



BANK OF ENGLAND

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Credit traps and macroprudential leverage

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Credit traps and macroprudential leverage

Angus Foulis,⁽¹⁾ Benjamin Nelson⁽²⁾ and Misa Tanaka⁽³⁾

Abstract

We construct a macroeconomic model with overlapping generations to study *credit traps* — prolonged periods of stagnant real activity accompanied by low productivity, financial sector undercapitalisation, and credit misallocation. Shocks to bank capital tighten banks' borrowing constraints causing them to allocate credit to easily collateralisable but low productivity projects. Low productivity weakens bank capital generation, reinforcing tight borrowing constraints, sustaining the credit trap steady state. Macroprudential policy to limit bank leverage can be welfare enhancing. In the presence of a credit trap, optimal leverage policy is countercyclical.

Key words: Leverage regulation, financial intermediation, financial crisis.

JEL classification: E58, G01, G21.

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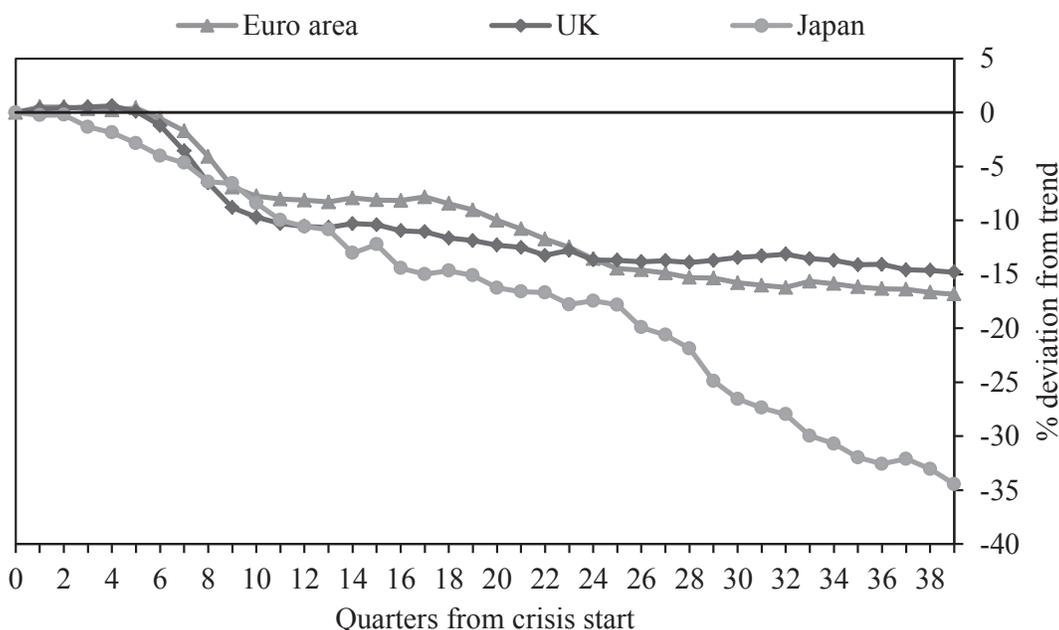
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1 Introduction

Financial crises tend to have severe negative effects on real activity, and recoveries following crises tend to be weak and slow (e.g. Claessens and Kose (2013)). In Japan, real GDP remained over 30 per cent below its pre-crisis trend 10 years after the onset of its financial sector distress in 1991 (Figure 1). In the United Kingdom, the gap between realised real GDP and the level implied by the pre-crisis trend was around 15 per cent a decade on from 2006, while in the euro area the gap was around 17 per cent. In each of these economies, credit collapsed following the onset of the crises (Figure 2). In Japan, it settled at a level around 40 per cent below its peak nearly a decade before, while the stocks of credit in the UK and euro area settled at levels between 20 and 30 per cent below their peaks a decade on.

Figure 1: Level of real GDP relative to the pre-crisis trend: UK, Japan, Euro area

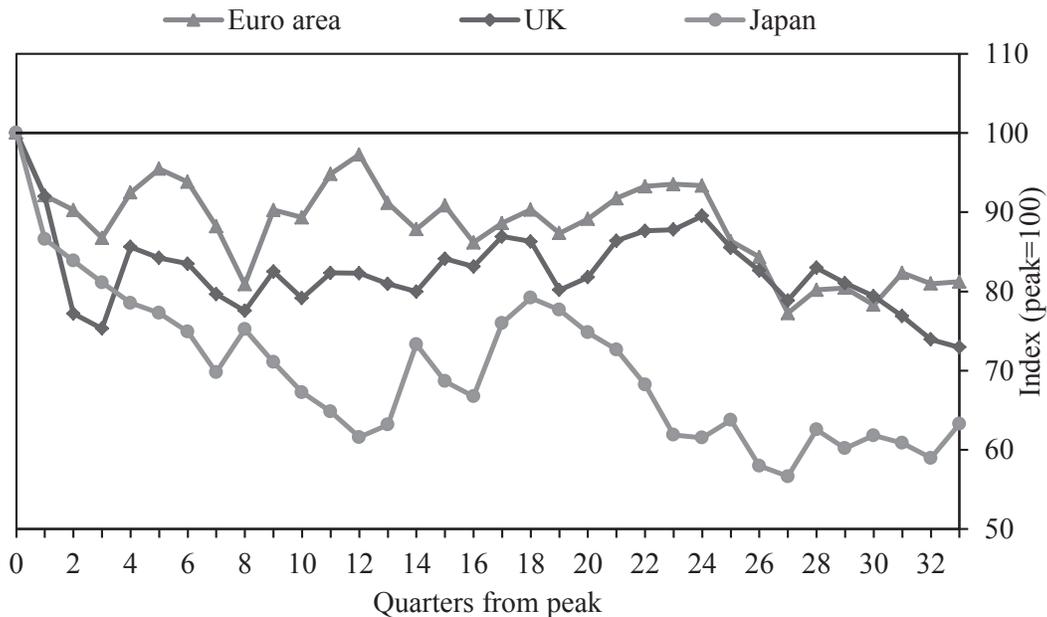


Note: Trend growth is calculated as mean quarterly increase in real GDP over 1955Q2–2006Q4 (UK); 1980Q2–1991Q4 (Japan); 1996Q2–2006Q4 (Euro area). Source: Datastream.

In the wake of the 2008-2009 Global Financial Crisis, several jurisdictions have established macroprudential policy authorities with powers to operate specific regulatory policy tools to maintain financial stability.¹ This paper develops a rich yet tractable overlapping generations (OLG) model

¹Leading examples include the Financial Policy Committee at the Bank of England, and the European Systemic Risk Board in the euro area.

Figure 2: Credit dynamics around crises: UK, Japan, Euro area



Note: Chart shows the stock of bank lending to the private sector, indexed to 100 in the quarter in which credit peaked (2008Q2 for UK and Euro area, and 1995Q2 for Japan). Source: Bank for International Settlements.

featuring financial intermediation and credit frictions in order to study how the macroprudential regulator could optimally set the leverage ratio for banks in these cases.

Our model is well-suited for this purpose as it can generate a ‘credit trap’ – a steady state of the economy that features low output, low productivity, low bank capital, and weak bank profitability. In our model, the borrowing constraints facing individual banks depend on the health of the banking system as a whole. In particular, when the level of aggregate bank capital is high, creditors are reassured that the liquidation value of bank assets is also high, raising the pledgeability of bank returns. This means that banks can maintain leverage easily and invest in productive assets; as a result, the economy remains in a ‘good’ steady state, characterised by high output, high productivity, high bank capital and high bank profitability. In that context, a downside shock to the level of bank capital generates two negative effects. First, lower bank capital reduces asset pledgeability from the perspective of bank creditors. As a result, banks are forced to delever by liquidating their assets to the point where they can credibly promise to repay their creditors. This liquidation of productive projects generates a welfare loss. Second, a negative shock to the level of banks’ capital also

tightens their funding conditions, forcing them to reduce their leverage ratio and contract lending, with knock-on consequences for capital generation and output in the economy.

A *credit trap* arises in our model when the initial shock is sufficiently large to reduce the aggregate level of bank capital below a critical threshold. If this happens, the pledgeability of the cash flows associated with productive lending falls to the point where banks can instead obtain pledgeable cash flows by holding more liquid, but less productive assets. This, in turn, means that aggregate profitability falls, inhibiting the rebuilding of bank capital and so sustaining the incentive to seek out highly pledgeable yet less productive assets. As such, this form of credit misallocation creates the possibility that a temporary shock to bank capital generates self-sustaining stagnation in credit and output.

We then use the model to examine what role macroprudential leverage regulation could play in such a setting. We articulate three main rationales for a leverage ratio restriction and examine how a macroprudential policy authority might optimally set this ratio in order to maximise the welfare of the agents in the economy.

First, procyclical fluctuations in banks' borrowing constraints (with the constraint tightening as the level of bank capital falls, and vice versa) induce excessive volatility in consumption and investment. This is not internalised by households or banks and so the leverage ratio restriction could be used to counteract this effect and smooth macroeconomic outcomes. In Section 4, we demonstrate analytically that, even in the absence of liquidation or the possibility of a credit trap, a welfare-maximising macroprudential authority will seek to maintain a constant leverage ratio and can implement this by applying a time-varying haircut to bank assets to offset the endogenous procyclicality of leverage.²

Second, when there is an additional possibility of asset liquidation, the macroprudential policy authority needs to deploy the leverage ratio restriction in a way that mitigates the incremental welfare costs that arise as a result. We demonstrate analytically that, if liquidation is possible, it is optimal for the macroprudential authority to require banks to be less leveraged than would otherwise be the case. Although forcing banks to maintain a lower leverage ratio reduces output in

²We note that, when other frictions are present, a macroprudential leverage ratio might need to be adjusted in a countercyclical manner even in the absence of liquidation or a credit trap. For example, using a model of incomplete information, Aikman, Nelson, and Tanaka (2015) have argued that a case for a countercyclical capital (or leverage) ratio can be made when low-ability banks have an incentive to make excessively risky investment decisions in an attempt to mimic high-ability banks during boom periods.

the ‘good’ steady state, it also reduces the welfare cost of liquidation because the higher is leverage, the greater is the amount of costly liquidation that must occur when the shock hits.

Third, when an economy faces the possibility of a credit trap, the macroprudential authority also needs to deploy the leverage ratio restriction as a tool to prevent the economy from descending into the trap in some states of the world. We illustrate using numerical simulations that a countercyclical leverage ratio restriction is optimal in ordinary times, when the level of bank capital is above the critical threshold, following the consumption smoothing logic described above. For example, as the level of bank capital falls, the macroprudential authority would optimally allow banks to increase their leverage ratio and borrow more, so as to enable them to expand lending. The resulting increase in output and bank profits allows banks to increase their net worth, which in turn expands lending and output and takes the economy away from the critical trap threshold. Thus, relaxing the leverage ratio restriction in response to a small negative shock to bank capital improves the economy’s resilience to a credit trap. We show that a policy authority would do this more aggressively the greater is the chance of experiencing a bad shock.

Two key features of the model contribute to its ability to generate credit traps. First, similar to Benmelech and Bergman (2012), the collateral value of bank assets depends on the aggregate health of the financial sector (see also e.g. Shleifer and Vishny (1992)). This makes bank equity capital the key state variable in the economy as it determines the severity of the credit friction that exists between banks and their creditors. As bank capital deteriorates, an intermediary’s ability to issue debt – to achieve leverage – is diminished, shrinking the supply of credit to finance productive activity. The focus of our paper is on the balance sheet health of the financial intermediary sector, whereas Benmelech and Bergman (2012) focus on collateral constraints in the real sector. The mechanism in our paper therefore bears resemblance to Benmelech and Bergman (2012) (and the Shleifer and Vishny (1992) channel they use), but also shares the focus of recent work by Gertler and Kiyotaki (2010) and Gertler and Karadi (2011) on credit-constrained financial intermediaries. This is natural given our focus on the impact of the leverage ratio restriction. Other papers examining the impact of bank leverage regulation in a macroeconomic context include Angeloni and Faia (2013), Christensen, Meh, and Moran (2011), Gertler, Kiyotaki, and Queralto (2012), Christiano and Ikeda (2016). The welfare implications of interventions in credit markets are examined in Bianchi (2011) and Lorenzoni (2008).

Second, similar to Matsuyama (2007), we allow for heterogeneity in the composition of credit, distinguishing investment opportunities – ‘projects’ – according to their inherent productivity and according to the pledgeability of the cash flows that they generate. The idea is that some assets, such as loans to small firms, have high productivity but, because of their relative opacity, have a collateral value that is particularly sensitive to the net worth of the intermediary sector. When the banking sector is healthy, bank creditors are willing to finance productive but opaque projects because they know that, if the worst came to the worst, they could seize assets and sell them to another lender to manage. When the banking sector has low net worth, by contrast, the resale value of these projects may be low because the ability of other buyers to absorb such assets is limited. In that case, creditors seek out pledgeable returns, and this encourages banks to invest in other, lower productivity assets, like loans to firms with established but less innovative technologies, or liquid assets like government bonds or central bank reserves. This, in turn, depresses output.

The mechanism which generates persistent credit misallocation in our model is different from ‘evergreening’ of loans examined by Peek and Rosengren (2005) and Caballero, Hoshi, and Kashyap (2008) in the context of Japan’s ‘lost decade’. According to their analysis, undercapitalised Japanese banks had the incentive to continue rolling over loans to weak and unproductive firms in order to prevent realisation of credit losses through foreclosure. In our model, by contrast, banks favour less productive but more collateralisable investments following a negative shock to their equity capital because their creditors demand more collateral.³ This phenomenon was widely observed after the recent global financial crisis which was triggered by system-wide dry-up of wholesale funding.⁴

The rest of this paper is organised as follows. Section 2 sets out the model, with Section 3 showing how the economy can become stuck in a credit trap and how the economy’s resilience to falling into a credit trap is linked to the initial leverage ratio. Section 4 examines the impact of

³The modelling approach in this paper is also very different to Caballero, Hoshi, and Kashyap (2008) who consider a micro model focusing on the dynamic path of firm entry and exit with and without subsidised lending to unprofitable *zombie* firms. In particular the bank lending decision is not modeled. By contrast, we develop a dynamic macro model in which an explicitly modeled banking sector chooses to invest in different sectors. This allows bank lending decisions to feed back into bank profitability and in turn the bank lending decisions, giving rise to the possibility of multiple steady states in the model and a permanent impact of temporary negative shocks (a *credit trap*). By contrast, in the model of Caballero, Hoshi, and Kashyap (2008) there is a unique steady state which the economy eventually returns to once a negative shock is unwound.

⁴Arguably, our model is less applicable to the case of Japan in the 1990s. Because all forms of deposits and uninsured debt were fully guaranteed in practice in all bank failures after 1996, there is little evidence that the credit misallocation there was driven by creditor’s demand for more collateral; rather, the existing research suggests that the credit misallocation was primarily driven by banks trying to prevent realisation of losses through evergreening. See Nelson and Tanaka (2014) for a summary.

macroprudential leverage policy on household welfare, whilst Section 5 concludes.

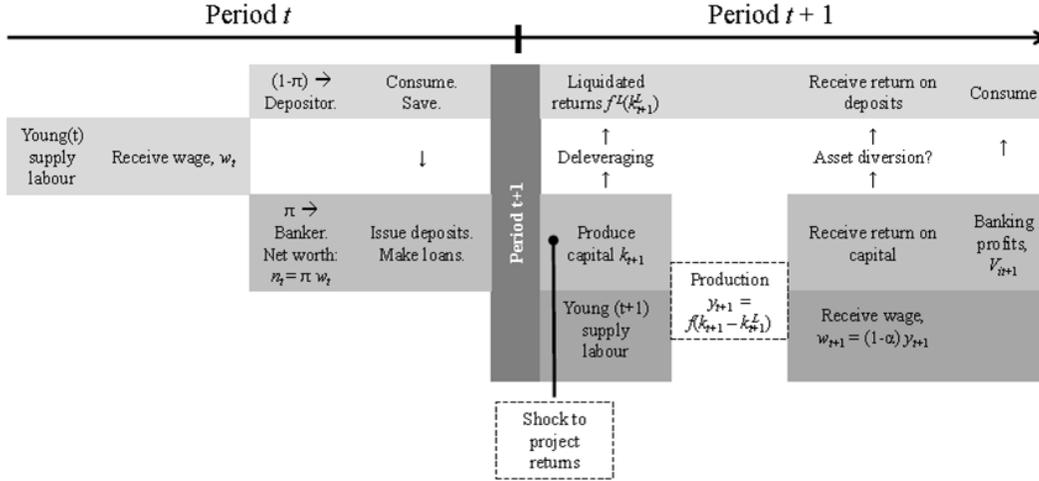
2 Model

2.1 Introduction

We begin with a brief overview of the overlapping generations model, with a timeline of the economy shown in Figure 3. Mass 1 of identical households are born each period and live for two periods. In the first period, when the household is *young*, each household receives a labour endowment of 1, which they sell inelastically in return for wage income w_t , denominated in final consumption goods. At the end of period 1 an exogenous fraction π of the household's income is used to start a household bank, with the remaining resources $(1 - \pi)w_t$ divided between consumption when young and saving via deposits in a bank (the households do not have access to a storage technology). The new banks combine their equity $n_t \equiv \pi w_t$ with deposits from other households to invest in one of two capital producing technologies. Banks are subject to a pledgeability constraint that limits how many deposits they can take. In the following period, capital is realised stochastically. If there is a negative shock, and less capital is produced than expected, banks may no longer satisfy their borrowing constraint, in which case they have to liquidate a portion of the capital to reduce their leverage. The remaining physical capital held by banks is combined with the labour endowment of the *next* generation, producing output goods. Banks receive the return on capital with which they pay back depositors, returning any profits lump-sum to the now *old* households, and the banks then exit. The new young workers, having received their wage, then form their own set of banks (which have no direct link to the previous banks) and the process summarised in Figure 3 repeats itself. As the wealth of each generation depends on the wage received when young, the liquidation decisions by banks can give rise to *intergenerational externalities*, as we discuss below.

In the following sub-sections the model is described in detail. There are ultimately two sectors that banks can invest in, denoted a and b . For ease of exposition, the model is first laid out for the case of banks only being able to invest in generic sector h . In Section 2.7 the choice between investment in $h = a$ versus $h = b$ is described, completing the model exposition.

Figure 3: Timeline of events in the model



Note: the figure shows the timeline of the economy covering the lifetime of a generation born at time t .

2.2 Households

A unit mass of households is born each period, and each generation lives for two periods in total: in the first period they are *young* (denoted y) and in the second period they become *old* (denoted o). Households are indexed with $i \in [0, 1]$. For tractability, their preferences are assumed to be a variant of those developed by Epstein and Zin (1989), taking the form:

$$U_t(i) = \log c_t^y(i) + \beta \log \mathbb{E}_t c_{t+1}^o(i) \quad (1)$$

in which $\beta < 1$ is the discount factor, c_s denotes consumption at time s , and \mathbb{E}_t is the mathematical expectations operator.⁵ These preferences combine risk neutrality with an intertemporal elasticity of substitution of 1.

In the first period of life, the household works, consumes and saves. Each household supplies its labour endowment $l_t(i) = 1$ inelastically to final goods firms, and earns a real wage of w_t in return. A proportion π of this wage is used to start a bank, as we discuss more in detail later. The remaining proportion $1 - \pi$ is available for consumption or saving. Households do not have access to storage technology to preserve output goods between when they are young and old, but can make deposits at a bank. When they choose to make deposits, the household receives interest income on

⁵See the Appendix for a derivation of these preferences from a more general Epstein-Zin form.

its bank deposits when old, together with lump-sum profits from banking, and uses this income to consume.

As a result, household i maximises (1) subject to the following budget constraints:

$$c_t^y(i) + d_t^h(i) \leq (1 - \pi)w_t \quad (2)$$

$$c_{t+1}^o(i) \leq R_t^{d,h} d_t^h(i) + V_{t+1}^h(i) \quad (3)$$

in which $d_t^h(i)$ is the household's deposits held with a bank investing in sector h , described further below, and $R_t^{d,h}$ is the return on these deposits. This return is non-state contingent, i.e. it is contracted at date t when the household makes its consumption-saving choices. Under conditions given below, banks do not default on deposits in equilibrium, thus deposits are riskless to households. The term $V_{t+1}^h(i)$ is household i 's profits from banking, also described in more detail later.

Household i 's optimal consumption path obeys the Euler equation:

$$\beta \frac{c_t^y(i)}{\mathbb{E}_t c_{t+1}^o(i)} R_t^{d,h} = 1 \quad (4)$$

Due to non-satiation, optimally the budget constraints when young and old (2, 3) hold with equality.

Combining these with the Euler equation gives the following supply of saving by household i :

$$d_t^h(i) = \frac{\beta}{1 + \beta} (1 - \pi)w_t - \frac{1}{1 + \beta} \frac{\mathbb{E}_t V_{t+1}^h(i)}{R_t^{d,h}} \quad (5)$$

Thus, the household's supply of deposits is upward sloping in $(d_t^h(i), R_t^{d,h})$ space for a given level of expected bank profits: a higher real return on bank deposits draws more saving into the banking system.

2.3 Banks

As with households, each generation of banks live for two periods. Each household injects an exogenous fixed fraction π of its labour income to create a new bank when they are young. As such, there is a unit mass of new banks in each period, indexed with $j \in [0, 1]$, each with initial equity capital, $n_t(j) = \pi w_t$ in period t . The objective of each bank is to maximise terminal expected value.

In period t , banks can use their equity capital along with any deposits it is profitable to raise from other households, in order to fund loans, $s_t^h(j)$. The balance sheet identity of a bank investing in sector h is as follows:

$$s_t^h(j) = d_t^h(j) + n_t(j) \quad (6)$$

The return on the bank's assets is stochastic and denoted R_{t+1}^h , with $R_{t+1}^h \sim [\underline{R}_{t+1}^h, \overline{R}_{t+1}^h]$, and mean, conditional on information at time t given by $\mathbb{E}_t(R_{t+1}^h) > 0$. Banks are atomistic, and take the return on assets, R_{t+1}^h , and the rate paid on deposits, $R_t^{d,h}$, as given. As discussed below, R_{t+1}^h is the bank's total return on its investment, including proceeds from any portion of the loan that is liquidated prior to the completion of the investment.

Credit constraint

In raising funding for its operations, each bank faces a borrowing limit. The return on its assets, R_{t+1}^h , is imperfectly pledgeable. In particular, when investing in sector h , only a fraction $\lambda^h \in (0, 1)$ of the bank's gross return can be pledged to depositors to cover expected deposit payouts. We maintain the following assumption:

Assumption 1: $\lambda^h = \lambda^h(n_t)$, $\partial \lambda^h / \partial n_t \geq 0$.

Thus, $\lambda^h(n_t)$ can be interpreted as the amount of money that households are willing to deposit in a bank, given that it invests in sector h and given the amount of aggregate bank net worth, $n_t \equiv \int_0^1 n_t(j) dj$. This captures the idea that with larger equity capital, households are more confident that the funds they deposit in banks will be repaid in full. This in turn captures the notion that when the aggregate banking sector is in better health, the 'second best' buyers of a bank's assets – namely other banks – are better able to ensure that a bank's assets remain liquid (e.g. Benmelech and Bergman (2012), Shleifer and Vishny (1992)). It can also serve as a reduced-form for other sources of externalities, such as the presence of strategic complementarities between banks' balance sheet choices, identified elsewhere in the literature as a source of cyclicity in balance sheet size and risk-taking (see e.g. De Nicolo, Favara, and Ratnovski (2012), Aikman, Haldane, and Nelson (2015)). An empirical implication of the assumption is procyclical bank leverage, consistent with the findings of Adrian, Colla, and Shin (2012) ⁶

⁶As banks are identical and the borrowing limit is based on initial capital levels, which are exogenous for each

The resulting *ex-ante* borrowing constraint for the bank is:

$$\lambda^h(n_t)(n_t(j) + d_t^h(j))\mathbb{E}_t(R_{t+1}^h) \geq R_t^{d,h}d_t^h(j) \quad (7)$$

The left-hand side of this expression is the bank's pledgeable returns, while the right-hand side is its repayment obligations to depositors. Equation (7) can also be interpreted as a constraint on bank leverage. Leverage for bank j , $\text{LEV}_t^h(j)$, defined as the ratio of assets to equity (mark-to-market) is given by

$$\text{LEV}_t^h(j) = \frac{(n_t(j) + d_t^h(j))\mathbb{E}_t(R_{t+1}^h)}{(n_t(j) + d_t^h(j))\mathbb{E}_t(R_{t+1}^h) - R_t^{d,h}d_t^h(j)} \leq \frac{1}{1 - \lambda^h(n_t)} \quad (8)$$

where the inequality follows from (7). Thus, for example, if $\lambda^h(n_t) = 0.9$ banks can borrow deposits up to 10 times the value of their equity.

