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Abstract

We provide conditions to allow modelling situations of asymmetric information in a tractable manner. In addition, we show a novel relationship between certain games of asymmetric information and corresponding games of symmetric information. This framework establishes links between certain games separately studied in the literature. The class of games considered is defined by scalable preference relations and a scalable information structure. We show that this framework can be used to solve asymmetric contests and auctions with loss aversion.

Key words: Asymmetric information, linear equilibria, global games.

JEL classification: D44, D82.

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1 Introduction

There are many economic situations in which each party knows something which the other parties do not know. For instance, a company may know its cost of producing a certain product but not know the costs faced by its competitors. Alternatively in a common value auction a bidder may know how much he thinks the object is worth but not know the estimates of other bidders. In our analysis we will refer to these situations where each party has some private information as *games of asymmetric information*.

The main contribution of this paper is to introduce an information structure that ensures situations of asymmetric information can be modelled in a tractable manner.

The key condition we consider is on the nature of players' private information - the scalable information structure. This condition ensures that a player's private information - his signal - does not provide him with information about how his signal compares to that of other players. That is to say after observing his signal a player cannot infer anything about whether the signal observed is a relatively high or relatively low compared to those of his opponents. We say that the scalable information structure exhibits *maximal rank uncertainty*, because a player's private information does not provide him with information about the rank of his payoff type.

To illustrate this information structure, consider the following examples. Abreu & Brunnermeier (2003) study the formation of asset price bubbles. Investors learn about the existence of a bubble, but they cannot infer whether they are likely to be among the first or the last to learn about the existence of a bubble. Similarly in the double auctions studied by Satterthwaite *et al.* (2014), bidders and the seller do not know whether their valuation is likely to be higher or lower than that of the other bidders and the seller. Their valuation does not provide them with information about the rank of their signal. The same information structure is also used in an application to contests and auctions with loss averse players considered in this paper.

All of the above are examples where a scalable information structure has been used to model a situation of asymmetric information which is intractable under alternative modelling assumptions. The equilibria in these examples are simple and easy to find. We refer to them as *constant strategy equilibria* as they are described by a single parameter for each player. In Abreu & Brunnermeier (2003)'s model of the formation of asset price bubbles, once he learns about the existence of the bubble, each investor decides to ride the bubble for a fixed amount of time independent of the time at which he learns. In the double auction example, all bidders shade their valuation by a constant amount, whatever their valuation. In our application to auctions and contests with loss averse bidders, players bid a constant proportion of their valuation or the effort exerted is a constant proportion of their ability respectively. The framework suggested here therefore allows us to study and make revenue comparisons for such auctions, which have received considerable attention in the literature, but cannot be solved under alternative modelling assumptions.[1]

Our entire analysis is restricted to games where players' preferences satisfy a mild condition, which is naturally satisfied in many situations and includes all cases where preferences are homogeneous of some degree k among others.

While the focus of our analysis lies on situations of asymmetric information, the theoretical contribution of our paper is to provide a link between games of asymmetric information and games of symmetric information.

One can think of a common situation where all parties share the same information, but there is some additional information which cannot be accessed by any of the parties. Extreme weather events are one example of such a situation, since all parties have the same information from a central weather forecast agency but still face uncertainty. In our analysis, these situations where all parties involved have the same information but nevertheless there is some uncertainty are referred to as *games of symmetric information*.

We show that under the scalable information structure and mild restrictions on players' preferences, games of asymmetric information are strategically equivalent to games of symmetric

[1] See Gill & Prowse (2012), Lange & Ratan (2010) and Eisenhuth & Ewers (2015) for example.

information. Although these games differ only in what is observed by the players, they are typically used to study very different situations. Our framework therefore provides a novel link between seemingly unrelated classes of games. We illustrate this link by demonstrating a connection between a second price auction with a reserve price where in one case participants know their valuation, but do not know the reserve price and in the other case they know the reserve price but are uncertain about their valuation. Another application using our framework to solve an otherwise complex situation considers the bargaining process to dissolve a business following bankruptcy.

The remainder of the paper is structured as follows. In the remainder of this section we relate the suggested approach to the literature. In section two, we illustrate the use of the framework in two simple examples. The key property of maximal rank uncertainty is discussed in detail section three. Section four introduces the model. The scalable structure is presented in section five, while section six contains the simplicity analysis. In section seven we present present an application to auctions with loss aversion to illustrate the simplicity and tractability of our model. Restricting attention to settings where players' preferences can be represented by utility functions, we introduce the equivalence between scalable games of asymmetric information and scalable games of symmetric information in section eight. The relevance of this link is illustrated in section nine. Section ten concludes.

1.1 Related Literature

In the literature, models which satisfy the scalable preference and scalable information structure considered in this paper have been used to model specific situations of uncertainty. As mentioned above, the formation of asset price bubbles studied by Abreu & Brunnermeier (2003) and the double auctions considered by Satterthwaite *et al.* (2014) are two such models. Other examples include the clock games considered by Brunnermeier & Morgan (2010), as well as supply function competition studied by Vives (2011). While these papers provide models for specific situations, we aim at providing a general tool to model situations of asymmetric information.

The information structure of the proposed class of games of asymmetric information - referred to as scalable games - has close links with the literature on *global games* introduced by Carlsson & Van Damme (1993) and considered in Morris & Shin (2002) and Morris & Shin (2003) among others. As in global games, players face uncertainty about the state of the world θ which is drawn from a diffuse prior. Moreover each player does not observe θ but instead receives a partially informative signal s_i about the state of the world, where $s_i = \theta + z_i$ and z_i can be interpreted as a noise term. However, in global games the main objective is equilibrium selection. This is achieved using the fact that coordination is more difficult when the state of the world is unknown. Moreover, the games considered in our paper do not necessarily have dominance regions and a player's signal typically enters his payoff function directly. Above all the focus of this paper lies on the *characterization* of equilibria in games of asymmetric information rather than equilibrium *selection* in games of complete information.

The framework proposed in our paper also has close ties with the literature on *quadratic utility models* [2] In these games there is also uncertainty about the state of the world and players receive a noisy signal of the state. Quadratic utility models typically focus on the social value of information and the role of information acquisition.[3] Applications to Cournot competition are provided by Vives (1988) and Myatt & Wallace (2013).

As in our paper, each player receives a signal about the state of the world which can be interpreted as his type and may enter his payoff function directly. The payoff functions in most quadratic utility models depend on the actions of others only through the aggregate. Scalable games of asymmetric information require weaker conditions on the preference structure - for example allowing for loss aversion - at the cost of making stronger distributional assumptions on the state and the signals. The information structure in a quadratic utility model is affine, satisfying the assumption that $E[\theta|s_i] = \alpha s_i + \beta$; in the related scalable game

[2] A comprehensive treatment of these games is provided in Angeletos & Pavan (2007) for a continuum of players, while Ui & Yoshizawa (2014) consider a discrete number of players.

[3] For models with endogenous information structures see for example Colombo & Pavan (2014) Myatt & Wallace (2012) and Pavan (2014).

in additive form we require the shape of the distribution to be the same for all types and hence $E[\theta|s_i] = s_i + \beta$.

In a recent paper, Morris *et al.* (2015) propose the concept of uniform rank belief. When there are two players, the authors say that players have a uniform rank belief if each of them assigns probability $\frac{1}{2}$ to having a higher payoff type than his opponent independent of his payoff type. Meanwhile the maximal rank uncertainty property suggested in this paper, says that the probability each player assigns to being in any particular rank is independent of his type, but it need not be equal to $\frac{1}{2}$ or $\frac{1}{n}$ in the case of n players.

Finally considering a translation from one game to a strategically equivalent game, which is easier to solve, has been proposed by Baye & Hoppe (2003) in the case of rent seeking and patent races. However they consider relationships between games of complete information, while we consider translations from a game of asymmetric information to a game of symmetric information. The aim to model situations of incomplete information in a tractable manner is also pursued by Compte & Postlewaite (2013) who consider a private value first price auction, where bidders shade their bid by a constant amount, independent of their valuation.

2 Illustrative Examples

We now introduce a simple example to illustrate the strategic equivalence of certain games that are closely related, but have a different information structure. Three cases are distinguished: (i) a game of asymmetric information, where a player faces uncertainty about the signals of other players, (ii) the case where players have symmetric information, but nevertheless there is some uncertainty, and (iii) a complete information game.

2.1 Single-player Example

Consider a game, where there is one buyer wanting to buy a product. His valuation for the product is given by $s \in (0, \infty)$. The reserve price for the product is given by $\theta \in (0, \infty)$.

The buyer offers to pay a fraction of his valuation $a \in \{\frac{1}{3}, \frac{1}{2}\}$. Hence, the suggested price is given by $p(s) = as$. In case the price offered by the player is higher than the reserve price, he obtains a payoff of $u(a, s, \theta) = s(1 - a)$ if $sa \geq \theta$. If the proposed price is below the reserve price the buyer obtains a payoff of zero: $u(a, s, \theta) = 0$ if $\theta > as$.

