sector's problem: this drives a wedge between the preferred levels of $x, k$ of the private sector versus the policymaker. The term $\mathcal{I}^a$denotes the information set of the private sector; we assume that the parameters $\beta_H^{pr}$, $a_H$ and $a_L$ are known with certainty and the private sector has the declared priors over $\omega$ and $\gamma$, with *priors* $\omega \sim N(0, \sigma_\omega^2), \gamma \sim iid(\gamma^*, \sigma_\gamma^2)$. Note that the private sector profit for a given level of risk-taking $x$, $\beta_H^{pr}$, will differ from that of the public sector. Furthermore, we will assume that the private sector is unsure of the level of crisis intolerance on the part of the policymaker: $C \sim N(C^*, \sigma_C^2)$. We further suppose for simplicity that $\mathbb{E}[C\omega] = 0$. This information is asymmetric however, and the public sector know $C$ (a preference parameter), and also observe $\omega$ before acting. The private sector can then infer information about $\omega$ from observing the policymaker's choice of $k$ ($C$ does not affect their problem so updating any prior over it does not affect their choice of $x$).

Under these circumstances, the first order condition on the private sector's problem is, given $k$:

$$\mathbb{E}\left[-(\beta_H^{pr} x - a_H k) + \beta_H^{pr}(1 - p^* - x + \gamma k - \omega) - (p^* + x - \gamma k + \omega) + (-x + a_L k)|\mathcal{I}^a\right] = 0$$

$$\mathbb{E}\left[a_H k + \beta_H^{pr}(1 - p^* + \gamma k - \omega) - (p^* - \gamma k + \omega) + a_L k)|\mathcal{I}^a\right] = 2\left(\beta_H^{pr} + 1\right) x$$

$$\left[\beta_H^{pr} - p^*\left(\beta_H^{pr} + 1\right)\right] + k\left(a_H + a_L + \gamma^*\left(1 + \beta_H^{pr}\right)\right) - \left(\beta_H^{pr} + 1\right)\mathbb{E}\left[\omega|\mathcal{I}^a\right] = 2\left(\beta_H^{pr} + 1\right) x$$

The second derivative of $x$ is given by

$$\mathbb{E}\left[-\beta_H^{pr} - \beta_H^{pr} - 1 - 1|\mathcal{I}^a\right] < 0$$

So the following is indeed the maximum and the solution to the private sector's problem:

$$x = \frac{1}{2}\left\{\frac{\beta_H^{pr}}{\left(\beta_H^{pr} + 1\right)} - \left(p^* + \mathbb{E}\left[\omega|k\right]\right) + k\left[\frac{a_H + a_L}{\left(\beta_H^{pr} + 1\right)} + \gamma^*\right]\right\} \tag{46}$$

This is the private sector's optimal choice of $x$ conditional on its expectations about $\omega$

where this also reflects any information gleaned from observing $k$. We note that $x$ is greater the higher $k$ : when capital requirements are higher the system will be more resilient and the probability of a crisis is low, resulting in greater risk taking by the private sector: *$x$ and $k$ are thus strategic complements.* We note that $x$ is lower the greater the probability of a crisis, represented by the joint terms $p^* + \mathbb{E}[\omega|\mathcal{I}^a]$ , which includes any information learned about the probability of the low state from observing the policy-maker's action.

We solve the model via the method of undetermined coefficients and suppose that $\mathbb{E}[\omega|\mathcal{I}^a]$ is a linear function of the signal $k$ :

$$\mathbb{E}[\omega|k] = \Gamma_0 + \Gamma_1 k$$

Written thus we have

$$x = \frac{1}{2}\left\{\frac{\beta_H^{pr}}{(\beta_H^{pr}+1)} - (p^* + \Gamma_0) + k\left[\frac{a_H + a_L}{(\beta_H^{pr}+1)} + \gamma^* - \Gamma_1\right]\right\}$$

When $\Gamma_1 > 0$, this reduces the sensitivity of $x$ to $k$ : in this case, when a greater $k$ is observed, the private sector believe that this is because $\omega$ is higher and thus there is a greater probability of a crisis. This partially offsets the response of $x$, reducing the degree of strategic complementarity between $x$ and $k$. For notational convenience we rewrite the private sector decision as:

$$x = \delta_0 + k\left[\delta_1 - \frac{\Gamma_1}{2}\right] \tag{47}$$

Where

$$\delta_0 := \frac{1}{2}\left(\frac{\beta_H^{pr}}{(\beta_H^{pr}+1)} - (p^* + \Gamma_0)\right)$$
$$\delta_1 := \frac{1}{2}\left(\frac{a_H + a_L + \gamma^*\left(1+\beta_H^{pr}\right)}{(\beta_H^{pr}+1)}\right)$$

We now turn to the policy maker's problem.