The bank's expected profits are given by:

$$\mathbb{E}_t V_{t+1}^h(j) = \mathbb{E}_t R_{t+1}^h s_t^h(j) - R_t^{d,h}d_t^h(j) \quad (9)$$

Each bank j maximises expected profits (9) subject to the balance sheet constraint (6) and the borrowing constraint (7).

Ex-post liquidation

At the start of period $t + 1$ asset returns are realised and publicly observed. The portion of a bank's returns that are pledgeable must be sufficiently high to repay depositors what they are promised. That is, the bank must satisfy an *ex-post* borrowing constraint:

$$\lambda^h(n_t)(n_t(j) + d_t^h(j))R_{t+1}^h \geq R_t^{d,h}d_t^h(j) \quad (10)$$

As banks face the *ex-ante* borrowing constraint (7) when raising deposits, if $R_{t+1}^h \geq \mathbb{E}_t R_{t+1}^h$, i.e. returns turn out higher than expected, the bank's *ex-post* borrowing constraint will also be satisfied and so they can credibly repay depositors what they are owed. It is then optimal for the depositors to leave their deposits in the bank and the bank's investment projects will be completed. By contrast,

bank, Assumption 1 could be replaced with bank-specific borrowing requirements based upon individual rather than aggregate bank capital, without changing the results of the model.

if $R_{t+1}^h < \mathbb{E}_t R_{t+1}^h$, and the realised return is less than the expected return, there will be a violation of the bank's *ex-post* borrowing constraint if the *ex-ante* borrowing constraint was binding initially. When the bank's *ex-post* borrowing constraint (10) is violated, deposits are withdrawn from each bank, with each depositor at a given bank withdrawing an equal fraction of deposits⁷, until the point where realised returns can support promised deposit returns. A *liquidation technology* is employed by the banks, which converts projects back into output goods before the full production process has been completed. The returns from the liquidation technology are fully pledgeable, so by deleveraging and repaying depositors, the bank can ensure that (10) holds again.

It is assumed that the liquidation technology ensures that the bank's net worth is invariant to the scale of deleveraging. Thus, the bank's total returns from investing in capital, R_{t+1}^h , which as discussed below, includes both the return to capital used in production, and output goods realised from any liquidated capital, is invariant to bank deleveraging. This assumption is a conservative one: high leverage will increase bank losses following a negative shock, however, the degree of deleveraging required does not result in banks incurring further losses. In spite of high leverage not having this additional potential drawback, it does impose different costs on the wider economy, as shall be seen.

Bank deposit demand

The following proposition, proved in the Appendix, characterises bank deposit demand and expected profits under a given set of conditions. In Proposition 3 below conditions are provided on primitive parameters of the model that ensure conditions (11), (12) hold in equilibrium.

Proposition 1. *Suppose the following conditions hold in equilibrium:*

$$\mathbb{E}_t R_{t+1}^h > R_t^{d,h} \tag{11}$$

$$R_t^{d,h} > \lambda^h(n_t) \mathbb{E}_t R_{t+1}^h \tag{12}$$

and bank net worth is invariant to deleveraging. Then, the deposit demand of bank j is decreasing

⁷Note, as will be shown below, in equilibrium banks will always be solvent for all return realisations. Thus, the possibility of depositor withdrawals will not generate bank runs.

in the deposit rate, $R_t^{d,h}$, and is given by

$$d_t^h(j) = \frac{\lambda^h(n_t) \mathbb{E}_t R_{t+1}^h}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t R_{t+1}^h} n_t(j) > 0 \quad (13)$$

Moreover, bank's j 's expected profits are given by

$$\mathbb{E}_t V_{t+1}^h(j) = \frac{1 - \lambda^h(n_t)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t R_{t+1}^h} \mathbb{E}_t \left(R_{t+1}^h \right) R_t^{d,h} n_t(j) > 0 \quad (14)$$

Therefore, the bank's participation constraint for taking deposits is always satisfied and banks can expect to make positive profits.

Condition (12) ensures that the cost of raising additional funds is greater than the pledgeable return those funds can generate. This ensures that (7) places a constraint upon bank's borrowing, and will eventually bind if demanded deposits are high enough. Condition (11) states that the bank's expected returns from investment are greater than the cost of raising funds (deposits). As banks face no costs from deleveraging, this ensures that they want to raise deposits from households, with their participation constraint for raising deposits satisfied. Moreover, given the linear nature of the problem, banks wish to raise as many deposits as they can, given these rates, thus the bank borrowing constraint will bind. Given that $\lambda^h(n_t) < 1$, the bank's profits are strictly positive. This, of course, reflects the imperfect pledgeability of the bank's returns: since some portion cannot be pledged to the bank's creditors, the remainder accrues to the bank's equity holders.

Optimality of household deposit supply

The household supply of deposits (5) was derived under the assumption that deposits are riskless to households. There are two potential threats to this. First, banks may be unable to repay deposits in full, if they are insolvent. Second, banks may be able to pay, but will abscond with a fraction of what they owe if the pledgeability constraint fails to hold after a negative shock. The following proposition, proved in the Appendix, provides conditions that ensures these issues don't arise and deposits are riskless to households.

Proposition 2. *Suppose the conditions of Proposition 1 hold and in addition the following holds in equilibrium:*

$$\underline{R}_{t+1}^h \geq \lambda^h(n_t) \mathbb{E}_t R_{t+1}^h \quad (15)$$

Then deposits are riskless for households and (5) is the optimal household supply of deposits.

When (12) holds, banks face the maximum leverage constraint (7). Given this, condition (15), and the assumption that bank net worth is invariant to the scale of bank deleveraging, ensures that banks are solvent for all realisations of R_{t+1}^h . Condition (15) ensures that for all realisations of R_{t+1}^h the gross investment returns are sufficient to repay depositors,⁸ whilst with no further losses incurred due to any deleveraging that may be required, banks will always be solvent. Moreover, given that households can withdraw deposits, forcing bank deleveraging following a negative shock, and this will not make banks insolvent, the pledgeability constraint (10) will always hold following adjustment after a shock, and banks will not abscond with household deposits. Households will then always receive the value of their deposits in full, making them riskless.

We next turn to the remainder of the economy.

2.4 Production

Final goods

There is a continuum of perfectly competitive final goods producers, $q \in [0, 1]$. Each employs labour $l_t(q)$, supplied inelastically by the young, together with capital $k_t(q)$ used to produce final goods. Firm q 's output is:

$$y_t(q) = l_t(q)^{1-\alpha} k_t(q)^\alpha \quad (16)$$

Total output produced by the final goods producers is given by

$$y_t = \int_0^1 l_t(q)^{1-\alpha} k_t(q)^\alpha dq = l_t^{1-\alpha} k_t^\alpha = k_t^\alpha \quad (17)$$

by symmetry of final goods producers, and the fact that one unit of labour is supplied inelastically by each young household (with the mass of young households equal to 1).

⁸The maximum leverage constraint ensures that *ex-ante* bank leverage is at most $(1 - \lambda^h(n_t))^{-1}$. Condition (15) then holds whenever the percentage shortfall in asset returns are lower than the reciprocal of the leverage ratio that held on the basis of expected returns: $\frac{\mathbb{E}_t R_{t+1}^h - R_{t+1}^h}{R_{t+1}^h} < (1 - \lambda^h(n_t))$. Thus, for example, if $\lambda^h(n_t) = 0.9$, bank leverage is at most 10 and banks will be solvent after shocks so long as returns are no more than 10% lower than expected.

Capital goods

Firms obtain capital from perfectly competitive capital goods producers, $z \in [0, 1]$, who use bank loans (of final goods) to produce capital goods. When one unit of output goods are invested in capital production in period t , x_{t+1}^h units of capital goods are produced in period $t + 1$, where x_{t+1}^h is stochastic and $x_{t+1}^h \sim [\underline{x}_{t+1}^h, \bar{x}_{t+1}^h]$, with mean, conditional on information at time t given by $\mathbb{E}_t(x_{t+1}^h) > 0$. We assume that this mean is constant over time. The total production of capital goods in the economy is then:

$$x_{t+1}^h \int_0^1 \int_0^1 s_t^h(z, j) dz dj = x_{t+1}^h s_t^h \quad (18)$$

where $s_t^h(z, j)$ are the loans made to capital producer z by bank j and $s_t^h = \int_0^1 s_t^h(j) dj$ is the total amount invested by banks across all capital producers. Under perfect competition, the return on capital earned by final goods producers is earned by capital goods producers, who in turn remit their earnings to banks. On the portion of capital used in the production of final output goods, denoted $k_{t+1}(j)$, banks earn the return to capital, R_{t+1}^k , equal to the productivity of capital times the marginal product of capital, given by:

$$R_{t+1}^k = x_{t+1}^h \times \alpha \frac{y_{t+1}}{k_{t+1}} \quad (19)$$

As such, project productivity x_{t+1}^h serves as a possible shock to the value of bank assets or, equivalently, the value of their collateral from the perspective of bank depositors. From (17), (18), and (19), when all of the capital generated is used to produce output with final goods producers (i.e. there is no liquidation by the banks) the realised return on bank assets is

$$\alpha (x_{t+1}^h)^\alpha (s_t^h)^{\alpha-1} \quad (20)$$

with total output goods returned to the banks given by

$$\alpha (x_{t+1}^h s_t^h)^\alpha \quad (21)$$

Liquidation technology

In the event of bank withdrawals, the bank must obtain final goods via *early liquidation* of the production project. This means final goods must be obtained *before* labour is employed. With banks behaving symmetrically, if an aggregate amount k_{t+1}^{liq} of capital is liquidated, the amount of output goods generated is given by:

$$y_{t+1}^{\text{liq}} = \mathbb{L}\left(k_{t+1}^{\text{liq}}, x_{t+1}^h\left(d_t^h + n_t\right)\right) \quad (22)$$

where $x_{t+1}^h\left(d_t^h + n_t\right)$ is the aggregate amount of capital generated by the banks' investments with capital producers, and $\mathbb{L}(\cdot, \cdot)$ is the liquidation technology. These proceeds are repaid to the old generation of depositors. In this case, the total amount of output goods returned to banks is given by:

$$y_{t+1} = \alpha\left(x_{t+1}^h\left(d_t^h + n_t\right) - k_{t+1}^{\text{liq}}\right)^\alpha + \mathbb{L}\left(k_{t+1}^{\text{liq}}, x_{t+1}^h\left(d_t^h + n_t\right)\right) \quad (23)$$

As discussed above, the output from liquidation is immediately repaid to depositors, and so is fully pledgeable. However, as before, only fraction $\lambda^h(n_t)$ of the output used in the production of final goods is pledgeable. Capital is liquidated until the point at which the *ex-post* borrowing constraint holds again (with banks behaving symmetrically, each liquidates the same fraction), which requires the following to hold, with the fraction of output goods that are pledgeable equal to the output goods promised to depositors:

$$\lambda^h(n_t) \alpha\left(x_{t+1}^h\left(d_t^h + n_t\right) - k_{t+1}^{\text{liq}}\right)^\alpha + \mathbb{L}\left(k_{t+1}^{\text{liq}}, x_{t+1}^h\left(d_t^h + n_t\right)\right) = R_t^{d,h} d_t^h \quad (24)$$

As discussed above, we make the conservative assumption that the liquidation technology is such that bank net worth is invariant to the degree of deleveraging. The functional form of the liquidation technology that delivers this is derived in the Appendix. Bank net worth will fall following a negative shock, however the subsequent deleveraging will not make the net worth fall further. Given this, the total final consumption of the old, which depends on returns from deposits and the value of the shares held in the household bank, is invariant to the size of the liquidation, k_{t+1}^{liq} . However, this will still entail a reduction in the consumption of future generations, with lower wages for the next generation of young workers, as less capital is used in the production of final output goods.

As a result, the realised equity available for next period's banks will be lower. This means that liquidation shocks transmit their effects across time: liquidation in period $t + 1$ means lower bank net worth in subsequent periods. This tightens bank's borrowing constraints and, as we shall see, can create the possibility that the economy falls into a credit trap.

2.5 Deposit market equilibrium

The profits of bank i are returned to household i , thus, from (5) and (14), under the conditions of Propositions 1 and 2, it follows that the deposit supply of household i is given by

$$d_t^h(i) = \frac{\beta}{1 + \beta}(1 - \pi)w_t - \frac{(1 - \lambda^h(n_t))}{1 + \beta} \frac{\mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)} n_t(i) \quad (25)$$

Aggregate bank deposits $d_t^h \equiv \int_0^1 d_t^h(i) di$ in banks are then:

$$d_t^h = \frac{\beta}{1 + \beta}(1 - \pi)w_t - \frac{1 - \lambda^h(n_t)}{1 + \beta} \frac{\mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)} \pi w_t \quad (26)$$

where $\pi w_t = n_t \equiv \int_0^1 n_t(i) di$ is aggregate bank capital.

Similarly, under the conditions of Proposition 1 aggregate deposit demand is:

$$d_t^h = \frac{\lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)} \pi w_t, \quad (27)$$

where (27) is the aggregate counterpart to (13). The following proposition, proved in the Appendix, characterises the resulting deposit market equilibrium from equating (26) and (27), given a set of parameter restrictions under which the conditions for Propositions 1 and 2 hold.

Proposition 3. *Suppose that bank net worth is invariant to deleveraging. Suppose that*

$$(\underline{x}_{t+1}^h)^\alpha > \lambda^h(n_t) \mathbb{E}_t\left((x_{t+1}^h)^\alpha\right) \quad (28)$$

which ensures that (15) holds. Further, suppose

$$\beta(1 - \pi) > \pi + \lambda^h(n_t) \beta \quad (29)$$

then in the unique deposit market equilibrium

$$\mathbb{E}_t \left(R_{t+1}^h \right) > R_t^{d,h} > \lambda^h (n_t) \mathbb{E}_t \left(R_{t+1}^h \right) \quad (30)$$

and the conditions for Propositions 1 and 2 hold.

Further, equilibrium deposits are given by

$$d_t^h = \frac{\beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} (1 - \pi) w_t \quad (31)$$

Banks' total investment in capital is given by

$$s_t^h = \frac{\pi + \beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} w_t \quad (32)$$

The equilibrium expected return on investment over the deposit rate is given by

$$\mathbb{E}_t \left(R_{t+1}^h \right) - R_t^{d,h} = \frac{\beta(1 - \pi) - \pi - \lambda^h (n_t) \beta}{\beta(1 - \pi)} \mathbb{E}_t \left(R_{t+1}^h \right) \quad (33)$$

Finally, the deposit rate is given by

$$R_t^{d,h} = \frac{\alpha (1 + \lambda^h (n_t) \beta)^{1-\alpha} (\pi + \lambda^h (n_t) \beta)^\alpha \mathbb{E}_t \left((x_{t+1}^h)^\alpha \right)}{\beta (1 - \pi) ((1 - \alpha) k_t^\alpha)^{1-\alpha}} \quad (34)$$

Equilibrium deposits (31) are increasing in $\lambda^h (n_t)$, the pledgeability of the bank's returns. That is, alleviating the financial friction would raise the amount of saving, and hence investment, in the economy. The intuition for this is that a greater degree of asset pledgeability reassures bank creditors that their deposits will be safe, so they are willing to expand the equilibrium quantity of saving.

2.6 Law of motion for capital

The key state variable in the economy is the total quantity of capital used in the production of output by final good producers, k_t . This determines the wage of each new generation, and so also the equity capital of new banks, with $w_t = (1 - \alpha) k_t^\alpha$, and $n_t = \pi w_t$ giving

$$n_t = \pi (1 - \alpha) k_t^\alpha \quad (35)$$

Key to the law of motion for k_t is whether banks are required to liquidate capital early to reduce leverage and repay depositors. Under the conditions of Proposition 3, the *ex-ante* borrowing constraint (7) binds, and banks will fail to satisfy the *ex-post* borrowing constraint (10) whenever investment returns are lower than expected: $R_{t+1}^h < \mathbb{E}_t(R_{t+1}^h)$. Given (20) this happens if and only if $(x_{t+1}^h)^\alpha < \mathbb{E}_t((x_{t+1}^h)^\alpha)$. The following proposition, proved in the Appendix, provides the law of motion for k_t given the realisation of x_{t+1}^h .

Proposition 4. *Suppose the conditions of Proposition 3 hold. Then the law of motion for capital used in production by final goods producers, k_{t+1} , is given by*

$$k_{t+1} = \left(\frac{(x_{t+1}^h)^\alpha - \lambda^h(n_t) \mathbb{E}_t((x_{t+1}^h)^\alpha)}{1 - \lambda^h(n_t)} \right)^{\frac{1}{\alpha}} \left(\frac{\pi + \beta \lambda^h(n_t)}{1 + \beta \lambda^h(n_t)} \right) (1 - \alpha) k_t^\alpha \text{ if } (x_{t+1}^h)^\alpha < \mathbb{E}_t((x_{t+1}^h)^\alpha) \quad (36)$$

$$k_{t+1} = x_{t+1}^h \left(\frac{\pi + \beta \lambda^h(n_t)}{1 + \beta \lambda^h(n_t)} \right) (1 - \alpha) k_t^\alpha \text{ if } (x_{t+1}^h)^\alpha \geq \mathbb{E}_t((x_{t+1}^h)^\alpha) \quad (37)$$

where

$$n_t = \pi (1 - \alpha) k_t^\alpha$$

Under our assumptions about the liquidation technology, which entails the consumption of the old being invariant to the extent of bank liquidation, the burden of liquidation is imposed entirely on the young and the subsequent generations, who need to work with less capital and thus face lower wages and consumption, and deposit savings at banks which have lower equity capital. Liquidation thus gives rise to negative intergenerational externalities.

2.7 Ex-ante choice: sector of investment

Two sectors

The above exposition of the model has assumed a single sector h in which banks can invest. We now extend this to the full model in which banks can invest in one of two sectors, $h \in \{a, b\}$, each of which produces a homogeneous capital good, for use in production, as its final output.⁹ The two sectors use different technology, and generate different pledgeable returns. In particular, sector a employs a capital-producing technology that is more productive in expectation, but its productivity

⁹In other words, there is a single type of capital, but two possible technologies for producing it.

is stochastic. Sector b employs a less productive capital-producing technology, and its productivity is fixed and known. Letting x_t^h denote the quantity of capital goods produced per unit of lending to sector h , we have:

Assumption 2: (Productivity): $x_{t+1}^a \sim [\underline{x}_{t+1}^a, \bar{x}_{t+1}^a]$, $x_{t+1}^b \equiv x^b$, with $\mathbb{E}_t((x_{t+1}^a)^\alpha) > (x^b)^\alpha$.

One could interpret sector a as loans to small firms, which use real estate as collateral, and b as an alternative use of bank funds such as holding cash, central bank reserves, or buying government bonds, which do not contribute as much to the growth of the economy.

We assume that, although sector- a investments are more productive, the pledgeability of the returns they generate is more sensitive to the aggregate balance sheet health of the banking sector:

Assumption 1': (Pledgeability): $\partial\lambda^a(n_t)/\partial n_t > \partial\lambda^b(n_t)/\partial n_t = 0$.

This assumption captures the fact that the resale value of some assets depends more on specialist ‘second-best’ buyers than others. For example, the liquidity of mortgages (collateralised with real estate) may depend more on the health of the banking system than the liquidity of government bonds, on account of the larger number of ‘second best’ buyers of the latter than the former. In particular, as less monitoring and expertise are required to hold government bonds than to service mortgages, there will be a greater number of potential buyers of government bonds, including many beyond the banking system, making their liquidity relatively less dependent on banking system health. As we saw during the financial crisis, the liquidity of government debt markets held up far better than the liquidity of the RMBS market as financial intermediaries balance sheet positions deteriorated.

To summarise, sector a assets are more productive, less pledgeable, and subject to risk; whereas sector b assets are less productive, more pledgeable, and have certain returns.

Choice of sector for investment

Which sector do banks invest in and which banks do households fund? In our model, banks compete for deposits from households, while households deposit their savings in the bank that offers the highest (incentive compatible) return. As such, a household invests in a sector- a bank if and only

if it offers at least as high a return on deposits as a sector- b bank¹⁰:

$$R_t^{d,a} \geq R_t^{d,b} \quad (38)$$

For this to be an equilibrium, the participation constraint for banks to take deposits from households must be satisfied: even if households wish to place deposits in a bank that invests in sector b , it may be optimal for banks to take no deposits and invest in sector a using only their own equity. The following proposition, proved in the Appendix, provides conditions under which it will be incentive-compatible for banks to invest in the sector that pays the highest return to depositors. This ensures that, at any point in time, only one sector will be invested in.

Proposition 5. *Suppose the conditions of Proposition 3 hold for both sectors $h = a, b$. Suppose further that*

$$x^b \left(1 - \lambda^b\right) \left(\frac{\pi + \beta \lambda^b}{1 + \beta \lambda^b}\right) \geq \pi \mathbb{E}_t \left(x_{t+1}^a\right) \quad (39)$$

$$\frac{\mathbb{E}_t \left(\left(x_{t+1}^a\right)^\alpha\right)}{F} \geq x^b \quad (40)$$

where¹¹

$$F := \int_{x_{t+1}^a}^{\left(\mathbb{E}_t \left(\left(x_{t+1}^a\right)^\alpha\right)\right)^{\frac{1}{\alpha}}} \left(\frac{\left(x_{t+1}^a\right)^\alpha - \lambda^a \left(n_t\right) \mathbb{E}_t \left(\left(x_{t+1}^a\right)^\alpha\right)}{1 - \lambda^a \left(n_t\right)}\right)^{\frac{\alpha-1}{\alpha}} f \left(x_{t+1}^a\right) dx_{t+1}^a \quad (41)$$

$$+ \int_{\left(\mathbb{E}_t \left(\left(x_{t+1}^a\right)^\alpha\right)\right)^{\frac{1}{\alpha}}}^{\bar{x}_{t+1}^a} \left(x_{t+1}^a\right)^{\alpha-1} f \left(x_{t+1}^a\right) dx_{t+1}^a$$

Then banks invest in sector a if and only if $R_t^{d,a} \geq R_t^{d,b}$. Moreover, given (34), banks invest in sector a if and only if

$$\left(1 + \lambda^a \left(n_t\right) \beta\right)^{1-\alpha} \left(\pi + \lambda^a \left(n_t\right) \beta\right)^\alpha \mathbb{E}_t \left(\left(x_{t+1}^a\right)^\alpha\right) \geq \left(1 + \lambda^b \beta\right)^{1-\alpha} \left(\pi + \lambda^b \beta\right)^\alpha \left(x^b\right)^\alpha \quad (42)$$

Condition (42) highlights the trade-off for investment between sectors a and b . Investment in sector a is always expected to be more productive, with $\mathbb{E}_t \left(\left(x_{t+1}^a\right)^\alpha\right) > \left(x^b\right)^\alpha$. However, if the

¹⁰It is assumed that in the knife-edge case of $R_t^{d,a} = R_t^{d,b}$, banks invest in sector a .

¹¹Note that, when x_{t+1}^a is deterministic, F reduces to $\left(x_{t+1}^a\right)^{\alpha-1}$ and the condition reduces to sector a being more productive than sector b : $x_{t+1}^a \geq x^b$.

health of the financial system is poor, and the maximum leverage permitted for investment in sector a is low, the return offered to depositors when investing in sector b can be greater, if the permitted maximum leverage for banks investing in sector b is sufficiently high.