Suppose that θ is determined according to an improper prior with density function $g(\theta) = \frac{1}{\theta}$ for all $\theta \in (0, \infty)$. Furthermore, assume that for any level of θ, the conditional distribution of s is given as follows:

$$f(s|\theta) = \begin{cases} \frac{1}{4} & \text{if } s = 2\theta \\ \frac{3}{4} & \text{if } s = 3\theta \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

First consider the game where the buyer observes his valuation s but does not know the reserve price θ. This is the game we refer to as a *scalable game of asymmetric information.* By Bayesian updating the buyer assigns a probability of $g(\theta|s) = \frac{2}{11}$ to the case $\theta = \frac{s}{2}$ and assigns the remaining probability $g(\theta|s) = \frac{9}{11}$ to the case $\theta = \frac{s}{3}$.

The buyer's expected payoff from offering $a = \frac{1}{2}$ is given by $E\left[u(\frac{1}{2}, s, \theta)|s\right] = s(1 - \frac{1}{2}) = \frac{s}{2}$. Meanwhile his expected payoff from choosing $a = \frac{1}{3}$ is given by $E\left[u(\frac{1}{3}, s, \theta)|s\right] = \frac{9}{11}s(1-\frac{1}{3}) = \frac{6s}{11}$. Hence, the buyer prefers to offer $a = \frac{1}{3}$ independent of his valuation.

Now consider instead the case where the buyer observes the reserve price - the price displayed at a shop - but does not know his valuation. This is the game we refer to as a *scalable game of symmetric information.*

His expected payoff from offering $a = \frac{1}{2}$ is given by $E\left[u(\frac{1}{2}, s, \theta)|\theta\right] = \frac{1}{4}2\theta(1-\frac{1}{2}) + \frac{3}{4}3\theta(1-\frac{1}{2}) = \frac{11\theta}{8}$. Meanwhile his expected payoff from choosing $a = \frac{1}{3}$ is given by $E\left[u(\frac{1}{3}, s, \theta)|\theta)\right] = \frac{3}{4}3\theta(1 - \frac{1}{3}) = \frac{3\theta}{2}$. Again the buyer prefers to offer $a = \frac{1}{3}$ independent of the reserve price.

Moreover, note that the ratio of payoffs is the same in both cases and is given by $\frac{12}{11}$. As will become clear later, this structure satisfies our scalability assumptions. We will show that these two games are equivalent.

Second, these games were very easy to solve. The optimal decision of a player does not depend on his valuation or the reserve price. In fact any games in the class of scalable games proposed in this paper can be solved by looking at the optimal action for a buyer with a valuation $s = 1$ (or a reserve price $\theta = 1$). It is not necessary to consider the optimal decision for each valuation (reserve price) separately. This also means that the game is strategically equivalent to a game of complete information which one could choose to solve instead.

2.2 Multi-player Example

To further illustrate this concept and show that the framework can also capture games with several players and using a different information structure, we now present a second example.

Consider a world with two competing countries labeled $\{1, 2\}$ who actively exert their influence in a certain region. At time θ a new militant group emerges, which threatens the security of one country but furthers the interests of the other.

Each country does not immediately learn of this new development, but rather finds out at some time s_i. After learning of existence of the militant group, each country must choose how long to wait until deciding upon a response. This waiting time is denoted by $a_i \geq 0$. It is assumed that decisions are immediately put into action. Since better intelligence will lead to more effective intervention, it is assumed that the payoff associated with executing an action after waiting for a time a_i is $\overline{u}_i(a_i, s_i) = a_i$. However so that the two countries do not enter into direct conflict, only the first action chosen is executed, and the second mover receives a payoff of $\underline{u}_i(a_i, s_i) = 0$. In case both countries move at the same time, we assume that country A's move is executed.[4] Moreover it is assumed that countries have no prior information about when the new group will emerge, and this is modelled by θ being drawn from a diffuse prior with $g(\theta) = 1$ for all $\theta \in \mathbb{R}$. Furthermore we assume that $s_i = \theta + z_i$, where each z_i is independent of θ and is distributed uniformly over the interval $[0, 1]$. Each country observes its signal, the time at which it learns s_i, but does not observe θ.

[4]This is a measure zero event and the choice of tie-breaking rule does not affect the results.

As will be clear from the formal definition that follows, this game is a scalable game of asymmetric information.

In order to solve this scalable game we look for a symmetric equilibrium in constant strategies of the form $\sigma_i(s_i) = a^*$. Since constant strategies directly imply monotonicity of reactions, the maximization problem of country i can be written as follows:

$$\max_{a_i} \int_{s_i-1}^{s_i} a_i g(\tilde{\theta}|s_i)(1 - F(s_i + a_i - a_j|\tilde{\theta}))d\tilde{\theta}$$

The key condition we consider is on the nature of players' private information - the scalable information structure. This condition ensures that a player's private information does not provide him with information about how his payoff type compares to that of other players. That is to say after observing his signal a player cannot infer anything about whether the signal observed is a relatively high or relatively low compared to those of his opponents. The scalable information structure exhibits *maximal rank uncertainty*, because a player's private information does not provide him with information about the rank of his payoff type.

$$\int_{s_i-1}^{s_i} (1 - F(s_i + a_i - a_j|\tilde{\theta}))d\tilde{\theta} = \int_{s_i-1}^{s_i} f(s_i + a_i - a_j|\tilde{\theta})d\tilde{\theta}$$

Since we are looking for a symmetric equilibrium $a_i = a_j$. Moreover from the information structure, we know that $\int_{s_i-1}^{s_i}(1 - F(s_i|\tilde{\theta}))d\tilde{\theta} = 0.5$ for all s_i. Independent of its signal, each country is always equally likely to have the lower or to have the higher signal. We also know that $\int_{s_i-1}^{s_i} f(s_i|\tilde{\theta})d\tilde{\theta} = 1$ and hence $\sigma_i(s_i) = 0.5$. It turns out that this strategy is an equilibrium in constant strategies.

Consider now that rather than after a delay, both countries learn of the emergence of the new militant group immediately and hence observe the state θ. Again, each country chooses how long to wait until deciding upon its response, a_i. However, in this version of the game there is a delay between the decision to act and the implementation of the action itself. This delay is given by $z_i = s_i - \theta$, where again z_i is drawn from a uniform distribution over the interval $[0, 1]$ for each $i \in \{1, 2\}$. Country i is the first mover only if $a_i + s_i < a_j + s_j$ and in this case country i receives a payoff of $\overline{u}_i(a_i, s_i) = a_i$. The second mover again receives a payoff of $\underline{u}(a_i, s_i) = 0$. The maximisation problem for each country looks as follows:

$$\max_{a_i} a_i\Big(1 - \int_{\theta}^{\theta+1} F_i(\tilde{s}_i + a_i - a_j|\theta)d\tilde{s}_i\Big)$$

Taking first order conditions leads to:

$$1 - \int_{\theta}^{\theta+1} F_i(\tilde{s}_i + a_i - a_j|\theta)d\tilde{s}_i = a_i \int_{\theta}^{\theta+1} f(\tilde{s}_i + a_i - a_j|\theta)$$

In a symmetric equilibrium we know that $a_i = a_j$. Moreover by the scalable information structure, $\int_{\theta}^{\theta+1} F_i(\tilde{s}_i + a_i - a_j|\theta))d\tilde{s}_i = 0.5$ and $\int_{\theta}^{\theta+1} f(\tilde{s}_i|\theta)d\tilde{s}_i = 1$. Therefore $\sigma_i = 0.5$ for i, j is an equilibrium of this game.

3 Key property: Maximal rank uncertainty

The key property driving the results in these examples is what we refer to as *maximal rank uncertainty*. We now consider this property in some more detail. Suppose there is a set of players $I = \{1, 2, ..., n\}$. Further suppose that each player i receives a signal $s_i \in \mathbb{R}$.

We now define the rank r_i of player i as follows. Let the set $\overline{I}_i = \{j : s_j \geq s_i\}$, so that $j \in \overline{I}_i$ only if the signal of player j is greater or equal to the signal of player i. With this notation in mind, define $r_i = |\overline{I}_i|$. This means that r_i captures the number of players with a signal greater than or equal to s_i. Hence if $r_i = 1$ then player i has the highest signal amongst all players and if $r_i = n$ then player i has the lowest signal amongst all players. The key property of our model can be informally stated as follows:

$$P(r_i = m|s_i) = P(r_i = m|s_i') \text{ for all } s_i, s_i', m, i$$

This equation captures that fact that the probability any player i assigns to having the n-highest signal is independent of his signal. It means that a player's signal does not give him information about his relative position compared to other players. This contrasts with a model where players have independent types. For instance consider a two player model where (i) $I = \{1, 2\}$, (ii) s_i and s_j are drawn from a uniform distribution over $[0, 1]$ and (iii) s_i and s_j are drawn independently. In this case:

$$P(r_i = 1|s_i) = s_i \text{ for all } s_i$$

This equation captures the fact that a player who observes a signal s_i close to 1 is very confident that he has the highest signal and $s_i = 1$, while a player who observes a signal s_i close to 0 is very confident that he has the lower signal and $r_i = 2$.

Hence in a model with independent types players gain *rank information* about their relative position compared to other players when they observe their signal. Meanwhile the key property in our model ensures that players do not gain *rank information* from observing their signal. We believe this to be a more appropriate way to model certain situations such as some auctions where a bidder's valuation may not help him decide whether he has the highest valuation or not (this would be the case in situations where having a higher valuation increases the likelihood that other bidders also have a high valuation).

