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Abstract

Multidimensional Value at Risk (MVaR) generalises VaR in a natural way as the intersection of univariate VaRs. We reduce the dimensionality of MVaRs which allows for adapting the techniques and applications developed for VaR to MVaR. As an illustration, we employ VaR forecasting and evaluation techniques. One of our forecasting models builds on the progress made in the volatility literature and decomposes multidimensional tail events into long-term trend and short-term cycle components. We compute short and long-term MVaR forecasts for several multidimensional time series and discuss their (un)conditional accuracy.

Key words: Multidimensional risk, multidimensional Value at Risk, two-factor decomposition, long-horizon forecasting.

JEL classification: C52, C53.

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1. Introduction

The interest in multidimensional tail (MT) events is driven by its importance in economics, finance, insurance and in many other areas of applied probability, statistics and decision theory. In economics and finance, modeling and forecasting MT events is paramount for many important applications such as portfolio decisions (e.g., Ang and Bekaert, 2002), risk management (e.g., Embrechts et al. 2002; Meine, et al. 2016), multidimensional options (e.g., Cherubini and Luciano, 2002), credit derivatives, collateralised debt obligations and insurance (e.g., Hull and White 2006; Kalemanova et al. 2007; Su and Spindler, 2013), contagion, spillovers and economic crises (Bae et al. 2003; Zheng, et al. 2012; Hautsch, Schaumburg and Schienle, 2015), systemic risk and financial stability (Adrian and Brunnermeier, 2009; Gonzáles-Rivera, 2014) and market integration (e.g., Bartram et al. 2006; Lehkonen, 2015).

Tail events are closely related to extreme risk that is generally defined as the potential for significant adverse deviation from expected results. In the univariate context, a measure of extreme risk widely used in practice is the Value at Risk (VaR). VaR is defined as the maximum loss on a portfolio over a certain period of time that can be expected with a nominal probability. However, modern risk management generally involves more than one risk factor and is particularly concerned with the evaluation and balancing of their impacts. For example, multifactor models (e.g., Chen et al., 1986; Ferson and Harvey, 1998) are used to measure and manage exposure to each of the multiple economy-wide risk factors.

This paper discusses a new angle of the literature on modeling and forecasting MT events (see also, Prékopa, 2012; Polanski and Stoja, 2012; Torres, et al., 2015). Building on this rapidly developing literature, we apply a generalized version of VaR, Multidimensional Value at Risk (MVaR), which is defined as the intersection of univariate VaRs with a nominal probability mass under a given density function. MVaR can be seen as an illustration of the multiple sources of risk: If VaR is a univariate risk measure, which instead of the variance takes into account the entire tail density, then MVaR is a measure of multidimensional risk that instead of the covariances takes into account the entire joint tail.

Why should we care about MVaR when in typical portfolio applications it is the portfolio VaR that matters and not the joint tail risk of the components of the portfolio? Although VaR might be the appropriate risk measure in portfolio applications, MVaR is useful in other circumstances where risk sources cannot be aggregated to form an informative risk measure or the portfolio interpretation of a collection of variables is not natural, useful or possible.

A prominent example of the importance of properly accounting for the distributional characteristics of the multiple sources of risk comes from stress testing of portfolios or financial systems. Typically, stress testing frameworks begin by developing scenarios with a negative outlook (tail events) for the evolution of certain economic drivers (e.g. GDP growth, interest rates, unemployment, stock market performance, investor sentiment) and then proceed to evaluate the impact of these on portfolios or systemically important institutions (e.g., Bank of England, 2015; European Banking Authority, 2016). Statistically speaking, the scenarios can be developed in three ways. The economic drivers could be projected into the future individually; they could be treated as a 'portfolio of risks' and aggregated into a single risk measure; or the economic drivers could be modelled jointly. Treating these drivers individually presents a problem[1] as they are obviously interdependent. Moreover, it would be difficult to construct a portfolio of these factors and use its VaR as a tail risk measure. For example, what are the appropriate weights and their interpretations for each source of risk in such a portfolio? Thus, a sensible alternative in this case is to consider the sources of risk jointly. In this case, MVaR can considerably simplify the task.

Another example, related to stress testing that highlights the importance of MVaR is systemic risk. This is the risk of collapse faced by the financial system as a whole when one of its constituent parts gets into financial distress. Due to the interconnectivity of the financial institutions, a shock faced by one institution in the form of a tail event, increases the probability other financial institutions experiencing similar tail events, leading to a domino effect (e.g., Gai and Kapadia, 2010; Rogers and Veraart, 2013; Hautsch, Schaumburg and Schienle, 2014). In this case, it would be inappropriate and uninformative to treat the financial system as portfolio of banks and compute its VaR.

[1] We emphasise that we are not referring to the approach the BoE or the EBA take. We are simply highlighting the obvious issue that treating dependent variables individually omits important information about their mutual dependence.

Therefore, while it is important to have an aggregate measure of the aggregate tail risk, often it is also important to know the direct dependence on, inter-relationships and co-dynamics of the specific sources of tail risk. By focusing on the joint distribution of the individual sources of tail risks, we provide a framework to characterize the co-dependence of these risks.

As operating directly on MVaRs might be cumbersome, we reduce their dimensionality to a univariate series. This simplification makes possible a number of applications which would be difficult otherwise. In this paper we apply it to short- and long-term MVaR forecasting and evaluation. First, we employ the Conditional Autoregressive Value at Risk (CAViaR) developed by Engle and Manganelli (2004) to obtain one-step ahead MVaR forecasts. However, CAViaR is a purely statistical model and does not distinguish between secular movements in the tails, driven perhaps by the macroeconomic and company fundamentals, and transitory movements due to investor sentiment or other short-lived effects. With this in mind, we develop a new Two-Factor forecasting model that we apply to MVaR after reducing their dimensionality. The model has several advantages. It is simple to estimate and it can easily produce multi-step ahead forecasts. The Two-Factor model decomposes MVaR into a long-term trend and short-term cycle which can then be examined for relationships with economic and other variables. Finally, we use the scaling property of financial and economic time series to forecast MVaR at different frequencies and horizons. We evaluate the MVaR forecasts by employing adapted conditional and unconditional evaluation techniques of VaR forecasts. This paper is, to the best of our knowledge, the first to raise these issues in relation to (multidimensional) tail events.

2. Multidimensional Value at Risk

For the unidimensional continuous CDF F (PDF f), the VaR at the nominal level α is the quantile q_α for which $F(q_\alpha) = \alpha$. The VaR definition implies that the probability mass under f of the interval $\{y \in R: y \leq q_\alpha\}$ is equal to α.

In analogy to VaR, the Multidimensional Value at Risk (MVaR) in direction $\boldsymbol{d} \in R^N$ at the nominal probability level α ($MVaR_\alpha^d$) is the N-dimensional region uniquely defined by the cut-off value $q_\alpha^{\boldsymbol{d}} \in R$,

$$\{\boldsymbol{y} \in R^N: y_i/d_i \geq q_\alpha^{\boldsymbol{d}},\ \forall d_i \neq 0\}, \quad (1)$$

with the probability mass under the N-dimensional PDF f equal to the nominal level α. As illustrated in Figure 1, $MVaR_\alpha^d$ can be seen as an intersection of univariate VaRs.