3 Credit trap

3.1 Definition of a credit trap

The economy can fall into a *credit trap* when a shock to sector a assets leads to a sharp deterioration in banks' net worth, leading to a severe tightening of their borrowing constraint against their sector a investment, such that they are incentivised to invest in the unproductive sector b . This can generate a steady state in which productivity, bank profits, and output remain persistently low.

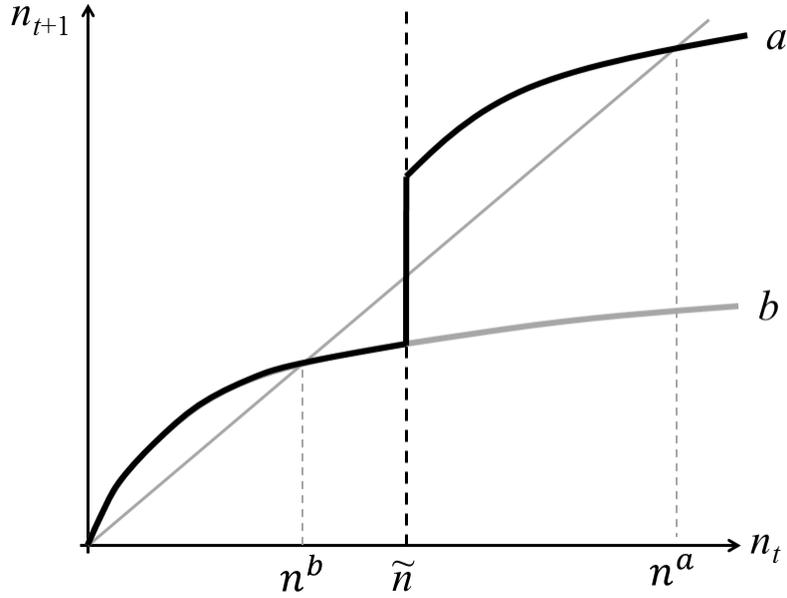
Definition. *A credit trap is a situation in which the banking sector perpetually invests in the unproductive sector (sector b).*

3.2 Condition for a credit trap

Banks will invest in sector a whenever condition (42) is satisfied. Note that the left-hand side of (42) is strictly increasing in bank equity, n_t , whilst the right-hand side is invariant to it. Therefore, under conditions on $\lambda^a(n_t)$ provided in the Appendix, there exists some threshold for bank net worth, \tilde{n} for which sector a is invested in whenever $n \geq \tilde{n}$, and sector b is invested in when $n < \tilde{n}$. When banks' net worth falls below a critical threshold \tilde{n} , creditors become unwilling and banks become unable to invest in sector a . Because sector a is more productive than sector b (with the assumption that $\mathbb{E}_t((x_{t+1}^a)^\alpha) > (x^b)^\alpha$), a higher return on sector b can only arise if there is more investment in it, i.e. a greater amount of leverage. When the banking system is healthy, high leverage when investing in sector a will be possible, making it more attractive. Only when the banking system is sufficiently impaired and banks cannot borrow enough to finance loans to sector a will investment flow to sector b . We next establish the aggregate consequences of these investment decisions.

Conditional upon investment flowing to sector b , given that the return is non-stochastic, from

Figure 4: Aggregate expected law of motion in an economy with a credit trap



Note: the figure the law of motion for banker net worth when the economy features a credit trap. For sector a the law of motion is shown when the realisation of x_{t+1}^a is at its expected value.

(37) it can be shown that the steady state level of banker net worth will converge to

$$n^b = \pi (1 - \alpha) \left(x^b \left(\frac{\pi + \lambda^b \beta}{1 + \lambda^b \beta} \right) (1 - \alpha) \right)^{\frac{\alpha}{1-\alpha}} \quad (43)$$

The following proposition, proved in the Appendix, then follows.

Proposition 6. *Suppose the conditions of Proposition 5 hold. Then the economy features a credit trap if*

$$\left(1 + \lambda^a (n^b) \beta \right)^{1-\alpha} \left(\pi + \lambda^a (n^b) \beta \right)^\alpha \mathbb{E}_t \left((x_{t+1}^a)^\alpha \right) < \left(1 + \lambda^b \beta \right)^{1-\alpha} \left(\pi + \lambda^b \beta \right)^\alpha \left(x^b \right)^\alpha \quad (44)$$

When condition (44) holds, banks will invest in sector b in the steady state of sector b . This is thus a steady state equilibrium, and given there are no shocks when investing in sector b , the economy will invest in sector b for the rest of time, so is struck in a credit trap.

An economy in which this holds is shown in Figure 4, abstracting from shocks to returns in

sector a .¹² The critical value of banking system net worth at which sector a is invested in is given by \tilde{n} . Above this level of banking system health, the economy invests exclusively in sector a , and the economy converges to the ‘good’ steady state (n^a), featuring high levels of capital, output, and income.¹³ If the banking system is sufficiently impaired with $n_t < \tilde{n}$, sector b is invested in, and the economy converges to the ‘bad’ steady state (n^b), featuring low levels of capital, output, and bank lending. This is indeed a steady state when banks invest in sector b when $n_t = n^b$, for which we require $n^b < \tilde{n}$, which is ensured by (44).

The intuition for the credit trap is as follows. When the net worth of the banking system is high, the collateral value of financial assets is also high. This allows banks to have large leverage when investing in sector a , making it more attractive than sector b (by allowing them to pay higher returns to depositors). Sector a is productive and so delivers high returns, resulting in high net worth in the banking system in the next period, which keeps them investing in a . Conversely, when the financial system is severely impaired, sector b is more attractive than sector a because banks can only achieve low leverage on sector a assets. Crucially, because the banks invest in the unproductive sector b , their net worth remains low in future periods, keeping them investing in b .

Because bank net worth is ultimately related to the physical capital stock of the economy, the credit trap threshold can easily and conveniently be re-cast as a threshold on k_t . Namely, using $n_t = \pi(1 - \alpha)k_t^\alpha$, we can write:

$$\tilde{k} = \left(\frac{\tilde{n}}{\pi(1 - \alpha)} \right)^{\frac{1}{\alpha}}$$

where \tilde{n} gives the threshold level of banker net worth at which sector a is invested in. It follows that the economy enters a credit trap for $k_t < \tilde{k}$, but does not if $k_t \geq \tilde{k}$.

¹²Note that there will always be a jump in the law of motion at \tilde{n} when the realisation of x_{t+1}^a is at its expected value. To see this, at the trap threshold, the return paid on deposits in a, b is the same, so after rearranging

$$\mathbb{E}_t((x_{t+1}^a)^\alpha) \left(\frac{\pi + \lambda^a(\tilde{n})\beta}{1 + \lambda^a(\tilde{n})} \right)^\alpha = (x^b)^\alpha \left(\frac{\pi + \lambda^b\beta}{1 + \lambda^b} \right)^\alpha \frac{(1 + \lambda^b)}{(1 + \lambda^a(\tilde{n}))} > (x^b)^\alpha \left(\frac{\pi + \lambda^b\beta}{1 + \lambda^b} \right)^\alpha$$

The inequality follows as, at the trap threshold, it must be that $\lambda^b > \lambda^a(\tilde{n})$, given $\mathbb{E}_t((x_{t+1}^a)^\alpha) > (x^b)^\alpha$, which is assumed. Applying (37) and (35) it’s clear that at \tilde{n} , n_{t+1} will be higher when sector a is invested in and the realisation of x_{t+1}^a is at its expected value.

¹³Depending upon the functional form for $\lambda^a(n_t)$ there may be multiple steady states when investing in sector a . The figure has been drawn on the assumption that sector a has a unique steady state.

3.3 Bank leverage and resilience to credit trap

How is the economy's vulnerability to falling into the trap linked to the initial leverage of the banking system? Here we turn to this question, analysing the link between initial leverage when banks are investing in sector a , and the range of shocks for which the economy will fall into a credit trap in the next period. Given initial pledgeability of bank returns λ^a , we define a threshold level of the productivity realisation, $\tilde{x}_{t+1}^a(\lambda^a)$, which reduces bank capital sufficiently in the next period such that banks start investing in sector b whenever the productivity realisation is below the threshold. In the Appendix we show that the threshold is given by:

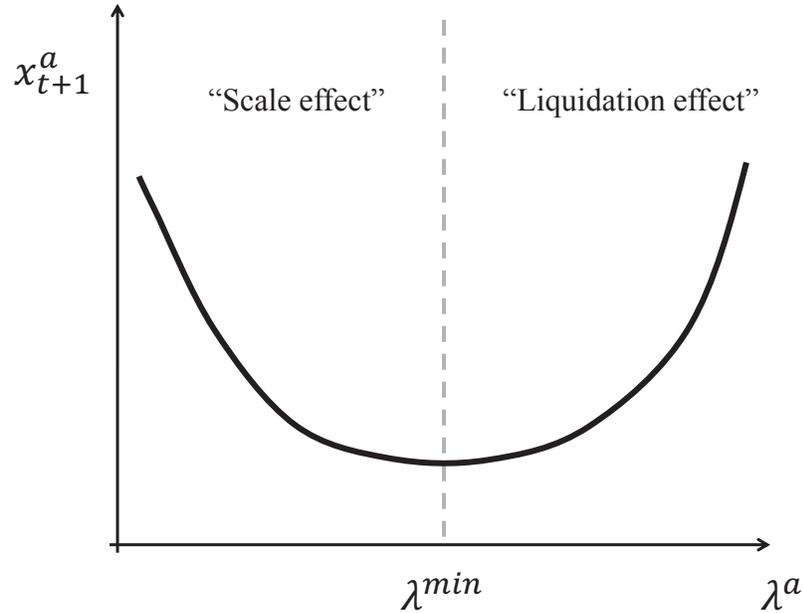
$$\tilde{x}_{t+1}^a(\lambda^a) = \left(\lambda^a \mathbb{E}_t \left((x_{t+1}^a)^\alpha \right) + \frac{\tilde{n}(1-\lambda^a)}{(1-\alpha)\pi \left[\frac{\pi+\lambda^a\beta}{1+\lambda^a\beta} (1-\alpha) k_t^\alpha \right]} \right)^{\frac{1}{\alpha}} \quad (45)$$

where \tilde{n} is the threshold level of bank net worth at which the economy falls into a credit trap. The economy falls into a credit trap whenever $x_{t+1}^a < \tilde{x}_{t+1}^a(\lambda^a)$. Thus, $\tilde{x}_{t+1}^a(\lambda^a)$ is a measure of the *resilience* of the financial system: the lower $\tilde{x}_{t+1}^a(\lambda^a)$, the more resilient the financial system, in the sense that the economy avoids getting into a credit trap for a larger range of negative productivity shocks.

In the Appendix we prove that, under mild conditions, $\tilde{x}_{t+1}^a(\lambda^a)$ is U-shaped, reaching its minimum at $\lambda^{min} \in (0, 1)$, i.e. λ^{min} is associated with the level of bank leverage that maximises the resilience of the financial system against credit traps.¹⁴ This is demonstrated in Figure (5). The U-shape reflects the two opposing effects of leverage on resilience. On the one hand, for any given initial level of capital k_t and productivity realisation x_{t+1}^a , banks would have produced more capital in period $t+1$ when leverage was high at time t (λ^a is high): other things equal, this puts the economy farther away from the credit trap threshold \tilde{n} and hence increases resilience. We call this the *scale effect*. On the other hand, depositors liquidate a greater proportion of the capital produced following the shock at $t+1$ the greater leverage at time t , because for a given negative shock, the reduction in net worth is greater the higher the initial leverage ratio. Other things equal, this makes it more likely the economy falls into a credit trap and hence reduces resilience. We call this the *liquidation effect*. When leverage is low ($\lambda^a < \lambda^{min}$), the scale effect dominates, such that

¹⁴Recall that when banks can pledge fraction λ of output, and the pledgeability constraint binds, bank leverage is given by $(1-\lambda)^{-1}$.

Figure 5: Resilience and leverage: the scale and liquidation effects



Note: the figure shows $\tilde{x}_{t+1}^a(\lambda^a)$ - the lowest productivity realisation for sector a under which the economy avoids the credit trap - as a function of the fraction of output that banks can pledge in the prior period, λ^a . The economy falls into the credit trap for all productivity realisations in the area below the black curve.

resilience will increase if banks have greater leverage. Over this range, *there is no trade-off between output and resilience*: increasing leverage increases both. But when leverage is high ($\lambda^a > \lambda^{min}$), the liquidation effect dominates, such that a further increase in leverage will reduce resilience.

Interestingly, the resilience-maximising leverage ratio is *countercyclical*. Formally, it is shown in the Appendix that

$$\frac{d\lambda^{min}}{dk_t} < 0 \quad (46)$$

When the state of the economy deteriorates - a decrease in k_t - the leverage ratio that maximises resilience *increases*. This is because the scale effect becomes relatively more important when k_t is lower. With n_t closer to the trap threshold \tilde{n} , banks are better able to improve their balance sheets with higher leverage.

The results of this section suggest that policy limiting bank leverage could make the financial system more resilient. This may not be desirable, however, if it places a large burden on the economy, and comes at the cost of low household consumption. In the next section we formally

turn to this and show that macroprudential leverage policy can improve *welfare*.

4 Bank leverage and welfare

4.1 Welfare

We first characterise the lifetime utility of a generation born at time t , as a function of the initial capital stock k_t , when the economy invests exclusively in sector h . This is covered in the following proposition, proved in the Appendix.

Proposition 7. *Suppose that the conditions of Proposition 3 hold and the economy invests exclusively in sector h . Then, for the generation born at time t , consumption when young, and expected consumption when old, are given respectively by:*

$$c_t^y = (1 - \alpha) k_t^\alpha \frac{(1 - \pi)}{1 + \lambda^h(n_t) \beta} \quad (47)$$

$$\mathbb{E}_t c_{t+1}^o = \mathbb{E}_t \left(\left(x_{t+1}^h \right)^\alpha \right) \alpha \left((1 - \alpha) k_t^\alpha \right)^\alpha \left(\frac{\pi + \lambda^h(n_t) \beta}{1 + \lambda^h(n_t) \beta} \right)^\alpha \quad (48)$$

whilst lifetime utility for the generation born at t is given by

$$U_t = (\alpha + \alpha^2 \beta) \log(k_t) + \alpha \beta \log(\pi + \lambda^h(n_t) \beta) - (1 + \alpha \beta) \log(1 + \lambda^h(n_t) \beta) + C_t \quad (49)$$

where C_t is independent of policy.

Households are risk-neutral within the two periods of their own lives and discount between periods at rate β . In keeping with this, we assume that the planner is risk-neutral *across* the lifetime utility of each generation, discounting between generations at rate β . The planner's objective at time t is then¹⁵

$$\mathbb{W}_t := \sum_{s=0}^{\infty} \beta^s \mathbb{E}_t U_{t+s} \quad (50)$$

The following proposition, proved in the Appendix, characterises the planner's time t welfare objective when the economy invests exclusively in sector h .

¹⁵Under the policy tool considered in the following section the planner cannot influence the utility of those who are old at time t , so it is excluded from this sum.

Proposition 8. *Suppose that the conditions of Proposition 3 hold and the economy invests exclusively in sector h . Then the planner's welfare function at time t is given by*

$$\begin{aligned} \mathbb{W}_t &= \sum_{s=0}^{\infty} \beta^s \mathbb{E}_t (\alpha + \alpha^2 \beta) \log(k_{t+s}) \\ &+ \sum_{s=0}^{\infty} \beta^s \mathbb{E}_t \left[\alpha \beta \log \left(\pi + \lambda^h (\pi (1 - \alpha) k_{t+s}^\alpha) \beta \right) - (1 + \alpha \beta) \log \left(1 + \lambda^h (\pi (1 - \alpha) k_{t+s}^\alpha) \beta \right) \right] + \bar{C}_t \end{aligned} \quad (51)$$

where \bar{C}_t is independent of policy.

We next turn to the policy tools the planner has at their disposal.

4.2 Policy to restrict leverage

Suppose the social planner (which could be the macroprudential policy authority) seeks to control bank leverage by subjecting pledgeable income to a ‘haircut’, $\tau_t \in [0, 1]$, in order to restrict banks’ leverage below levels permitted by the market. Thus, the social planner uses the haircut in order to set the leverage ratio restriction, which can be time-varying. Under the haircut policy, the amount of deposits that banks can raise against expected returns from sector- h loans is restricted to $(1 - \tau_t) \lambda^h (n_t) \mathbb{E}_t R_{t+1}^h (n_t(j) + d_t^h(j))$, such that the *ex-ante* borrowing constraint faced by the bank becomes:

$$(1 - \tau_t) \lambda^h (n_t) \mathbb{E}_t \left(R_{t+1}^h \right) (n_t(j) + d_t^h(j)) \geq R_t^{d,h} d_t^h(j) \quad (52)$$

It is assumed that the policymaker can only set the haircut once per period, after output goods have been produced, and prior to the investment choices of banks. In particular, the policymaker cannot change the haircut at the start of a period after capital has been realised, for example to offset liquidation pressure on banks. In particular then, the *ex-post* borrowing constraint faced by banks continues to mirror the *ex-ante* borrowing constraint:

$$(1 - \tau_t) \lambda^h (n_t) R_{t+1}^h (n_t(j) + d_t^h(j)) \geq R_t^{d,h} d_t^h(j) \quad (53)$$

In what follows, we refer to $(1 - \tau_t) \lambda^h (n_t)$ as the *effective pledgeability* of banks for making loans to sector h . The leverage restriction, given τ_t set by the social planner, is then given by

$(1 - (1 - \tau_t)\lambda^h(n_t))^{-1}$, with a higher haircut τ_t reducing both the effective pledgeability and leverage restriction. The equilibrium conditions remain as before, but $(1 - \tau_t)\lambda^h(n_t)$ replaces $\lambda^h(n_t)$ throughout.

4.3 Optimal leverage restriction in the absence of a credit trap

As discussed in the Introduction, the model can articulate three distinct macroprudential rationales for a leverage ratio requirement: offsetting procyclical fluctuations in banks' borrowing constraints, counteracting the liquidation costs associated with high leverage, and counteracting the possibility of a credit trap. In this subsection we examine the first two motivations in the absence of a credit trap, focusing on an economy in which only sector a exists. This allows analytical characterisation of the solution to the planner's problem. Following this, the next subsection turns to numerical analysis of macroprudential policy when the economy can fall into a credit trap.

Planner's problem

When there is no possibility of a credit trap, and only sector a exists, the planner's problem at time t is to choose a sequence of haircuts $\{\tau_{t+s}\}_{s=0}^{\infty}$ to maximise (51) subject to the law of motion for capital given by (36) and (37), where $h = a$, and $(1 - \tau_t)\lambda^h(n_t)$ replaces $\lambda^h(n_t)$. When there is no possibility of a credit trap, the following proposition, proved in the Appendix, characterises the solution to the planner's problem.

Proposition 9. *Suppose the conditions of Proposition 3 hold. Then the solution to the planner's problem satisfies the following condition:*

$$\frac{(1 + \alpha\beta)}{1 + (1 - \tau_t)\lambda^a(n_t)\beta} - \frac{2\alpha\beta}{\pi + (1 - \tau_t)\lambda^a(n_t)\beta} \quad (54)$$

$$+ \int_{(x_{t+1}^a)}^{\left(\mathbb{E}_t((x_{t+1}^a)^\alpha)\right)^{\frac{1}{\alpha}}} \left[\frac{(1 + \alpha\beta)\mathbb{E}_t((x_{t+1}^a)^\alpha)}{(x_{t+1}^a)^\alpha - (1 - \tau_t)\lambda^a(n_t)\mathbb{E}_t((x_{t+1}^a)^\alpha)} - \frac{(1 + \alpha\beta)}{1 - (1 - \tau_t)\lambda^a(n_t)} \right] f(x_{t+1}^a) dx_{t+1}^a = 0$$

This solution has the following properties:

(i) *The leverage ratio restriction $(1 - (1 - \tau_t)\lambda^h(n_t))^{-1}$ is independent of k_t , constant across all states, and haircut policy is procyclical to ensure this, offsetting the procyclical market leverage.*

(ii) When there is no liquidation (i.e. no shocks to x_{t+1}^a), the leverage ratio restriction is given by

$$\left(1 - (1 - \tau_t)\lambda^h(n_t)\right)^{-1} = \frac{\beta(1 - \alpha\beta)}{\beta(1 - \alpha\beta) + (1 + \alpha\beta)\pi - 2\alpha\beta} \quad (55)$$

(iii) The leverage ratio restriction is lower when there is liquidation, and lower the greater the potential for downside risks, as given by $E_t((x_{t+1}^a)^\alpha) - (\underline{x}_{t+1}^a)^\alpha$.

Proposition 9 clarifies the trade-offs that the planner faces in setting the leverage restriction in the absence of a credit trap. First, the leverage ratio restriction affects the inter-temporal allocation of consumption *within* a given generation. From (49) it can be shown that if policy was just set to maximise the lifetime utility of the generation born at t , it would satisfy

$$\left(1 - (1 - \tau_t)\lambda^h(n_t)\right)^{-1} = \frac{\beta}{\beta(1 - \alpha) + (1 + \alpha\beta)\pi} \quad (56)$$

Due to the logarithmic form of household utility, and the lump sum redistribution of bank profits, the planner would optimally set the haircut so that households consume a fixed portion of their income when young, regardless of the state of the economy. Thus, when banks' (market) borrowing constraint fluctuates in a procyclical manner with the level of bank capital (with the constraint tightening as the level of bank capital falls), the planner's optimal policy is to counteract this by also adjusting the haircut in a procyclical manner in order to smooth consumption and investment, keeping the leverage ratio restriction constant.

Second, the leverage ratio restriction also affects the allocation of consumption *across* generations. When there is no liquidation (i.e. there are no shocks to x_{t+1}^a), the integral term in (54) is zero and the planner will unambiguously ensure higher leverage when considering the welfare of many generations: that is, it can be shown that the leverage ratio restriction in (55) is higher than in (56). When the planner only considers the welfare of one generation, they don't take the impact of policy on the future capital stock into account. When considering the welfare of all future generations, the planner finds it optimal to increase the leverage ratio restriction, with the current young consuming less and saving more. This comes as a welfare cost to the young, but benefits future generations.

Third, when liquidation is possible (i.e. there are shocks to x_{t+1}^a), the integral term in (54)

appears in the planner’s solution and they will reduce the leverage ratio restriction to enhance the resilience of the economy.¹⁶ Under the assumptions of the proposition, the current generation face no costs due to liquidation, as banks remain solvent and can delever without a fall in their net worth. However, liquidations impose costs on future generations, with less capital used in the production of final output, and as a result lower future wages and consumption. To offset this the planner reduces the leverage ratio restriction, with the reduction greater the bigger the potential for downside risk, as given by $E_t((x_{t+1}^a)^\alpha) - (x_{t+1}^a)^\alpha$, which increases the range of negative shocks integrated over.

To illustrate the above results, we parametrise the model with the values contained in Table 1. This calibration is also used for the numerical work on optimal policy in the presence of a credit trap. The calibration we employ is intended only to be illustrative. We assume a capital share of income of 0.3. To capture the longer time-horizon implied by the OLG model, we choose a discount rate of 0.8, such that supposing each period is ten years, the annual real interest rate implied by this would be around 2.25%. We set the household’s bank equity injection π to 0.1, which implies an equity-asset ratio of around 15% given other parameter values. Table 1 contains the functional form we employ to capture the endogeneity of λ^a with respect to aggregate banking net worth n_t , via k_t . This smoothly increasing function takes a value of $\lambda_0 = 0.6$ when the aggregate capital stock goes to zero, and a value of $\lambda_0 + \lambda_1 = 0.9$ when the aggregate capital stock becomes unboundedly large. The parameter λ_2 governs the speed with which λ^a rises with k_t . We set this to 0.05 to generate a smooth increase in λ^a over the domain of the capital stock we consider. Because sector b is assumed to generate returns that are more pledgeable than sector a , we set λ^b to 0.95, in excess of $\lambda_0 + \lambda_1$. Finally, we specify the properties of the productivity processes. Productivity in sector b is set at 0.775, a 22.5% discount to expected productivity in sector a , which has expected productivity of 1, with a standard deviation of 1.5%.¹⁷

Figure 6 plots the optimal haircut as a function of the capital stock given the parameters in Table 1. The solid line shows the optimal haircut in the baseline case in which no liquidation is possible and there can be no credit traps. As shown analytically, the optimal haircut in this case rises in a procyclical manner with the capital stock, k_t , so as to stabilise the leverage ratio restriction

¹⁶With the assumption that x_{t+1}^a is i.i.d. over time, the leverage ratio restriction is also constant when there is liquidation.