4 The Model

We now introduce the general model and formally define a class of games with maximal rank uncertainty, capturing the illustrative examples above.

Consider a finite set of players $I = \{1, ..., n\}$. The state is denoted by $\theta \in \Theta = (\underline{\theta}, \overline{\theta})$ and each player is associated with a signal $s_i \in S_i = (\underline{s}_i, \overline{s}_i)$. In most applications, this signal can be thought of as a player's type and hence describing his preferences. For simplicity we consider $S_i = \Theta$ for all $i \in I$. We expect the results to hold for any open interval S_i.

Each player i simultaneously and independently chooses an action $a_i \in A_i \subseteq \mathbb{R}$. Action sets may be player specific. To ease notation we use $\mathbf{s} = (s_1, \ldots, s_n)$ to denote the vector of players' signals and $\mathbf{a} = (a_1, \ldots, a_n)$ to denote the vector of players' actions. Moreover we define $\omega = (\mathbf{a}, \theta, \mathbf{s})$ to be an outcome described by a vector of actions $\mathbf{a}$ the state θ and the vector of all players signals $\mathbf{s}$. Let Ω denote the set of outcomes.

In order to cover both expected and certain non-expected utility frameworks, we state players' preference relations in terms of lotteries over outcomes. A lottery $L \in \mathbb{L}$ is a cumulative distribution over the outcomes Ω, $L : \Omega \mapsto [0,1]$, where an outcome ω is given by $(\mathbf{a}, \theta, \mathbf{s})$. Each player has a preference relation over the set of lotteries $\mathbb{L}$. It is assumed that these preference relations are complete, continuous and transitive. But crucially, we do not assume the independence axiom.

The information structure is given as follows. The distribution of each player's signal s_i is given by $F_i(s_i^*|\theta)$. These distributions are assumed to be independent conditional on θ and $F(\mathbf{s}|\theta)$ denotes the distribution of all players' signals conditional on θ. In addition we assume that these conditional distributions have a density, which we denote by $f_i(s_i|\theta)$.

It is crucial in our analysis that we allow θ to have an improper prior distribution. To this aim, we define the prior over θ with a function $g : \Theta \to [0, \infty)$. The probability that $\theta \leq \theta^*$ given s_i, $G(\theta^*|s_i)$ is given as follows:

$$G_i(\theta^*|s_i) = \int_{\underline{\theta}}^{\theta^*} \frac{f_i(s_i|\theta)g(\theta)}{\int_\Theta f_i(s_i|\tilde{\theta})g(\tilde{\theta})d\tilde{\theta}} d\theta \tag{2}$$

The case of a proper prior corresponds to the case in which (i) g plays the role of a density and (ii) (2) is the standard Bayes rule. However, the above formulation also allows for improper priors in which $\int_\Theta g(\theta)d\theta = \infty$. We use $g_i(\theta^*|s_i)$ to denote the probability density function corresponding to $G_i(\theta^*|s_i)$.

To simplify notation, we use Γ to summarise the primitives of the model:

$$\Gamma \equiv \{I, \Theta, (A_i)_{i \in I}, (\succeq_i)_{i \in I}, g, (F_i)_{i \in I}\}$$

We study two cases of this basic model. The difference between the two lies in the source of the uncertainty. First, we consider the case where each player privately observes his signal s_i, but does not observe the state. This is the game we refer to as a *game of asymmetric information.* Second, we consider the case where all players observe the state, but do not observe their private signals s_i. This is the game we refer to as a *game of symmetric information*.

We write $\mathcal{A}(\Gamma)$ to denote the game of asymmetric information where each player privately observes s_i while θ is not observed. Similarly $\mathcal{S}(\Gamma)$ is used to denote the game of symmetric information where θ is commonly known among all players, but the signals s_i are not observed.

4.1 Strategies

In order to avoid introducing additional notation, we will jointly define the strategies used in games of asymmetric information and games of symmetric information, despite the differences in what is observed by the players.

A strategy for player i is described by a cumulative distribution function over actions, conditional on the state θ and on the player's signal s_i and is denoted by $\sigma_i(a_i, \theta, s_i)$. This notation allows us to capture both mixed strategies and pure strategies succinctly. We use σ to denote $(\sigma_1, \ldots, \sigma_n)$.

In a game of asymmetric information $\mathcal{A}(\Gamma)$, players do not observe the state θ and hence feasible strategies are constant in θ. The set of strategies which are constant in θ is denoted by $\Sigma^{\mathcal{A}}$, as it is the set of feasible strategies under asymmetric information: $\sigma_i^{\mathcal{A}}(a_i, \theta, s_i) = \sigma_i^{\mathcal{A}}(a_i, \theta', s_i)$ for all θ, θ', s_i and a_i. A typical element in this set for player i is denoted by $\sigma_i^{\mathcal{A}}$.

Similarly, in a game of symmetric information $\mathcal{S}(\Gamma)$, players do not observe their signal s_i, and we require the strategy to be constant in s_i. The set of such strategies is denoted by $\Sigma^{\mathcal{S}}$ and describes the feasible strategies in a game of symmetric information: $\sigma_i^{\mathcal{S}}(a_i, \theta, s_i) = \sigma_i^{\mathcal{S}}(a_i, \theta, s_i')$ for all θ, s_i, s_i' and a_i. A typical element in this set for player i is denoted by $\sigma_i^{\mathcal{S}}$.

Our analysis makes use of strategies which are constant in both s_i and θ. We refer to these strategies as *constant strategies*. The set of these strategies is given by Σ^C and a typical element in this set for player i is denoted σ_i^C: $\sigma_i^C(a_i, \theta, s_i) = \sigma_i^{\mathcal{C}}(a_i, \theta', s_i')$ for all $\theta, \theta', s_i, s_i'$ and a_i. A typical element in this set for player i is denoted by $\sigma_i^{\mathcal{C}}$.

In the examples mentioned in the introduction, these constant strategies correspond to all investors riding the bubble for a fixed amount of time, shading their valuation by a constant amount in the double auction or bidding a constant fraction of their valuation respectively.

4.2 Equilibria

Suppose (i) player i has observed a signal s_i^*, (ii) player i chooses an action $a_i \in A_i$ and (iii) other players play according to the strategy profile $\sigma^{\mathcal{A}}$ where $\sigma^{\mathcal{A}}(\mathbf{a}|\mathbf{s}) = \prod_{j\in I} \sigma^{\mathcal{C}}(a_j)$. We define $L_i^{\mathcal{A}}[s_i^*; a_i, (\sigma_j^{\mathcal{C}})_{j\neq i}]$ to capture the weights that player i assigns to different outcomes in this situation. Hence:

$$L_i^{\mathcal{A}}[s_i^*; a_i, (\sigma_j^{\mathcal{C}})_{j\neq i}](\mathbf{a}, \mathbf{s}, \theta) = \begin{cases} \left(\prod_{i\neq j} \sigma_j^{\mathcal{C}}(a_j)\right) \int_{\underline{\theta}}^{\theta} \int_{\underline{\mathbf{s}}_{-\mathbf{i}}}^{\mathbf{s}_{-i}} g(\tilde{\theta}|s_i^*) f(\tilde{\mathbf{s}}_{-i}|\tilde{\theta}) d\tilde{\mathbf{s}}_{-i} d\tilde{\theta} \text{ if } s_i \geq s_i^* \text{ and } a_i \geq a_i^* \\ 0 \text{ otherwise} \end{cases}$$

The equilibrium for a game of asymmetric information $\mathcal{A}(\Gamma)$ can now be defined as follows:

Definition 1 (Constant strategy equilibrium: Game of asymmetric information)**.** *A strategy profile $\sigma^{\mathcal{A}} \in \Sigma^{\mathcal{A}}$ is a constant strategy equilibrium of the game $\mathcal{A}(\Gamma)$ if for all players $i \in I$ and for all signals $s_i^* \in S_i$ (i) $\sigma^{\mathcal{A}}(\mathbf{a}|\mathbf{s}^*) = \prod_{j\in I} \sigma_j^{\mathcal{C}}(a_j)$ and (ii) for all actions $a_i^* \in supp(\sigma_i^c)$ and all deviations $\hat{a}_i \in A_i$ it holds that:*

$$L_i^{\mathcal{A}}[s_i^*; a_i^*, (\sigma_j^{\mathcal{C}})_{j\neq i}] \succeq_i L_i^{\mathcal{A}}[s_i^*; \hat{a}_i, (\sigma_j^{\mathcal{C}})_{j\neq i}]$$

This definition says that the constant strategy profile $\sigma^{\mathcal{A}}$ is an equilibrium, if each player i - given that he observes signal s_i^* - weakly prefers the lottery generated by choosing any optimal action $a_i^* \in \text{supp}\sigma_i^c$ compared to the lottery generated by choosing any alternative action $\hat{a}_i$. Although this definition only considers constant strategy profiles, it allows for arbitrary deviations. Hence a constant strategy equilibrium of $\mathcal{A}(\Gamma)$ is also a Bayesian Nash equilibrium of $\mathcal{A}(\Gamma)$.

