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The calm policymaker
John Barrdear

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Abstract

Determinacy is ensured in the New Keynesian model when firms face imperfect common knowledge, regardless of whether the Taylor principle is satisfied. Strategic complementarity in pricing and idiosyncratic noise in firms' signals, however small, are together sufficient to eliminate backward-looking solutions without appealing to the assumptions of Blanchard and Kahn (1980). Standard solutions emerge when the Taylor principle is followed, but when the policymaker demurs, the price level — and not just inflation — is stationary. A unique and stable solution also emerges with the interest rate pegged to its steady-state value, in contrast to Sargent and Wallace (1975).

Key words: Dispersed information, imperfect common knowledge, New Keynesian, indeterminacy, Blanchard-Kahn, Taylor rules, Taylor principle, interest rate peg.

JEL classification: D84, E31, E52.

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1 Introduction

A long literature has studied the question of price level determinacy, dating (in the modern sense of the word) to the rise of the rational expectations paradigm,[1] with Sargent and Wallace's (1975) demonstration of indeterminacy in a model with rational expectations under an interest rate peg. It is now commonly accepted that when monetary policy is set via interest rates, determinacy and stability rely critically on the Taylor principle: that when inflation rises, the nominal interest rate should be raised sufficiently – usually by more than one-for-one – to ensure that the real interest rate will rise, thus damping demand and lowering inflation. More formally, when a New Keynesian model is closed with an interest rate rule and solved with the assumptions first introduced by Blanchard and Kahn (1980), a lower bound emerges on the central bank's marginal response to inflation for the solution to be unique.

This paper challenges this narrative by demonstrating that it is not strictly necessary for a central bank to respond to temporary deviations of the economy from its long run trend. This is not to suggest that policy ought not respond, or that if policy does respond it will be ineffective. The model below adopts the canonical New Keynesian framework, with monetary policy operating through the same channels, and with equal effect. Nevertheless, this paper's results partially confound such discussion by demonstrating the determinacy of (deviations from trend in) the price level when arbitrarily small amounts of noise are introduced into firms' information sets, regardless of the strength of the central bank's response to inflation.[2]

Extending the three-equation model of Galí (2008) to impose *Imperfect Common Knowledge* (ICK) on firms – each rationally combining idiosyncratically noisy signals of the underlying state of the economy while facing strategic complementarity in their price-setting – I establish the following results:

[1] The broader question of what *determines* an economy's price level is clearly far older, dating (at least) to Hume's (1748) advocacy of the quantity theory of money.

[2] The model I present is linearised around a deterministic trend, implying an assumption that long-run inflation expectations remain anchored throughout.

1. *Uniqueness.* So long as firms never discover past values of the price level with certainty, backward-looking solutions and extrinsic bubbles are eliminated without appealing to the famous conditions of Blanchard and Kahn (1980).

2. *Standard results remain.* The solution is a purtubation from the forward solution under full information and nests the canonical solution when the Taylor principle is satisfied and firms' noise is taken towards zero.

3. *Interest rate peg.* In partial contrast to the results of Sargent and Wallace (1975), a unique and stable solution exists when the nominal interest rate remains pegged at its steady-state level.

4. *Stationary prices.* When the central bank declines to satisfy the Taylor principle, the price level – and not just the rate of inflation – is stationary around its trend path, with policymaker-determined persistence.

5. *The real interest rate.* The real interest rate rises following a positive demand shock, regardless of the strength of the central bank's response.[3]

6. *Output volatility.* Demand-driven deviations of output from trend are larger under a 'passive' regime than an 'active' one, but also less persistent. Unconditional volatility is generally larger in a passive regime.

7. *Inflation volatility.* The unconditional volatility of inflation peaks at the Taylor threshold, falling as the central bank's marginal response to inflation moves in either direction.

The elimination of backward-looking solutions poses challenges to a number of applications of the New Keynesian model that have relied on full information, including the 'backward stable' approach, and subsequent neo-Fisherian results, of Cochrane (2016), and studies of inflation dynamics that rely on the possibility of sunspot shocks, such as Ascari, Bonomolo and Lopes (2016).