[Figure 1]

We often refer to the region (1) as multidimensional tails in direction $\boldsymbol{d}$ and to the cut-off value $q_\alpha^{\boldsymbol{d}}$ as $MVaR_\alpha^d$-value or, when there is no risk of confusion, simply as $MVaR_\alpha^d$. We also say that $\boldsymbol{x} \in R^N$ is an extreme observation when $\boldsymbol{x} \in MVaR_\alpha^d$. The directional vector d has a distinct financial interpretation. For example, suppose a hypothetical financial system contains only two banks with market capitalizations of one and two units respectively. Then, a directional vector of particular interest for the regulator of this financial system is $d = (d_1, d_2)' = -(1,2)'$ as it succinctly represents the exposure of the domestic economy to the financial system in R^2 space.

In spite of their conceptual simplicity, working directly with MVaRs can prove challenging in higher dimensions. However, the relevant MVaR inference can be easily obtained by transforming points in the domain of f into scalars. Specifically, we define the projection $\boldsymbol{x}^d$ of the point $\boldsymbol{x} \in R^N$ on the line along the directional vector $\boldsymbol{d} \in R^N$ as follows,

$$\boldsymbol{x}^d = v^d(x) \cdot \boldsymbol{d}, \text{ where } v^d(x) = \min_{d_i \neq 0}\{x_i/d_i\}. \tag{2}$$

We illustrate in Figure 2 and show in the Appendix that,

$$v^d(\boldsymbol{x}) \geq q_a^{\boldsymbol{d}} \Leftrightarrow \boldsymbol{x} \in MVaR_a^d \tag{3}$$

Intuitively, observation $\boldsymbol{x}$ being in the $MVaR_a^d$ implies that its projection $v^d(\boldsymbol{x})$ exceeds $q_a^{\boldsymbol{d}}$ and vice-versa. Therefore, for convenience we sometimes refer to the threshold value $q_a^{\boldsymbol{d}}$ as $MVaR_a^d$.

[Figure 2]

3. Forecasting MVaR

A quantile computed from a series of *i.i.d.* observations at the frequency of k-steps (frequency-k quantile for short) is the natural estimator of the corresponding frequency-k VaR. For example, frequency-*1*, frequency-*5* and frequency-*20* quantiles computed from daily financial data estimate the daily, weekly and monthly VaR, respectively. Similarly, frequency-k $MVaR_{\alpha}^{d}$ can be estimated as the frequency-k a-quantile of the projections $v^d(\mathbf{x}_t)$ (2) of multidimensional of observations $\mathbf{x}_t$. As we use daily observations in the empirical section, the frequency-*1* MVaR corresponds to daily MVaR.

In the reminder of this section, we apply three different MVaR forecasting methods to obtain forecasts over a horizon of k-steps ahead.[2] The methods presented in Subsections 3.1 and 3.2 are useful for forecasting daily MVaR one-step ahead and k-step ahead, where $k = 1$ and $k \geq 5$ refer to short- and long-term horizon forecasts respectively. The method presented in Subsection 3.3 allows for forecasting low frequency (e.g., monthly) MVaR which due to the limited number of such observations in practice would be difficult otherwise.

3.1. Conditional Autoregressive Value at Risk

Several approaches to short-term VaR forecasting have been proposed (e.g., Kuester et al., 2006; Nieto and Ruiz, 2016 for surveys of the VaR forecasting techniques). Some estimate the volatility of the time series first (e.g., by a GARCH model) and then compute VaR,

[2] We also apply these techniques to VaR and find that the models do a similarly good job at forecasting VaR. As MVaR encompasses VaR, in order to preserve space we do not report these results. They are available upon request.

often under the assumption of normality. Others use rolling historical quantiles (e.g., Boudoukh et al., 1998) or rely on extreme value theory (e.g., Danielsson and de Vries, 2000).

Engle and Manganelli (2004) propose a different approach to quantile estimation and forecasting. Instead of modeling the whole distribution, they model the quantile directly. As VaR is closely linked to volatility and the latter is often autocorrelated in financial data, a natural way to model VaR is to use an autoregressive process. Engle and Manganelli (2004) specify the evolution of the quantile over time by the Conditional Autoregressive Value at Risk (CAViaR) model and estimate its parameters by quantile regression. CAViaR allows for many specifications of the autoregressive process which can be used for MVaR forecasting. In our empirical exercise in Section 4, we use their asymmetric slope function,

$$q_{a,t+1} = \beta_1 + \beta_2 q_{a,t} + \beta_3 \max\left(v_t^d, 0\right) - \beta_4 \min\left(v_t^d, 0\right), \quad (4)$$

where the next period quantile $q_{a,t+1}$ is a function of the current period quantile $q_{a,t}$ and projection v_t^d.

The quantile regression estimation of the parameter vector $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4)$ in (4) boils down to the solution of the minimization problem,

$$\min_{\boldsymbol{\beta}} \frac{1}{T} \sum_{t=1}^{T} \left[a - I(v_t^d < q_{a,t})\right]\left[v_t^d - q_{a,t}\right], \quad (5)$$

where $q_{a,t}$ is computed by (4), $I(.)$ is the indicator function and a is the nominal probability. In our empirical study, we use CAViaR not as a competing, but as a complementary short-term MVaR forecasting model and obtain the k-step ahead forecasts of MVaR $q_{a,t+k}$ with a technique that we present next.

3.2. Two-Factor Model

Similar to GARCH, CAViaR is a purely statistical model which cannot be easily related to macroeconomic or company fundamentals. However, tail events – similar to volatility – must be connected to fundamentals (see, for example, Bloom, 2009; Massacci, 2016). Moreover, evidence increasingly suggests that volatility is characterised by a multi-factor structure, with different dynamic processes governing its long-term and short-term dynamics. Engle and Lee (1999) introduce a component GARCH model which decomposes volatility into a permanent long-run trend component and a transitory short-run component that is mean-reverting towards the long-run trend. They find that a two-factor model provides a better fit to the data than an equivalent one-factor model (see also Alizadeh et al. 2002; Brandt and Jones, 2006). Importantly, the two-factor specification makes possible linking the long-term trend of volatility to macroeconomic variables (e.g., Engle and Rangel, 2008). There is a significant number of VaR forecasting models in the literature but models that link VaR to macroeconomic fundamentals are as yet elusive. While perhaps the spline-GARCH model of Engle and Rangel (2008) may be extended to MVaR, it would be computationally-demanding. The Two-Factor model that we present here offers a simple and efficient way to decompose MVaR into a long-term trend and a short-term cycle. This decomposition would then allow for the linking of the long-term trend to macroeconomic and company fundamentals while the short-term cyclical component may be related to transient investor sentiment or other short-lived effects. For brevity, we do not pursue this idea in this paper but are investigating it in a separate project.