Suppose now that (i) the state is known to be θ^*, (ii) player i chooses action a_i^* and (iii) other players play according to the strategy profile $\sigma^S(\mathbf{a}|\theta^*)$ where $\sigma^{\mathcal{S}}(\mathbf{a}|\theta^*) = \prod_{j\in I} \sigma_j^{\mathcal{C}}(a_j)$. We define $L_i^{\mathcal{S}}[\theta^*; a_i^*, (\sigma_j^{\mathcal{C}})_{j \neq i}]$ to capture the weights that player i assigns to different outcomes in this situation. Therefore:

$$L_i^{\mathcal{S}}[\theta^*; a_i, (\sigma_j^{\mathcal{C}})_{j \neq i}](\mathbf{a}, \mathbf{s}, \theta) = \begin{cases} \left(\prod_{i \neq j} \sigma_j^{\mathcal{C}}(a_j) \right) \int_{\underline{\mathbf{s}}}^{\mathbf{s}} f(\tilde{\mathbf{s}}|\theta^*) d\tilde{\mathbf{s}} \text{ if } \theta \geq \theta^* \text{ and } a_i \geq a_i^* \\ 0 \text{ otherwise} \end{cases}$$

A constant strategy equilibrium in this game of symmetric information $\mathcal{S}(\Gamma)$, can now be defined as follows:

Definition 2 (Constant strategy equilibrium: Game of symmetric information)**.** *A strategy profile $\sigma^{\mathcal{S}} \in \Sigma^{\mathcal{S}}$ is a constant strategy equilibrium of the game $\mathcal{S}(\Gamma)$ if for all states $\theta^* \in \Theta$ and for all players $i \in I$ (i) $\sigma^{\mathcal{S}}(\mathbf{a}|\theta^*) = \prod_{j \in I} \sigma_j^{\mathcal{C}}(a_j)$ and (ii) for all actions $a_i^* \in supp(\sigma_i)$ and all deviations $\hat{a}_i \in A_i$ it holds that:*

$$L_i^{\mathcal{S}}[\theta^*; a_i^*, (\sigma_j^{\mathcal{C}})_{j \neq i}] \succeq_i L_i^{\mathcal{S}}[\theta^*; \hat{a}_i, (\sigma_j^{\mathcal{C}})_{j \neq i}]$$

This definition says that the constant strategy profile $\sigma^{\mathcal{S}}$ is an equilibrium, if each player i - given that the state is θ - weakly prefers the lottery generated by choosing choosing any optimal action $a_i^* \in \text{ supp}\sigma_i$ compared to the lottery generated by choosing any alternative action $\hat{a}_i$. Again note that while this definition only considers constant strategy profiles, it allows for arbitrary deviations and hence constant strategy equilibria of $\mathcal{S}(\Gamma)$ are also Bayesian Nash equilibria of $\mathcal{S}(\Gamma)$. In the next section we propose conditions on the preference relation and information structure which help ensure that constant strategy equilibria exist.

5 Scalable primitives

We now propose conditions on the players' preference relations and the information structure which lead to games with the desired scalability properties.

In order to define scalable games in a general framework, we use a generator and an operator to state the required conditions. A *generator*, denoted by H, is a strictly increasing bijection

Θ	$H(\theta)$	$a \oplus_H b$	$a \ominus_H b$
$\mathbb{R}$	θ	$a+b$	$a-b$
$\mathbb{R}_{++}$	$ln(\theta)$	$a \times b$	$a \div b$

Table 1: The generator

from Θ to $\mathbb{R}$. We also assume that it is differentiable.[5] Secondly the *operator* associated to the generator H, denoted by $\oplus_H$, maps any two numbers $(a, b) \in \Theta^2$ into the unique number $a \oplus_H b \in \Theta$ that solves:

$$a \oplus_H b \equiv H^{-1}\Big(H(a) + H(b)\Big)$$

The operator $\ominus_H$ is defined symmetrically as $a \ominus_H b \equiv H^{-1}\Big(H(a) - H(b)\Big)$. [6] An obvious example is $H(x) = x$. In this case, the operators $\oplus_H$ and $\ominus_H$ are the usual sum and subtraction, respectively. Another example is the case when $H(x) = \ln(x)$. Here the operators $\oplus_H$ and $\ominus_H$ are the usual multiplication and division, respectively.

In some cases it is useful to consider a reference point for either the signal of player i or the state. For a generator H, we use $0_H := x$ such that $H(x) = 0$.[7] Returning to the illustrative examples, the reference point in the case of a single buyer wanting to buy an object would be his valuation $s = 1$, or the reserve price $\theta = 1$, while in the two country example, the reference point corresponds to the case where a firm learns about the existence of the military group at time zero or where the military group is formed at time zero.

5.1 Scalable preference relations

Given an outcome $\omega = (\mathbf{a}, \theta, \mathbf{s})$,let $\omega \oplus_H k \equiv (\mathbf{a}, \theta \oplus_H k, \mathbf{s} \oplus_H k)$ and let $[L \oplus_H k](w) \equiv L(w \oplus_H k)$. This allows us to introduce scalable preference relations.

Definition 3 (Scalable preference relations)**.** *A preference relation $\succeq_i$ is scalable with respect*

[5]The assumption of differentiability is made in order to simplify calculations. We believe that this assumption is not necessary.

[6]The terms $\oplus_H$ and $\ominus_H$ can be thought of the the normal $+$ and $-$ after a projection of the state space from $\mathbb{R}$ to Θ.

[7]The choice of this reference point is arbitrary and has no particular meaning.

to H if whenever:

$$L_i \succeq_i L'_i$$

then,

$$[L_i \oplus_H k] \succeq_i [L'_i \oplus_H k]$$

This definition says that if player i prefers lottery L_i to the lottery L'_i, then he will also prefer the lottery corresponding to scaling up the signals of all players and the state by some constant k using the notion of scalability given by H and keeping all actions constant, in lottery L, to a similarly scaled version of the lottery L'. This preference structure is naturally satisfied in many situations. All examples mentioned in the introduction - including auctions and contests with loss averse players - exhibit scalable preference relations. Moreover this general preference structure allows us to capture and hence study situations which cannot be modelled using expected utility, such as the auctions with loss averse players. However, such a general structure is not always necessary. We will later present a simple sufficient condition for it to be satisfied.

5.2 Scalable information structure

The second key element of our analysis is the information structure. A scalable information structure is defined as follows:

Definition 4 (Scalable information structure)**.** *The information structure $\{g, \{F_i(s_i|\theta)\}_{i\in I}\}$ is scalable with respect to H if:*

1. *$g(\theta) = H'(\theta)$ for all $\theta \in \Theta$*
2. *For all $\theta, k, s_i \in \Theta$*

$$F_i(s_i|\theta) = F_i(s_i \oplus_H k|\theta \oplus_H k)$$

The first part of this definition ensures that the notion of scalability used in the information structure corresponds is appropriate for the primitives.

The second part captures the fact that the conditional distribution of signals has a similar shape when θ is changed. When $a \oplus_H b = a + b$ this implies that conditional beliefs are additively invariant: that is to say the shape of the distribution is common knowledge but players do not know their position in the distribution. For instance this holds when players know that the distribution is uniform over the interval $[\theta - 1, \theta + 1]$, but do not necessarily know the value of the state θ. This is illustrated in Figure 1.

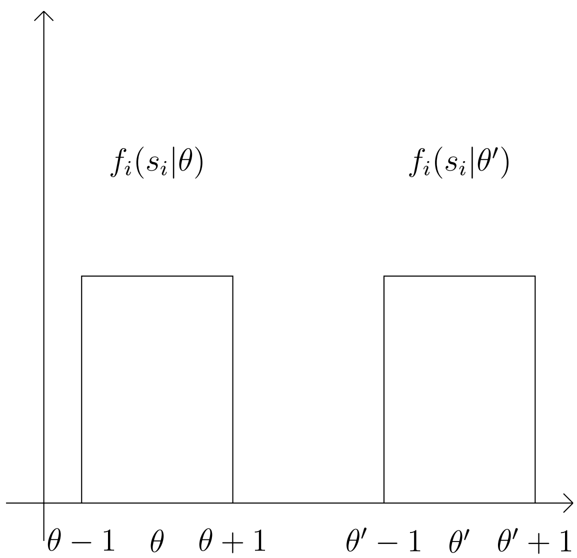

Figure 1: Uniform:Additive

Meanwhile when $a \oplus_H b = a \times b$ this definition implies that conditional beliefs are homogeneous of degree 0. For instance this holds when players know that the distribution is uniform over the interval $[0, 2\theta]$, but do not necessarily know the value of the median θ. This is illustrated in Figure 2.

5.3 Scalable Games

Considering a structure with primitives given by Γ, where players simultaneously choose an action, we say that the structure is *scalable* if the preference relations are scalable (see definition 3) and the information structure is scalable (see definition 4).