Methodologically, this paper adds to the ICK literature by deriving an exact finite-state representation that accommodates both dynamic elements

[3] The real interest falls on impact under an interest rate peg, but subsequently rises above, and remains above, trend thereafter, with the integral over time being positive.

in agents' decision rules and endogenous signals. By contrast, earlier work has either (i) approximated the solution by granting agents full knowledge of the state with a T-period lag (e.g. Lorenzoni, 2009) or by truncating the hierarchy of beliefs (e.g. Nimark, 2011); or (ii) produced a finite-state representation only when agents face a sequence of static problems with exogenous signals (e.g. Woodford, 2003). More recently, Huo and Takayama (2016) have demonstrated a finite-state representation in models with dynamic choices when agents' signals are exogenous and proven the impossibility of a finite representation when agents observe contemporaneous endogenous signals. The method used here is simpler than that of Huo and Takayama (2016) and successfully includes endogenous signals by having them be observed with a lag.

This is by no means the first paper to apply ICK to the study of monetary business cycles.[4] Woodford (2003) first introduced Townsend's (1983) hierarchy of expectations to a nominal economy, using a reduced-form expression for demand and demonstrating sluggish aggregate behaviour following a shock to nominal spending, despite price flexibility. Nimark (2008) extends Woodford's approach to include a standard demand side to the economy, but grants firms perfect knowledge of the previous period's price level. This maintains the possibility of indeterminacy and so requires approaches like the Taylor principle to address it. Melosi (2014) estimates a similar model for the US economy. More recently, Kohlhas (2014) has re-explored the 'anti-disclosure' result of Morris and Shin (2005), while Angeletos and Lian (2016*b*) have demonstrated that the absence of perfect common knowledge can address the forward guidance 'puzzle' of Del Negro, Giannoni and Patterson (2016).

The rest of the paper is arranged as follows. Section 2 first provides context for the paper, presenting a simple illustration of the indeterminacy problem in New Keynesian models. Section 3 next presents the model, before section 4 presents the solution. Section 5 presents a variety of testable implications that follow, conditional on the model, and section 6 concludes.

[4] Angeletos and Lian (2016*a*) provide a recent overview of models of incomplete information, including imperfect common knowledge.

2 Some Context

Before examining the New Keynesian model under imperfect common knowledge, it is helpful to first consider the question of determinacy in the following simple model of a log-linearised Euler equation (1a) and Taylor rule (1b). It is nested in the full model below by supposing full price flexibility and full information on the part of all agents, so that output remains on its trend path in every period.

$$0 = i_t - E_t^{\Omega}\left[\pi_{t+1}\right] - x_t \tag{1a}$$

$$i_t = \phi_\pi \pi_t \tag{1b}$$

$$x_t = \rho x_{t-1} + \sigma_u u_t \quad \text{with } \rho \in (0,1) \text{ and } u_t \sim N(0,1) \tag{1c}$$

where i_t is the nominal interest rate, $\pi_t \equiv p_t - p_{t-1}$ is inflation, p_t is the aggregate price level, x_t is a persistent shock to the natural interest rate and $E_t^{\Omega}\left[\cdot\right] \equiv E\left[\cdot|\Omega_t\right]$ is the mathematical expectation conditional on all information available in period t. Combining (1a) and (1b) gives a single equilibrium condition for the model, written in terms of inflation:

$$\pi_t = \frac{1}{\phi_\pi} E_t^{\Omega}\left[\pi_{t+1}\right] + \frac{1}{\phi_\pi} x_t \tag{2}$$

Following Blanchard (1979), Ascari, Bonomolo and Lopes (2016) show that the complete set of rational solutions to (2) may be written as a linear combination of a *purely forward-looking solution* (a function of only current or expected future values of the structural shock) and a *purely backward-looking solution* (a function of only past values of the structural shock), together with an *extrinsic bubble* in the style of Flood and Garber (1980):

$$\pi_t = (1-\xi)\pi_t^{(F)} + \xi \pi_t^{(B)} + w_t \tag{3a}$$

where $\xi \in \mathbb{R}$ and

$$\pi_t^{(F)} = \left(\frac{1}{\phi_\pi} \sum_{s=0}^{\infty} \left(\frac{\rho}{\phi_\pi} \right)^s \right) x_t \tag{3b}$$

$$\pi_t^{(B)} = \phi_\pi \pi_{t-1} - x_{t-1} \tag{3c}$$

$$E_t^{\Omega}\left[w_{t+1}\right] = \phi_\pi w_t \quad \text{and} \quad \mathrm{Cov}\left(w_t, u_s\right) = 0 \;\forall\, t, s \tag{3d}$$