The finding that volatility has both a highly persistent factor and a strongly stationary factor has important implications for modeling and forecasting VaR. As VaR is closely related to volatility (e.g., Takahashi, Watanabe and Omori, 2016), any improvements in volatility forecasts are inherited by VaR forecasts. Motivated by the interpretation of two-factor volatility models, we explore an alternative, simple approach to modeling and forecasting MVaR over both short and long horizons. Specifically, we hypothesize that MVaR-values follows a two-factor process given by

$$q_{a,t} = \varrho_{\alpha,t} + \varphi\left(q_{a,t-1} - \varrho_{\alpha,t-1}\right) + \varepsilon_t, \tag{6}$$

where $\varrho_{a,t}$ is the long-term trend component of MVaR, $q_{a,t} - \varrho_{\alpha,t}$ is the short-term cyclical deviation from the long-term trend and ε_t is a random error term with zero mean and constant variance. We assume that the long-term trend $\varrho_{a,t}$ is a stationary but highly persistent process but leave its precise dynamics unspecified. The parameter φ measures the speed of reversion of the cyclical component of MVaR to the long-term trend.

We implement the Two-Factor model (2FM) given by (6) in two steps. In the first step, we extract the long-run component $\varrho_{a,t}$ non-parametrically from the historical estimate of the a-quantile $\tilde{q}_{a,t}$. To do this, we use the low-pass filter of Hodrick and Prescott (1997) which extracts a low frequency non-linear trend from a time-series and is often employed in applied macroeconomics. To implement the Two-Factor MVaR model with the Hodrick-Prescott filter, we set the smoothing parameter to the commonly used value of 100 multiplied by the squared frequency of the data, which for daily data (assuming 240 trading days per year) is 5,760,000 (see, for example, Baxter and King, 1999).

In the second step, we estimate an autoregressive model for the cyclical component:

$$\tilde{q}_{a,t} - \tilde{\varrho}_{a,t} = \varphi\left(\tilde{q}_{a,t-1} - \tilde{\varrho}_{a,t-1}\right) + e_t, \tag{7}$$

where e_t is a zero mean random error. In order to forecast MVaR using the Two-Factor model, we assume that the long-term trend follows a random walk over the forecast horizon, so that the k-steps ahead forecast $\hat{\varrho}_{a,t+k} = \tilde{\varrho}_{a,t}$ for all $k > 0$, and use the estimated autoregressive parameter from (7) to forecast the cyclical component. The k-step ahead forecast of MVaR is therefore given by

$$\hat{q}_{a,t+k} = (1 - \tilde{\varphi}^k)\tilde{\varrho}_{a,t} + \tilde{\varphi}^k \tilde{q}_{a,t} \tag{8}$$

This is a weighted average of the current estimate of the long-term trend $\tilde{\varrho}_{\alpha,t}$ and the current estimate of MVaR $\tilde{q}_{\alpha,t}$. For the very long-term horizon, i.e., as $k \to \infty$, $\hat{q}_{\alpha,t+k} \to \tilde{\varrho}_{\alpha,t}$, with a speed that is determined by the estimated coefficient $\tilde{\varphi}$.

3.3. Scaled MVaR

So far, we have focused on forecasting frequency-*1* (daily) MVaR. However, often risk forecasts at lower frequencies are needed. For example, Basel Accords require financial institutions to model risk using a 10-day (i.e., frequency-10) holding period. It has become the industry standard to estimate daily VaR and then scale it up by $10^{1/2}$ in order to get the 10-day VaR. This is known as the square-root-of-time rule (SQRT-rule). The SQRT-rule originates in the scaling property of *i.i.d.* Gaussian variables $X_1, \dots, X_k$,

$$X_1 + X_2 + \cdots + X_k \overset{d}{=} k^{1/2} \cdot X_1$$

As the financial asset returns strongly violate the assumption of normality, neither moments of distributions (such as volatility) nor their quantiles should be scaled according to the SQRT-rule.[3]

Generally, the distribution of the random variables $X_1, \dots, X_k$ displays a scaling behavior if it holds that,

[3] Indeed, the Basel Committee in its technical guidance paper (Basel Committee on Banking Supervision, 2002) no longer suggests that the SQRT-rule be used, but that "in constructing VaR models estimating potential quarterly losses, institutions may use quarterly data or convert shorter period data to a quarterly equivalent using an analytically appropriate method supported by empirical evidence".

$$X_1 + X_2 + \cdots + X_k \overset{d}{=} k^{\delta} \cdot X_1,$$

where δ is the scaling exponent. Then, the α-quantile satisfies,

$$q_\alpha\left(\sum_{i=1}^{k} X_i\right) = k^{\delta} \cdot q_\alpha(X_1) \tag{9}$$

For many empirical distributions, the scaling property (9) is a good approximation only for nominal probability α sufficiently close to zero. For these distributions, one can estimate an extreme event at high frequencies for which there is an abundance of data (e.g., daily) and then use the scaling laws to estimate the extreme event at the lower frequency of interest (e.g., monthly; see Mandelbrot, 1997; McNeil and Frey, 2000; Gabaix, 2009). Taking the logarithm of (9),

$$\ln\left(q_\alpha\left(\sum_{i=1}^{k} X_i\right)\right) = \ln(q_\alpha(X_1)) + \delta \cdot \ln(k), \tag{10}$$

makes it obvious why a straight line on the log-log plot is called a signature of scaling law.

4. Empirical Evaluation of MVaR Forecasts

4.1. Statistical Evaluation of MVaR forecasts

There is a vast number of alternative methods for evaluating VaR forecasts (see, for example, Nieto and Ruiz, 2016 for a recent review). Due to their intuitive appeal and popularity among practitioners, we focus in what follows on three simple and mutually complementary tests. Although these tests have been designed for testing VaR accuracy, they clearly also apply to the univariate projection series $(v^d(\mathbf{x}_t))_{t=1}^{T}$.

Under the correct forecasting model, the proportion of $MVaR_\alpha^d$ violations, i.e., the proportion of projections $v^d(\mathbf{x}_t)$ of observation $\mathbf{x}_t$ that verify (3) should approach the nominal probability α for a sufficiently large sample. We refer to this procedure as unconditional accuracy. On the other hand, the conditional accuracy requires that the number of projections exceeding the MVaR-value should be unpredictable when conditioned on past violations. In other words, the MVaR violations should be serially uncorrelated. To assess both types of accuracy, we resort to the original unconditional accuracy test of Kupiec (1995) and the conditional accuracy test of Christoffersen (1998).

The test statistic of the unconditional accuracy test of Kupiec (1995) is given by,

$$\mathrm{t_u} = (\hat{\alpha} - \alpha)/\sqrt{\hat{\alpha}(\hat{\alpha} - \alpha)/T} \qquad (11)$$

where $\hat{\alpha}$ is the percentage of actual MVaR exceptions (violations), α is the nominal probability of exceptions and T is the number of observations. Intuitively, an unconditionally accurate model has an exception rate $\hat{\alpha}$ that is close to α.