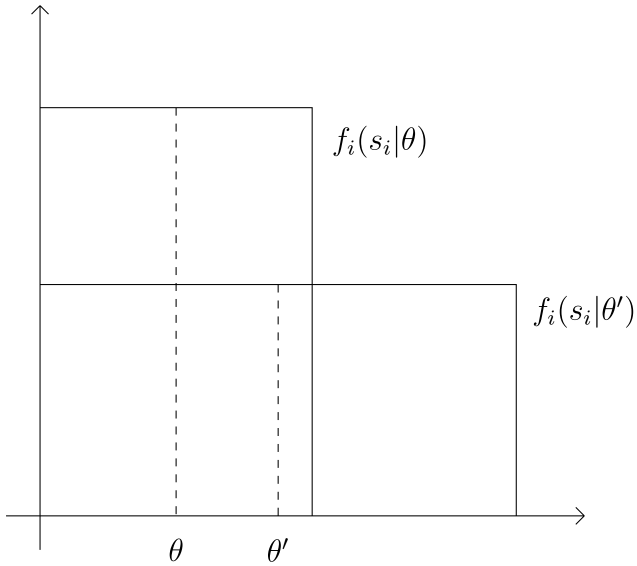

Figure 2: Uniform: Multiplicative

More precisely, combining the notion of a scalable information structure with the definition of $\mathcal{S}(\Gamma)$ and $\mathcal{A}(\Gamma)$, we define the following:

Definition 5 (Scalable game of asymmetric information)**.** *We say that the game of asymmetric information $\mathcal{A}(\Gamma)$ is a scalable game of asymmetric information, if the preferences $(\succeq_i)_{i\in I}$ are scalable (see definition 3) and the information structure $\{g,(F_i)_{i\in I}\}$ is scalable (see definition 4).*

Definition 6 (Scalable game of symmetric information)**.** *We say that the game of symmetric information $\mathcal{S}(\Gamma)$ is a scalable game of symmetric information, if the preferences $(\succeq_i)_{i\in I}$ are scalable (see definition 3) and the information structure $\{g,(F_i)_{i\in I}\}$ is scalable (see definition 4).*

These are the two types of games to which we apply our framework.

6 Analysis: Simplicity

In this section we show that scalable games are particularly tractable. This is demonstrated by drawing the connection between scalable games of asymmetric information $\mathcal{A}(\Gamma)$ and an associated game of complete information.

Informally, scalable games are particularly easy to solve, because in order to determine the optimal strategy for each player, it is sufficient to look at one particular signal or state for each player. The optimal actions are the same when he has a different signal or the state is different. In the case of pure strategies, the problem is reduced from solving for a fixed point in the space of functions to solving for a fixed point in the space of vectors, one for each player.

This simplicity can also be demonstrated by considering a related game of complete information.

Let $L_i^{\mathcal{A}}[\sigma^{\mathcal{C}}, 0_H]$ represent the lottery that player i assigns to possible outcomes when (i) player i observes signal $s_i = 0_H$ and (ii) players play according to constant strategy profile $\sigma(\mathbf{a}|\mathbf{s}) = \sigma^{\mathcal{C}}(\mathbf{a})$. As a reminder:

$$L_i^{\mathcal{A}}[\sigma^{\mathcal{C}}, 0_H](\mathbf{a}, s_i, s_{-i}, \theta) = \begin{cases} \sigma^{\mathcal{C}}(\mathbf{a}) \int_\theta \int_{s_{-i}} g(\theta|0_H) f(s_{-i}|\theta) ds_{-i} d\theta \text{ when } s_i \geq 0_H \\ 0 \text{ otherwise} \end{cases}$$

Using this notation we can now define the complete information game $\mathcal{C}(\Gamma)$:

Definition 7 (Complete information game $\mathcal{C}(\Gamma)$)**.** *The complete information game corresponding to the primitives* Γ *is given by* $\mathcal{C}(\Gamma) := \{I, (A_i)_{i\in I}, (\succeq_i^c)_{i\in I}\}$ *where:*

$$\sigma^{\mathcal{C}} \succeq_i^c \hat{\sigma}^c \text{ if and only if } L^{\mathcal{A}}_{i\,[\sigma^{\mathcal{C}}]} \succeq_i L^{\mathcal{A}}_{i\,[\hat{\sigma}^c]}$$

This leads us to the following result:

Proposition 6.1 (Game of complete information)**.** *For goven primitives* Γ*, the constant strategy profile* $\sigma^{\mathcal{C}}$ *is a Bayesian Nash equilibrium of the scalable game* $\mathcal{A}(\Gamma)$*, if and only if it is a Nash equilibrium of the complete information game* $\mathcal{C}(\Gamma)$*.*

Hence a scalable game where players have a symmetric information - denoted by $\mathcal{A}(\Gamma)$ - is particularly easy to solve because it is sufficient to study a corresponding game of complete information $\mathcal{C}(\Gamma)$. If we also require that (i) the independence axiom holds (so that preferences can be represented by a utility function) and (ii) each action space A_i is finite then it is a standard result that the complete information game $\mathcal{C}(\Gamma)$ must have an equilibrium (possibly in mixed strategies). Then, by appealing to proposition 6.1, we can ensure that an equilibrium in constant strategies also exists in the game of asymmetric information $\mathcal{A}(\Gamma)$. Even when these conditions do not hold, an equilibrium can often be found by studying the game of complete information $\mathcal{C}(\Gamma)$.

7 Applications: Simplicity

To show that the simplicity of the scalable game framework allows to study situations which are difficult to model under alternative assumptions, we now present an application to auctions and contests with loss averse participants.

7.1 Auctions and Contests with loss averse players

We now study how loss aversion affects the bidding behaviour - or respectively the effort exerted - in auctions or contests. In particular we compare the effects of loss aversion in first price auctions and all pay auctions.

Consider a contest with I participants, where one prize is to be handed out. Player i's valuation of the prize is given by his signal s_i. Each contestant's effort is denoted by a_i and is interpreted as the proportion of his valuation he spends. The outcome function is given as follows:

$$\phi_i(\mathbf{a}, \mathbf{s}, \theta) = \begin{cases} 1 & \text{if } a_i s_i \geq a_j s_j \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

We consider information structures $\{g, F(\mathbf{s}|\theta)\}$ which are scalable according to definition 3. In the scalable game of asymmetric information the probability player i assigns to winning the contest and hence the contest success function is denoted $\psi(a_i, a_{-i})$. Assuming that contestants choose constant efforts, this function is independent of s_i and is given as follows:

$$\psi_i(a_i, a_{-i}) = \int_{-\infty^n}^{\mathbf{s}} (\phi_i(\mathbf{a}, \tilde{\mathbf{s}}, \theta)|\theta) d\tilde{\mathbf{s}} \tag{4}$$

Note that this function is strictly increasing in a_i given a_{-i}. Given a vector of actions $\mathbf{a}$, a contestant is equally likely to win, independent of his signal.

The analysis extends to any contest success function $\psi_i(a_i, a_{-i})$ which is strictly increasing in a_i given a_{-i} and does not depend on s_i, but is not necessarily derived from a deterministic allocation rule.

In addition, we assume that players are loss averse. In particular, players feel a loss whenever their true payoff is lower than their expected payoff. This loss is given by β times the expected payoff minus the actual payoff whenever this is positive:

$$u_i = \begin{cases} \pi_i - \beta(E(\pi_i) - \pi_i) & \text{if } \pi_i \leq E(\pi_i) \\ \pi_i & \text{otherwise} \end{cases} \tag{5}$$

A related definition of expectation based loss aversion is considered in Koszegi & Rabin (2006).

7.2 Loss Aversion in standard contests

First we consider a standard contest, where each contestant pays his effort. Player i's expected utility is denoted $V(a_i, a_{-i}|\beta)$ and is given as follows:

$$V(a_i, a_{-i}|\beta) = \psi_i(a_i, a_{-i})s_i - as_i - \beta\Big[1 - \psi_i(a_i, a_{-i})\Big]\Big[\psi_i(a_i, a_{-i})s_i\Big] \tag{6}$$

Note that the corresponding distribution over outcomes is scalable (see definition 4). Since we have assumed a scalable information structure (see definition 3), this is a scalable game of asymmetric information. We now show that this problem is indeed tractable.

First differentiating with respect to a_i observing that in equilibrium $\frac{\delta V(a_i,a_{-i}|\beta)}{\delta a_i} = 0$, we consider the effect of changes in the degree of loss aversion β:

$$\frac{\delta V}{\delta\beta\delta a_i} = 2\Big[\psi_i(a_i, a_{-i}) - \frac{1}{2}\Big] \tag{7}$$

Since V is a single-peaked function. Hence at equilibrium $\frac{\delta V}{\delta a_i} = 0$. If $\psi_i(a_i, a_{-i}) > \frac{1}{2}$, then an increase in β will increase $\frac{\delta V}{\delta a_i} = 0$. In order to remain in equilibrium a_i will increase. On the other hand if $\psi_i(a_i, a_{-i}) < \frac{1}{2}$, the opposite effect prevails and a_i will decrease. This leads to the following proposition:

Proposition 7.1. *In a contest where all players pay their effort and which is a scalable game of asymmetric information $\mathcal{A}(\Gamma)$, if players become more loss averse (i.e. β increases), then*

- *Players with over a half chance of winning will bid higher.*
- *Bidders with under a half chance of winning will bid lower.*