**Policymaker's Problem**

The policymaker's problem has the following form (with specific $\beta_H^{pub}$):

$$max_k \mathbb{E}\left[(1-p(x,k))\pi^H(x,k) + p(x,k)\left(\pi^L(x,k) - C\right)|\mathcal{I}^p\right] : x = \delta_0 + k\left[\delta_1 - \frac{\Gamma_1}{2}\right]$$

$$p(x,k) = p^* + x - \gamma k + \omega$$
$$\pi^H(x,k) = \beta_H^{pub} x - a_H k$$
$$\pi^L(x,k) = -x + a_L k$$

Where $\omega$ is observed by the public sector before acting, with the only uncertainty over $\gamma$, with mean $\gamma^*$. Substituting in Equation 47 for $x$ gives for the functional forms

$$p(k) = p^* + \delta_0 + \omega + k\left[\delta_1 - \frac{\Gamma_1}{2} - \gamma\right]$$
$$\pi^H(k) = \beta_H^{pub}\delta_0 + k\left[\beta_H^{pub}\left(\delta_1 - \frac{\Gamma_1}{2}\right) - a_H\right]$$
$$\pi^L(k) = -\delta_0 + k\left[a_L - \left(\delta_1 - \frac{\Gamma_1}{2}\right)\right]$$

For convenience write this as

$$p(k) = \alpha_p + \varepsilon_p k$$
$$\pi^H(k) = \alpha_H + \varepsilon_H k$$
$$\pi^L(k) = \alpha_L + \varepsilon_L k$$

Then the FOC to the problem can be written as

$$\mathbb{E}\left\{-\varepsilon_p\left(\alpha_H + \varepsilon_H k\right) + \left(1 - \alpha_p - \varepsilon_p k\right)\varepsilon_H + \varepsilon_p\left(\alpha_L + \varepsilon_L k - C\right) + \left(\alpha_p + \varepsilon_p k\right)\varepsilon_L\right\} = 0$$

The SOC for a maximum is given by

$$\mathbb{E}\left\{-\varepsilon_p\varepsilon_H - \varepsilon_p\varepsilon_H + \varepsilon_p\varepsilon_L + \varepsilon_p\varepsilon_L\right\} < 0$$
$$\mathbb{E}\left\{-\varepsilon_p\left(\varepsilon_H - \varepsilon_L\right)\right\} < 0$$
$$\mathbb{E}\left\{-\varepsilon_p\left(\varepsilon_L - \varepsilon_H\right)\right\} > 0$$

Now, the public sector know $\varepsilon_L - \varepsilon_H$ with certainty thus the SOC condition reduces to

$$-\varepsilon_p^*\left(\varepsilon_L - \varepsilon_H\right) > 0$$

Now

$$
\begin{aligned}
-\varepsilon_p^* &= \mathbb{E}\left[\gamma - \delta_1 + \frac{\Gamma_1}{2}\right] \\
&= \gamma^* - \frac{1}{2}\left(\frac{a_H + a_L + \gamma^*\left(1+\beta_H^{pr}\right)}{\left(\beta_H^{pr}+1\right)}\right) + \frac{\Gamma_1}{2} \\
&= \frac{1}{2}\left[\gamma^* - \frac{a_H + a_L}{\left(\beta_H^{pr}+1\right)}\right] + \frac{\Gamma_1}{2}
\end{aligned}
$$

If $\gamma^* > \frac{a_H+a_L}{\left(\beta_H^{pr}+1\right)}$, then we'll have $-\varepsilon_p^* > 0$ if $\Gamma_1 > 0$, which we'll verify below.

Under this condition, for an interior solution we'll need $(\varepsilon_L - \varepsilon_H) > 0$. The following lemma provides such a sufficient condition.

**Lemma 23** *Suppose $\Gamma_1 > 0$, so that the private sector sector believes that $\omega$ is higher when the observed $k$ is higher. Then a sufficient condition for an interior policy solution for the Stackelberg game is given by:*

$$
\frac{a_H + a_L}{\beta_H^{pr}+1} < \gamma^* < \frac{a_H + a_L}{\beta_H^{pr}+1}\left[\frac{2\left(\beta_H^{pr} - \beta_H^{pub}\right)}{\left(1+\beta_H^{pr}\right)} + 1\right] \tag{48}
$$

**Proof.** From above the required condition for an interior solution is $-\varepsilon_p^*\left(\varepsilon_L - \varepsilon_H\right) > 0$. Given $\Gamma_1 > 0$ and $\gamma^* > \frac{a_H+a_L}{\left(\beta_H^{pr}+1\right)}$ we have $-\varepsilon_p^* > 0$. Thus, a sufficient condition for an interior solution is given by $(\varepsilon_L - \varepsilon_H) > 0$.