¹⁷Whilst, being normally distributed, expected productivity in sector a has infinite support, in the numerical solutions the distribution is discretised. Given the discretisation chosen, condition (28) holds, and banks will always be solvent.

Table 1: Parameter values for numerical experiments

<i>Parameter</i>		<i>Value</i>
α	Capital share	0.3
β	Discount rate	0.8
π	Bank equity injection	0.1
$\lambda^a(n) = \lambda_0 + \lambda_1 \frac{k}{\lambda_2 + k}$	Sector <i>a</i> pledgeability	$(\lambda_0, \lambda_1, \lambda_2) = (0.6, 0.3, 0.05)$
λ^b	Sector <i>b</i> pledgeability	0.95
$x_t^a \sim \mathcal{N}(\bar{x}^a, \sigma^2)$	Sector <i>a</i> productivity	$(\bar{x}^a, \sigma^2) = (1, 0.015^2)$
\bar{x}^b	Sector <i>b</i> productivity	0.775

Note: the table gives parameter values for the numerical welfare experiments, with the choice of parameters discussed in the text. Note that we write $\lambda^a(n)$ as a function of k , the state variable. This is a shorthand for writing the relationship $k = \left(\frac{n}{\pi(1-\alpha)}\right)^{\frac{1}{\alpha}}$.

for sector *a*.

Figure 6 also shows the optimal haircut policy in the case in which liquidation is possible (the dotted line). In this case, higher leverage risks a larger liquidation following a negative shock, thus amplifying the effects of a negative shock on the future productive capacity of the economy. For this reason, the planner sets a higher haircut (and hence a lower leverage ratio restriction) across all levels of capital stock, k_t .

In the cases considered in this sub-section, the planner varies the haircut cyclically to offset changes in the market-based borrowing constraint and ensure a constant leverage ratio restriction over time and different states of the world. In the next section we show that, in the presence of a credit trap, the leverage ratio restriction will no longer be constant, but will vary in a *countercyclical* manner outside of the credit trap.

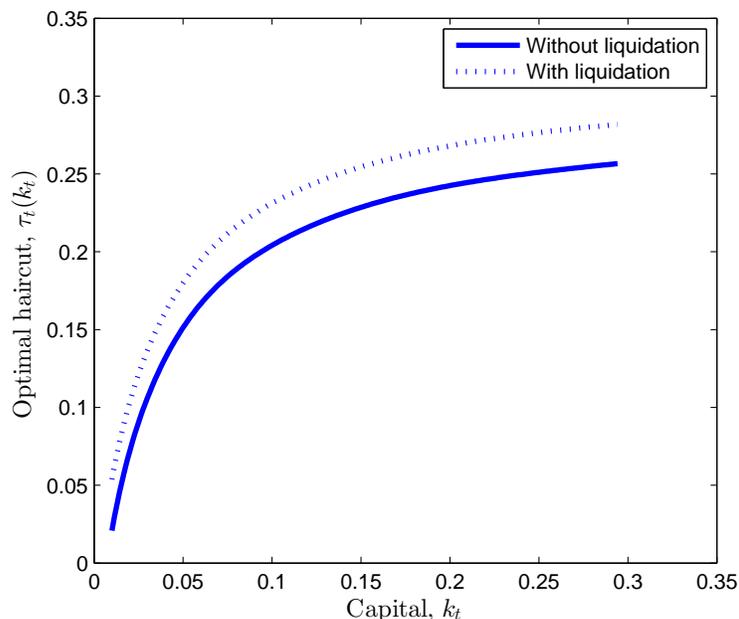
4.4 Optimal leverage policy with credit traps

When there are multiple sectors, the haircut can influence the choice of sector invested in. To avoid the planner trivially being able to ensure investment always flows to sector *a*, the haircut is applied uniformly to both sectors *a, b*.¹⁸ In particular, the trap threshold is now implicitly defined by:

$$\begin{aligned} & (\pi + \beta(1 - \tau_t) \lambda^a(\tilde{n}))^\alpha (1 + \beta(1 - \tau_t) \lambda^a(\tilde{n}))^{1-\alpha} \mathbb{E}_t((x_{t+1}^a)^\alpha) \\ & = \left(\pi + \beta(1 - \tau_t) \lambda^b\right)^\alpha \left(1 + \beta(1 - \tau_t) \lambda^b\right)^{1-\alpha} (x^b)^\alpha \end{aligned} \quad (57)$$

¹⁸If haircuts could be set differentially for sectors *a, b* then the planner could set a haircut of $\tau_t^b = 1$, preventing deposits flowing to any bank investing in sector *b*.

Figure 6: Optimal haircut policy as a function of the capital stock, with and without liquidation, without credit traps



Note: the figure displays the haircut that maximises the welfare function when there are no credit traps. The solid blue line is the case when liquidation does not occur; the dashed line is the case with liquidation.

where \tilde{n} is the bank net worth threshold below which the economy falls into a credit trap. As is clear from (57), through setting the haircut τ_t , the planner can influence the trap threshold itself. Indeed, as shown by the following proposition, proved in the Appendix, an increase in the haircut, which reduces the permitted leverage, reduces the critical level of bank net worth below which the economy falls into a credit trap.

Proposition 10. *Suppose that the conditions of Proposition 5 hold. Then*

$$\frac{d\tilde{n}}{d\tau_t} < 0$$

so an increase in the haircut (i.e. the lowering of the leverage ratio restriction) makes sector b relatively less attractive compared to sector a , and thus lowers the level of bank net worth at which the economy falls into a credit trap.

The intuition behind the effects of a higher haircut (i.e. a lower leverage ratio restriction) on the trap threshold \tilde{n} is as follows. As τ_t rises, the borrowing constraint facing a bank tightens. The

marginal tightening is greatest where pledgeable returns are greatest, which is in the sector for which λ^h is high. This is sector b , where the assets generate relatively low returns but are highly liquid and therefore more easily pledgeable. Because a tighter haircut penalises leverage, it discourages investment in sector b at the margin. As a result, it becomes more preferable for bank investors to fund sector a assets, thus lowering the threshold \tilde{n} below which banks start investing in sector b .

When the planner’s problem also features a credit trap, a discontinuity appears in the policy-maker’s payoff: in the absence of policy intervention, the economy enters a credit trap and resides there forever once the level of capital stock falls below a threshold. We assume that the planner always uses their policy tool to avoid this damaging outcome by setting the haircut sufficiently high to avoid the credit trap. The presence of a credit trap then modifies the planners problem from Section 4.3 by imposing an effective lower bound on the optimal haircut:

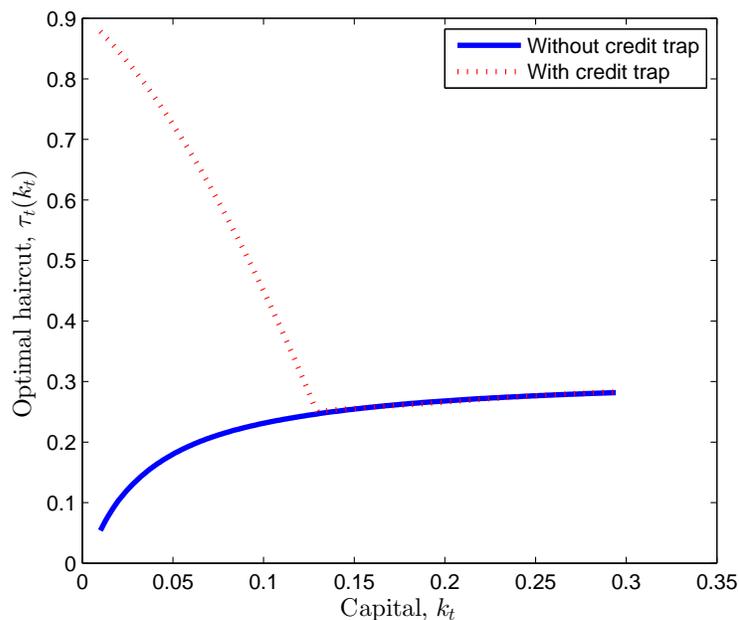
$$\tau_t \in [\tau^*(k_t), 1]$$

The effective lower bound $\tau^*(k_t)$ is in turn determined by the solution to the credit trap threshold condition (57) above. Key to solving the planner’s problem without a credit trap analytically was that the optimal leverage restriction was independent of k_t . With a credit trap this is no longer the case, and the model is instead solved numerically via dynamic programming, as described in the Appendix.

Imposing this constraint on the planner results in the optimal policy rule shown in Figure 7. Compared to the case without the credit trap, the possibility of a credit trap induces a kink in the optimal policy function such that, below a critical level of capital, the optimal haircut no longer falls as capital falls, but instead *rises* (the dotted line). In other words, as the economy deteriorates and faces a severe credit trap threat, the planner *tightens*, rather than loosens, the leverage ratio restriction on banks. The reason for this is that, in these states of the world, tighter leverage policy discourages the allocation of capital towards highly liquid yet unproductive investments. The policymaker can discourage a switch into this kind of liquidity hoarding by making it sufficiently unprofitable by restricting banks’ ability to borrow against these liquid but unproductive assets, and thus can maintain the flow of investment into productive lending.

We note that this counter-intuitive result arises because the planner in our model does not have

Figure 7: Optimal haircut policy as a function of the capital stock when credit traps are possible



Note: the figure displays the haircut that maximises the welfare function when liquidation is possible. The solid blue line shows the case in which there are no credit traps, whilst the dashed red line shows the case where credit traps are possible.

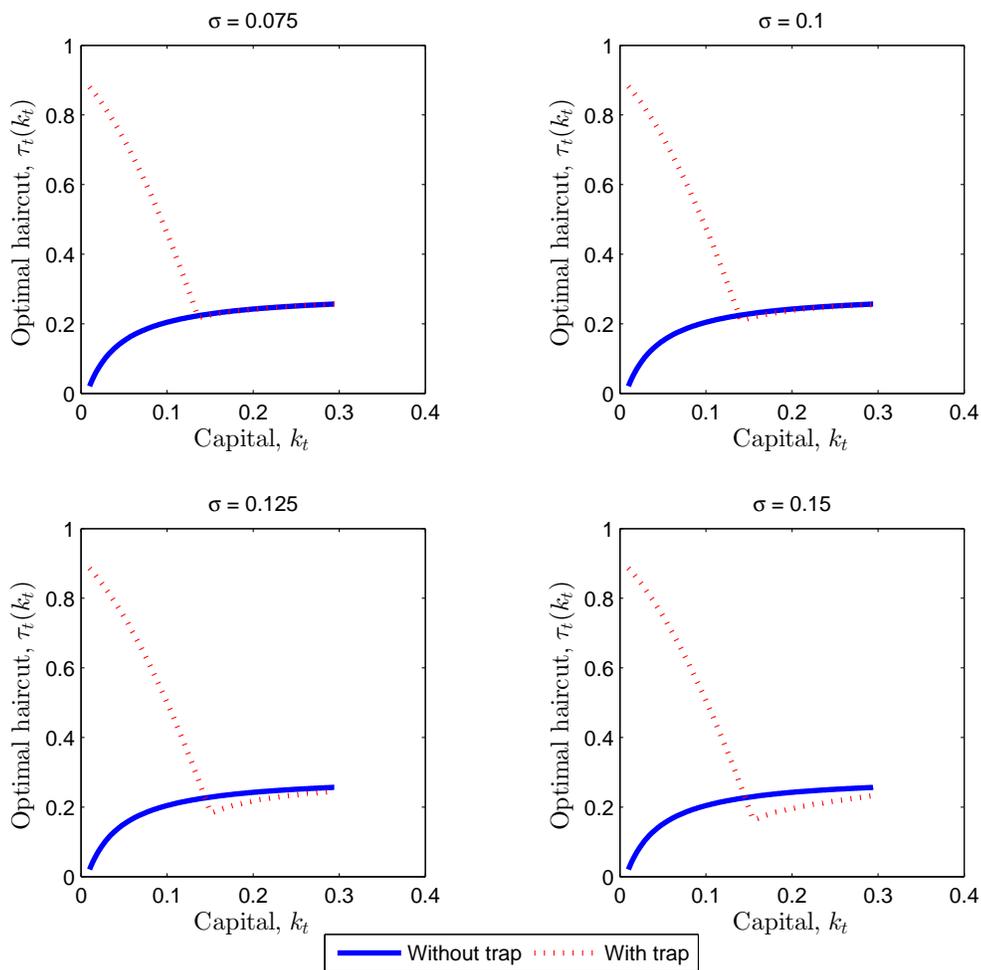
any other alternative policy instrument, e.g. bank recapitalisation. In reality, the macroprudential policy authority is more likely to request banks to raise new capital rather than tighten a leverage ratio restriction when the economy is at the brink of a credit trap, and loosening the leverage ratio restriction is no longer effective in stimulating lending to the productive sectors. We nevertheless think that our results provide a key insight into the limitation of countercyclical macroprudential policy: while countercyclical macroprudential policy is welfare-enhancing in normal periods, it ceases to be effective once the economy is on the verge of a financial crisis and banks have a strong incentive to hoard safe, liquid assets instead of making loans to productive sectors.

More generally, the response of policy in the presence of the credit trap captures the severe welfare cost of a financial crisis, with a sharp tightening in borrowing constraints and associated reductions in output and consumption. This resembles other globally solved models of crises with occasionally binding constraints, such as Bianchi and Mendoza (Forthcoming). When volatility in the economy is sufficiently large, this can lead to precautionary behaviour by the planner. To show this, we explore how the planner's policy in the presence of a credit trap responds to the volatility

of the economy. In particular, we ask what effect a higher value of σ , the standard deviation of the sector- a productivity shock, has on the optimal haircut policy. To focus on the implications of higher volatility in the presence of a credit trap, in this experiment we assume that ex-post liquidation does not occur, i.e. the *ex-post* borrowing constraint is not imposed. This also allows us to explore larger shock values while retaining stability in the numerical routines. In particular, we solve for the optimal policy rule for a grid of values for σ , given by $\sigma \in \{0.075, 0.10, 0.125, 0.15\}$, all considerably above the baseline value. The optimal policy rules that result are shown in the panels of Figure 8: the solid line shows the optimal policy without a credit trap, while the dotted line shows the optimal policy with a trap. We see from the top-left panel that for low values of volatility, optimal policy when a credit trap is possible resembles closely the optimal policy without a credit trap when the economy resides at a point sufficiently far away from the trap threshold. In other words, when volatility is sufficiently low, the planner does not behave in a particularly precautionary fashion. However, as volatility rises, policy with and without a credit trap begin to diverge. For example, when volatility is high, as in the bottom-right panel (with $\sigma = 0.15$), the optimal policy in the presence of a credit trap exhibits some precautionary behaviour: as the economy deteriorates and capital falls closer to the trap threshold, the planner relaxes the haircut restriction more aggressively than would be the case in the absence of a trap. She does this because lowering the haircut (and thus allowing banks to operate with a higher leverage) today helps to increase bank profits, and thus boosts bank capital and hence the state of the economy tomorrow. This in turn makes it less likely that the economy will be faced with a credit trap tomorrow, allowing the planner to avoid a drastic hike in the haircut. In other words, as volatility rises, the policymaker acts in a more precautionary way to boost the economy and reduce the probability of becoming constrained by a credit trap in future periods. Optimal leverage policy under the threat of a credit trap thus bears resemblance to the resilience-maximising leverage ratio of Section 3, which is also countercyclical.

Finally, using the baseline optimal policy rules, it is possible to perform stochastic simulations of the model's dynamics in order to approximate the ergodic distribution of the capital stock and the optimal haircut policy, given the parameters of the economy and its shock processes. Figure 9 shows estimated densities from one experiment simulating 10,000 paths for the model economy under different assumptions about liquidation and the presence of a credit trap. In the top panel we see the baseline case in which no liquidation and no trap are possible in the solid (blue) density.

Figure 8: Impact of higher volatility on the optimal haircut policy



Note: the figures display the model when liquidation is not possible for different values of σ . The solid blue line shows the case in which there are no credit traps, whilst the dashed red line shows the case where credit traps are possible.

This configuration generates a relatively tightly centred distribution for the capital stock. Next, the figure shows that adding liquidation in the fine-dotted (red) density lowers the mean value of the capital stock – even though the fundamental shock processes hitting the economy have not changed – while also making it more volatile. Finally, compared to this, adding the possibility of credit traps in the thick-dotted (green) density changes the distribution of the capital stock rather little.

What is the reason for the last of these observations? Note that in the bottom panel, estimates of the ergodic distribution of the haircut differ between the fine-dotted (red) density where no trap is possible and the thick-dotted (green) density differ; in particular, when the credit trap is possible, the haircut is on average higher, and the policymaker is much less likely to lower the haircut compared to the case where traps are not possible. Thus, it is as a result of the optimal policy that the distributions of *capital* do not differ markedly in the case when there is and is not a trap. In order to deliver this, the distribution of the optimal *haircut* must, however, shift to the right, becoming more conservative on average in order to avoid credit traps.

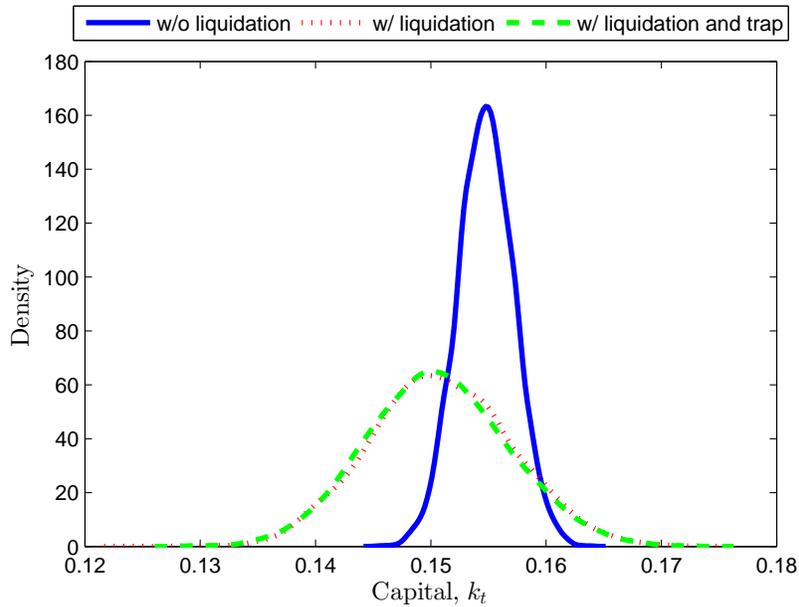
5 Concluding remarks

The key contribution of our paper is to develop a rich yet tractable framework to examine optimal macroprudential leverage policy. Contrary to most modern macroeconomic models, in which the economy quickly recovers following a shock, our model allows for the possibility that a sufficiently severe financial shock can tip the economy into a *credit trap*, i.e. a steady state featuring permanently lower output, bank credit, and productivity. By including a state of the world that resembles the aftermath of a financial crisis, this framework is ideally suited for examining how the leverage ratio should be optimally set outside of a crisis.

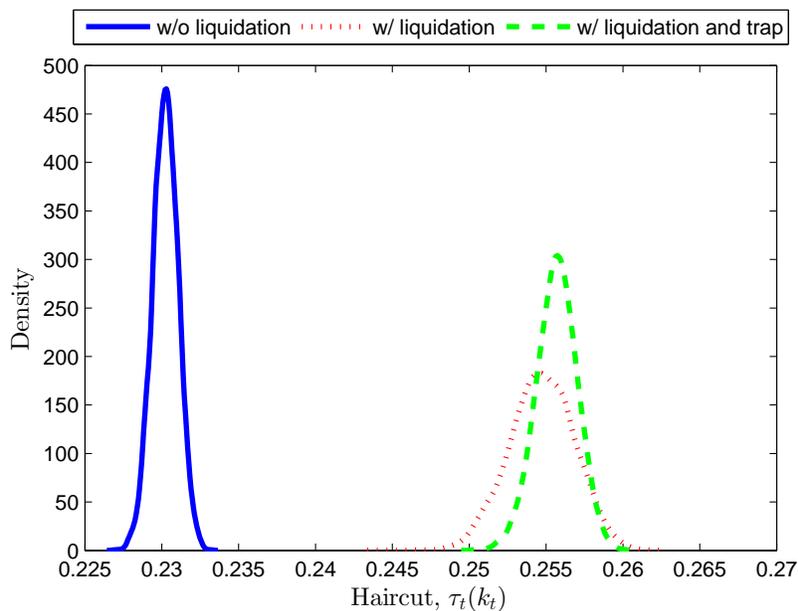
Our analysis offers three key policy results. First, when banks' borrowing constraints fluctuate in a procyclical manner with the health of the banking system, the regulatory leverage ratio restriction can be used to counteract this to stabilise banks' leverage, and thereby smooth consumption and investment. Second, when there is a possibility of costly asset liquidation, the macroprudential policy authority should set a tighter leverage ratio restriction, to reduce the amplification of negative shocks, and reduce the associated negative welfare costs. In our simple model, these motivations result in a constant leverage ratio restriction. Finally, when faced with the possibility of a credit

Figure 9: Densities of the capital stock and optimal haircut

(a) Capital



(b) Haircut



Note: the figures display the distribution of the capital stock and optimal haircut respectively across 10,000 simulated paths for the economy. In both cases the solid blue line represents the density without liquidation or a credit trap, the dotted red line the case with liquidation but no credit trap, and the dashed green line the case with liquidation and a credit trap.

trap, the macroprudential authority will set a countercyclical leverage ratio outside of the trap, to reduce the chance of the economy falling into this damaging state.

References

- ADRIAN, T., P. COLLA, AND H. S. SHIN (2012): “Which Financial Frictions? Parsing the Evidence from the Financial Crisis of 2007-9,” in *NBER Macroeconomics Annual 2012, Volume 27*. University of Chicago Press.
- AIKMAN, D., A. G. HALDANE, AND B. D. NELSON (2015): “Curbing the credit cycle,” *The Economic Journal*, 125(585), 1072–1109.
- AIKMAN, D., B. NELSON, AND M. TANAKA (2015): “Reputation, risk-taking, and macroprudential policy,” *Journal of Banking & Finance*, 50, 428–439.
- ANGELONI, I., AND E. FAIA (2013): “Capital regulation and monetary policy with fragile banks,” *Journal of Monetary Economics*, 60(3), 311–324.
- BENMELECH, E., AND N. K. BERGMAN (2012): “Credit Traps,” *The American Economic Review*, 102(6), 3004–3032.
- BIANCHI, J. (2011): “Overborrowing and systemic externalities in the business cycle,” *American Economic Review*, 101(7), 3400–3426.
- BIANCHI, J., AND E. MENDOZA (Forthcoming): “Optimal Time-Consistent Macroprudential Policy,” *Journal of Political Economy*.
- CABALLERO, R. J., T. HOSHI, AND A. K. KASHYAP (2008): “Zombie Lending and Depressed Restructuring in Japan,” *The American Economic Review*, pp. 1943–1977.
- CHRISTENSEN, I., C. MEH, AND K. MORAN (2011): “Bank Leverage Regulation and Macroeconomic Dynamics,” *Bank of Canada Working Paper No. 32*.
- CHRISTIANO, L., AND D. IKEDA (2016): “Bank Leverage and Social Welfare,” *The American Economic Review*, 106(5), 560–564.
- CLAESSENS, S., AND M. A. KOSE (2013): “Financial Crises: Explanations, Types, and Implications,” *IMF Working Paper No. 13/28*.