7.3 First price auction with loss aversion

Now consider the case of a contest, where players do not have to pay their cost and hence a generalised first price auction with loss averse players. The distribution over outcomes remains scalable and the problem is still tractable. In this case, the expected utility is given as follows:

$$\begin{aligned} V(a_i, a_j|\beta) &= \psi_i(a_i, a_j)\Big[s_i - as_i\Big] - \beta\Big[1 - \psi_i(a_i, a_j)\Big]\Big[\psi_i(a_i, a_j)(s - as_i)\Big] \\ &= \Big[(1 - a_i)\psi_i(a_i, a_j)\Big]\Big[1 - \beta\psi_i(a_i, a_j)\Big] \end{aligned}$$

Differentiating with respect to a_i and noting that in equilibrium $\frac{\delta V}{\delta a_i} = 0$ gives:

$$a_i = 1 - \frac{\psi_i(a_i, a_j) - \beta\psi_i(a_i, a_j)^2}{\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}} \tag{8}$$

Moreover in a symmetric first price auction, the player with highest bid receives the object and $a_i = a_j$. Hence:

$$a_i = 1 - \frac{\frac{1}{2} - \beta\frac{1}{4}}{\delta\psi_i(a_i, a_i)\delta a_i} \tag{9}$$

An increase in β leads to an increase of the right hand side. Therefore for the first order condition to continue to hold, a_i must increase. This leads to the following result:

Proposition 7.2. *In a generalised first price auction which is a scalable game of asymmetric information, where players are loss averse, then: If players become more loss averse (ie β increases) they choose a higher effort (bid).*

Using propositions 7.1 and 7.2 we can determine the optimal strategy for a loss averse bidder participating in a general contest or a first price auction respectively. Although due to the improper prior, the seller's expected revenue cannot be computed under the scalable information structure, it can be calculated for any given state θ. Propositions 7.1 and 7.2 can therefore be used to determine a seller's - and buyers' - preferred auction mechanism when players are loss averse.

8 Analysis: Equivalence

In many applications players' preferences satisfy the independence axiom. This means that players have expected utility and their preferences can be represented by von Neumann-Morgenstern utility functions. We now provide a sufficient condition for preferences to satisfy definition 3 when players are expected utility maximisers. This expected utility representation will also be used to show the equivalence between games of asymmetric information and games of symmetric information.

We denote the von Neumann-Morgenstern utility function of player i by $u_i(\mathbf{a}, \theta, \mathbf{s})$, where $\mathbf{a}$ is the vector of players' actions, θ is the state and $\mathbf{s}$ is the vector of players' signals.

Attention is limited to utility functions which satisfy the following:

Assumption 1 (Scalable payoff structure)**.**

$$g(\theta)u_i(\mathbf{a},\mathbf{s},\theta) = g(\theta \oplus_H k)u_i(\mathbf{a},\mathbf{s} \oplus_H k, \theta \oplus_H k) \text{ for all } i \in I$$

It is clear that if a utility function satisfies assumption 1, then the corresponding preference relation over lotteries are scalable (see definition 3). Therefore utility functions that satisfy assumption 1 are a special case of the more general class of preferences studied in the previous section. Assumption 1 is satisfied when $H(\theta) = \theta$, the operator $\oplus_H$ represents $+$ and:

$$u_i(\mathbf{a},\mathbf{s},\theta) = u_i(\mathbf{a},\mathbf{s}+k,\theta+k)$$

Moreover, assumption 1 is also satisfied when $H(\theta) = \ln(\theta)$, the operator $\oplus_H$ represents $\times$ and:

$$u_i(\mathbf{a},\mathbf{s},\theta) = \frac{1}{k}u_i(\mathbf{a},\mathbf{s}.k,\theta.k)$$

Hence assumption 1 holds for utility functions which are (i) homogeneous of degree 0 in the log transform and (ii) homogeneous of degree 1. In particular, it is satisfied by all the examples using utility functions given in this paper. This includes (i) beauty contests and quadratic utility models where utility functions are homogeneous of degree 0 in the log transform and (ii) first price, second price and all pay auctions with risk neutral bidders where utility functions are homogeneous of degree 1. Using this assumption, we can now state the main result of this paper:

Theorem 8.1. *Suppose $\mathcal{A}(\Gamma)$ is a scalable game of asymmetric information and $\mathcal{S}(\Gamma)$ is a scalable game of symmetric information with the same primitives Γ. If preferences in $\Gamma^{\mathcal{S}}$ satisfy assumption 1, then the constant strategy profile σ^C is a Nash equilibrium of $\mathcal{A}(\Gamma)$ if and only if it is a Nash equilibrium of $\mathcal{S}(\Gamma)$.*

The proof can be found in the appendix.

This result shows that there is a correspondence between the equilibria of (i) the game $\mathcal{A}(\Gamma)$ where each player i observes some private information s_i and (ii) the game $\mathcal{S}(\Gamma)$, where all players observe some public information θ and have no private information. Therefore this result provides a deeper understanding of certain strategic situations, where the equilibrium outcomes are the same when (i) each player i observes private information s_i and (ii) players all observe the same piece of public information θ.

9 Applications: Symmetric information and asymmetric information

To illustrate the relevance of the link between $\mathcal{A}(\Gamma)$ and $\mathcal{S}(\Gamma)$, we now present two applications. The first application focuses on second price auctions, while the second application studies creditors bargaining in a bankruptcy situation. We then provide two examples to show that the equivalence of asymmetric and symmetric games is indeed important.

9.1 Second Price Auctions

First we consider a second price auction where players have valuations s_i and there is an unknown reserve price θ. In the first situation, there are two collectors interested in buying a first edition book. They are labeled $\{1, 2\}$. It could be the case that each collector knows how much he values the book (ie the value of s_i) but does not know the reserve price set by the seller (ie the value of θ). Collectors may then choose to enter the auction with $a_i = E$ or choose not to enter the auction with $a_i = NE$. To order the decisions, we assign $NE = 0$ and $E = 1$. Each collector who enters submits a secret bid (a collector who does not enter submits a bid of 0), and if the reserve price is not met then there is no sale. Crucially there is a cost to attending the auction given by c. Therefore a collector may be put off attending the auction because of the cost involved in participating. Assuming that when a collector chooses to enter he bids his valuation, the following utility function represents this situation.

$$u_i(\mathbf{a},\theta,\mathbf{s}) = \begin{cases} s_i - \max\{s_j,\theta\} - c & \text{if} \quad a_i = P \qquad a_j = P \qquad s_i > \max\{s_j,\theta\} \\ -c & \text{if} \quad a_i = E \quad a_j = \{E, NE\} \quad s_i < \max\{s_j,\theta\} \\ 0 & \text{if} \quad a_i = NE \end{cases} \tag{10}$$

Secondly we consider a second price auction for oil tracts. Now consider a situation with two oil firms labeled $\{1,2\}$. It could well be the case that the buyer knows the reserve price (denoted by θ), but does not can only estimate how much oil the tract contains and hence the value of the oil tract (denoted by s_i). Firms may then choose to enter the auction with $a_i = E$ or choose not to enter the auction with $a_i = NE$. Each firm who enters submits a secret bid (a firm who does not enter submits a bid of 0), and if the reserve price is not met then there is no sale. Again there is a cost to attending the auction of c. Therefore as before a firm may be put off attending the auction because of the cost involved in participating. This situation is represented by exactly the same utility function as above, but instead of observing s_i firms observe θ.

Note that the utility function is the same in both cases and players are risk neutral expected utility maximisers. Assuming that in both situations $g(\theta) = 1$ for all $\theta \in \mathbb{R}$ ensures assumption 1 holds. In addition assuming that the distribution of valuations such that definition 3 is satisfied, we can apply theorem 8.1. From the theorem it follows directly, that the two games described have the same set of constant equilibria.

Since the second game is a game of symmetric information, by proposition 6.1 it is possible to average over the uncertainty to form a complete information game $\mathcal{C}(\Gamma)$. We define

$$\pi_i = P(s_i > \max\{s_j,\theta\})E[s_i - \max\{s_j,\theta\}|s_i > \max\{s_j,\theta\}]$$

to be the expected payoff of player i given that both players participate in the auction:

	E	NE
E	$\left(\pi_1 - c, \pi_2 - c\right)$	$\left(\frac{E[s_1-\theta \vert s_1>\theta]}{P(s_1>\theta)} - c, 0\right)$
NE	$\left(0, \frac{E[s_2-\theta \vert s_2>\theta]}{P(s_2>\theta)} - c\right)$	$\left(0, 0\right)$

Having tackled the problem using a general distribution, to fix ideas we now consider a specific example. Say s_1 is drawn uniformly from $[\theta, \theta+6]$, while s_2 is drawn uniformly form $[\theta, \theta+4]$. The table above now reduces to:

	E	NE
E	$(2-c, \frac{2}{3}-c)$	$(3-c, 0)$
NE	$(0, 2-c)$	$(0, 0)$

If $c \leq \frac{2}{3}$, then it is a dominant strategy for each player to enter the auction. This is because player 1 receives (at worst) an expected payoff of $2 - c > 0$, while player 2 receives (at worst) an expected payoff of $\frac{2}{3} - c \geq 0$.[8] Hence an auctioneer can guarantee himself revenue $R = \min\{s_1, s_2\} + \frac{4}{3}$ by setting the entry cost c to be $\frac{2}{3}$. This is better than simply running a second price auction where the auctioneer raises revenue $\min\{s_1, s_2\}$.