Now

$$
\begin{aligned}
\varepsilon_L-\varepsilon_H &= \left[a_L-\left(\delta_1-\frac{\Gamma_1}{2}\right)\right]-\left[\beta_H^{pub}\left(\delta_1-\frac{\Gamma_1}{2}\right)-a_H\right]\\
&=(a_H+a_L)-\left(1+\beta_H^{pub}\right)\left(\delta_1-\frac{\Gamma_1}{2}\right)\\
&=(a_H+a_L)-\frac{\left(1+\beta_H^{pub}\right)}{2}\left(\frac{a_H+a_L}{(1+\beta_H^{pr})}+\gamma^*-\Gamma_1\right)\\
&=(a_H+a_L)\left[1-\frac{\left(1+\beta_H^{pub}\right)}{2\left(1+\beta_H^{pr}\right)}\right]-\frac{\left(1+\beta_H^{pub}\right)}{2}\left(\gamma^*-\Gamma_1\right)\\
&=\frac{(a_H+a_L)}{2\left(1+\beta_H^{pr}\right)}\left[2\left(1+\beta_H^{pr}\right)-\left(1+\beta_H^{pub}\right)\right]-\frac{\left(1+\beta_H^{pub}\right)}{2}\left(\gamma^*-\Gamma_1\right)\\
&=\frac{(a_H+a_L)}{2\left(1+\beta_H^{pr}\right)}\left[2\left(1+\beta_H^{pr}\right)-2\left(1+\beta_H^{pub}\right)+\left(1+\beta_H^{pub}\right)\right]-\frac{\left(1+\beta_H^{pub}\right)}{2}\left(\gamma^*-\Gamma_1\right)\\
&=\frac{(a_H+a_L)}{2\left(1+\beta_H^{pr}\right)}\left[2\left(\beta_H^{pr}-\beta_H^{pub}\right)+\left(1+\beta_H^{pub}\right)\right]-\frac{\left(1+\beta_H^{pub}\right)}{2}\left(\gamma^*-\Gamma_1\right)\\
&=\frac{\left(1+\beta_H^{pub}\right)}{2}\left[\frac{(a_H+a_L)}{\left(1+\beta_H^{pr}\right)}\left[\frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)}{\left(1+\beta_H^{pub}\right)}+1\right]-\left(\gamma^*-\Gamma_1\right)\right]
\end{aligned}
$$

Thus, when $\Gamma_1>0$,

$$
\frac{(a_H+a_L)}{\left(1+\beta_H^{pr}\right)}\left[\frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)}{\left(1+\beta_H^{pub}\right)}+1\right]>\gamma^*
$$

will ensure that $\varepsilon_L-\varepsilon_H>0$. This completes the proof of the lemma. ■

We will verify below in Proposition 24 that under these conditions we indeed have $\Gamma_1>0$ and hence a solution to the problem.

Returning to the FOC for the policymaker, we have

$$
\begin{aligned}
k2\mathbb{E}\left[\varepsilon_p\left(\varepsilon_H-\varepsilon_L\right)\right] &= \mathbb{E}\left\{-\varepsilon_p\alpha_H+\left(1-\alpha_p\right)\varepsilon_H+\varepsilon_p\left(\alpha_L-C\right)+\alpha_p\varepsilon_L\right\}\\
k2\mathbb{E}\left[\varepsilon_p\left(\varepsilon_H-\varepsilon_L\right)\right] &= \mathbb{E}\left\{\varepsilon_p\left(\alpha_L-\alpha_H-C\right)+\varepsilon_H-\alpha_p\left(\varepsilon_H-\varepsilon_L\right)\right\}\\
k &= \frac{\mathbb{E}\left\{\varepsilon_p\left(\alpha_L-\alpha_H-C\right)+\varepsilon_H-\alpha_p\left(\varepsilon_H-\varepsilon_L\right)\right\}}{2\mathbb{E}\left[\varepsilon_p\left(\varepsilon_H-\varepsilon_L\right)\right]}
\end{aligned}
$$

Now all parameters are known by the policymaker except for $\gamma$, which only enters $\varepsilon_p$ so this solution can be written as

$$
k=\frac{\varepsilon_H-\varepsilon_p^*\left(C+\alpha_H-\alpha_L\right)+\alpha_p\left(\varepsilon_L-\varepsilon_H\right)}{2\left[-\varepsilon_p^*\left(\varepsilon_L-\varepsilon_H\right)\right]}
$$