- DE NICOLO, G., G. FAVARA, AND L. RATNOVSKI (2012): “Externalities and Macroprudential Policy,” *IMF Staff Discussion Notes*, (12/05).
- EPSTEIN, L., AND S. ZIN (1989): “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework,” *Econometrica*, 57(4), 937–969.
- GERTLER, M., AND P. KARADI (2011): “A model of unconventional monetary policy,” *Journal of Monetary Economics*, 58(1), 17–34.
- GERTLER, M., AND N. KIYOTAKI (2010): “Financial intermediation and credit policy in business cycle analysis,” *Handbook of monetary economics*, 3(3), 547–599.
- GERTLER, M., N. KIYOTAKI, AND A. QUERALTO (2012): “Financial crises, bank risk exposure and government financial policy,” *Journal of Monetary Economics*, 59, S17–S34.
- LORENZONI, G. (2008): “Inefficient credit booms,” *The Review of Economic Studies*, 75(3), 809–833.
- MATSUYAMA, K. (2007): “Credit traps and credit cycles,” *The American Economic Review*, 97(1), 503–516.
- MIRANDA, M., AND P. FACKLER (2002): *Applied Computational Economics and Finance*. MIT Press.
- NELSON, B., AND M. TANAKA (2014): “Dealing with a banking crisis: what lessons can be learned from Japan’s experience?,” *Bank of England Quarterly Bulletin*, 54(1), 36–48.
- PEEK, J., AND E. S. ROSENGREN (2005): “Unnatural Selection: Perverse Incentives and the Misallocation of Credit in Japan,” *The American Economic Review*, 95(4), 1144–1166.
- SHLEIFER, A., AND R. W. VISHNY (1992): “Liquidation values and debt capacity: A market equilibrium approach,” *The Journal of Finance*, 47(4), 1343–1366.

A Household preferences

A commonly used form for Epstein-Zin preferences to take is:

$$V_t = [c_t^{1-\rho} + \beta(\mathbb{E}_t V_{t+1}^{1-\alpha})^{\frac{1-\rho}{1-\alpha}}]^{\frac{1}{1-\rho}},$$

in which ρ governs the degree of intertemporal substitution and α the degree of risk aversion. As risk aversion goes to zero ($\alpha \rightarrow 0$),

$$V_t = [c_t^{1-\rho} + \beta(\mathbb{E}_t V_{t+1})^{1-\rho}]^{\frac{1}{1-\rho}},$$

and as $\rho \rightarrow 1$,

$$V_t = c_t(\mathbb{E}_t V_{t+1})^\beta,$$

such that taking logs gives:

$$U_t \equiv \log V_t = \log c_t + \beta \log \mathbb{E}_t V_{t+1}.$$

In the two period case, $V_{t+1} = c_{t+1}$, so we get:

$$U_t = \log c_t + \beta \log \mathbb{E}_t c_{t+1},$$

which is the case of intra-temporal risk neutrality combined with an intertemporal elasticity of substitution of unity.

B Deposit market equilibrium

B.1 Proposition 1

Proof of Proposition 1. When $R_t^{d,h} > \lambda^h(n_t) \mathbb{E}_t R_{t+1}^h$ the bank is constrained by the *ex-ante pledgeability constraint* (7). Given that banks are risk neutral, when $\mathbb{E}_t R_{t+1}^h > R_t^{d,h}$ they wish to raise as many deposits as they can, given they face no costs from deleveraging, in the face of a negative shock. The *ex-ante pledgeability constraint* then holds with equality giving

$$\lambda^h(n_t) (n_t(j) + d_t^h(j)) \mathbb{E}_t (R_{t+1}^h) = R_t^{d,h} d_t^h(j) \quad (58)$$

Rearranging gives the deposit demand for banks as

$$d_t^h(j) = \frac{\lambda^h(n_t) \mathbb{E}_t (R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t (R_{t+1}^h)} n_t(j) > 0 \quad (59)$$

Turning to the terminal pay-off for banks, we have, given there are no costs from deleveraging, and using (6)

$$V_{t+1}^h(j) = (R_{t+1}^h - R_t^{d,h}) d_t^h(j) + R_{t+1}^h n_t(j)$$

Substituting in (59) gives

$$V_{t+1}^h(j) = \left[(R_{t+1}^h - R_t^{d,h}) \frac{\lambda^h(n_t) \mathbb{E}_t (R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t (R_{t+1}^h)} + R_{t+1}^h \right] n_t(j)$$



This can be rearranged to $V_{t+1}^h(j) =$

$$\left[R_{t+1}^h \left(\frac{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h) + \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)} \right) - \frac{R_t^{d,h} \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)} \right] n_t(j)$$

And then

$$V_{t+1}^h(j) = \left[\frac{R_{t+1}^h - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)} \right] R_t^{d,h} n_t(j) \quad (60)$$

Taking time t expectations gives the required formula:

$$\mathbb{E}_t V_{t+1}^h(j) = \frac{1 - \lambda^h(n_t)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)} \mathbb{E}_t(R_{t+1}^h) R_t^{d,h} n_t(j) > 0 \quad (61)$$

□

B.2 Proposition 2

Proof of Proposition 2. Given that the conditions of Proposition 1 hold, from (60), the realised profits for banks are given by

$$V_{t+1}^h(j) = \left[\frac{R_{t+1}^h - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)} \right] R_t^{d,h} n_t(j)$$

Thus, given $R_t^{d,h} > \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)$, banks will be solvent following a shock so long as $R_{t+1}^h \geq \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)$, and are thus will be solvent *ex-ante* with certainty when they are solvent under the lowest possible return for investment in sector h , i.e., when $R_{t+1}^h \geq \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)$.

Given that banks are then solvent following all return realisations, and there are no costs from deleveraging, households can always withdraw deposits following a negative shock until the point at which the *ex-post* pledgeability constraint (10) holds, without pushing the bank into insolvency. Thus banks will not abscond with household deposits and so household deposits are indeed riskless.

□

B.3 Proposition 3

Proof of Proposition 3. We proceed in a number of steps.

Step 1: In any equilibrium we must have $R_t^{d,h} > \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)$.

To show this, suppose for a contradiction that it doesn't hold: $\lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h) \geq R_t^{d,h}$. Then, given that $\lambda^h(n_t) \in (0, 1)$, and $\mathbb{E}_t(R_{t+1}^h) > 0$ (as $\mathbb{E}_t(x_{t+1}^h) > 0$) we must have $\mathbb{E}_t(R_{t+1}^h) > \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h) \geq R_t^{d,h}$. The *ex-ante* pledgeability constraint requires that

$$\lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h) (n_t(j) + d_t^h(j)) \geq R_t^{d,h} d_t^h(j)$$



Under the supposition, this always holds as

$$\lambda^h(n_t) \mathbb{E}_t \left(R_{t+1}^h \right) \left(n_t(j) + d_t^h(j) \right) \geq R_t^{d,h} \left(n_t(j) + d_t^h(j) \right) \geq R_t^{d,h} d_t^h(j)$$

Hence, in this case the pledgeability constraint is satisfied for all $d_t^h(j)$. Further, there is a positive spread and the bank wants to take as many deposits as possible. This cannot be an equilibrium as there is a finite amount of potential deposits available from households. Thus we have a contradiction, and so we must have $R_t^{d,h} > \lambda^h(n_t) \mathbb{E}_t \left(R_{t+1}^h \right)$ in any equilibrium, completing the proof of Step 1.

Thus, given Step 1, banks face the pledgeability constraint. Moreover, given (20) and $(\underline{x}_{t+1}^h)^\alpha > \lambda^h(n_t) \mathbb{E}_t \left((\underline{x}_{t+1}^h)^\alpha \right)$, we also have

$$\left(\underline{R}_{t+1}^h \right)^\alpha > \lambda^h(n_t) \mathbb{E}_t \left(\left(R_{t+1}^h \right)^\alpha \right) \quad (62)$$

and so deposits are riskless for households.

Step 2: Given $\beta(1-\pi) \geq \pi$, in any equilibrium we must have $\mathbb{E}_t \left(R_{t+1}^h \right) \geq R_t^{d,h}$.

To show this, suppose for a contradiction that it does not hold. Then $\mathbb{E}_t \left(R_{t+1}^h \right) < R_t^{d,h}$ and banks expect to lose money on every unit of deposits taken. Optimally the bank's deposit demand is then zero. Given this, $\mathbb{E}_t \left(V_{t+1}^h(j) \right) = n_t(j) \mathbb{E}_t \left(R_{t+1}^h \right) = \pi w_t \mathbb{E}_t \left(R_{t+1}^h \right)$ with the banks just investing their own equity. With bank deposits riskless, from (5) it then following that household deposit demand of household j is given by

$$d_t^h(j) = \frac{w_t}{1+\beta} \left(\beta(1-\pi) - \frac{\pi \mathbb{E}_t \left(R_{t+1}^h \right)}{R_t^{d,h}} \right) > \frac{w_t}{1+\beta} (\beta(1-\pi) - \pi) \geq 0 \quad (63)$$

where the last inequality follows from the condition and $w_t > 0$. Thus, household deposit supply is positive, and so the economy is not in equilibrium as bank deposit demand is zero. This is a contradiction, so we must have $\mathbb{E}_t \left(R_{t+1}^h \right) \geq R_t^{d,h}$, completing the proof of Step 2.

Step 3: Given $\beta(1-\pi) > \pi + \lambda^h(n_t)\beta$, in any equilibrium we must have $\mathbb{E}_t \left(R_{t+1}^h \right) > R_t^{d,h}$.

To show this, suppose for a contradiction that it does not hold. Then, given the result from Step 2, we must have $\mathbb{E}_t \left(R_{t+1}^h \right) = R_t^{d,h}$. Hence, with bank deposits riskless, from (63) we have household deposit supply given by $d_t^h(j) = \frac{w_t}{1+\beta} (\beta(1-\pi) - \pi) > 0$. Given $\mathbb{E}_t \left(R_{t+1}^h \right) = R_t^{d,h}$, the *ex-ante* pledgeability constraint becomes

$$\lambda^h(n_t) \left(n_t(j) + d_t^h(j) \right) \geq d_t^h(j)$$

Rearranging and substituting in for $d_t^h(j)$, and using $n_t(j) = \pi w_t$ this constraint becomes

$$\lambda^h(n_t) \pi \geq \frac{1}{1+\beta} (\beta(1-\pi) - \pi) \left(1 - \lambda^h(n_t) \right)$$

$$\lambda^h(n_t) (\pi(1+\beta) + (\beta(1-\pi) - \pi)) \geq (\beta(1-\pi) - \pi)$$

$$\lambda^h(n_t) \beta + \pi \geq \beta(1-\pi)$$

This is a contradiction, so we must have $\mathbb{E}_t \left(R_{t+1}^h \right) > R_t^{d,h}$, completing the proof of Step 3.

Thus, under the conditions of the proposition, in any equilibrium we must have

$$\mathbb{E}_t \left(R_{t+1}^h \right) > R_t^{d,h} > \lambda^h(n_t) \mathbb{E}_t \left(R_{t+1}^h \right)$$

Thus, with bank net worth invariant to deleveraging, and (62) holding, the conditions of Propositions 1 and 2 hold. Thus (27) and (26) in the text hold. Equating the two for equilibrium in the deposit market gives

$$\frac{\lambda^h(n_t) \mathbb{E}_t \left(R_{t+1}^h \right)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t \left(R_{t+1}^h \right)} \pi w_t = \frac{\beta}{1 + \beta} (1 - \pi) w_t - \frac{(1 - \lambda^h(n_t))}{1 + \beta} \frac{\mathbb{E}_t \left(R_{t+1}^h \right)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t \left(R_{t+1}^h \right)} \pi w_t$$

Cancelling common factors and rearranging gives

$$\frac{\mathbb{E}_t \left(R_{t+1}^h \right)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t \left(R_{t+1}^h \right)} \left(\lambda^h(n_t) \pi + \pi \frac{(1 - \lambda^h(n_t))}{1 + \beta} \right) = \frac{\beta}{1 + \beta} (1 - \pi)$$

$$\frac{\mathbb{E}_t \left(R_{t+1}^h \right)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t \left(R_{t+1}^h \right)} = \frac{\beta(1 - \pi)}{\pi(\beta\lambda^h(n_t) + 1)}$$

Substituting this into (27) and tidying gives

$$d_t^h = \frac{\beta\lambda^h(n_t)}{1 + \beta\lambda^h(n_t)} (1 - \pi) w_t \quad (64)$$

Thus, the total amount invested by banks is given by

$$s_t^h(j) = \left(\frac{\beta\lambda^h(n_t)}{1 + \beta\lambda^h(n_t)} (1 - \pi) + \pi \right) w_t = \frac{\pi + \beta\lambda^h(n_t)}{1 + \beta\lambda^h(n_t)} w_t \quad (65)$$

giving the result in the text when aggregating across banks.

Given $\mathbb{E}_t \left(R_{t+1}^h \right) > R_t^{d,h} > \lambda^h(n_t) \mathbb{E}_t \left(R_{t+1}^h \right)$, from the proof of Proposition 1, in equilibrium the *ex-ante* pledgeability constraint holds with equality, so

$$R_t^{d,h} = \lambda^h(n_t) \left(\frac{n_t(j)}{d_t^h(j)} + 1 \right) \mathbb{E}_t \left(R_{t+1}^h \right)$$

Now, from (64)

$$\frac{n_t(j)}{d_t^h(j)} + 1 = \frac{(1 + \beta\lambda^h(n_t)) \pi}{\beta\lambda^h(n_t) (1 - \pi)} + 1 = \frac{\pi + \beta\lambda^h(n_t)}{\beta\lambda^h(n_t) (1 - \pi)}$$

Thus,

$$R_t^{d,h} = \mathbb{E}_t \left(R_{t+1}^h \right) \left(\frac{\pi + \beta\lambda^h(n_t)}{\beta(1 - \pi)} \right) \quad (66)$$

Given this, $\mathbb{E}_t \left(R_{t+1}^h \right) > R_t^{d,h} > \lambda^h(n_t) \mathbb{E}_t \left(R_{t+1}^h \right)$ indeed holds and we have a valid equilibrium. To see this, note that, given $\beta(1 - \pi) > \pi + \lambda^h(n_t)\beta$, from (66) we indeed have $R_t^{d,h} < \mathbb{E}_t \left(R_{t+1}^h \right)$.

Further, given $1 + \beta\lambda^h(n_t) > 0$, $\pi + \beta\lambda^h(n_t) > \lambda^h(n_t)\beta(1 - \pi)$ and so we indeed have

$$R_t^{d,h} = \mathbb{E}_t \left(R_{t+1}^h \right) \left(\frac{\pi + \beta\lambda^h(n_t)}{\beta(1 - \pi)} \right) > \lambda^h(n_t) \mathbb{E}_t \left(R_{t+1}^h \right)$$

Now, turning to the spread, from (66) we have

$$\mathbb{E}_t \left(R_{t+1}^h \right) - R_t^{d,h} = \mathbb{E}_t \left(R_{t+1}^h \right) \left(\frac{\beta(1 - \pi) - (\pi + \beta\lambda^h(n_t))}{\beta(1 - \pi)} \right)$$

It remains to derive the equilibrium deposit rate. From (20), (65), and using $w_t = (1 - \alpha)k_t^\alpha$ the equilibrium return on bank's investment, absent deleveraging, is given by

$$R_{t+1}^h = \frac{\alpha (x_{t+1}^h)^\alpha}{\left(\frac{\pi + \beta\lambda^h(n_t)}{1 + \beta\lambda^h(n_t)} (1 - \alpha) k_t^\alpha \right)^{1-\alpha}} \quad (67)$$

Note that as total bank returns R_{t+1}^h , which include the return on capital used in the production of final output goods and liquidated capital goods, are invariant to deleveraging, (67) holds both when there is bank deleveraging and when there is not. Thus, from (66) and taking time t expectations of (67) we have

$$R_t^{d,h} = \frac{\alpha \mathbb{E}_t \left((x_{t+1}^h)^\alpha \right)}{\left(\frac{\pi + \beta\lambda^h(n_t)}{1 + \beta\lambda^h(n_t)} (1 - \alpha) k_t^\alpha \right)^{1-\alpha}} \left(\frac{\pi + \beta\lambda^h(n_t)}{\beta(1 - \pi)} \right)$$

Rearranging gives

$$R_t^{d,h} = \frac{\alpha (1 + \beta\lambda^h(n_t))^{1-\alpha} (\pi + \beta\lambda^h(n_t))^\alpha \mathbb{E}_t \left((x_{t+1}^h)^\alpha \right)}{\beta(1 - \pi) ((1 - \alpha) k_t^\alpha)^{1-\alpha}}$$

This completes the proof of the proposition. □

C Liquidation technology

In this section of the Appendix we derive the function form of liquidation technology that ensures that bank net worth is invariant to deleveraging.

Proposition 11. *Let $x_{t+1}^h (d_t^h + n_t)$ be the aggregate amount of capital generated by the banks' investments with capital producers. Let k_{t+1}^{liq} be the aggregate amount of liquidation of capital by banks. Let the liquidation technology be given by $y_{t+1}^{liq} = \mathbb{L} \left(k_{t+1}^{liq}, x_{t+1}^h (d_t^h + n_t) \right)$, so that the amount of output goods produced when banks liquidate k_{t+1}^{liq} units of capital is dependent upon both the initial amount of aggregate capital and the amount of capital liquidated.*

Suppose the liquidation technology has the following functional form:

$$\mathbb{L}\left(k_{t+1}^{liq}, x\left(d_t^h + n_t\right)\right) := \alpha\left(x_{t+1}^h\left(d_t^h + n_t\right)\right)^\alpha - \alpha\left(x_{t+1}^h\left(d_t^h + n_t\right) - k_{t+1}^{liq}\right)^\alpha$$

This functional form satisfies $\mathbb{L}\left(0, x\left(d_t^h + n_t\right)\right) = 0$, and is increasing in k_{t+1}^{liq} .

Moreover, bank returns, R_{t+1}^h , and bank net worth, $V_{t+1}^h(j)$, are independent of the degree of deleveraging.

Proof. It's clear that $\mathbb{L}\left(0; x_{t+1}^h\left(d_t^h + n_t\right)\right) = 0$, and so the liquidation technology generates no output goods when no capital is liquidated. Moreover,

$$\frac{\partial \mathbb{L}(\cdot, \cdot)}{\partial k_{t+1}^{liq}} = \alpha^2\left(x_{t+1}^h\left(d_t^h + n_t\right) - k_{t+1}^{liq}\right)^{\alpha-1} > 0$$

so more output goods are produced as more capital goods are liquidated.

Turning to the second part of the proposition, in aggregate, the value of old banks at time $t + 1$ is given by

$$V_{t+1}^h = R_{t+1}^h\left(d_t^h + n_t\right) - R_t^{d,h}d_t^h$$

The banks' investment with the capital producers generates $x_{t+1}^h\left(d_t^h + n_t\right)$ units of capital goods. Portion k_{t+1} is used in the production of final output goods, with the remainder, $k_{t+1}^{liq} := x_{t+1}^h\left(d_t^h + n_t\right) - k_{t+1}$ liquidated to produce output goods. The output returned to banks from production of final output goods is given by αk_{t+1}^α . Thus, the value of old banks at $t + 1$, given aggregate liquidation of k_{t+1}^{liq} units of capital, with each bank behaving symmetrically, is given by

$$V_{t+1}^h = \alpha\left(x_{t+1}^h\left(d_t^h + n_t\right) - k_{t+1}^{liq}\right)^\alpha + \mathbb{L}\left(k_{t+1}^{liq}, x_{t+1}^h\left(d_t^h + n_t\right)\right) - R_t^{d,h}d_t^h$$

Using the functional form in the proposition, the value of old banks is then

$$V_{t+1}^h = \alpha\left(x_{t+1}^h\left(d_t^h + n_t\right) - k_{t+1}^{liq}\right)^\alpha + \alpha\left(x_{t+1}^h\left(d_t^h + n_t\right)\right)^\alpha - \alpha\left(x_{t+1}^h\left(d_t^h + n_t\right) - k_{t+1}^{liq}\right)^\alpha - R_t^{d,h}d_t^h$$

$$V_{t+1}^h = \alpha\left(x_{t+1}^h\left(d_t^h + n_t\right)\right)^\alpha - R_t^{d,h}d_t^h$$

Thus, the aggregate value of banks, and their return on investment, is independent of the degree of deleveraging. Thus, by symmetry across banks, the value of each individual bank $V_{t+1}^h(j)$ is also independent of the degree of deleveraging. Thus so too are the returns on bank's investments, R_{t+1}^h .

□

D Law of motion for capital used in production of final goods

Proposition 4

Proof of Proposition 4. From (65) the total amount of capital generated by the banks capital investments is given by

$$x_{t+1}^h s_t^h = x_{t+1}^h \frac{\pi + \beta \lambda^h(n_t)}{1 + \beta \lambda^h(n_t)} w_t = x_{t+1}^h \frac{\pi + \beta \lambda^h(n_t)}{1 + \beta \lambda^h(n_t)} (1 - \alpha) k_t^\alpha$$

If $(x_{t+1}^h)^\alpha \geq \mathbb{E}_t((x_{t+1}^h)^\alpha)$ then $R_{t+1}^h \geq \mathbb{E}_t(R_{t+1}^h)$ and the pledgeability constraint holds for each bank *ex-post*, and so there is no deleveraging and all the generated capital is used in the production of final output goods, giving $k_{t+1} = x_{t+1}^h (s_t^h)$.

Suppose instead, the capital producing technology is less productive than expected: $(x_{t+1}^h)^\alpha < \mathbb{E}_t((x_{t+1}^h)^\alpha)$. Then the bank needs to reduce its leverage until the point at which the ex-post constraint holds. From (24), the required condition is

$$\lambda^h(n_t) \alpha \left(x_{t+1}^h s_t^h - k_{t+1}^{liq} \right)^\alpha + \mathbb{L} \left(k_{t+1}^{liq}, x_{t+1}^h \left(d_t^h + n_t \right) \right) = R_t^{d,h} d_t^h$$

Now, given the *ex-ante* borrowing constraint holds, we have, on the basis of expected returns

$$\lambda^h(n_t) \alpha \mathbb{E}_t \left(\left(x_{t+1}^h \right)^\alpha \right) \left(s_t^h \right)^\alpha = R_t^{d,h} d_t^h$$

Thus, substituting this in, alongside the functional form for $\mathbb{L}(\cdot, \cdot)$ from Proposition 11 gives

$$\begin{aligned} \lambda^h(n_t) \alpha \left(x_{t+1}^h s_t^h - k_{t+1}^{liq} \right)^\alpha + \alpha \left(x_{t+1}^h s_t^h \right)^\alpha - \alpha \left(x_{t+1}^h s_t^h - k_{t+1}^{liq} \right)^\alpha \\ = \lambda^h(n_t) \alpha \mathbb{E}_t \left(\left(x_{t+1}^h \right)^\alpha \right) \left(s_t^h \right)^\alpha \end{aligned}$$

Gathering terms gives

$$\left(x_{t+1}^h s_t^h - k_{t+1}^{liq} \right)^\alpha \left(1 - \lambda^h(n_t) \right) = \left(x_{t+1}^h s_t^h \right)^\alpha - \lambda^h(n_t) \mathbb{E}_t \left(\left(x_{t+1}^h \right)^\alpha \right) \left(s_t^h \right)^\alpha$$

Tidying then gives

$$\left(x_{t+1}^h s_t^h - k_{t+1}^{liq} \right)^\alpha = \frac{\left(x_{t+1}^h s_t^h \right)^\alpha - \lambda^h(n_t) \mathbb{E}_t \left(\left(x_{t+1}^h \right)^\alpha \right) \left(s_t^h \right)^\alpha}{\left(1 - \lambda^h(n_t) \right)}$$

Now, the capital used in the production of final output goods, $k_{t+1} := x_{t+1}^h s_t^h - k_{t+1}^{liq}$. Using (65) gives the aggregate capital used in the production of final output goods when $(x_{t+1}^h)^\alpha < \mathbb{E}_t((x_{t+1}^h)^\alpha)$ as

$$k_{t+1} = \left(\frac{\left(x_{t+1}^h \right)^\alpha - \lambda^h(n_t) \mathbb{E}_t \left(\left(x_{t+1}^h \right)^\alpha \right)}{\left(1 - \lambda^h(n_t) \right)} \right)^{\frac{1}{\alpha}} \frac{\pi + \beta \lambda^h(n_t)}{1 + \beta \lambda^h(n_t)} (1 - \alpha) k_t^\alpha$$

□

E Choice of investment between sectors

Proposition 5

Proof of Proposition 5. As the conditions for Proposition 3 hold for both sectors, there would be a positive spread in each sector if it were invested in alone. Moreover, (31), (32), (34), and (67) all hold.