The second, more stringent criterion regards the conditional accuracy. The likelihood ratio test of Christoffersen (1998) examines the serial independence of MVaR violations and is given by

$$\mathrm{LR_c} = 2(\mathrm{lnL_A} - \mathrm{lnL_0}) \tag{12}$$

where,

$$L_A = (1 - \Pi_{01})^{T_{00}} \Pi_{01}{}^{T_{01}} (1 - \Pi_{11})^{T_{10}} \Pi_{11}{}^{T_{11}},$$

$$L_0 = (1 - \Pi)^{T_{00}+T_{10}} \Pi^{T_{01}+T_{11}} (1 - \Pi_{11})^{T_{10}} \Pi_{11}{}^{T_{11}},$$

and

$$\Pi_{ij} = \frac{T_{ij}}{T_{i0} + T_{i1}},$$

$$\Pi = \frac{T_{01} + T_{11}}{T_{00} + T_{01} + T_{10} + T_{11}}.$$

T_{ij} is the number of times that state *j* follows state *i*. Here, state 0 obtains if no exceedence of MVaR forecast occurs and state 1 if such exceedence occurs. This statistic has an asymptotic χ^2 distribution with one degree of freedom, $\mathrm{LR}_c \to \chi^2(1)$.

Engle and Manganelli (2004) remark that unconditional and conditional accuracy are necessary but not sufficient conditions to assess the performance of a quantile forecasting model. They construct an example where unconditional exceedances are correct and serially uncorrelated but the conditional probability of violation, given the quantile forecast, differs dramatically from the nominal level. Their dynamic quantile (DQ) test aims at avoiding such errors. Complementary to Kupiec (1995) and Christoffersen (1998) tests, we use a version of the DQ statistic to test the null that the conditional coverage, given the MVaR forecast, is equal to the nominal level a,

$$DQ = \frac{\boldsymbol{hit}'\boldsymbol{q_a}\boldsymbol{q'_a}\boldsymbol{hit}}{a(1-a)\boldsymbol{q_a}'\boldsymbol{q_a}} \qquad (13)$$

where $\boldsymbol{hit}$ and $\boldsymbol{q_a}$ are $T \times 1$ column vectors containing $hit_t = I(v_t^d < q_{a,t}) - a$ and the MVaR forecasts $q_{a,t}$, respectively. This statistic has an asymptotic χ^2 distribution with one degree of freedom, $DQ \rightarrow \chi^2(1)$.

4.2. Data

We use three different datasets to evaluate the performance of the MVaR forecasting models: the main US and European stock indices as well as EU bond indices. The US stock index dataset contains daily closing prices for S&P 500, Dow Jones and Nasdaq considered here as proxies for the performance of the underlying general sectors; the European stock index dataset contains daily closing prices of FTSE100 (UK), DAX (Germany), CAC40 (France) and MIB30 (Italy) used here as proxies for the health of respective economies. Finally, the European bond index dataset contains daily closing prices of 10 year government bonds considered here as proxies for country risk. From the raw prices, we compute the continuously compounded daily returns covering the period from 21 February 2002 to 31 October 2015, 5000 daily observations for each return series. We use the first 2000 observations for the initial estimation and the remaining 3000 observations for evaluating the out-of-sample forecasts in which the estimation window is rolled forward daily.

For each set of returns, we compute the corresponding vector of standard deviations $\boldsymbol{SD}$. The projection $v^d(\mathbf{x}_t)$ for each observation $\mathbf{x}_t$ in this set is then computed by (2) for the directional vector $\boldsymbol{d} = -\boldsymbol{SD}$. This vector implicitly adjusts the projections for different volatilities. For consistency with the VaR literature, we multiply each projection $v^d(\boldsymbol{x})$ by -1 so that more extreme negative returns correspond to lower values of $-v^d(\boldsymbol{x})$.

Table 1 reports summary statistics for the daily log return series for the sample. Panel A reports the mean, standard deviation, skewness, excess kurtosis and the Bera-Jarque statistic for the log returns and their projections. Panel B reports the first six autocorrelation coefficients, the Ljung-Box Q statistic for autocorrelation up to six lags for the projections and the p-values. All series are highly non-normal with negative skewness and positive excess kurtosis. The excess kurtosis for bond returns is almost half that of the stock returns. The projected series are highly autocorrelated and in addition have also different empirical properties from the returns from which they originate due to the projection which for any set of returns obtains the minimum.

[Table 1]

Panel A of Figure 3 plots the projected US stock index returns (US Projections) and their "realized" daily MVaR over the period 2 January 2012 to 31 October 2015. The "realized" MVaR is estimated as the historical fifth quantile in the estimation window rolled forward daily. It is clear that the "realized" MVaR is slowly evolving. Panel B plots the same "realized" MVaR (note the different scale from Panel A) together with its long-term trend estimated using the Hodrick-Prescott filter over the sample. The trend is a smoothed version of the "realized" MVaR and closely tracks it although there are periods, for example during 2013, when the deviation is evident. Panel C of Figure 3 plots the resulting cyclical component of the "realized" MVaR using the Hodrick-Prescott filter. It is clear that the long-term trend in MVaR is time-varying and highly persistent, while the cyclical component is strongly mean-reverting, lending support to the two-factor representation of MVaR.

[Figure 3]

Figure 4 shows log-log plots of the frequencies $\{2^i\}_{i=0}^{7}$ days (x-axis) vs. the empirical frequency-2^i MVaR estimates for US Projections (y-axis) and the corresponding fitted straight lines. Estimates of frequency-2^i MVaRs have been computed from non-overlapping intervals of length 2^i, $i = \{0, \ldots, 7\}$ (i.e. one day to 6.4 months) spanning the whole sample of 5,000 observations. We find a good linear fit for all our datasets which indicates scaling in the tails of the projected return distributions. For the US (EU) Projections and for $\alpha = 1\%$, 2.5% and 5% the scaling exponents δ are 0.52 (0.53), 0.56 (0.57) and 0.59 (0.55) respectively, implying that the underlying distributions have fat tails. These estimates differ markedly from the estimates of around 0.42 in Hauksson et al. (2001) for the univariate VaRs which imply, counterfactually, thin tails for asset returns.

[Figure 4]

5. Results

The out-of-sample MVaR estimation is performed using the last 3000 observations. For the out-of-sample forecasts, we moved a window of $\mathrm{T} = 2000$ observations along the time axis. For each window $\mathbf{v}_t^d = (v_{t-T+1}^d, \dots, v_t^d)$ where $t = T, \dots, 3000 + T$, we first estimate the parameters $\boldsymbol{\beta}$ in (4) by solving the minimization problem (5) numerically and φ in (7) by a simple regression of the deviations $\hat{q}_{\alpha,t} - \tilde{\varrho}_{\alpha,t}$ on their one-lagged values. For each window, we compute also frequency-k returns in non-overlapping intervals of length $k = 2^i$, $i = \{0, \dots, 4\}$, within this window. From these returns, we estimate the frequency-k MVaR by the relevant quantiles and the scaling exponent by regressing the frequency-k log-MVaR on the log-frequencies log(k).