Now we have

$$-\varepsilon_p^* = \frac{1}{2}\left[\gamma^* - \frac{a_H + a_L}{\left(\beta_H^{pr}+1\right)}\right] + \frac{\Gamma_1}{2}$$

$$\varepsilon_L - \varepsilon_H = \frac{\left(1+\beta_H^{pub}\right)}{2}\left[\frac{(a_H+a_L)}{\left(1+\beta_H^{pr}\right)}\left[\frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)}{\left(1+\beta_H^{pub}\right)}+1\right] - (\gamma^* - \Gamma_1)\right]$$

$$\varepsilon_H = \beta_H^{pub}\left(\delta_1 - \frac{\Gamma_1}{2}\right) - a_H = \frac{\beta_H^{pub}}{2}\left[\frac{a_H + a_L}{\left(\beta_H^{pr}+1\right)} + \gamma^* - \Gamma_1\right]$$

$$\alpha_H - \alpha_L = \delta_0\left(1+\beta_H^{pub}\right) = \frac{\left(1+\beta_H^{pub}\right)}{2}\left(\frac{\beta_H^{pr}}{\left(\beta_H^{pr}+1\right)} - (p^* + \Gamma_0)\right)$$

$$\alpha_p = p^* + \delta_0 + \omega = (p^* + \omega) + \frac{1}{2}\left(\frac{\beta_H^{pr}}{\left(\beta_H^{pr}+1\right)} - (p^* + \Gamma_0)\right)$$

$$= \omega + \frac{1}{2}\left(\frac{\beta_H^{pr}}{\left(\beta_H^{pr}+1\right)} + p^* - \Gamma_0\right)$$

And so, substituting in, we have the optimal $k$ given by

$$k = \frac{\frac{\beta_H^{pub}}{2}\left[\frac{a_H+a_L}{\left(\beta_H^{pr}+1\right)} + \gamma^* - \Gamma_1\right]}{\left(\left[\gamma^* - \frac{a_H+a_L}{\left(\beta_H^{pr}+1\right)}\right] + \Gamma_1\right)\frac{\left(1+\beta_H^{pub}\right)}{2}\left[\frac{(a_H+a_L)}{\left(1+\beta_H^{pr}\right)}\left[\frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)}{\left(1+\beta_H^{pub}\right)}+1\right] - (\gamma^* - \Gamma_1)\right]}$$
$$+ \frac{C + \frac{\left(1+\beta_H^{pub}\right)}{2}\left(\frac{\beta_H^{pr}}{\left(\beta_H^{pr}+1\right)} - (p^* + \Gamma_0)\right)}{\left(1+\beta_H^{pub}\right)\left[\frac{(a_H+a_L)}{\left(1+\beta_H^{pr}\right)}\left[\frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)}{\left(1+\beta_H^{pub}\right)}+1\right] - (\gamma^* - \Gamma_1)\right]}$$
$$+ \frac{\omega + \frac{1}{2}\left(\frac{\beta_H^{pr}}{\left(\beta_H^{pr}+1\right)} + p^* - \Gamma_0\right)}{\left[\gamma^* - \frac{a_H+a_L}{\left(\beta_H^{pr}+1\right)}\right] + \Gamma_1}$$

To complete the solution we need to solve for $\Gamma_1, \Gamma_0$.

We can write the solution for $k$ as

$$k = \phi_0 + \phi_1 C + \phi_2 \omega$$

Where

$$\phi_0 := \frac{\frac{\beta_H^{pub}}{2}\left[\frac{a_H+a_L}{\left(\beta_H^{pr}+1\right)}+\gamma^*-\Gamma_1\right]}{\left(\left[\gamma^*-\frac{a_H+a_L}{\left(\beta_H^{pr}+1\right)}\right]+\Gamma_1\right)\frac{\left(1+\beta_H^{pub}\right)}{2}\left[\frac{(a_H+a_L)}{\left(1+\beta_H^{pr}\right)}\left[\frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)}{\left(1+\beta_H^{pub}\right)}+1\right]-(\gamma^*-\Gamma_1)\right]}$$

$$+\frac{\frac{1}{2}\left(\frac{\beta_H^{pr}}{\left(\beta_H^{pr}+1\right)}-(p^*+\Gamma_0)\right)}{\left[\frac{(a_H+a_L)}{\left(1+\beta_H^{pr}\right)}\left[\frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)}{\left(1+\beta_H^{pub}\right)}+1\right]-(\gamma^*-\Gamma_1)\right]}+\frac{\frac{1}{2}\left(\frac{\beta_H^{pr}}{\left(\beta_H^{pr}+1\right)}+p^*-\Gamma_0\right)}{\left[\gamma^*-\frac{a_H+a_L}{\left(\beta_H^{pr}+1\right)}\right]+\Gamma_1}$$

$$\phi_1 := \frac{1}{\left(1+\beta_H^{pub}\right)\left[\frac{(a_H+a_L)}{\left(1+\beta_H^{pr}\right)}\left[\frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)}{\left(1+\beta_H^{pub}\right)}+1\right]-(\gamma^*-\Gamma_1)\right]}$$

$$\phi_2 := \frac{1}{\left[\gamma^*-\frac{a_H+a_L}{\left(\beta_H^{pr}+1\right)}\right]+\Gamma_1}$$