We first calculate the terminal value of bank j when sector h is invested in. It is assumed sector h offers the highest return to deposits, so the bank raises deposits when investing in sector h . Its expected terminal value is given by (recalling that it's unaffected by deleveraging)

$$V_{t+1}^h(j) = R_{t+1}^h \left(d_t^h(j) + n_t(j) \right) - R_t^{d,h} d_t^h(j)$$

From, (32) and (67), and using $w_t = (1 - \alpha) k_t^\alpha$, we have

$$R_{t+1}^h \left(d_t^h(j) + n_t(j) \right) = \frac{\alpha (x_{t+1}^h)^\alpha \frac{\pi + \beta \lambda^h(n_t)}{1 + \beta \lambda^h(n_t)} (1 - \alpha) k_t^\alpha}{\left(\frac{\pi + \beta \lambda^h(n_t)}{1 + \beta \lambda^h(n_t)} (1 - \alpha) k_t^\alpha \right)^{1-\alpha}} = \alpha \left(x_{t+1}^h \right)^\alpha \left(\frac{\pi + \beta \lambda^h(n_t)}{1 + \beta \lambda^h(n_t)} (1 - \alpha) k_t^\alpha \right)^\alpha$$

Further, from (31) and (34), and using $w_t = (1 - \alpha) k_t^\alpha$, we have

$$R_t^{d,h} d_t^h(j) = \frac{\alpha (1 + \beta \lambda^h(n_t))^{1-\alpha} (\pi + \beta \lambda^h(n_t))^\alpha \mathbb{E}_t \left((x_{t+1}^h)^\alpha \right)}{\left((1 - \alpha) k_t^\alpha \right)^{1-\alpha} 1 + \beta \lambda^h(n_t)} \lambda^h(n_t) (1 - \alpha) k_t^\alpha$$

$$R_t^{d,h} d_t^h(j) = \alpha \mathbb{E}_t \left(\left((x_{t+1}^h)^\alpha \right) \lambda^h(n_t) \left(\frac{\pi + \beta \lambda^h(n_t)}{1 + \beta \lambda^h(n_t)} (1 - \alpha) k_t^\alpha \right)^\alpha \right) \quad (68)$$

Thus, we have that

$$V_{t+1}^h(j) = \alpha \left(\frac{\pi + \beta \lambda^h(n_t)}{1 + \beta \lambda^h(n_t)} (1 - \alpha) k_t^\alpha \right)^\alpha \left[\left(x_{t+1}^h \right)^\alpha - \mathbb{E}_t \left(\left(x_{t+1}^h \right)^\alpha \right) \lambda^h(n_t) \right] \quad (69)$$

And hence, the expected terminal value of bank j is given by

$$\mathbb{E}_t V_{t+1}^h(j) = \alpha \mathbb{E}_t \left(\left(x_{t+1}^h \right)^\alpha \right) \left[1 - \lambda^h(n_t) \right] \left(\frac{\pi + \beta \lambda^h(n_t)}{1 + \beta \lambda^h(n_t)} (1 - \alpha) k_t^\alpha \right)^\alpha \quad (70)$$

Now consider the expected terminal value of bank j if it switched to investment in sector $g \neq h$, taking no deposits (as it's not the sector that offers the highest deposit rate to households), with every other bank investing in sector h . The terminal value of this bank is given by

$$V_{t+1}^g(j) = R_{t+1}^g n_t(j)$$

Now, $n_t(j) = \pi (1 - \alpha) k_t^\alpha$, and the total capital goods bank j will have at the start of $t + 1$ will be $x_{t+1}^g \pi (1 - \alpha) k_t^\alpha$. Note that the bank faces no pressure to de-lever as they take no deposits, so the capital generated will be used to produce output goods. When the aggregate capital used in production is given by k_{t+1} the marginal product of capital is given by

$$\alpha k_{t+1}^{\alpha-1}$$

thus the expected total amount of output goods returned to bank j is equal to

$$V_{t+1}^g(j) = \mathbb{E}_t \left(\frac{\alpha x_{t+1}^g \pi (1 - \alpha) k_t^\alpha}{k_{t+1}^{1-\alpha}} \right)$$

Crucially, as the deviating bank is infinitesimal, the capital produced next period is unaltered: it is the level of investment in capital from banks investing in sector h that determines returns next period.

We consider two cases:

(i) $R_t^{d,a} \geq R_t^{d,b}$. Then it is an equilibrium for bank j not to deviate to sector b iff $\mathbb{E}_t V_{t+1}^a(j) \geq \mathbb{E}_t V_{t+1}^b(j)$ iff (noting the return on sector b is non-stochastic)

$$\alpha \mathbb{E}_t \left((x_{t+1}^a)^\alpha \right) [1 - \lambda^a(n_t)] \left(\frac{\pi + \beta \lambda^a(n_t)}{1 + \beta \lambda^a(n_t)} (1 - \alpha) k_t^\alpha \right)^\alpha \geq \alpha x^b \pi (1 - \alpha) k_t^\alpha \mathbb{E}_t (k_{t+1}^{\alpha-1})$$

Now from (36) and (37)

$$\mathbb{E}_t (k_{t+1}^{\alpha-1}) = \left(\frac{\pi + \beta \lambda^h(n_t)}{1 + \beta \lambda^h(n_t)} \right)^{\alpha-1} ((1 - \alpha) k_t^\alpha)^{\alpha-1} F$$

with F given as in the proposition. This expression is complicated, reflecting that deleveraging by the *other* banks in the face of a negative shock will affect the return on capital for the deviating bank. They are not liquidating any capital goods themselves, as they have invested in sector b and thus didn't face any shocks. Indeed, in the face of a negative shock to sector a returns, the deviating bank would not wish to liquidate: as the return on capital increases with less capital, their return *increases* when there is a negative shock to the other sector.

Now $\mathbb{E}_t V_{t+1}^a(j) \geq \mathbb{E}_t V_{t+1}^b(j)$ iff

$$\mathbb{E}_t \left((x_{t+1}^a)^\alpha \right) [1 - \lambda^a(n_t)] \frac{\pi + \beta \lambda^a(n_t)}{1 + \beta \lambda^a(n_t)} \geq x^b \pi F \quad (71)$$

Now,

$$\begin{aligned} [1 - \lambda^a(n_t)] \frac{\pi + \beta \lambda^a(n_t)}{1 + \beta \lambda^a(n_t)} &\geq \pi \text{ iff} \\ [1 - \lambda^a(n_t)] (\pi + \beta \lambda^a(n_t)) &\geq (1 + \beta \lambda^a(n_t)) \pi \text{ iff} \\ \pi - \pi \lambda^a(n_t) + \beta \lambda^a(n_t) - \beta (\lambda^a(n_t))^2 &\geq \pi + \pi \beta \lambda^a(n_t) \text{ iff} \\ -\pi + \beta - \beta \lambda^a(n_t) &\geq \pi \beta \text{ iff} \\ \beta (1 - \pi) &\geq \pi + \beta \lambda^a(n_t) \end{aligned}$$

which is a condition of the proposition.

Thus

$$\mathbb{E}_t \left((x_{t+1}^a)^\alpha \right) [1 - \lambda^a(n_t)] \frac{\pi + \beta \lambda^a(n_t)}{1 + \beta \lambda^a(n_t)} \geq \mathbb{E}_t \left((x_{t+1}^a)^\alpha \right) \pi \geq x^b \pi F$$

given that $\mathbb{E}_t \left((x_{t+1}^a)^\alpha \right) \geq x^b F$ is a condition of the proposition. Hence, it's optimal for banks not to deviate to sector b .

(ii) $R_t^{d,a} < R_t^{d,b}$. Then it is an equilibrium for bank j not to deviate to sector a iff $\mathbb{E}_t V_{t+1}^b(j) \geq$



$\mathbb{E}_t V_{t+1}^a(j)$ iff

$$\alpha (x^b)^\alpha [1 - \lambda^b] \left(\frac{\pi + \beta \lambda^b}{1 + \beta \lambda^b} (1 - \alpha) k_t^\alpha \right)^\alpha \geq \mathbb{E}_t \left(\frac{\alpha x_{t+1}^a \pi (1 - \alpha) k_t^\alpha}{k_{t+1}^{1-\alpha}} \right)$$

noting that investment in sector b is non-stochastic. Given this, the total capital used in investment by the banks investing in sector b is given by $k_{t+1} = x_{t+1}^b \left(\frac{\pi + \beta \lambda^b}{1 + \beta \lambda^b} \right) (1 - \alpha) k_t^\alpha$, with certainty: as the returns are non-stochastic, there will be no need for deleveraging. Thus, the condition for non-deviation can be rewritten as

$$\begin{aligned} (x^b)^\alpha [1 - \lambda^b] \left(\frac{\pi + \beta \lambda^b}{1 + \beta \lambda^b} (1 - \alpha) k_t^\alpha \right)^\alpha &\geq \left(\frac{\mathbb{E}_t (x_{t+1}^a) \pi (1 - \alpha) k_t^\alpha}{(x_{t+1}^b \left(\frac{\pi + \beta \lambda^b}{1 + \beta \lambda^b} \right) (1 - \alpha) k_t^\alpha)^{1-\alpha}} \right) \\ [1 - \lambda^b] \left(x^b \frac{\pi + \beta \lambda^b}{1 + \beta \lambda^b} \right) &\geq \mathbb{E}_t (x_{t+1}^a) \pi \end{aligned}$$

which holds given the condition in the proposition. Thus, it is an equilibrium for bank j not to deviate to sector a .

The proof of (42) then follows immediately from (34). □

F Existence of credit trap

Conditions for existence of investment threshold \tilde{n}

Proposition 12. *Suppose the conditions of Proposition 5 hold. Moreover, suppose that*

$$\frac{d\lambda^a(n_t)}{dn_t} > 0 \text{ for } n_t \geq 0$$

$$\mathbb{E}_t ((x_{t+1}^a)^\alpha) > (x^b)^\alpha$$

$$\lambda^a(0) = \underline{\lambda}^a \in [0, \lambda^b)$$

$$\lim_{n_t \rightarrow \infty} \lambda^a(n_t) = \bar{\lambda}^a \in (\underline{\lambda}^a, 1)$$

$$(1 + \underline{\lambda}^a \beta)^{1-\alpha} (\pi + \underline{\lambda}^a \beta)^\alpha \mathbb{E}_t ((x_{t+1}^a)^\alpha) < (1 + \lambda^b \beta)^{1-\alpha} (\pi + \lambda^b \beta)^\alpha (x^b)^\alpha$$

$$(1 + \bar{\lambda}^a \beta)^{1-\alpha} (\pi + \bar{\lambda}^a \beta)^\alpha \mathbb{E}_t ((x_{t+1}^a)^\alpha) > (1 + \lambda^b \beta)^{1-\alpha} (\pi + \lambda^b \beta)^\alpha (x^b)^\alpha$$

Then there exists level of banker net worth \tilde{n} such that banks invest in a iff $n_t > \tilde{n}$, where \tilde{n} is



time-invariant so long as $\mathbb{E}_t((x_{t+1}^a)^\alpha)$ is constant over time. This is defined implicitly by

$$(1 + \lambda^a(\tilde{n})\beta)^{1-\alpha} (\pi + \lambda^a(\tilde{n})\beta)^\alpha \mathbb{E}_t((x_{t+1}^a)^\alpha) = (1 + \lambda^b\beta)^{1-\alpha} (\pi + \lambda^b\beta)^\alpha (x^b)^\alpha$$

Proof. Under the conditions of the proposition, Proposition (5) holds and banks invest in sector a iff $R_t^{d,a} \geq R_t^{d,b}$. Let

$$g(n_t) := (1 + \lambda^a(n_t)\beta)^{1-\alpha} (\pi + \lambda^a(n_t)\beta)^\alpha \mathbb{E}_t((x_{t+1}^a)^\alpha) - (1 + \lambda^b\beta)^{1-\alpha} (\pi + \lambda^b\beta)^\alpha (x^b)^\alpha$$

Then, by Proposition (5), banks invest in sector a iff $g(n_t) \geq 0$. By the above conditions, $g(0) < 0$ and $\lim_{n_t \rightarrow \infty} g(n_t) > 0$. Thus, for sufficiently large n_t , $g(n_t) > 0$. As $\lambda^a(\cdot)$ is differentiable on $[0, \infty)$ it is continuous on the same interval and hence so too is $g(\cdot)$. Thus, by the Intermediate Value Theorem, $\exists \tilde{n} : g(\tilde{n}) = 0$. Further, as $\frac{d\lambda^a(n_t)}{dn_t} > 0$ for all $n_t \geq 0$, $\frac{dg(n_t)}{dn_t} > 0$ for all $n_t \geq 0$. Hence, \tilde{n} is unique, and $g(n_t) \geq 0$ iff $n_t > \tilde{n}$. □

Proposition 6

Proof of Proposition 6. Given the conditions for Proposition 5 hold, banks invest in sector b iff

$$(1 + \lambda^a(n_t)\beta)^{1-\alpha} (\pi + \lambda^a(n_t)\beta)^\alpha \mathbb{E}_t((x_{t+1}^a)^\alpha) < (1 + \lambda^b\beta)^{1-\alpha} (\pi + \lambda^b\beta)^\alpha (x^b)^\alpha$$

Let n^b be the steady state bank net worth when sector b is exclusively invested in, as given by (43). Then given that under the condition of the proposition

$$(1 + \lambda^a(n^b)\beta)^{1-\alpha} (\pi + \lambda^a(n^b)\beta)^\alpha \mathbb{E}_t((x_{t+1}^a)^\alpha) < (1 + \lambda^b\beta)^{1-\alpha} (\pi + \lambda^b\beta)^\alpha (x^b)^\alpha$$

at n^b , banks invest in sector b . Thus, given there are no shocks to investment returns in sector b , there is a steady state in which banks invest perpetually in sector b . □

G Resilience to credit trap

Derivation of productivity threshold for trap

From Section 3, when the economy features a credit trap, it will fall into the trap whenever bank equity falls below a critical value \tilde{n} . We assume that the economy is originally investing in sector a and so is outside the credit trap. The economy will fall into a credit trap if there is a

sufficiently large negative shock to the realised bank returns and consequent liquidation, leading to bank net worth falling below \tilde{n} . In this case, using $n_{t+1} = (1 - \alpha) \pi k_{t+1}^\alpha$ and (36) applied to $h = a$, we have the threshold productivity realisation for entering a trap as a function of λ^a , $\tilde{x}_{t+1}^a(\lambda^a)$, given implicitly by

$$\tilde{n} = (1 - \alpha) \pi \left(\frac{(\tilde{x}_{t+1}^a(\lambda^a))^\alpha - \lambda^a(n_t) \mathbb{E}_t((x_{t+1}^a)^\alpha)}{1 - \lambda^a(n_t)} \right) \left(\frac{\pi + \beta \lambda^a(n_t)}{1 + \beta \lambda^a(n_t)} (1 - \alpha) k_t^\alpha \right)^\alpha$$

Rearranging for $\tilde{x}_{t+1}^a(\lambda^a)$ gives (45) in the text.

Productivity trap threshold U-shaped in leverage

The following proposition demonstrates the shape of Figure 5 in the text.

Proposition 13. *Suppose the conditions of Proposition 5 hold. Further suppose that*

$$(1 - \alpha) \pi \mathbb{E}_t((x_{t+1}^a)^\alpha) \left\{ \frac{\pi + \beta}{1 + \beta} (1 - \alpha) k_t^\alpha \right\}^\alpha > \tilde{n}$$

and

$$\mathbb{E}_t((x_{t+1}^a)^\alpha) < \frac{\tilde{n}}{(1 - \alpha) \pi ((1 - \alpha) k_t^\alpha)^\alpha} \left(\frac{\alpha \beta (1 - \pi) + \pi}{\pi^{1+\alpha}} \right)$$

Then

$$\exists \lambda^{min} \in (0, 1) : \frac{d\tilde{x}_{t+1}^a(\lambda^a)}{d\lambda^a} \left\{ \begin{array}{l} < 0 \text{ for } \lambda^a \in (0, \lambda^{min}) \\ = 0 \text{ for } \lambda^a = \lambda^{min} \\ > 0 \text{ for } \lambda^a \in (\lambda^{min}, 1) \end{array} \right\}$$

Further, λ^{min} is unique and $\tilde{x}_{t+1}^a(\lambda^a)$ reaches a unique minimum at $\lambda^a = \lambda^{min}$.

The first condition states that when there are no shocks ($\mathbb{E}_t((x_{t+1}^a)^\alpha) = (x_{t+1}^a)^\alpha$) and $\lambda^a = 1$ the economy avoids the credit trap, i.e. if there are no shocks and leverage is high enough, it's always possible to avoid the credit trap. The second condition ensures that when there is no borrowing ($\lambda^a = 0$), increasing leverage increases the resilience of the economy.

Proof of Proposition 13. We first introduce some notation to simplify the exposition of the proof. Let

$$z(\lambda) := \lambda \mathbb{E}_t((x_{t+1}^a)^\alpha) + \frac{\tilde{n}(1 - \lambda)}{(1 - \alpha) \pi \left[\frac{\pi + \lambda \beta}{1 + \lambda \beta} (1 - \alpha) k_t^\alpha \right]^\alpha}$$

Then $\tilde{x}_{t+1}^a(\lambda) \equiv (z(\lambda))^{\frac{1}{\alpha}}$. Now $\frac{d\tilde{x}_{t+1}^a(\lambda)}{d\lambda} = \frac{1}{\alpha} (z(\lambda))^{\frac{1}{\alpha}-1} z'(\lambda) > 0$ iff $z'(\lambda) > 0$. Further, $\frac{d^2\tilde{x}_{t+1}^a(\lambda)}{d\lambda^2} = \frac{1}{\alpha} \left(\frac{1}{\alpha} - 1 \right) (z(\lambda))^{\frac{1}{\alpha}-2} (z'(\lambda))^2 + \frac{1}{\alpha} (z(\lambda))^{\frac{1}{\alpha}-1} z''(\lambda)$. Hence, if $z''(\lambda) > 0$ then $\frac{d^2\tilde{x}_{t+1}^a(\lambda)}{d\lambda^2} > 0$. Given these results, in the following steps of the proof we can work with $z(\lambda)$. We introduce further notation: let $h(\lambda) := \frac{(1-\lambda)}{\left[\frac{\pi+\lambda\beta}{1+\lambda\beta} \right]^\alpha}$. Then $z(\lambda) = \lambda \mathbb{E}_t((x_{t+1}^a)^\alpha) + \frac{\tilde{n}h(\lambda)}{(1-\alpha)\pi \left[\frac{\pi+\lambda\beta}{1+\lambda\beta} \right]^\alpha}$. The proof now proceeds via a series of steps.

(i) $\frac{d\tilde{x}_{t+1}^a(\lambda)}{d\lambda} > 0$ for λ close to 1. We show $z'(\lambda) > 0$ for λ close to 1. We have $z'(\lambda) = \mathbb{E}_t((x_{t+1}^a)^\alpha) + \frac{\tilde{n}h'(\lambda)}{(1-\alpha)\pi \left[\frac{\pi+\lambda\beta}{1+\lambda\beta} \right]^\alpha}$.



Now

$$h'(\lambda) = -(1 + \lambda\beta)^\alpha (\pi + \lambda\beta)^{-\alpha} - \alpha\beta(1 - \pi)(1 - \lambda)(1 + \lambda\beta)^{\alpha-1} (\pi + \lambda\beta)^{-\alpha-1}$$

Thus $\lim_{\lambda \rightarrow 1} z'(\lambda) = \mathbb{E}_t \left((x_{t+1}^a)^\alpha \right) - \frac{\tilde{n}}{(1-\alpha)\pi[(1-\alpha)k_t^\alpha]^\alpha} \left(\frac{1+\beta}{\pi+\beta} \right)^\alpha$. This is positive so long as $\mathbb{E}_t \left((x_{t+1}^a)^\alpha \right) (1 - \alpha)\pi [(1 - \alpha)k_t^\alpha]^\alpha \left(\frac{\pi+\beta}{1+\beta} \right)^\alpha > \tilde{n}$. Thus, given the first condition of the proposition, $\lim_{\lambda \rightarrow 1} z'(\lambda) > 0$.

However, $z'(\lambda)$ is continuous so $\exists \lambda^* < 1 : z'(\lambda) > 0 \forall \lambda \in [\lambda^*, 1)$. Thus $\frac{d\tilde{x}_{t+1}^a(\lambda)}{d\lambda} > 0 \forall \lambda \in [\lambda^*, 1)$.

(ii) $\frac{d^2\tilde{x}_{t+1}^a(\lambda)}{d\lambda^2} > 0 \forall \lambda \in [0, 1]$. It is sufficient to show that $z''(\lambda) > 0 \forall \lambda \in [0, 1]$. Now $z''(\lambda) = \frac{\tilde{n}h''(\lambda)}{(1-\alpha)\pi[(1-\alpha)k_t^\alpha]^\alpha}$. Using the expression for $h'(\lambda)$ from step (i) it can be shown that

$$\begin{aligned} \frac{h''(\lambda)}{\alpha\beta} &= \left(\frac{1 + \lambda\beta}{\pi + \lambda\beta} \right)^\alpha \left[\frac{-1}{1 + \lambda\beta} + \frac{1}{\pi + \lambda\beta} \right] \\ &+ (1 - \pi) \left(\frac{1 + \lambda\beta}{\pi + \lambda\beta} \right) \left[\frac{1}{(1 + \lambda\beta)(\pi + \lambda\beta)} + \frac{(1 - \lambda)\beta(1 - \alpha)}{(1 + \lambda\beta)^2(\pi + \lambda\beta)} + \frac{(1 - \lambda)\beta(1 + \alpha)}{(1 + \lambda\beta)(\pi + \lambda\beta)^2} \right] \end{aligned}$$

This expression is positive-for the first term note that $1 > \pi$. Hence $z''(\lambda) > 0 \forall \lambda \in [0, 1]$.

(iii) We now use steps (i), (ii) to prove the proposition. We have that $\frac{d\tilde{x}_{t+1}^a(0)}{d\lambda} < 0$ iff $z'(0) < 0$.

With

$$h'(0) = -\pi^{-\alpha} - \alpha\beta(1 - \pi)\pi^{-\alpha-1}$$

the condition becomes

$$\mathbb{E}_t \left((x_{t+1}^a)^\alpha \right) < \frac{\tilde{n}\pi^{-\alpha-1} (\pi + \alpha\beta(1 - \pi))}{(1 - \alpha)\pi [(1 - \alpha)k_t^\alpha]^\alpha}$$

Thus by the second condition of the proposition we have $\frac{d\tilde{x}_{t+1}^a(0)}{d\lambda} < 0$.