Subsequently, we use the estimated parameters to obtain MVaR forecasts as follows. For the CAViaR and Two-Factor models, the k-day ahead forecast $\hat{q}_{\alpha,t+k}$ of the daily MVaR-value is given directly by (4) and (8) respectively, where in the case of CAViaR $k = 1$. Finally, for the Scaling model the formula (9) delivers at date t a forecast of the frequency-k MVaR-value for the period $(t + 1, \dots, t + k)$.

The performance statistics for the MVaR forecasting models are presented in Tables 2 – 4. These tables report the actual exception rates ($\hat{\alpha}$) as well as the t_u, LR_c and DQ statistics to test the null hypotheses of unconditional and conditional accuracy for different MVaR specifications and nominal probability levels across the three datasets.

In line with previous evidence, CAViaR performs well for stock indices for one-day ahead forecasts both, conditionally and unconditionally. Indeed, the t_u statistics cannot reject the null of unconditional accuracy for all three nominal probabilities. Further, the LR_c and DQ statistics suggest that the conditional accuracy performance is satisfactory.

The results for the bond return projections are the exception. In all three cases, CAViaR generates exceptions that are considerably below the required nominal probability α. Perhaps, this should be expected as CAViaR is a model for forecasting the quantiles of series that are more prone to tail events. Focusing on the stock indices datasets (Tables 2 and 3), there are differences in performance for different levels of α: it appears that CAViaR is more accurate for higher α. For example, in the case of US indices for $\alpha = 5\%$ the actual exceedance rate $\hat{\alpha}$ is 5.2%, whereas for $\alpha = 1\%$ the actual exceedance rate is 1.3%. This finding is similar to findings in the VaR literature (e.g., Kuester et al. 2006).

[Table 2]

[Table 3]

[Table 4]

The Two-Factor model on the other hand appears to perform well for all three portfolios and at all nominal levels. At the longer end of the forecast horizon (60-day, i.e. approximately three months ahead), the forecast errors start to become considerable and the LR_c statistics suggest that the conditional accuracy performance of the model is inadequate. However, for the shorter horizons, the performance on balance, seems acceptable. Interestingly, the performance of the Two-Factor model appears more balanced with regard to α relative to CAViaR. For example, in the case of one-day ahead MVaR forecasts for US indices and $\alpha = 5\%$, the actual exceedance rate is $\hat{\alpha} = 5.6\%$, whereas for $\alpha = 1\%$ it is 1.7. However, in the case of European indices and for $\alpha = 5\%$

and 1%, these statistics are 5.4% and 1.3% respectively. This pattern can be observed for the longer horizon forecasts, although the relative errors of forecasts increase with horizon. For example, in the case of 60-day ahead MVaR forecasts for US indices and $\alpha = 5\%$ and 1%, the actual exceedance rates $\hat{\alpha}$ are 6% and 2.2%, while for the European indices these statistics are 5.8% and 1.7% respectively.

Importantly, the Two-Factor model performs remarkably well unconditionally for the bond indices and it would appear that the forecasts are more accurate than in the case of stock indices. Moreover, the accuracy does not deteriorate substantially with horizon (Table 4). For example, in the case of one-day ahead forecasts for $\alpha = 5\%$ the actual exceedance rate $\hat{\alpha}$ is 5%, whereas for $\alpha = 1\%$ the exceedance rate is 1.2%. In the case of a 60-day ahead forecasts these statistics are 5.1% and 1.4% respectively. The errors are smaller for the shorter horizons. However, the conditional accuracy tests suggest that violations are serially correlated for the one-day and 60-day ahead forecasts for $\alpha = 1\%$ but they improve for the intermediate horizons. For $\alpha = 2.5\%$ the conditional accuracy does not appear to change much with horizon and for $\alpha = 5\%$ it improves slightly with horizon. Thus, on balance the Two-Factor model produces unconditionally accurate MVaR forecasts for all datasets.

The Scaling model delivers frequency-k MVaR forecasts of reasonable unconditional accuracy, especially for shorter periods, except perhaps for the bond return projections. However, the Christoffersen (1998) test indicates that MVaR violations are highly serially correlated. This is not surprising given that we move a relatively long window of

2000 observations one day at each step. As a result, the resulting scaling forecasts change very slowly and cannot anticipate clusters of turbulence.

There is also an interesting performance discrepancy between bonds and stocks. For bonds, the Scaling model consistently generates pessimistic forecasts with actual exception rates below the nominal ones. For stocks, on the other hand, the Scaling model generates optimistic forecasts that are violated more often than they should. Somewhat surprisingly, the actual exception rate for $\alpha = 5\%$ tends to increase for longer periods. For example, for US indices the actual exception rates are 0.40, 0.45, 0.48, 0.53 and 0.52 for horizons of 1, 5, 10, 20 and 60 days ahead respectively. However, the Scaling model forecasts in this empirical exercise should be treated with caution as the scaling exponents (slopes of the regression lines in the log-log plots) have been estimated in each window from five 2^i −MVaRs ($i = 0, \dots ,4$) only.

For all three models, we observe that the DQ and the t_u statistics are well aligned (except in a few instances as e.g. for the 1% Scaling forecast 10-days ahead). However, there is no obvious relationship between the DQ and LR_c statistics. The intuition for this regularity is exemplified by a constant forecast. If this forecast generates a correct unconditional coverage, then the DQ statistic (13) takes on the value of zero even if violations are serially correlated. On the other hand, an unconditional actual coverage that deviates significantly from the nominal level will lead to a large value of the DQ statistic (13).

In line with the Two-Factor model (cf., equation (6)) we argue that an MVaR forecast has two components. We conjecture that the first component is slowly evolving and captures the evolution of macroeconomic or other (e.g., company) fundamentals. The second component captures the fast and occasionally violent but transitory movements perhaps reflecting investor sentiment or other short-lived effects. Changes in sentiment can trigger strong liquidity shocks with a significant impact on volatility (Campbell, Grossman and Wang, 1993). In the short run, a change in one set of prices may influence investor sentiment triggering changes in a seemingly unrelated set of prices (Eichengreen and Mody, 1998), thus leading to joint tail risk.

In this context, unconditional Kupiec (1995) and conditional Christoffersen (1998) tests can be intuitively linked to these two components of forecasts. The unconditional accuracy test effectively examines whether a model is consistent with the fundamentals and generates, over the long term, the correct exception rates. The conditional accuracy test, on the other hand, examines how well a forecasting model responds to the twists and turns of the market "animal spirits" which, by definition, are of a behavioral nature with little or no relationship to the long-term fundamentals.

This decomposition highlights the difficulty of long-term (M)VaR forecasting. A comprehensive forecasting model should not only capture the long-term general movements in fundamentals but also anticipate short-lived bursts of turbulence. As it is almost impossible to accurately forecast, well in advance, the latter component, it is too demanding to expect any long-term (M)VaR forecasting model to be conditionally accurate. Therefore, we argue that the adequacy of long-term (M)VaR forecasts should be judged primarily on the basis of the unconditional accuracy test. The conditional accuracy test, on the other hand, is relevant mainly for short-term (M)VaR forecasts. The practical implication of these observations is that institutions can only get an indication of average long-term exposures from these models but need to monitor their short-term exposures with short-term, conditionally accurate forecasting models such as CAViaR.