Thus, *from the perspective of the private sector*, with their priors over $\omega, C$ being normal and independent, their prior over $k$ will be normally distributed. i.e $k \sim N\left(\mu_k, \sigma_k^2\right)$.

Thus, we have that

$$\mathbb{E}[\omega|k] = \omega^* + \frac{Cov(\omega,k)}{\sigma_k^2}(k-\mu_k)$$

Now

$$\mu_k = \phi_0 + \phi_1 C^* + \phi_2\omega^* = \phi_0 + \phi_1 C^*$$
$$\sigma_k^2 = \phi_1^2\sigma_C^2 + \phi_2^2\sigma_\omega^2$$
$$Cov(\omega,k) = Cov(\omega, \phi_0+\phi_1 C+\phi_2\omega) = \phi_2\sigma_\omega^2$$

Thus, we have that

$$\mathbb{E}[\omega|k] = -(\phi_0+\phi_1 C^*)\frac{\phi_2\sigma_\omega^2}{\phi_1^2\sigma_C^2+\phi_2^2\sigma_\omega^2} + \frac{\phi_2\sigma_\omega^2}{\phi_1^2\sigma_C^2+\phi_2^2\sigma_\omega^2}k$$

Thus, the functional form assumption is verified, and equating coefficients we have

$$\Gamma_0 = -(\phi_0+\phi_1 C^*)\frac{\phi_2\sigma_\omega^2}{\phi_1^2\sigma_C^2+\phi_2^2\sigma_\omega^2}$$
$$\Gamma_1 = \frac{\phi_2\sigma_\omega^2}{\phi_1^2\sigma_C^2+\phi_2^2\sigma_\omega^2}$$

Note that, for a given solution for $\Gamma_1$, $\phi_1, \phi_2$ are pinned down independently of $\Gamma_0$. Further, $\Gamma_0$ enters $\phi_0$ linearly. Hence, for a given solution for $\Gamma_1$, there is a unique solution for

$\Gamma_0$. We thus focus on solving for $\Gamma_1$.

Thus, solving for $\Gamma_1$ we have

$$\Gamma_1\sigma_C^2\phi_1^2 + \Gamma_1\sigma_\omega^2\phi_2^2 = \phi_2\sigma_\omega^2$$

$$\frac{\Gamma_1\sigma_C^2}{\left(1+\beta_H^{pub}\right)^2\left[\frac{(a_H+a_L)}{(1+\beta_H^{pr})}\left[\frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)}{\left(1+\beta_H^{pub}\right)}+1\right]-(\gamma^*-\Gamma_1)\right]^2} + \frac{\Gamma_1\sigma_\omega^2}{\left[\gamma^*+\Gamma_1-\frac{a_H+a_L}{(\beta_H^{pr}+1)}\right]^2} = \frac{\sigma_\omega^2}{\left[\gamma^*+\Gamma_1-\frac{a_H+a_L}{(\beta_H^{pr}+1)}\right]}$$

Rearranging this can be written as

$$\frac{\Gamma_1\sigma_C^2}{\left(1+\beta_H^{pub}\right)^2\left[\frac{(a_H+a_L)}{(1+\beta_H^{pr})}\left[\frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)}{\left(1+\beta_H^{pub}\right)}+1\right]-(\gamma^*-\Gamma_1)\right]^2} + \frac{\sigma_\omega^2\left[\Gamma_1-\left[\gamma^*+\Gamma_1-\frac{a_H+a_L}{(\beta_H^{pr}+1)}\right]\right]}{\left[\gamma^*+\Gamma_1-\frac{a_H+a_L}{(\beta_H^{pr}+1)}\right]^2} = 0$$

$$\frac{\sigma_C^2}{\sigma_\omega^2\left(1+\beta_H^{pub}\right)^2}\frac{\Gamma_1}{\left[\frac{(a_H+a_L)}{(1+\beta_H^{pr})}\left[\frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)}{\left(1+\beta_H^{pub}\right)}+1\right]-\gamma^*+\Gamma_1\right]^2} + \frac{\left[\frac{a_H+a_L}{(1+\beta_H^{pr})}-\gamma^*\right]}{\left[\gamma^*+\Gamma_1-\frac{a_H+a_L}{(1+\beta_H^{pr})}\right]^2} = 0$$