Now, from step (i) $\exists \lambda^* < 1 : \frac{d\tilde{x}_{t+1}^a(\lambda^*)}{d\lambda} > 0$. We must have $\lambda^* > 0$, for otherwise, we'd have $\frac{d\tilde{x}_{t+1}^a(0)}{d\lambda} > 0$, a contradiction. As $\frac{d\tilde{x}_{t+1}^a(\lambda)}{d\lambda}$ is continuous, by the Intermediate Value Theorem, $\exists \lambda^{min} \in (0, 1) : \frac{d\tilde{x}_{t+1}^a(\lambda^{min})}{d\lambda} = 0$. Further, as $\frac{d^2\tilde{x}_{t+1}^a(\lambda)}{d\lambda^2} > 0$, λ^{min} is unique. The following then holds

$$\frac{d\tilde{x}_{t+1}^a(\lambda)}{d\lambda} \begin{cases} < 0 \text{ for } \lambda \in (0, \lambda^{min}) \\ = 0 \text{ for } \lambda = \lambda^{min} \\ > 0 \text{ for } \lambda \in (\lambda^{min}, 1) \end{cases}$$

And so $\frac{d\tilde{x}_{t+1}^a(\lambda)}{d\lambda}$ reaches a unique minimum at $\lambda = \lambda^{min}$. This completes the proof of the proposition. □



Resilience maximising leverage ratio is countercyclical

Proposition 14. *Suppose the conditions of Proposition 13 hold. Then, with λ^{min} defined as in Proposition 13, we have that λ^{min} is countercyclical:*

$$\frac{d\lambda^{min}}{dk_t} < 0$$

Proof of Proposition 14. Using the notation from the proof of Proposition 13 we have $\frac{d\bar{x}_{t+1}^a(\lambda)}{d\lambda} = 0$ iff $z'(\lambda) = 0$ iff

$$\begin{aligned} \mathbb{E}_t \left((x_{t+1}^a)^\alpha \right) &= \frac{-\tilde{n}h'(\lambda)}{(1-\alpha)\pi [(1-\alpha)k_t^\alpha]^\alpha} \\ \frac{\mathbb{E}_t \left((x_{t+1}^a)^\alpha \right) (1-\alpha)\pi [(1-\alpha)k_t^\alpha]^\alpha}{\tilde{n}} &= \left(\frac{1 + \lambda^{min}\beta}{\pi + \lambda^{min}\beta} \right)^\alpha \left[1 + \frac{\alpha\beta(1-\pi)(1-\lambda^{min})}{(1 + \lambda^{min}\beta)(\pi + \lambda^{min}\beta)} \right] \end{aligned}$$

This last equation implicitly defines λ^{min} . The RHS is decreasing in λ^{min} . Increasing k_t increases the LHS, so λ^{min} must decrease to maintain equality between the two sides of the equation. Thus $\frac{d\lambda^{min}}{dk_t} < 0$, completing the proof of the proposition. □

H Welfare functions

Proposition 7

Proof of Proposition 7. Under the conditions of the proposition, the deposits by the young are given by (31), thus the consumption of the young household i is given by

$$c_t^y(i) = (1-\pi)w_t - d_{i,t} = (1-\pi)w_t \left(1 - \frac{\beta\lambda^h(n_t)}{1 + \beta\lambda^h(n_t)} \right)$$

Thus, using $w_t = (1-\alpha)k_t^\alpha$, and noting that deposits are identical across households, we have

$$c_t^y = (1-\alpha)k_t^\alpha \frac{(1-\pi)}{1 + \beta\lambda^h(n_t)}$$

Turning the the consumption of the old, for household i we have

$$c_{t+1}^o(i) = R_t^{d,h} d_t^h(i) + V_{t+1}^h(i)$$

Now, under the conditions of the proposition, $R_t^{d,h} d_t^h(i)$ is given by (68) whilst $V_{t+1}^h(j)$ is given by (69), thus

$$c_{t+1}^o(i) = \alpha \left(\frac{\pi + \beta\lambda^h(n_t)}{1 + \beta\lambda^h(n_t)} (1-\alpha)k_t^\alpha \right)^\alpha \left[(x_{t+1}^h)^\alpha - \mathbb{E}_t \left((x_{t+1}^h)^\alpha \right) \lambda^h(n_t) + \mathbb{E}_t \left((x_{t+1}^h)^\alpha \right) \lambda^h(n_t) \right]$$



Noting that this is identical across households, and taking time t expectations, we have

$$\mathbb{E}_t (c_{t+1}^o) = \alpha \left(\frac{\pi + \beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} (1 - \alpha) k_t^\alpha \right)^\alpha \mathbb{E}_t \left((x_{t+1}^h)^\alpha \right)$$

Thus lifetime utility for the generation born at t , $U_t = \log (c_t^o) + \beta \log (\mathbb{E}_t (c_{t+1}^o))$, is given by

$$U_t = \log ((1 - \alpha) (1 - \pi)) + \alpha \log (k_t) - \log (1 + \beta \lambda^h (n_t)) \\ + \beta \left\{ \log \left(\alpha \mathbb{E}_t \left((x_{t+1}^h)^\alpha \right) (1 - \alpha)^\alpha \right) + \alpha \log \left(\frac{\pi + \beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} \right) + \alpha^2 \log (k_t) \right\}$$

Thus,

$$U_t = (\alpha + \alpha^2 \beta) \log (k_t) + \alpha \beta \log (\pi + \beta \lambda^h (n_t)) - (1 + \alpha \beta) \log (1 + \beta \lambda^h (n_t)) + C_t$$

with C_t independent of policy and given by

$$C_t := \log ((1 - \alpha) (1 - \pi)) + \beta \left\{ \log \left(\alpha \mathbb{E}_t \left((x_{t+1}^h)^\alpha \right) (1 - \alpha)^\alpha \right) \right\}$$

□

Proposition 8

Proof of Proposition 8. As discussed in the text, the planner's objective at time t is to maximise the discounted expected lifetime utility of each generation (excluding the old at time t , whose utility the planner can't affect with the tools they have at their disposal): $\sum_{s=0}^{\infty} \beta^s \mathbb{E}_t U_{t+s}$. Using the results of Proposition 7 this is given by

$$\sum_{s=0}^{\infty} \beta^s \mathbb{E}_t \left\{ (\alpha + \alpha^2 \beta) \log (k_{t+s}) + \alpha \beta \log (\pi + \beta \lambda^h (n_{t+s})) - (1 + \alpha \beta) \log (1 + \beta \lambda^h (n_{t+s})) \right\} \\ + \sum_{s=0}^{\infty} \beta^s \mathbb{E}_t \left(\log ((1 - \alpha) (1 - \pi)) + \beta \left\{ \log \left(\alpha \mathbb{E}_{t+s} \left((x_{t+s+1}^h)^\alpha \right) (1 - \alpha)^\alpha \right) \right\} \right)$$

Using $n_{t+s} = \pi (1 - \alpha) k_{t+s}^\alpha$ and defining the second term as \bar{C} gives the result.

□

I Welfare solution without credit traps

In this section of the Appendix we prove Proposition 9. The proof is involved, so we proceed in several steps, first establishing sub-results.

I.1 Lemma

We first state & prove a useful lemma.

Lemma 15. *Suppose the conditions of Proposition 3 hold and the economy invests exclusively in sector h . Suppose further that the planner uses the haircut τ_s and*

$$(1 - \tau_{t+j}) \lambda^h(n_{t+j}) \text{ is independent of } k_{t+j} \forall j \geq 1$$

Then, for $j \geq 1$

$$\mathbb{E}_t \frac{\partial \log(k_{t+j})}{\partial \tau_t} = \alpha^{j-1} \mathbb{E}_t \frac{\partial \log(k_{t+1})}{\partial \tau_t} \quad (72)$$

Proof. As the conditions of Proposition 3 hold, we have, using (36) and (37), and $\int_{\underline{x}_{t+1}^h}^{\bar{x}_{t+1}^h} f(x_{t+1}^h) dx_{t+1}^h = 1$, that, when the planner uses the haircut tool

$$\mathbb{E}_t \log(k_{t+1}) = \alpha \log(k_t)$$

$$\begin{aligned} & + \log \left\{ \left(\frac{\pi + \beta(1 - \tau_t) \lambda^h(n_t)}{1 + \beta(1 - \tau_t) \lambda^h(n_t)} \right) (1 - \alpha) \right\} + \int_{(\mathbb{E}_t((x_{t+1}^h)^\alpha))^{\frac{1}{\alpha}}}^{\bar{x}_{t+1}^h} \log(x_{t+1}^h) f(x_{t+1}^h) dx_{t+1}^h \\ & + \int_{\underline{x}_{t+1}^h}^{(\mathbb{E}_t((x_{t+1}^h)^\alpha))^{\frac{1}{\alpha}}} \frac{1}{\alpha} \log \left(\frac{(x_{t+1}^h)^\alpha - (1 - \tau_t) \lambda^h(n_t) \mathbb{E}_t((x_{t+1}^h)^\alpha)}{1 - (1 - \tau_t) \lambda^h(n_t)} \right) f(x_{t+1}^h) dx_{t+1}^h \end{aligned}$$

Using this, we prove the lemma by induction on j .

Base Case: $j = 1$. Then LHS (72) = $\mathbb{E}_t \frac{\partial \log(k_{t+1})}{\partial \tau_t}$, whilst RHS (72) = $\alpha^{1-1} \mathbb{E}_t \frac{\partial \log(k_{t+1})}{\partial \tau_t} = \mathbb{E}_t \frac{\partial \log(k_{t+1})}{\partial \tau_t}$. Thus, the base case holds.

Inductive Step. Suppose (72) holds $\forall j = 1, \dots, m$. Thus, in particular, as it holds for $j = m$,

$$\mathbb{E}_t \frac{\partial \log(k_{t+m})}{\partial \tau_t} = \alpha^{m-1} \mathbb{E}_t \frac{\partial \log(k_{t+1})}{\partial \tau_t} \quad (73)$$

Consider $j = m + 1$. By using the above expression for $\mathbb{E}_{t+m} \log(k_{t+m+1})$, and taking time t expectations of it, we have

$$\begin{aligned} & \mathbb{E}_t \{ \mathbb{E}_{t+m} \log(k_{t+m+1}) \} = \alpha \mathbb{E}_t \log(k_{t+m}) \\ & + \mathbb{E}_t \log \left\{ \left(\frac{\pi + \beta(1 - \tau_{t+m}) \lambda^h(n_{t+m})}{1 + \beta(1 - \tau_{t+m}) \lambda^h(n_{t+m})} \right) (1 - \alpha) \right\} \\ & + \mathbb{E}_t \int_E^{\bar{x}_{t+m+1}^h} \log(x_{t+m+1}^h) f(x_{t+m+1}^h) dx_{t+m+1}^h \end{aligned}$$

$$+ \mathbb{E}_t \int_{\underline{x}_{t+m+1}^h}^E \frac{1}{\alpha} \log \left(\frac{(x_{t+m+1}^h)^\alpha - (1 - \tau_{t+m}) \lambda^h (n_{t+m}) \mathbb{E}_{t+m} ((x_{t+m+1}^h)^\alpha)}{1 - (1 - \tau_{t+m}) \lambda^h (n_{t+m})} \right) f(x_{t+m+1}^h) dx_{t+m+1}^h$$

where $E := (\mathbb{E}_{t+m} ((x_{t+m+1}^h)^\alpha))^{\frac{1}{\alpha}}$.

Now, by the condition of the lemma, $(1 - \tau_{t+j}) \lambda^h (n_{t+j})$ is independent of $k_{t+j} \forall j \geq 1$. Thus, in particular, it holds for $t + m$ and future periods, with $m \geq 1$. Thus, the only impact of τ_t on $\mathbb{E}_t \{\mathbb{E}_{t+m} \log(k_{t+m+1})\}$ will come through its direct impact on k_{t+m} , with no indirect impact on $(1 - \tau_{t+m}) \lambda^h (n_{t+m})$, and no impact on the exogenous realisations of capital productivity x_{t+m+1}^h . It then follows, using (73), that

$$\frac{\partial \mathbb{E}_t \{\mathbb{E}_{t+m} \log(k_{t+m+1})\}}{\partial \tau_t} = \alpha \frac{\partial \mathbb{E}_t \log(k_{t+m})}{\partial \tau_t} = \alpha \alpha^{m-1} \mathbb{E}_t \frac{\partial \log(k_{t+1})}{\partial \tau_t} = \alpha^m \mathbb{E}_t \frac{\partial \log(k_{t+1})}{\partial \tau_t}$$

By L.I.E., $\mathbb{E}_t \{\mathbb{E}_{t+m} \log(k_{t+m+1})\} = \mathbb{E}_t \log(k_{t+m+1})$, and hence $\mathbb{E}_t \frac{\partial \log(k_{t+m+1})}{\partial \tau_t} = \alpha^m \mathbb{E}_t \frac{\partial \log(k_{t+1})}{\partial \tau_t}$. Thus, the inductive step is complete, hence by the Principal of Mathematical Induction, the proof of the lemma is complete. □

I.2 Optimal policy with finite horizon

We first establish optimal policy when there is a finite horizon, before turning in the next subsection to the case of an infinite horizon.

Proposition 16. *Suppose the conditions of Proposition 3 hold, but that the planner has a finite horizon, with terminal period T . Then, in all periods up to and including T , optimally effective pledgeability $(1 - \tau_{T-s}) \lambda^h (n_{T-s})$ is independent of $k_{T-s} \forall s \geq 0$. Moreover, the optimal haircut policy in period $T - s$ satisfies*

$$\begin{aligned} & \frac{(1 + \alpha\beta) \left(1 - (\alpha\beta)^{s+1}\right)}{1 + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta} - \frac{\alpha\beta (2 - (1 + \alpha\beta) (\alpha\beta)^s)}{\pi + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta} \tag{74} \\ & + [1 - (\alpha\beta)^s] \int_{\underline{x}_{T-s+1}^h}^{(\mathbb{E}_{T-s}((x_{T-s+1}^h)^\alpha))^{\frac{1}{\alpha}}} \frac{(1 + \alpha\beta) \mathbb{E}_{T-s} ((x_{T-s+1}^h)^\alpha) f(x_{T-s+1}^h)}{(x_{T-s+1}^h)^\alpha - (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \mathbb{E}_{T-s} ((x_{T-s+1}^h)^\alpha)} dx_{T-s+1}^h \\ & - [1 - (\alpha\beta)^s] \int_{\underline{x}_{T-s+1}^h}^{(\mathbb{E}_{T-s}((x_{T-s+1}^h)^\alpha))^{\frac{1}{\alpha}}} \frac{(1 + \alpha\beta) f(x_{T-s+1}^h)}{1 - (1 - \tau_{T-s}) \lambda^h (n_{T-s})} dx_{T-s+1}^h = 0 \end{aligned}$$

Proof. We first prove that $(1 - \tau_{T-s}) \lambda^h (n_{T-s})$ is independent $k_{T-s} \forall s \geq 0$ by induction on s , the number of periods prior to the end of the planner's horizon at time T .

Base Case: $s = 0$.



The end of the planner's horizon is not the end of the world, but the last generation whose welfare they take into account when setting policy. In the terminal period the planner's objective is to choose τ_T to maximise U_T as given by (49). Noting that at time T , k_T is given, and C_T is independent of policy, their objective is

$$\alpha\beta \log\left(\pi + (1 - \tau_T)\lambda^h(n_T)\beta\right) - (1 + \alpha\beta) \log\left(1 + (1 - \tau_T)\lambda^h(n_T)\beta\right)$$

The FOC w.r.t. τ_T is given by

$$\frac{-\alpha\beta\lambda^h(n_T)\beta}{\pi + (1 - \tau_T)\lambda^h(n_T)\beta} + \frac{(1 + \alpha\beta)\lambda^h(n_T)\beta}{1 + (1 - \tau_T)\lambda^h(n_T)\beta} = 0 \quad (75)$$

Note that this is of the same form as (74). Setting $s = 0$ in (74) gives (with the integral term disappearing)

$$\frac{(1 + \alpha\beta)(1 - (\alpha\beta))}{1 + (1 - \tau_T)\lambda^h(n_T)\beta} - \frac{\alpha\beta(2 - (1 + \alpha\beta))}{\pi + (1 - \tau_T)\lambda^h(n_T)\beta} = 0$$

noting that $2 - (1 + \alpha\beta) = 1 - (\alpha\beta)$, and cancelling gives the same condition as (75), when the common factor of $\lambda^h(n_T)\beta$ has been cancelled.

Returning to the FOC, rearranging and cancelling the common $\lambda^h(n_T)\beta$ term gives

$$(1 + \alpha\beta)\left(\pi + (1 - \tau_T)\lambda^h(n_T)\beta\right) = \alpha\beta\left(1 + (1 - \tau_T)\lambda^h(n_T)\beta\right) \quad (76)$$

Which can be rearranged to give

$$(1 - \tau_T)\lambda^h(n_T) = \frac{\alpha\beta - (1 + \alpha\beta)\pi}{\beta} \quad (77)$$

This is a valid maximum, as the second derivative is negative at the FOC. To see this, note that the second derivative of the objective w.r.t. τ_T is given by

$$\frac{-\alpha\beta(\lambda^h(n_T)\beta)^2}{(\pi + (1 - \tau_T)\lambda^h(n_T)\beta)^2} + \frac{(1 + \alpha\beta)(\lambda^h(n_T)\beta)^2}{(1 + (1 - \tau_T)\lambda^h(n_T)\beta)^2}$$

The second derivative is then negative whenever

$$(1 + \alpha\beta)\left(\pi + (1 - \tau_T)\lambda^h(n_T)\beta\right)^2 < \alpha\beta\left(1 + (1 - \tau_T)\lambda^h(n_T)\beta\right)^2$$

Using (76), this follows as $(\pi + (1 - \tau_T)\lambda^h(n_T)\beta) < (1 + (1 - \tau_T)\lambda^h(n_T)\beta)$, with $\pi < 1$.

Thus, in the base case, optimal effective pledgeability (and hence effective leverage), as given by (77) is independent of k_T .

Inductive Step

Suppose $(1 - \tau_{T-s})\lambda^h(n_{T-s})$ is independent of $k_{T-s} \forall s = 0, \dots, k$.

Consider $s = k + 1$. Then, from (51) the objective function of the policymaker at time $T - k - 1$, ignoring constants independent of policy, is given by

$$\sum_{s=T-k-1}^T \beta^{s-(T-k-1)} \mathbb{E}_{T-k-1} (\alpha + \alpha^2\beta) \log(k_s)$$

$$+ \sum_{s=T-k-1}^T \beta^{s-(T-k-1)} \mathbb{E}_{T-k-1} \left[\alpha \beta \log \left(\pi + (1 - \tau_s) \lambda^h(n_s) \beta \right) - (1 + \alpha \beta) \log \left(1 + (1 - \tau_s) \lambda^h(n_s) \beta \right) \right]$$

Noting that, by the inductive hypothesis, $(1 - \tau_{T-s}) \lambda^h(n_{T-s})$ is independent of $k_{T-s} \forall s = 0, \dots, k$, and hence unaffected by τ_{T-k-1} , and k_{T-k-1} is given at time $T - k - 1$ and so independent of policy, the FOC w.r.t. τ_{T-k-1} is given by

$$\begin{aligned} & (\alpha + \alpha^2 \beta) \sum_{s=T-k}^T \beta^{s-(T-k-1)} \mathbb{E}_{T-k-1} \frac{\partial \log(k_s)}{\partial \tau_{T-k-1}} \\ & + \frac{(1 + \alpha \beta) \lambda^h(n_{T-k-1}) \beta}{1 + (1 - \tau_{T-k-1}) \lambda^h(n_{T-k-1}) \beta} - \frac{\alpha \beta \lambda^h(n_{T-k-1}) \beta}{\pi + (1 - \tau_{T-k-1}) \lambda^h(n_{T-k-1}) \beta} = 0 \end{aligned}$$

Moreover, using the inductive hypothesis and Lemma 15, for $s \geq T - k$

$$\mathbb{E}_{T-k-1} \frac{\partial \log(k_s)}{\partial \tau_{T-k-1}} = \alpha^{s-(T-k)} \mathbb{E}_{T-k-1} \frac{\partial \log(k_{T-k})}{\partial \tau_{T-k-1}}$$

Thus

$$\sum_{s=T-k}^T \beta^{s-(T-k-1)} \mathbb{E}_{T-k-1} \frac{\partial \log(k_s)}{\partial \tau_{T-k-1}} = \beta \mathbb{E}_{T-k-1} \frac{\partial \log(k_{T-k})}{\partial \tau_{T-k-1}} \sum_{s=T-k}^T (\alpha \beta)^{s-(T-k)}$$

Now

$$\sum_{s=T-k}^T (\alpha \beta)^{s-(T-k)} = \sum_{s=0}^k (\alpha \beta)^s = \frac{1 - (\alpha \beta)^{k+1}}{1 - \alpha \beta}$$

And so, we have

$$(\alpha + \alpha^2 \beta) \sum_{s=T-k}^T \beta^{s-(T-k-1)} \mathbb{E}_{T-k-1} \frac{\partial \log(k_s)}{\partial \tau_{T-k-1}} = \frac{\alpha \beta (1 + \alpha \beta)}{1 - \alpha \beta} \left[1 - (\alpha \beta)^{k+1} \right] \frac{\partial \mathbb{E}_{T-k-1} \log(k_{T-k})}{\partial \tau_{T-k-1}}$$

Thus, the FOC can be written as

$$\begin{aligned} & \frac{(1 + \alpha \beta) \lambda^h(n_{T-k-1}) \beta}{1 + (1 - \tau_{T-k-1}) \lambda^h(n_{T-k-1}) \beta} - \frac{\alpha \beta \lambda^h(n_{T-k-1}) \beta}{\pi + (1 - \tau_{T-k-1}) \lambda^h(n_{T-k-1}) \beta} \\ & + \frac{\alpha \beta (1 + \alpha \beta)}{1 - \alpha \beta} \left[1 - (\alpha \beta)^{k+1} \right] \frac{\partial \mathbb{E}_{T-k-1} \log(k_{T-k})}{\partial \tau_{T-k-1}} = 0 \end{aligned}$$

From the proof of Lemma 15, we have that the terms that τ_{T-k-1} can influence in $\mathbb{E}_{T-k-1} \log(k_{T-k})$ are given by

$$\begin{aligned} & \log \left(\pi + \beta (1 - \tau_{T-k-1}) \lambda^h(n_{T-k-1}) \right) - \log \left(1 + \beta (1 - \tau_{T-k-1}) \lambda^h(n_{T-k-1}) \right) \\ & + \int_{x_{T-k}^h}^{\left(\mathbb{E}_{T-k-1} \left((x_{T-k}^h)^\alpha \right) \right)^{\frac{1}{\alpha}}} \frac{1}{\alpha} \log \left(\frac{\left((x_{T-k}^h)^\alpha - (1 - \tau_{T-k-1}) \lambda^h(n_{T-k-1}) \mathbb{E}_{T-k-1} \left((x_{T-k}^h)^\alpha \right) \right)}{1 - (1 - \tau_{T-k-1}) \lambda^h(n_{T-k-1})} \right) f \left(x_{T-k}^h \right) dx_{T-k}^h \end{aligned}$$



Thus, letting $\Delta := \frac{\alpha\beta(1+\alpha\beta)}{1-\alpha\beta} \left[1 - (\alpha\beta)^{k+1}\right]$, the FOC is given by

$$\begin{aligned} & \frac{(1 + \alpha\beta) \lambda^h (n_{T-k-1}) \beta}{1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} - \frac{\alpha\beta \lambda^h (n_{T-k-1}) \beta}{\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} \\ & + \Delta \left[\frac{\lambda^h (n_{T-k-1}) \beta}{1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} - \frac{\lambda^h (n_{T-k-1}) \beta}{\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} \right] \\ & + \frac{\Delta}{\alpha} \int_{\underline{x}_{T-k}^h}^{(\mathbb{E}_{T-k-1}((x_{T-k}^h)^\alpha))^{\frac{1}{\alpha}}} \frac{\lambda^h (n_{T-k-1}) \mathbb{E}_{T-k-1}((x_{T-k}^h)^\alpha) f(x_{T-k}^h)}{(x_{T-k}^h)^\alpha - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \mathbb{E}_{T-k-1}((x_{T-k}^h)^\alpha)} dx_{T-k}^h \\ & - \frac{\Delta}{\alpha} \int_{\underline{x}_{T-k}^h}^{(\mathbb{E}_{T-k-1}((x_{T-k}^h)^\alpha))^{\frac{1}{\alpha}}} \frac{\lambda^h (n_{T-k-1}) f(x_{T-k}^h)}{1 - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1})} dx_{T-k}^h = 0 \end{aligned}$$

Now, we have that

$$(1 + \alpha\beta) + \Delta = (1 + \alpha\beta) \left(\frac{1 - \alpha\beta + \alpha\beta \left[1 - (\alpha\beta)^{k+1}\right]}{1 - \alpha\beta} \right) = (1 + \alpha\beta) \left(\frac{1 - (\alpha\beta)^{k+2}}{1 - \alpha\beta} \right)$$

Whilst

$$\alpha\beta + \Delta = \alpha\beta \left(\frac{(1 - \alpha\beta) + (1 + \alpha\beta) \left[1 - (\alpha\beta)^{k+1}\right]}{1 - \alpha\beta} \right) = \alpha\beta \left(\frac{2 - (1 + \alpha\beta) (\alpha\beta)^{k+1}}{1 - \alpha\beta} \right)$$