6. Conclusions

Aggregation of multiple sources of risk sidelines questions which are paramount for hedging, risk management and financial stability. Interesting answers can be obtained by considering the individual sources of risks jointly. We propose a simple and flexible framework to capture multidimensional tail risk. This framework allows for adapting the techniques and applications developed for unidimensional tail risk which is relatively straightforward even in higher dimensions.

We apply this framework to forecast multidimensional tail events out-of-sample at different horizons and evaluate them statistically. While short horizon forecasts are both conditionally and unconditionally accurate, we find that long horizon forecasts are unconditionally accurate but fail the conditional accuracy tests. However, we argue that this is to be expected. Conditional accuracy is too demanding a criterion for any long horizon (multidimensional) tail event forecasting model. Given our understanding of, and ability to model (multidimensional) tail events, only short horizon forecasts should be subjected to conditional accuracy tests. Long horizon forecasting models of (multidimensional) tail events should be judged primarily on their ability to generate unconditionally accurate forecasts. In this context, it would be interesting to understand the relationship of the long-term trend and short-term cycle of MVaR to macroeconomic and other fundamentals and investor sentiment, respectively.

Appendix

Proof of (3):

$$v^d(\boldsymbol{x}) \geq q_a^{\boldsymbol{d}} \Leftrightarrow \boldsymbol{x} \in MVaR_a^d$$

$$\Rightarrow: v^d(\boldsymbol{x}) = \min_{i:d_i \neq 0} \frac{x_i}{d_i} \geq q_a^{\boldsymbol{d}} \Rightarrow \frac{x_i}{d_i} \geq q_a^{\boldsymbol{d}}, \qquad \forall\, i: d_i \neq 0 \Rightarrow \boldsymbol{x} \in MVaR_a^d.$$

$$\Leftarrow: \boldsymbol{x} \in MVaR_a^d \Rightarrow \frac{x_i}{d_i} \geq q_a^{\boldsymbol{d}}, \qquad \forall\, i: d_i \neq 0 \Rightarrow \min_{i:d_i \neq 0} \frac{x_i}{d_i} = v^d(\boldsymbol{x}) \geq q_a^{\boldsymbol{d}}.$$

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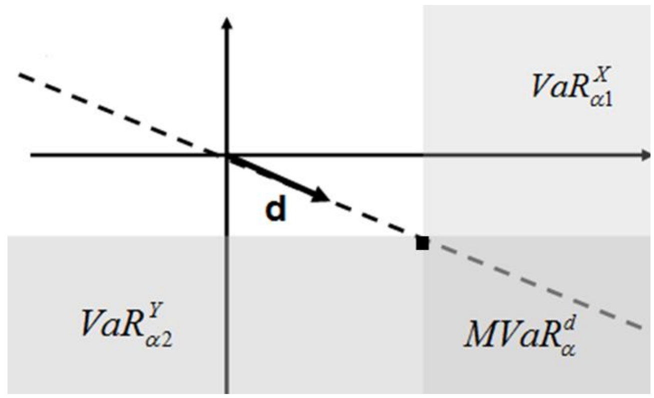
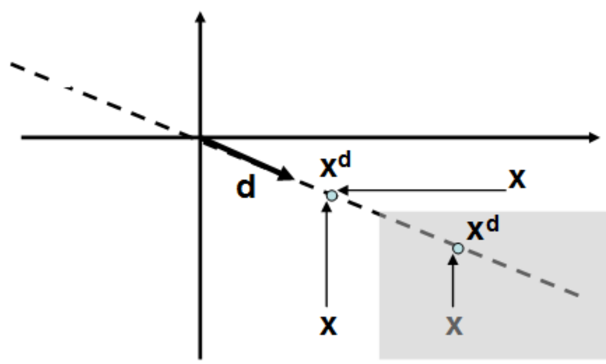
Figure 1: MVaR and its Decomposition into VaRs for *N*=2



Notes: $MVaR^d_a$ (dark shaded area) in the direction of the vector $\boldsymbol{d}$. Note that the upper left corner of $MVaR^d_a$ (indicated by the small black square) corresponds to the point $q^{\boldsymbol{d}}_a \cdot \boldsymbol{d}$ and that $MVaR^d_a$ is the intersection of univariate VaRs (light shaded areas).

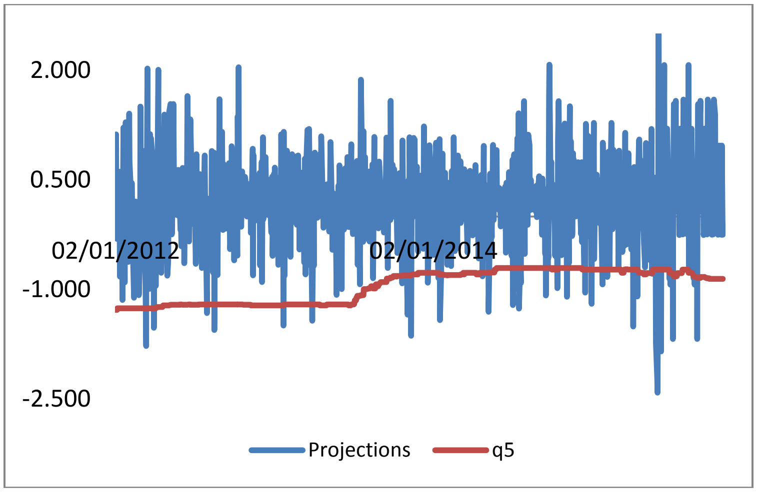
Figure 2: Projections (2) of Observations Inside and Outside of MVaR



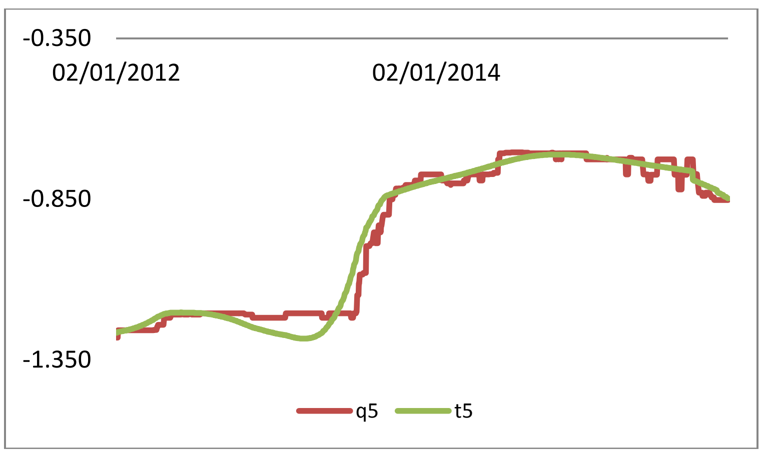
Notes: All points inside (outside) $MVaR^d_a$ (shaded area) are projected inside (outside) $MVaR^d_a$.