$$\frac{\sigma_C^2}{\sigma_\omega^2\left(1+\beta_H^{pub}\right)^2}\frac{\Gamma_1}{\left[\frac{(a_H+a_L)}{(1+\beta_H^{pr})}-\gamma^*+\Gamma_1+\frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)(a_H+a_L)}{\left(1+\beta_H^{pub}\right)(1+\beta_H^{pr})}\right]^2} + \frac{\left[\frac{a_H+a_L}{(1+\beta_H^{pr})}-\gamma^*\right]}{\left[\frac{a_H+a_L}{(1+\beta_H^{pr})}-\gamma^*-\Gamma_1\right]^2} = 0$$

For notational convenience we can write this as

$$g(\Gamma_1) := d\frac{\Gamma_1}{[e+\Gamma_1+f]^2} + \frac{e}{[e-\Gamma_1]^2}$$

With solutions satisfying $g(\Gamma_1)=0$, where

$$d := \frac{\sigma_C^2}{\sigma_\omega^2\left(1+\beta_H^{pub}\right)^2}$$

$$e := \frac{(a_H+a_L)}{(1+\beta_H^{pr})} - \gamma^*$$

$$f := \frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)(a_H+a_L)}{\left(1+\beta_H^{pub}\right)(1+\beta_H^{pr})} > 0$$

**Proposition 24** *Suppose*

$$\frac{a_H+a_L}{\beta_H^{pr}+1} < \gamma^* < \frac{a_H+a_L}{\beta_H^{pr}+1}\left[\frac{2\left(\beta_H^{pr}-\beta_H^{pub}\right)}{\left(1+\beta_H^{pub}\right)}+1\right] \tag{49}$$

*Then all solutions for $\Gamma_1$ are positive.*

*Further, suppose that*

$$\frac{a_H + a_L}{\beta_H^{pr} + 1} < \gamma^* < \frac{a_H + a_L}{\beta_H^{pr} + 1}\left[\frac{3\left(\beta_H^{pr} - \beta_H^{pub}\right)}{2\left(1 + \beta_H^{pub}\right)} + 1\right] \tag{50}$$

*Then there is a unique solution for $\Gamma_1$, $\Gamma_1^*$. Furthermore this solution satisfies $g'(\Gamma_1^*) > 0$. Finally, this also implies a unique solution for $\Gamma_0$ and hence a unique solution to the model.*

**Remark 25** *Given that all solutions satisfy $\Gamma_1 > 0$, the conditions of Lemma 23 are satisfied and the solution for $k$ is indeed an interior maximum.*

**Remark 26** *Note that these conditions are independent of $\sigma_C^2, \sigma_\omega^2$*

**Proof.** We have

$$g(\Gamma_1) := d\frac{\Gamma_1}{[e + \Gamma_1 + f]^2} + \frac{e}{[e - \Gamma_1]^2}$$

where a solution for $\Gamma_1$ satisfies $g(\Gamma_1) = 0$. By assumption $d, f > 0$ and condition (49) gives $e < 0, e + f > 0$.

Hence if $\Gamma_1 \leq 0$ we have $\frac{e}{[e-\Gamma_1]^2} < 0$ and $\frac{\Gamma_1 d}{[e+\Gamma_1+f]^2} < 0$. Thus $g(\Gamma_1)$ for $\Gamma_1 \leq 0$, so any solution must satisfy $\Gamma_1 > 0$.

We next show there must be a solution. The solution condition can be written (with $x = \Gamma_1$)

$$\begin{aligned}
dx[e - x]^2 + e[(e + f) + x]^2 &= 0 \\
dx\left[e^2 + x^2 - 2ex\right] + e\left[(e + f)^2 + x^2 + 2(e + f)x\right] &= 0 \\
x^3 d + x^2(-2ed + e) + x\left(de^2 + 2e(e + f)\right) + e(e + f)^2 &= 0 \\
x^3(d) + x^2 e(1 - 2d) + xe(2(e + f) + ed) + e(e + f)^2 &= 0
\end{aligned}$$

Hence it's a cubic, thus there must be at least one real solution. Hence, we have at least one solution for $\Gamma_1$ and it must be positive.

We next show that given (50), at any solution we must have $g'(\Gamma_1) > 0$.