If $c \in (\frac{2}{3}, 3)$ only player 1 will participate in the auction. Hence the auctioneer will sell the object at the reserve price of $\theta \leq \min\{s_1, s_2\}$. Hence in this case the optimal entry cost is $\frac{2}{3}$. This analysis shows that an auctioneer can set an entry cost players are always willing to pay. Importantly the entry cost *does not jeopardise the chance that the object is sold*. This is true even though the reserve price is included in the support of all the players. This effect is driven by the fact that no player knows he has a valuation close to the reserve price and so each player is willing to pay an entry fee in the hope that he has a valuation significantly above the reserve price. This strikingly differs from the standard model, where players with low valuations are unwilling to pay entry fees.[9]

[8]Despite the weak inequality it is still a dominant strategy because if player 1 chooses NE then $2 - c > 0$

[9]A resulting effect of the standard model is that entry fees typically mean that the object may not be sold.

This simple example gives an indication of how the modelling tool proposed in our paper can be applied to second price auctions to help set either the entry cost or the reserve price. The next section looks at how this modelling tool can be used to uncover links between games which have been studied in the literature.

9.2 Bankruptcy and Bargaining

Consider a company going bankrupt. There are two senior creditors numbered $\{1, 2\}$. Creditor i is owed s_i. However there is only θ to distribute and it may be the case that $s_1 + s_2 > \theta$ and the company does not have enough money to fully repay its senior creditors.

Each creditor demands part of his money. Hence the strategy set for each player is given by $A_i = [0, 1]$, where $a_i = 1$ captures a creditor demanding all his money and $a_i < 1$ captures a creditor demanding only some of his money.

If there the company has enough money to satisfy both demands then each creditor is paid the amount he demanded (any surplus is divided between junior creditors). However if the company does not have enough money to satisfy both demands then creditors enter arbitration. Each creditor is awarded a fraction β_i of the surplus. However since $\beta_1 + \beta_2 < 1$, there is always an agreement that Pareto dominates disagreement.

$$u_i(\mathbf{a}, \theta, \mathbf{s}) = \begin{cases} a_i s_i & \text{if} \quad a_1 s_1 + a_2 s_2 \leq \theta \\ \beta_i s_i & \text{otherwise} \end{cases} \tag{11}$$

We assume a scalable information structure with $G(\theta) = \ln(\theta)$ and $g(\theta) = \frac{1}{\theta}$. Moreover, it can be checked that the preference relation is scalable with respect to this G satisfying assumption 1 and hence theorem 8.1 applies.

One typical situation in a bankruptcy case is where the value of the assets of the company is unknown and players have private information about how much they are owed. This would be captured by the game $\mathcal{A}(\Gamma)$. Another situation is where the assets of the company are known, but players do not know exactly how much they will gain in arbitration. In case players choose which proportion of their claim to demand, this would be captured by the game $\mathcal{S}(\Gamma)$. This correspondence unifies much of the literature on bargaining.

10 Conclusion

In this paper we proposed a framework for modelling situations of asymmetric information. This framework can be used to model such situations in a tractable way and establishes a close connection between certain games of asymmetric information and games of symmetric information. We illustrate the relevance of both points using examples of games frequently studied in the literature.

In future work we would like to use this framework to study particular situations in more detail and provide additional key applications. Such applications consider the risk estimates banks provide to the regulator as well as a general model for the formation of asset price bubbles. Furthermore, an extension to multi dimensional signal spaces and action spaces may increase the relevance of this framework.

In addition, we are interested in comparing the scalable information structure proposed here to other information structures typically used in the literature. In particular we would like to draw the link with the independent types assumption and consider intermediate cases. The goal is to develop a full comprehensive framework covering a range of information structures for the general class of payoffs considered in this paper.

11 Appendix A: Proofs Analysis: Simplicity

11.1 Proof of proposition 6.1

Proof. Suppose the strategy profile σ^* is a Nash equilibrium of $\mathcal{C}(\Gamma)$. If $a_i^* \in \text{supp}(\sigma_i^*)$, then $(a_i^*, \sigma_{-i}^*) \succeq_i^{\mathcal{C}} (\hat{a}_i, \sigma_{-i}^*)$ for all $\hat{a}_i$.

Hence $L_i[a_i^*, \sigma_{-i}^*, 0_H] \succeq_i^{\mathcal{C}} {}_i L_i[\hat{a}_i, \sigma_{-i}^*, 0_H]$.

By the scalability of payoffs (definition 3) and the scalability of the information structure (definition 4):

$$L_i[a_i^*, \sigma_{-i}^*, 0_H] \succeq_i^{\mathcal{C}} L_i[\hat{a}_i, \sigma_{-i}^*, 0_H] \Rightarrow L_i[a_i^*, \sigma_{-i}^*, s_i^*] \succeq^{\mathcal{A}} L_i[\hat{a}_i, \sigma_{-i}^*, s_i^*]$$

Hence player i has no incentive to deviate from the strategy profile σ_i^* when he observes s_i^*.

□

12 Appendix B: Proofs Applications: Simplicity

12.1 Proof of proposition 7.1: Loss Aversion in an all pay contest

Proof.

$$V(a_i, a_{-i}|\beta) \;=\; \psi_i(a_i, a_{-i})s_i - as_i - \beta\Big[1 - \psi_i(a_i, a_{-i})\Big]\Big[\psi_i(a_i, a_{-i})s_i\Big]$$

Differentiating with respect to a_i gives:

$$\begin{aligned}
\frac{\delta V}{\delta a_i} &= \frac{\delta\psi_i(a_i, a_{-i})}{\delta a_i} - 1 - \beta\Big[1 - \psi_i(a_i, a_{-i})\Big]\Big[\frac{\delta\psi_i(a_i, a_{-i})}{\delta a_i}\Big] + \beta\Big[\frac{\delta\psi_i(a_i, a_{-i})}{\delta a_i}\Big]\Big[\psi_i(a_i, a_{-i})\Big] \\
&= \frac{\delta\psi_i(a_i, a_{-i})}{\delta a_i} - 1 - \beta\frac{\delta\psi_i(a_i, a_{-i})}{\delta a_i}\Big[1 - \psi_i(a_i, a_{-i})\Big] + \beta\Big[\psi_i(a_i, a_{-i})\Big] \\
&= \frac{\delta\psi_i(a_i, a_{-i})}{\delta a_i} - 1 + \beta\frac{\delta\psi_i(a_i, a_{-i})}{\delta a_i}\Big[2\psi_i(a_i, a_{-i}) - 1\Big]
\end{aligned}$$

$$\frac{\delta V}{\delta a_i} \;=\; \frac{\delta\psi_i(a_i, a_{-i})}{\delta a_i} - 1 + \beta\frac{\delta\psi_i(a_i, a_{-i})}{\delta a_i}\Big[2\psi_i(a_i, a_{-i}) - 1\Big]$$

Differentiating with respect to β yields:

$$\frac{\delta V}{\delta\beta\delta a_i} = 2\frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i}\Big[\psi_i(a_i,a_{-i}) - \frac{1}{2}\Big]$$

Now it is assumed that V is a single-peaked function and we use the condition that $\psi_i(a_i, a_{-i})$ is strictly increasing in a_i. Hence at equilibrium $\frac{\delta V}{\delta a_i} = 0$. □

12.2 Proof of proposition 7.2: Loss aversion in a first price contest

Proof.

$$\begin{aligned} V(a_i, a_j|\beta) &= \psi_i(a_i,a_j)\Big[s_i - as_i\Big] - \beta\Big[1-\psi_i(a_i,a_j)\Big]\Big[\psi_i(a_i,a_j)(s-as_i)\Big] \\ &= \Big[(1-a_i)\psi_i(a_i,a_j)\Big]\Big[1-\beta\psi_i(a_i,a_j)\Big] \end{aligned}$$

$$\frac{\delta V}{\delta a_i} = \Big[(1-a_i)\psi_i(a_i,a_j)\Big]\Big[-\beta\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}\Big] + \Big[(1-a_i)\frac{\delta\psi_i(a_i,a_j)}{\delta a_i} - \psi_i(a_i,a_j)\Big]\Big[1-\beta\psi_i(a_i,a_j)\Big]$$

Looking at just the terms involving β:

$$\begin{aligned} \frac{\delta V'}{\delta a_i} &= -\beta\Big[\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}\Big((1-a_i)\psi_i(a_i,a_j)\Big) + \psi_i(a_i,a_j)\Big((1-a_i)\frac{\delta\psi_i(a_i,a_j)}{\delta a_i} - \psi_i(a_i,a_j)\Big)\Big] \\ &= -\beta\Big[\psi_i(a_i,a_j)^2\Big] \end{aligned}$$

Looking at terms not involving β:

$$\frac{\delta V''}{\delta a_i} = \Big[(1-a_i)\frac{\delta\psi_i(a_i,a_j)}{\delta a_i} - \psi_i(a_i,a_j)\Big]$$

In equilibrium $\frac{\delta V}{\delta a_i} = \frac{\delta V'}{\delta a_i} + \frac{\delta V''}{\delta a_i} = 0$. Hence:

$$0 = \Big[(1-a_i)\frac{\delta\psi_i(a_i,a_j)}{\delta a_i} - \psi_i(a_i,a_j)\Big] + \beta\Big[\psi_i(a_i,a_j)^2\Big]$$

Collecting terms:

$$\frac{\beta\Big[\psi_i(a_i,a_j)^2\Big]\Big] + \psi_i(a_i,a_j)}{\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}} = \Big[(1-a_i)\Big]$$

$$a_i = 1 - \frac{\beta\Big[\psi_i(a_i,a_j)^2\Big]\Big] + \psi_i(a_i,a_j)}{\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}}$$

$$a_i = 1 - \psi_i(a_i,a_j)\left[\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}\right]^{-1}\Big[\beta\psi_i(a_i,a_j)+1\Big]$$

An increase in β leads to an increase in the right hand side of the equation (if a_i is held constant). Hence for the FOC to continue to hold a_i must increase.