Figure 3 Decomposition of US Return Projections MVaR into Trend and Cycle Components

Panel A: Projected Returns and their "Realized" Fifth Quantile



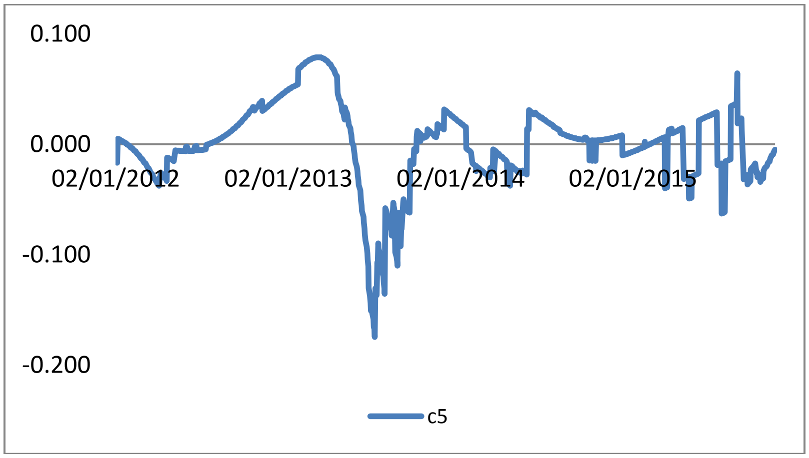
Panel B: Trend of the "Realized" Fifth Quantile Estimated from HP Filter



Notes: Panel A shows the "realized" MVaR estimator (q5) of the US stock indices return projections computed by equation (2). The sample period in the figure is 02/01/2012 to 31/10/2015 (1000 observations). Panel B shows the "realized" MVaR estimator (q5) and its long-run trend (t5) estimated with a Hodrick-Prescott filter with a smoothing parameter of 5,760,000. Panel C shows the cyclical component of the MVaR (c5) defined as the difference between the original series and the trend.

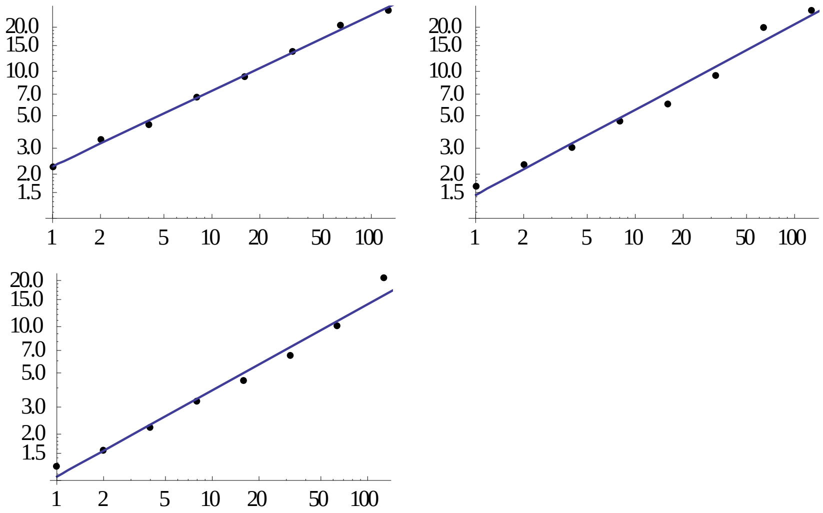
Figure 3 Decomposition of US Return Projections MVaR into Trend and Cycle Components

Panel C: The Cyclical Component of the "Realized" Fifth Quantile



Notes: Panel A shows the "realized" MVaR estimator of the US stock indices return projections computed by equation (2). The sample period in the figure is 02/01/2012 to 31/10/2015 (1000 observations). Panel B shows the "realized" MVaR estimator (q5) and its long-run trend (t5) estimated with a Hodrick-Prescott filter with a smoothing parameter of 5,760,000. Panel C shows the cyclical component of the MVaR (c5) defined as the difference between the original series and the trend.

Figure 4: MVaR Scaling for US Stock Indices



Notes: A log-log plot of empirical k-day MVaR (y-axis) at 1% (top left), 2.5% (top right) and 5% (bottom) computed from the returns of US stock indices at different frequencies (x-axis, k days). The respective scaling parameters (slopes) are 0.52, 0.56 and 0.59. The sample period is 21 February 2002 to 31 October 2015 (5000 observations).

Table 1: Summary Statistics and Autocorrelations

Panel A: Summary Statistics

	Mean	Standard Deviation	Skewness	Excess Kurtosis	Bera-Jarque
DJ30	0.023%	1.155%	-0.143	7.871	12924.742
SP500	0.023%	1.227%	-0.231	7.994	13356.985
NASDAQ	0.030%	1.615%	-0.050	5.391	6056.037
US Projections	-24.84%	97.88%	-0.634	7.698	12681.444
FTSE100	0.027%	1.122%	-0.215	6.071	7716.688
DAX	0.031%	1.265%	0.078	8.855	16339.224
CAC40	0.035%	1.294%	-0.086	4.524	4269.501
MIB30	0.026%	1.371%	-0.162	4.243	3772.200
E-S Projections	-40.59%	98.06%	-0.813	9.600	19752.610
UK Bonds	0.027%	0.381%	-0.006	2.146	959.166
German Bonds	0.025%	0.338%	-0.260	2.411	1267.072
French Bonds	0.025%	0.345%	-0.224	2.993	1908.178
Italian Bonds	0.031%	0.427%	0.537	3.414	1977.693
E-B Projections	-61.01%	98.67%	-1.634	13.544	40437.863

Panel B: Autocorrelations
Projected Returns

	1	2	3	4	5	6	Q	P-value
US Projections	-0.046	-0.031	0.009	0.004	-0.008	-0.005	16.356	0.012
E-S Projections	0.056	-0.019	-0.032	0.056	-0.017	0.009	40.056	0.000
E-B Projections	0.215	0.127	0.106	0.132	0.103	0.098	557.405	0.000

Notes: Panel A reports the mean, standard deviation, skewness, excess kurtosis and the Bera-Jarque statistic for daily log close-to-close returns for US stock indices DJ30, SP500 and Nasdaq, European stock indices FTSE100, DAX, CAC40 and MIB30 and 10 year bond prices for UK, Germany, France and Italy. The corresponding projections are computed for the directional vector of standard deviations of the relevant variables. The sample period is 21/02/2002 to 31/10/2015 (5000 observations). Panel B reports the first six autocorrelation coefficients and the Ljung-Box Q statistic for autocorrelation up to six lags, for projected US stock, EU stock and EU bond returns. P-values are also reported.