Recall:

$$g(\Gamma_1) := \frac{\Gamma_1 d}{[e + \Gamma_1 + f]^2} + \frac{e}{[e - \Gamma_1]^2}$$

Hence:

$$
\begin{aligned}
g'(\Gamma_1) &= d\frac{[e+f+\Gamma_1]^2 - \Gamma_1 2[e+f+\Gamma_1]}{[e+f+\Gamma_1]^4} + \frac{-2e[e-\Gamma_1](-1)}{[e-\Gamma_1]^4} \\
&= \frac{d[e+f+\Gamma_1]\{[e+f+\Gamma_1]-2\Gamma_1\}}{[e+f+\Gamma_1]^4} + \frac{2e}{[e-\Gamma_1]^3} \\
&= \frac{d[e+f+\Gamma_1][e+f-\Gamma_1]}{[e+f+\Gamma_1]^4} + \frac{2e}{[e-\Gamma_1]^3} \\
&= \frac{d[e+f-\Gamma_1]}{[e+f+\Gamma_1]^3} + \frac{2e}{[e-\Gamma_1]^3}
\end{aligned}
$$

At any solution for $\Gamma_1$ we have $g(\Gamma_1) = 0$ and hence

$$
\begin{aligned}
\frac{-\Gamma_1 d}{[e+\Gamma_1+f]^2} &= \frac{e}{[e-\Gamma_1]^2} \\
\frac{-2\Gamma_1 d}{[e+\Gamma_1+f]^2[e-\Gamma_1]} &= \frac{2e}{[e-\Gamma_1]^3} \\
\frac{2\Gamma_1 d}{[e+\Gamma_1+f]^2[\Gamma_1-e]} &= \frac{2e}{[e-\Gamma_1]^3}
\end{aligned}
$$

Thus, at any solution we have

$$
g'(\Gamma_1) = \frac{d[e+f-\Gamma_1]}{[e+f+\Gamma_1]^3} + \frac{d2\Gamma_1}{[e+\Gamma_1+f]^2[\Gamma_1-e]}
$$

Then $g'(\Gamma_1) > 0$ iff

$$
\begin{aligned}
\frac{d[e+f-\Gamma_1]}{[e+f+\Gamma_1]^3} + \frac{2\Gamma_1 d}{[e+\Gamma_1+f]^2[\Gamma_1-e]} &> 0 \\
[e+f-\Gamma_1][\Gamma_1-e] + 2\Gamma_1[e+f+\Gamma_1] &> 0
\end{aligned}
$$

Where we note that, given $\Gamma_1 > 0$ and $e < 0, e+f > 0$, we have $e+f+\Gamma_1 > 0, \Gamma_1 - e > 0$

Rearranging further we have

$$
\begin{aligned}
-\Gamma_1^2 - e(e+f) + \Gamma_1(2e+f) + 2\Gamma_1^2 + 2\Gamma_1(e+f) &> 0 \\
\Gamma_1^2 + \Gamma_1(4e+3f) - e(e+f) &> 0
\end{aligned}
$$

Now, we have $\Gamma_1 > 0$ and $-e(e+f) > 0$. Hence if $4e+3f > 0$ we have $g'(\Gamma_1) > 0$ at all solutions $\Gamma_1$. But this is precisely condition (50). Hence under that condition we have $g'(\Gamma_1) > 0$.

Finally, we show that this condition also implies there is a unique solution for $\Gamma_1$. Suppose for a contradiction that there is more than one solution: $\exists x_1, x_2 : g(x_1) = g(x_2) = 0$, $x_1 \neq x_2$. By the above we must have both solutions positive. Without loss of generality (as they can be relabeled) let $x_1 < x_2$. Then, given (50) we have $g'(x_1) > 0, g'(x_2) > 0$. Given $g(.)$ is continuous for $x \geq 0$, $\exists \varepsilon > 0 : g(x_1 + \varepsilon) > 0, g(x_2 - \varepsilon) < 0$. Hence, by the Intermediate Value Theorem (IVT), $\exists x^* \in (x_1 + \varepsilon, x_2 - \varepsilon) : g(x^*) = 0$. Hence, if there is not a unique solution, there are at least three solutions. Furthermore, there can be at most three solutions as the roots are those of a cubic. Thus, there are no solutions for $x \in (x_1 + \varepsilon, x^*)$. Suppose $g'(x^*) > 0$. Then $\exists \varepsilon_{x^*} \in (0, x^* - x_1 - \varepsilon) : g(x^* - \varepsilon_{x^*}) < 0$. But then by IVT there is a solution in $(x_1 + \varepsilon, x^* - \varepsilon_{x^*})$, a contradiction. Suppose $g'(x^*) = 0$ Then $\exists \varepsilon_{x'} \in (0, x^* - x_1 - \varepsilon) : g(x^* - \varepsilon_{x'}) = 0$, but this is then a further solution, a contradiction. Thus we must have $g'(x^*) < 0$. This is a contradiction to the result that $g'(\Gamma_1) > 0$ at all solutions $\Gamma_1$. Hence, the original assumption is incorrect and we must have a unique solution for $\Gamma_1$. Finally, as discussed above, as $\phi_1, \phi_2$ are pinned down uniquely for a given $\Gamma_1$, independently of $\Gamma_0$ and $\Gamma_0$ enters $\phi_0$ linearly, a unique solution for $\Gamma_1$ implies a unique solution for $\Gamma_0$ and hence a unique solution to the model. This completes the proof of the proposition. ■