□

13 Appendix C: Proofs Analysis: Equivalence

13.1 Proof of Theorem 8.1

Proof. We first prove this for the case where $f(s_i|\theta)$ is a discrete distribution.

Suppose $\sigma(\mathbf{a}|\mathbf{s}) = \sigma^C(\mathbf{a})$ is a constant strategy profile. Suppose also that σ is a pure strategy so that for some $\mathbf{a}^* = (a_1^*, ..., a_n^*)$, it holds that $\sigma^C(\mathbf{a}) = 1$ whenever $a_i \geq a_i^*$ for all $i \in I$ and $\sigma^C(\mathbf{a}) = 0$ otherwise. Suppose further that σ is a BNE of $A(\Gamma)$. This means that when player i has signal 0_H he has no incentive to deviate. Hence for all deviations $\hat{a}_i \in A_i$ it holds that $V_1(a_i^*, a_{-i}^*; 0_H) \geq V_1(\hat{a}, a_{-i}^*; 0_H)$ where:

$$V_1^A(a_i, a_{-i}) = \sum_{(\theta, s_{-i})} g(\theta|0_H) f_{-i}(\mathbf{s}_{-i}|\theta) u_i(a_i, a_{-i}; 0_H, \mathbf{s}_{-i}; \theta)$$

Note that:

$$g(\theta|0_H) = \frac{g(\theta) f(0_H|\theta)}{\sum_{\tilde{\theta}} g(\tilde{\theta}) f(0_H|\tilde{\theta})}$$

Now define:

$$V_2^A(a_i, a_{-i}) = \sum_{(\theta, s_{-i})} g(\theta) f(0_H|\theta) f_{-i}(\mathbf{s}_{-i}|\theta) u_i(a_i, a_{-i}; 0_H, \mathbf{s}_{-i}; \theta)$$

Substituting $g(\theta|0_H) = \Big[g(\theta) f(0_H|\theta)\Big] \Big[\int g(\tilde{\theta}) f(0_H|\tilde{\theta}) d\tilde{\theta}\Big]^{-1}$ and multiplying each side by the constant in the second set of square brackets it follows that $V_2(a_i^*, a_{-i}^*; 0_H) \geq V_2(\hat{a}, a_{-i}^*; 0_H)$. Now define:

$$V_3^A(a_i, a_{-i}) = \sum_{(\theta, s_{-i})} f(0_H|\theta) f_{-i}(\mathbf{s}_{-i}|\theta) u_i(a_i, a_{-i}; 0_H \ominus \theta, \mathbf{s}_{-i} \ominus \theta; 0_H)$$

Note that by the extra condition imposed it follows that:

$$g(\theta) u_i(a_i, a_{-i}; \theta; 0_H, \mathbf{s}_{-i}) = g(0_H) u_i(a_i, a_{-i}; 0_H \ominus \theta, \mathbf{s}_{-i} \ominus \theta; 0_H)$$

It follows from this equation that $V_2^A(a_i, a_{-i}) = g(0_H) V_3^S(a_i, a_{-i})$ and hence $V_3^S(a_i^*, a_{-i}^*) \geq V_3^S(\hat{a}_i, a_{-i}^*)$.

Define also:

$$V_4^S(a_i, a_{-i}) = \sum_{(\hat{s}_i, \hat{s}_{-i})} f(\hat{s}_i|0_H) f_{-i}(\hat{\mathbf{s}}_{-i}|0_H) u_i(a_i, a_{-i}; \hat{s}_i, \hat{\mathbf{s}}_{-i}; 0_H)$$

Define $\hat{s}_i = 0_H \ominus_H \theta$ and $\hat{s}_j = s_j \ominus_H \theta$. Note that from the assumption that the distribution is scalable it follows that (i) $f(\hat{s}_i|0_H) = f(0_H|\theta)$ and (ii) $f(\hat{s}_i|0_H) = f(s_j|\theta)$. Using these facts and substitutions it follows that $V_3^S(a_i, a_{-i}) = V_4^S(a_i, a_{-i})$. Hence $V_4^S(a_i^*, a_{-i}^*) \geq V_4^S(\hat{a}_i, a_{-i}^*)$. This shows that $\sigma^C(\mathbf{a})$ is also a Nash equilibrium of the game of symmetric information. The reverse direction can easily be seen by repeating the steps above. Finally it is clear that the case where $f(s_i|\theta)$ is a continuous distribution (although needing more notation) can be proved along similar lines.

□

14 Appendix D: Additional Material

14.1 Alternative specification of preferences: scalable actions

We show that our framework can also be used to model situations of asymmetric information, where preferences over lotteries are unchanged when scaling the actions of all players, the signals of all players and the state. For instance, consider the case of an auction, where payoffs are homogeneous of degree one: multiplying the valuations, the bids of all players by a constant their payoffs are multiplied by the same constant. If the game satisfies this alternative definition of scalable preferences and the information structure is scalable, these games are strategically equivalent to a scalable game of asymmetric information $\mathcal{A}(\Gamma)$, where $a_i = \hat{a}_i \ominus_H s_i$ and $\hat{a}_i$ is the action chosen in this alternative game. These strategies are of the form $\sigma_i(s_i) = e_i^* \oplus_H s_i$, where e_i^* is a player specific constant. These games can be studied in their original form. However the translation to the scalable game form is useful for studying the link between asymmetric information and symmetric information.

Consider a game of asymmetric information, $\mathcal{A}(\Gamma) = I, (\hat{A}_i)_{i\in I}, (\succeq_i)_{i\in I}, g, (F_i)_{i\in I}$, where $\hat{A}_i = \Theta$ for all $i \in I$ and the information structure is scalable with respect to G (see definition 3). If the game $\mathcal{A}(\Gamma) = I, (A_i)_{i\in I}, (\succeq_i)_{i\in I}, g, (F_i)_{i\in I}$

Given $\omega = (\hat{\mathbf{a}}, \theta, \mathbf{s})$, let $\omega \hat{\oplus}_G k \equiv (\mathbf{a} \oplus_H, \theta \oplus_H k, \mathbf{s} \oplus_H k)$ and let $[L \hat{\oplus}_G k](w) \equiv L(w \hat{\oplus}_G k)$. Suppose the preference relations satisfy the following definition:

Definition 8 (Alternative scalable preference relations). *A preference relation* $\succeq_i$ *is alternatively scalable with respect to G if whenever:*

$$L \succeq_i L'$$

then,

$$[L \hat{\oplus}_H k] \succeq_i [L' \hat{\oplus}_H k]$$

This definition says that if a player prefers lottery L to lottery L' then, when all the actions, the state and the signals are scaled up by a constant, he continues to prefer the scaled up lottery arising from L to the one arising from L'. This definition differs from the standard definition of a scalable preference structure in that all the elements of ω are scaled including the actions.

In some applications, the preference structure satisfies this alternative definition of scalable preference relations. In these cases, there exists a transformation $a_i = \hat{a}_i \ominus_H$, which redefines the action space from $\hat{A}_i = \Theta$ to A_i, such that when the actions are changed from $\hat{a}_i$ to a_i, the transformed preference relations satisfy definition 4. Formally, this can be stated as follows:

Proposition 14.1. *In a game* $\hat{\Gamma}^{\mathcal{A}} = \{I, (\hat{A}_i)_{i\in I}, (\succeq_i)_{i\in I}, g, F\}$*, if* $A_i = \Theta$ *for all* $i \in I$ *and players' preferences satisfy definition 8, there exists a strategically equivalent game* $\mathcal{A}(\Gamma) = \{I, (A_i)_{i\in I}, (\succeq_i)_{i\in I}, g, F\}$*: If* $\sigma_i^{\mathcal{A}}$ *is an equilibrium of* $\mathcal{A}(\Gamma)$*, then* $\hat{\sigma}_i^{\mathcal{A}}$ *where* $\hat{a}_i = a_i \oplus_H s_i$*, is an equilibrium of* $\hat{\Gamma}^{\mathcal{A}}$*.*

The proof follows immediately from substitutions.[10]

As an example consider the case of an auction. The true actions players take are their bids $\hat{a}_i$ in $[0, \infty)$. A first price auction clearly satisfies definition 8. However instead of considering the bids directly, we can consider the case where players choose the proportion of their valuation they want to bid $a_i = \hat{a}_i \ominus_H s_i$. Using these proportions to describe a player's preferences, these satisfy definition 3.

[10] A similar result exists for $\mathcal{S}(\Gamma)$, but it is slightly more complicated, because actions require scaling by s_i which is not observed in $\mathcal{S}(\Gamma)$.

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