Thus, cancelling the common $\lambda^h (n_{T-k-1})$ term and rearranging gives the FOC for τ_{T-k-1} as

$$\begin{aligned} & \frac{(1 + \alpha\beta) \left(1 - (\alpha\beta)^{k+2}\right)}{1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} - \frac{\alpha\beta \left(2 - (1 + \alpha\beta) (\alpha\beta)^{k+1}\right)}{\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} \tag{78} \\ & + \left[1 - (\alpha\beta)^{k+1}\right] \int_{\underline{x}_{T-k}^h}^{(\mathbb{E}_{T-k-1}((x_{T-k}^h)^\alpha))^{\frac{1}{\alpha}}} \frac{(1 + \alpha\beta) \mathbb{E}_{T-k-1}((x_{T-k}^h)^\alpha) f(x_{T-k}^h)}{(x_{T-k}^h)^\alpha - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \mathbb{E}_{T-k-1}((x_{T-k}^h)^\alpha)} dx_{T-k}^h \\ & - \left[1 - (\alpha\beta)^{k+1}\right] \int_{\underline{x}_{T-k}^h}^{(\mathbb{E}_{T-k-1}((x_{T-k}^h)^\alpha))^{\frac{1}{\alpha}}} \frac{(1 + \alpha\beta) f(x_{T-k}^h)}{1 - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1})} dx_{T-k}^h = 0 \end{aligned}$$

Note that the combined integral terms are positive. This follows as, over the range of the integral, $(\mathbb{E}_{T-k-1}((x_{T-k}^h)^\alpha))^{\frac{1}{\alpha}} > (x_{T-k}^h)$ and so $\mathbb{E}_{T-k-1}((x_{T-k}^h)^\alpha) > (x_{T-k}^h)^\alpha$. Thus

$$\mathbb{E}_{T-k-1} \left((x_{T-k}^h)^\alpha \right) \left(1 - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \right) \tag{79}$$

$$> \left(x_{T-k}^h\right)^\alpha - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \mathbb{E}_{T-k-1} \left(\left(x_{T-k}^h\right)^\alpha\right)$$

Given insolvency doesn't occur under the conditions of the proposition, the RHS > 0, and hence the LHS is also positive. This expression can then be rearranged to show that the combined terms being integrated are positive, and so the integral as a whole is positive. Thus, from the FOC, we must have

$$\frac{(1 + \alpha\beta) \left(1 - (\alpha\beta)^{k+2}\right)}{1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} < \frac{\alpha\beta \left(2 - (1 + \alpha\beta) (\alpha\beta)^{k+1}\right)}{\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} \quad (80)$$

To verify the FOC is indeed a maximum, we now turn to the second derivative of the planner's objective w.r.t. τ_{T-k-1} . Ignoring constant terms that don't affect the overall sign, it's given by

$$\begin{aligned} & \frac{(1 + \alpha\beta) \left(1 - (\alpha\beta)^{k+2}\right) \lambda^h (n_{T-k-1}) \beta}{(1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta)^2} - \frac{\alpha\beta \left(2 - (1 + \alpha\beta) (\alpha\beta)^{k+1}\right) \lambda^h (n_{T-k-1}) \beta}{(\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta)^2} \\ & - \int_{\underline{x}_{T-k}^h}^{\left(\mathbb{E}_{T-k-1} \left(\left(x_{T-k}^h\right)^\alpha\right)\right)^{\frac{1}{\alpha}}} \frac{\left[1 - (\alpha\beta)^{k+1}\right] \left(\mathbb{E}_{T-k-1} \left(\left(x_{T-k}^h\right)^\alpha\right)\right)^2 (1 + \alpha\beta) \lambda^h (n_{T-k-1}) f \left(x_{T-k}^h\right)}{\left(\left(x_{T-k}^h\right)^\alpha - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \mathbb{E}_{T-k-1} \left(\left(x_{T-k}^h\right)^\alpha\right)\right)^2} dx_{T-k}^h \\ & + \int_{\underline{x}_{T-k}^h}^{\left(\mathbb{E}_{T-k-1} \left(\left(x_{T-k}^h\right)^\alpha\right)\right)^{\frac{1}{\alpha}}} \frac{\left[1 - (\alpha\beta)^{k+1}\right] (1 + \alpha\beta) \lambda^h (n_{T-k-1}) f \left(x_{T-k}^h\right)}{\left(1 - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1})\right)^2} dx_{T-k}^h \end{aligned}$$

We note that the integral is negative. Throughout the range of the integral we have, squaring both sides of (79) which maintains the inequality as both sides are positive,

$$\begin{aligned} & \left(\mathbb{E}_{T-k-1} \left(\left(x_{T-k}^h\right)^\alpha\right)\right)^2 \left(1 - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1})\right)^2 \\ & > \left(\left(x_{T-k}^h\right)^\alpha - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \mathbb{E}_{T-k-1} \left(\left(x_{T-k}^h\right)^\alpha\right)\right)^2 \end{aligned}$$

from which it follows, after rearranging, that the combined integral is negative. Now, noting that, as $\pi < 1$,

$$\left(1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta\right)^{-1} < \left(\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta\right)^{-1}$$

and using (80), we have that, at the FOC

$$\frac{(1 + \alpha\beta) \left(1 - (\alpha\beta)^{k+2}\right) \lambda^h (n_{T-k-1}) \beta}{(1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta)^2} < \frac{\alpha\beta \left(2 - (1 + \alpha\beta) (\alpha\beta)^{k+1}\right) \lambda^h (n_{T-k-1}) \beta}{(\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta)^2}$$

Hence, with the integral term in the second derivative negative, we have that the second derivative is negative at the FOC. Thus, the FOC as given by (78) is indeed an optimal solution.

It follows that the optimal effective pledgeability $(1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1})$ is then equal to a constant, independent of k_{T-k-1} . This completes the inductive step, hence by the Principle of Mathematical Induction, $(1 - \tau_{T-s}) \lambda^h (n_{T-s})$ is independent $k_{T-s} \forall s \geq 0$. Moreover, (74) follows

from (78), when $k = s - 1$, completing the proof. □

I.3 Optimal policy with infinite horizon: Proposition 9

Proof of Proposition 9. Using Proposition 16, and supposing T is infinitely far into the future, equivalently, $s \rightarrow \infty$, we have, noting that $\alpha\beta \in (0, 1)$ and so $\lim_{s \rightarrow \infty} (\alpha\beta)^s = 0$, the FOC w.r.t τ_{T-s} is given by

$$\begin{aligned} & \frac{(1 + \alpha\beta)}{1 + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta} - \frac{2\alpha\beta}{\pi + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta} \\ & + \int_{\underline{x}_{T-s+1}^h}^{(\mathbb{E}_{T-s}((x_{T-s+1}^h)^\alpha))^{\frac{1}{\alpha}}} \frac{(1 + \alpha\beta) \mathbb{E}_{T-s}((x_{T-s+1}^h)^\alpha) f(x_{T-s+1}^h)}{(x_{T-s+1}^h)^\alpha - (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \mathbb{E}_{T-s}((x_{T-s+1}^h)^\alpha)} dx_{T-s+1}^h \\ & - \int_{\underline{x}_{T-s+1}^h}^{(\mathbb{E}_{T-s}((x_{T-s+1}^h)^\alpha))^{\frac{1}{\alpha}}} \frac{(1 + \alpha\beta) f(x_{T-s+1}^h)}{1 - (1 - \tau_{T-s}) \lambda^h (n_{T-s})} dx_{T-s+1}^h = 0 \end{aligned}$$

Letting $t := T - s$ and applying to sector $h = a$ gives the result in the text. We next turn to the other parts of the proposition.

(i) It's clear from the formula that effective pledgeability $(1 - \tau_t) \lambda^h (n_t)$, and hence effective leverage, is independent of k_t and constant across states. Thus, with $\lambda^h (n_t)$ procyclical, we must have τ_t also procyclical to ensure this is constant.

(ii) When there is no liquidation, with no shocks to x_{t+1}^h , the integral term is zero (with zero support) and the FOC satisfies

$$\frac{(1 + \alpha\beta)}{1 + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta} = \frac{2\alpha\beta}{\pi + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta}$$

Rearranging this gives

$$\begin{aligned} (1 + \alpha\beta) \left(\pi + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta \right) &= 2\alpha\beta \left(1 + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta \right) \\ (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta (1 + \alpha\beta - 2\alpha\beta) &= 2\alpha\beta - (1 + \alpha\beta) \pi \end{aligned}$$

Effective pledgeability is then

$$(1 - \tau_{T-s}) \lambda^h (n_{T-s}) = \frac{2\alpha\beta - (1 + \alpha\beta) \pi}{\beta (1 - \alpha\beta)}$$

and rearranging gives the result in the text for effective leverage.

(iii) Let the FOC (54) be given by $g(\underline{x}_{t+1}^h, \tau_t(\underline{x}_{t+1}^h)) \equiv 0$, where we consider the optimal policy as a function of the lower bound of the stochastic distribution, where we hold $\mathbb{E}_t((x_{t+1}^h)^\alpha)$ fixed (e.g. by increasing the upper bound). Thus, when $\underline{x}_t^h = (\mathbb{E}_t((x_{t+1}^h)^\alpha))^{\frac{1}{\alpha}}$ we have the case of no



shocks, and no liquidation. Consider a change to this lower bound. Taking the total derivative of $g(\cdot, \cdot)$ w.r.t. \underline{x}_{t+1}^h we have

$$\frac{\partial g(\underline{x}_{t+1}^h, \tau_t(\underline{x}_{t+1}^h))}{\partial \underline{x}_{t+1}^h} + \frac{\partial g(\underline{x}_{t+1}^h, \tau_t(\underline{x}_{t+1}^h))}{\partial \tau_t(\underline{x}_{t+1}^h)} \frac{d\tau_t}{d\underline{x}_{t+1}^h} = 0$$

and so

$$\frac{d\tau_t}{d\underline{x}_{t+1}^h} = \frac{\partial g(\underline{x}_{t+1}^h, \tau_t(\underline{x}_{t+1}^h))}{\partial \underline{x}_{t+1}^h} / \frac{-\partial g(\underline{x}_{t+1}^h, \tau_t(\underline{x}_{t+1}^h))}{\partial \tau_t(\underline{x}_{t+1}^h)}$$

Now, from the proof of Proposition 16 we have that $\frac{\partial g(\underline{x}_{t+1}^h, \tau_t(\underline{x}_{t+1}^h))}{\partial \tau_t(\underline{x}_{t+1}^h)} < 0$, with the second order condition for a maximum satisfied. Moreover, the proof of Proposition 16 establishes that each term in the integral in the FOC, given by (54), is positive. Thus, it follows from Leibniz's rule that $\frac{\partial g(\underline{x}_{t+1}^h, \tau_t(\underline{x}_{t+1}^h))}{\partial \underline{x}_{t+1}^h} < 0$, as positive terms are integrated over a smaller support. Combining these results we have that $\frac{d\tau_t}{d\underline{x}_{t+1}^h} < 0$. Hence, with $\mathbb{E}_t((x_{t+1}^h)^\alpha)$ fixed, *lowering* the lower bound of the distribution, \underline{x}_{t+1}^h , and so increasing the range over which there is costly liquidation, *raises* the haircut τ_t , *lowering* effective leverage. Moreover, as the case of no liquidation coincides with no shocks and $\underline{x}_t^h = (\mathbb{E}_t((x_{t+1}^h)^\alpha))^{\frac{1}{\alpha}}$, when there is liquidation, optimal effective leverage will be lower than when there is not.

This completes the proof of the proposition. □

J Welfare solution with credit traps

Proposition 10

Proof of Proposition 10. Under the conditions of Proposition 10, and with the planner applying haircut τ_t , from (57) the trap threshold is defined implicitly by

$$\begin{aligned} & (\pi + \beta(1 - \tau_t)\lambda^a(\tilde{n}))^\alpha (1 + \beta(1 - \tau_t)\lambda^a(\tilde{n}))^{1-\alpha} \mathbb{E}_t((x_{t+1}^a)^\alpha) \\ & = \left(\pi + \beta(1 - \tau_t)\lambda^b\right)^\alpha \left(1 + \beta(1 - \tau_t)\lambda^b\right)^{1-\alpha} (x^b)^\alpha \end{aligned}$$

Note that, by assumption, $\mathbb{E}_t((x_{t+1}^a)^\alpha) > (x^b)^\alpha$, and hence, we must have, with all expressions in brackets positive:

$$(\pi + \beta(1 - \tau_t)\lambda^a(\tilde{n}))^\alpha (1 + \beta(1 - \tau_t)\lambda^a(\tilde{n}))^{1-\alpha} < \left(\pi + \beta(1 - \tau_t)\lambda^b\right)^\alpha \left(1 + \beta(1 - \tau_t)\lambda^b\right)^{1-\alpha}$$

As the LHS is strictly increasing in $\lambda^a(\tilde{n})$, we must then have $\lambda^a(\tilde{n}) < \lambda^b$.

We now turn to showing $\frac{d\tilde{n}}{d\tau_t} < 0$. Taking the logarithm of both sides of (57) gives

$$\begin{aligned} & \alpha \log(\pi + \beta(1 - \tau_t)\lambda^a(\tilde{n})) + (1 - \alpha) \log(1 + \beta(1 - \tau_t)\lambda^a(\tilde{n})) \\ &= \alpha \log(\pi + \beta(1 - \tau_t)\lambda^b) + (1 - \alpha) \log(1 + \beta(1 - \tau_t)\lambda^b) + D \end{aligned}$$

where $D := \log((x^b)^\alpha) - \log(\mathbb{E}_t((x_{t+1}^a)^\alpha))$ is independent of τ_t . Thus, with \tilde{n} a function of τ_t , \tilde{n} is implicitly defined by

$$h(\tilde{n}(\tau_t), \tau_t) \equiv D$$

where

$$\begin{aligned} h(\tilde{n}(\tau_t), \tau_t) : &= \alpha \left[\log(\pi + \beta(1 - \tau_t)\lambda^a(\tilde{n})) - \log(\pi + \beta(1 - \tau_t)\lambda^b) \right] \\ &+ (1 - \alpha) \left[\log(1 + \beta(1 - \tau_t)\lambda^a(\tilde{n})) - \log(1 + \beta(1 - \tau_t)\lambda^b) \right] \end{aligned}$$

Taking the total derivative of this w.r.t. τ_t and rearranging, gives

$$\frac{d\tilde{n}}{d\tau_t} = - \frac{\partial h(\tilde{n}(\tau_t), \tau_t)}{\partial \tau_t} / \frac{\partial h(\tilde{n}(\tau_t), \tau_t)}{\partial \tilde{n}(\tau_t)}$$

Now

$$\frac{\partial h(\tilde{n}(\tau_t), \tau_t)}{\partial \tilde{n}(\tau_t)} = \left\{ \frac{\alpha\beta(1 - \tau_t)}{(\pi + \beta(1 - \tau_t)\lambda^a(\tilde{n}))} + \frac{(1 - \alpha)\beta(1 - \tau_t)}{1 + \beta(1 - \tau_t)\lambda^a(\tilde{n})} \right\} \frac{d\lambda^a(\tilde{n})}{d\tilde{n}} > 0$$

as $\frac{d\lambda^a(\tilde{n})}{d\tilde{n}} > 0$ by Assumption 1' in the text. As this is non-zero, it also verifies the conditions for the Implicit Function Theorem.

Turning to the other term, we have

$$\begin{aligned} - \frac{\partial h(\tilde{n}(\tau_t), \tau_t)}{\partial \tau_t} &= \alpha \left[\frac{\beta\lambda^a(\tilde{n})}{\pi + \beta(1 - \tau_t)\lambda^a(\tilde{n})} - \frac{\beta\lambda^b}{\pi + \beta(1 - \tau_t)\lambda^b} \right] \\ &+ (1 - \alpha) \left[\frac{\beta\lambda^a(\tilde{n})}{1 + \beta(1 - \tau_t)\lambda^a(\tilde{n})} - \frac{\beta\lambda^b}{1 + \beta(1 - \tau_t)\lambda^b} \right] \end{aligned}$$

Rearranging gives

$$\begin{aligned} - \frac{\partial h(\tilde{n}(\tau_t), \tau_t)}{\partial \tau_t} &= \alpha\beta \left[\frac{\pi(\lambda^a(\tilde{n}) - \lambda^b)}{(\pi + \beta(1 - \tau_t)\lambda^a(\tilde{n}))(\pi + \beta(1 - \tau_t)\lambda^b)} \right] \\ &+ (1 - \alpha)\beta \left[\frac{(\lambda^a(\tilde{n}) - \lambda^b)}{(1 + \beta(1 - \tau_t)\lambda^a(\tilde{n}))(1 + \beta(1 - \tau_t)\lambda^b)} \right] \end{aligned}$$

As shown above, $\lambda^a(\tilde{n}) < \lambda^b$, and hence $-\frac{\partial h(\tilde{n}(\tau_t), \tau_t)}{\partial \tau_t} < 0$. It thus follows that $\frac{d\tilde{n}}{d\tau_t} < 0$, completing the proof. □

K Numerical welfare analysis - method

We formulate the policy problem as a discrete time, continuous action, continuous state dynamic programming problem and apply the solution methods developed by Miranda and Fackler (2002), their `dpsolve` routine in particular. We formulate the policymaker's problem as to solve the Bellman equation:

$$V(k_t) = \max_{\tau_t \in [0,1]} \{W(k_t, \tau_t) + \beta V(k_{t+1}(k_t, \tau_t, x_{t+1}^a))\}$$

where $V(k_t)$ is the value function to be solved for, with an associated optimal policy rule as a function of the state of the economy, $\tau_t(k_t)$. An approximate solution to the Bellman equation can be found by constructing a value function approximant as a linear combination of a collection of basis functions whose coefficients are to be determined. The coefficients are determined, in turn, by requiring the approximant to satisfy the Bellman equation at n collocation nodes. As described in Miranda and Fackler, this transforms the problem from one in which a functional equation must be solved for to one involving n nonlinear equations in n unknowns. This nonlinear problem can be solved using a number of different methods.

Practically this involves choices over how the stochastic process at work in the model economy is discretised and how the function space is approximated. Here there is generally a trade-off between approximation accuracy and computational efficiency. Having experimented with various formulations, the results we report in the text employ a 50-bin discretisation of the sector a productivity shock process and a 20-point approximation to the value function. After convergence has been achieved, we compute the size of the residuals that remain between the left- and right-hand side of the Bellman equation at each value of the state, evaluated using the value function approximants. The Bellman equation holds exactly at the collocation nodes; we check that the residuals away from these nodes are sufficiently small.

Figures 10–13 show the optimised value function, policy rule, and Bellman equation residuals for four specifications of the model: a baseline case with no liquidation and no credit trap possible (Figure 10); next a case adding in the possibility of liquidation (Figure 11); next a case in which there is no liquidation but credit traps are possible (Figure 12); and finally a case in which both liquidation and a credit trap are possible (Figure 13). From each of these Figures, note that the approximation residuals are at most of the order of 10^{-4} , an approximation tolerance we deem acceptable given the scale of the value function itself (which is of the order 10^1). We discuss the properties of the optimal policy rules shown in panel (b) of each of the figure in more detail in the text.

Figure 10: Value function, optimal policy, and approximation residuals: case of no liquidation and no credit trap

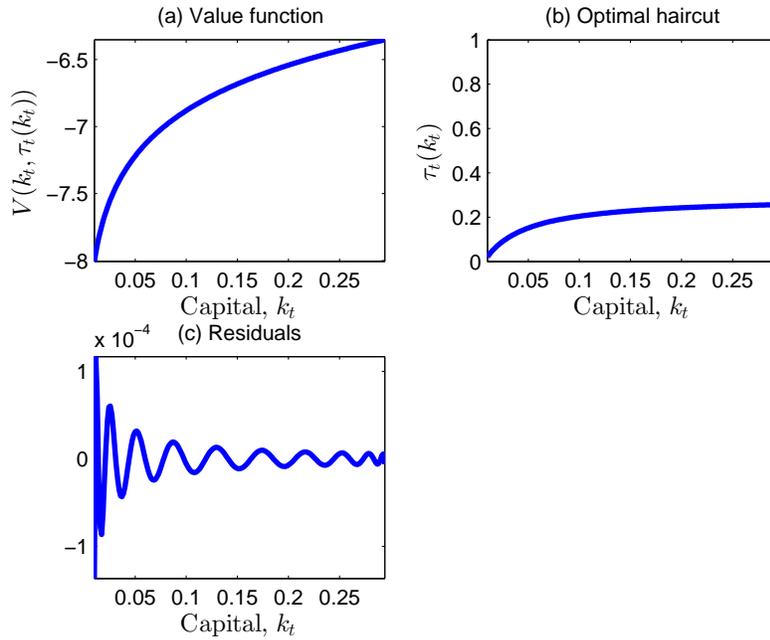


Figure 11: Value function, optimal policy, and approximation residuals: case with liquidation and no credit trap

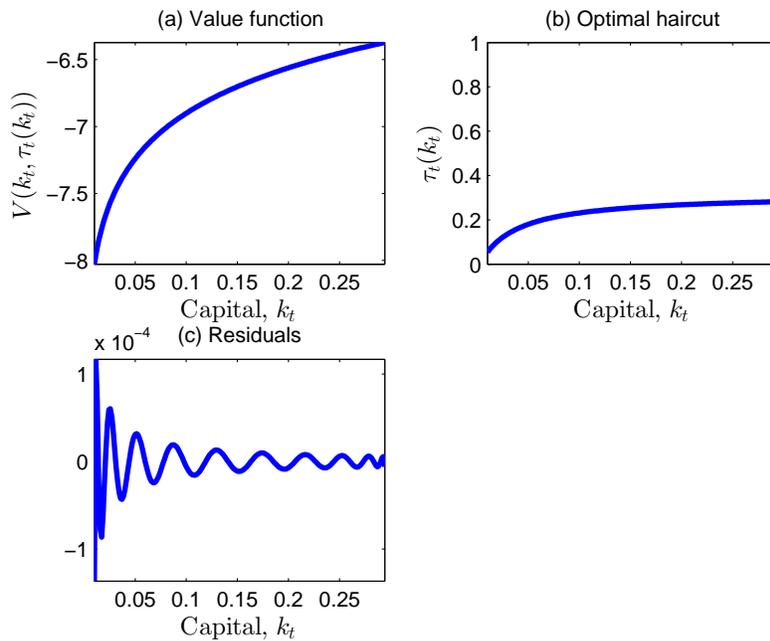


Figure 12: Value function, optimal policy, and approximation residuals: case of no liquidation with credit trap

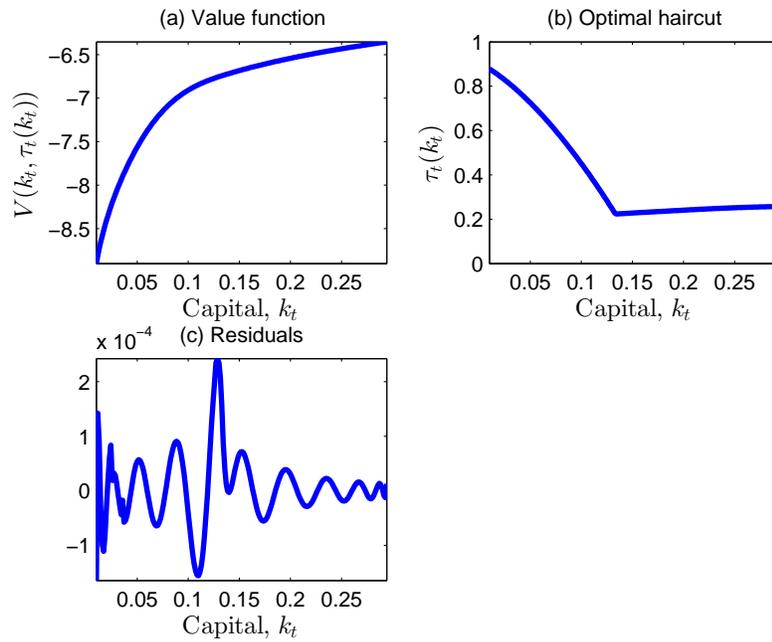


Figure 13: Value function, optimal policy, and approximation residuals: case with liquidation and credit trap