Table 2: MVaR Out-of-Sample Forecasting Results for US Stock Indices

k	**Model**	$\alpha = 1\%$				$\alpha = 2.5\%$				$\alpha = 5\%$			
		$\hat{\alpha}$	t_u	LR_c	DQ	$\hat{\alpha}$	t_u	LR_c	DQ	$\hat{\alpha}$	t_u	LR_c	DQ
1	CAViaR	0.013	1.592	1.081	1.121	0.027	0.567	0.009	1.095	0.052	0.493	1.521	1.773
	2FM	0.017	3.078	3.191	6.885	0.030	1.605	5.168	0.025	0.056	1.505	1.292	0.154
	Scaling	0.009	-0.186	14.83	0.388	0.023	-0.731	7.727	1.070	0.04	-2.795	6.533	3.123
5	2FM	0.018	3.408	2.689	8.362	0.032	2.096	8.493	2.196	0.057	1.595	1.935	0.093
	Scaling	0.006	-2.522	48.88	6.561	0.016	-3.731	91.78	21.16	0.454	-1.211	313.6	1.871
10	2FM	0.018	3.308	5.556	8.355	0.033	2.566	9.458	3.679	0.057	1.689	1.774	0.135
	Scaling	0.009	-0.770	118.0	2.355	0.015	-4.472	165.7	20.04	0.048	-0.474	533.0	0.997
20	2FM	0.018	3.322	5.493	6.419	0.034	2.680	9.025	1.705	0.057	1.655	1.790	0.021
	Scaling	0.011	0.558	215.7	1.283	0.014	-4.482	230.3	22.42	0.053	0.811	642.0	1.211
60	2FM	0.022	4.555	6.696	10.966	0.038	3.707	5.151	3.686	0.060	2.251	2.916	2.014
	Scaling	0.019	3.690	511.9	5.641	0.019	-2.210	511.9	7.965	0.052	0.494	793.4	1.910

Notes: The table reports the actual exception rate ($\hat{\alpha}$) for each MVaR forecasting model out of 3000 observations, (i.e. the proportion of times the forecasted MVaR is exceeded), the t-statistic to test the null hypothesis of unconditional accuracy (formula (11)) and the LR and DQ statistics (formulas (12) and (13), respectively) to test the null hypothesis of conditional accuracy for different confidence levels. The out-of-sample period of 3000 observations is 14 August 2007 to 31 October 2015. For CAViaR and 2FM models the daily MVaR forecasts are *k*-day ahead, while for Scaling the forecasts are for *k*-day period (*k*-day MVaR).

Table 3: MVaR Out-of-Sample Forecasting Results for European Stock Indices

k	**Model**	$\alpha = 1\%$				$\alpha = 2.5\%$				$\alpha = 5\%$			
		$\hat{\alpha}$	t_u	LR_c	DQ	$\hat{\alpha}$	t_u	LR_c	DQ	$\hat{\alpha}$	t_u	LR_c	DQ
	CAViaR	0.012	1.157	0.336	0.115	0.026	0.344	0.671	0.992	0.048	-0.512	0.973	1.564
	2FM	0.013	1.451	2.62	0.171	0.027	0.784	1.193	0.314	0.054	0.969	16.48	0.015
1	Scaling	0.009	-0.788	0.455	1.777	0.023	-0.731	2.745	1.981	0.046	-1.046	7.765	1.118
	2FM	0.013	1.313	2.771	0.087	0.027	0.795	1.178	0.472	0.055	1.217	17.42	0.053
5	Scaling	0.011	0.532	61.61	0.097	0.024	-0.346	218.2	1.655	0.049	-0.322	250	1.004
	2FM	0.013	1.606	2.436	0.407	0.028	1.125	2.235	0.307	0.056	1.314	19.54	0.256
10	Scaling	0.015	2.391	241.1	7.322	0.025	-0.142	346.3	0.531	0.051	0.449	560.4	.8821
	2FM	0.014	2.026	2.113	0.312	0.031	1.946	2.797	0.009	0.056	1.354	20.06	0.321
20	Scaling	0.021	4.226	366.4	12.891	0.034	2.68	459.1	1.879	0.057	1.58	749.2	1.231
	2FM	0.017	2.822	3.843	1.868	0.032	2.052	12.637	2.290	0.058	1.887	14.45	0.969
60	Scaling	0.026	4.496	449.5	9.281	0.036	3.128	707	2.989	0.058	1.887	890.2	1.525

Notes: The table reports the actual exception rate ($\hat{\alpha}$) for each MVaR forecasting model out of 3000 observations, (i.e. the proportion of times the forecasted MVaR is exceeded), the t-statistic to test the null hypothesis of unconditional accuracy (formula (11)) and the LR and DQ statistics (formulas (12) and (13), respectively) to test the null hypothesis of conditional accuracy for different confidence levels. The out-of-sample period of 3000 observations is 14 August 2007 to 31 October 2015. For CAViaR and 2FM models the daily MVaR forecasts are *k*-day ahead, while for Scaling the forecasts are for *k*-day period (*k*-day MVaR).

Table 4: MVaR Out-of-Sample Forecasting Results for European Bond Indices

k	**Model**	$\alpha = 1\%$				$\alpha = 2.5\%$				$\alpha = 5\%$			
		$\hat{\alpha}$	t_u	LR_c	DQ	$\hat{\alpha}$	t_u	LR_c	DQ	$\hat{\alpha}$	t_u	LR_c	DQ
1	CAViaR	0.006	-2.532	0.242	2.769	0.0187	-2.563	2.557	1.923	0.039	-3.001	3.546	5.011
	2FM	0.012	1.006	6.87	0.005	0.025	0.116	2.652	1.967	0.05	0	3.734	1.069
	Scaling	0.007	-1.97	0.296	0.889	0.019	-2.254	0.584	3.809	0.042	-2.085	2.246	3.588
5	2FM	0.012	1.013	3.12	0.009	0.026	0.242	3.455	1.608	0.05	0.1	5.027	1.898
	Scaling	0.007	-1.703	33.39	2.329	0.013	-5.786	103.2	20.46	0.041	-2.037	285.7	8.538
10	2FM	0.012	1.021	3.104	0.071	0.026	0.37	3.423	1.293	0.051	0.204	4.763	1.857
	Scaling	0.005	-3.859	46.19	6.105	0.013	-5.101	147	22.97	0.041	-2.546	377.2	7.152
20	2FM	0.012	0.882	3.251	0.910	0.026	0.286	3.359	1.179	0.05	-0.004	2.513	2.390
	Scaling	0.007	-1.929	138.4	3.921	0.011	-7.269	188.7	27.12	0.046	-0.963	594.8	2.005
60	2FM	0.014	1.823	10.96	1.259	0.025	0.056	2.738	1.123	0.051	0.164	1.634	1.391
	Scaling	0.004	-4.561	62.68	7.120	0.009	-11.50	111.1	33.62	0.053	0.896	892.0	0.592

Notes: The table reports the actual exception rate ($\hat{\alpha}$) for each MVaR forecasting model out of 3000 observations, (i.e. the proportion of times the forecasted MVaR is exceeded), the t-statistic to test the null hypothesis of unconditional accuracy (formula (11)) and the LR and DQ statistics (formulas (12) and (13), respectively) to test the null hypothesis of conditional accuracy for different confidence levels. The out-of-sample period of 3000 observations is 14 August 2007 to 31 October 2015. For CAViaR and 2FM models the daily MVaR forecasts are *k*-day ahead, while for Scaling the forecasts are for *k*-day period (*k*-day MVaR).