**Lemma 27** *Suppose*

$$\frac{a_H + a_L}{\beta_H^{pr} + 1} < \gamma^* < \frac{a_H + a_L}{\beta_H^{pr} + 1} \left[ \frac{3\left(\beta_H^{pr} - \beta_H^{pub}\right)}{2\left(1 + \beta_H^{pub}\right)} + 1 \right]$$

*Then*

$$\frac{d\Gamma_1}{d\left(\frac{\sigma_C^2}{\sigma_\omega^2\left(1+\beta_H^{pub}\right)^2}\right)} < 0$$

**Proof.** As above let

$$g(\Gamma_1) := \frac{\Gamma_1 d}{[e + \Gamma_1 + f]^2} + \frac{e}{[e - \Gamma_1]^2}$$

Where

$$d := \frac{\sigma_C^2}{\sigma_\omega^2\left(1 + \beta_H^{pub}\right)^2}$$

Under the condition of the Lemma, there is a unique solution for $\Gamma_1$. Then, the solution for $\Gamma_1$ is defined implicity by $g(\Gamma_1) \equiv 0$. .

Totally differentiating $g(\Gamma_1(d), d) \equiv 0$ wrt the parameter $d$ we have

$$\frac{\partial g}{\partial \Gamma_1}\Gamma_1'(d) + \frac{\partial g}{\partial d}1 = 0$$

Thus,

$$\Gamma_1'(d) = \frac{\frac{\partial g}{\partial d}}{-\frac{\partial g}{\partial \Gamma_1}} = \frac{\frac{\Gamma_1}{[e+\Gamma_1+f]^2}}{-\frac{\partial g}{\partial \Gamma_1}}$$

Under the condition in the Lemma, at the unique solution for $\Gamma_1$ we have $\frac{\partial g}{\partial \Gamma_1} > 0$ and $\Gamma_1 > 0$. Hence $\frac{\Gamma_1}{[e+\Gamma_1+f]^2} > 0, -\frac{\partial g}{\partial \Gamma_1} < 0$ and hence $\Gamma_1'(d) < 0$.

This completes the proof of the lemma. ■

**Corollary 28** *Suppose that*

$$\frac{a_H + a_L}{\beta_H^{pr} + 1} < \gamma^* < \frac{a_H + a_L}{\beta_H^{pr} + 1}\left[\frac{3\left(\beta_H^{pr} - \beta_H^{pub}\right)}{2\left(1 + \beta_H^{pub}\right)} + 1\right]$$

*holds. Then*

$$\frac{d^2 k}{d\omega d\left(\frac{\sigma_C^2}{\sigma_\omega^2}\right)} > 0$$

*In other words*

$$\frac{d\phi_2}{d\left(\frac{\sigma_C^2}{\sigma_\omega^2}\right)} > 0$$

*and the policy maker responds more to changes in* $\omega$ *when* $\left(\frac{\sigma_C^2}{\sigma_\omega^2}\right)$ *is higher.*

*Further*

$$\frac{d\phi_1}{d\left(\frac{\sigma_C^2}{\sigma_\omega^2}\right)} > 0$$

*and the policy maker responds more to their own preference parameter* $C$ *when* $\left(\frac{\sigma_C^2}{\sigma_\omega^2}\right)$ *is higher.*

**Proof.** This follows from the prior results noting that under this condition the unique solution to the problem has

$$k = \phi_0 + \phi_1 C + \phi_2 \omega$$

with $\Gamma_1 > 0$ and

$$\phi_1 := \frac{1}{\left(1 + \beta_H^{pub}\right)\left[\frac{(a_H + a_L)}{\left(1+\beta_H^{pr}\right)}\left[\frac{2\left(\beta_H^{pr} - \beta_H^{pub}\right)}{\left(1+\beta_H^{pub}\right)} + 1\right] - \gamma^* + \Gamma_1\right]} > 0$$

$$\phi_2 := \frac{1}{\left[\gamma^* - \frac{a_H + a_L}{\left(\beta_H^{pr} + 1\right)}\right] + \Gamma_1} > 0$$

■