The parameter ξ may take any value on the real line, but two special cases are clear: $\xi = 0$, when the backward-looking solution is excluded, and $\xi = 1$, when the forward-looking solution is excluded. Without further assumptions, the model is therefore indeterminate, with two elements of the solution as yet unspecified: ξ and w_t. To complete the solution, it is necessary to select between the infinite number of eligible values or processes for these elements.[5] Note that substituting (2) forward gives:

$$\pi_t = \pi_t^{(F)} + \lim_{s\to\infty} \left(\frac{1}{\phi_\pi}\right)^s E_t^{\Omega}\left[\pi_{t+s+1}\right] \tag{4}$$

Since (4) is a restatement of (2), all solutions must satisfy it. Indeed, substituting (3) forward, it is easy to confirm that:

$$\lim_{s\to\infty} \left(\frac{1}{\phi_\pi}\right)^s E_t^{\Omega}\left[\pi_{t+s+1}\right] = \xi\left(\pi_t^{(B)} - \pi_t^{(F)}\right) + w_t \tag{5}$$

from which it follows that the transversality condition $\lim_{s\to\infty}\left(\frac{1}{\phi_\pi}\right)^s E_t^{\Omega}\left[\pi_{t+s+1}\right] = 0$ is achieved only if backward-looking solutions and extrinsic bubbles can be eliminated. In particular, the assumptions of Blanchard and Kahn (1980)[6] – manifested here as (i) $\phi_\pi > 1$ and (ii) π_t must be stationary – together serve to eliminate all but the forward-looking solution (forcing $\xi = w_t = 0\ \forall\, t$), since $\phi_\pi > 1$ renders both $\pi_t^{(B)}$ (3c) and w_t (3d) explosive. The first assumption is also sufficient to ensure that $\pi_t^{(F)}$ is finite, so that we are left with $\pi_t = \left(\frac{1}{\phi_\pi - \rho}\right) x_t$ or, expanding π_t,

$$p_t = p_{t-1} + \left(\frac{1}{\phi_\pi - \rho}\right) x_t \tag{6}$$

which is to say that the (log) price level has a unit root in the standard solution to the NK model. It is important to note, however, that this feature comes

[5] Rather than finding assumptions that pin down specific values for ξ and w_t, another approach is to suppose that they are chosen by extrinsic shocks – *sunspots* – that serve to determine how agents coordinate their beliefs. Ascari, Bonomolo and Lopes, 2016) fall in this literature, imposing $w_t = 0$, but supposing that ξ follows a random walk. Since their model still satisfies the assumptions of Muth (1961), they label this a 'rational sunspot'.

[6] The same two assumptions also underlie more recent solution techniques such as Klein (2000) and Sims (2002).

from the parameter restrictions imposed by the Blanchard-Kahn assumptions and is not always a feature of the forward-looking solution *per se*. To appreciate this point, rewrite (2) in terms of the price level instead of inflation:

$$p_t = \left(\frac{\phi_\pi}{1+\phi_\pi}\right) p_{t-1} + \left(\frac{1}{1+\phi_\pi}\right) E_t^\Omega \left[p_{t+1}\right] + \left(\frac{1}{1+\phi_\pi}\right) x_t \tag{7}$$

Solving such a model with a lag in the equilibrium condition is slightly more involved, but ultimately quite straightfoward. In the appendix, I demonstrate the following set of solutions:

Proposition 1. *The full set of rational solutions to* (1c) *and* (7) *is:*

$$p_t = (1-\xi) p_t^{(F)} + \xi\, p_t^{(B)} + w_t \tag{8a}$$

where $\xi \in \mathbb{R}$ *and*

$$p_t^{(F)} = \lambda p_{t-1} + \gamma x_t \tag{8b}$$

$$p_t^{(B)} = (1+\phi_\pi)\, p_{t-1} - \phi_\pi p_{t-2} - x_{t-1} \tag{8c}$$

$$E_t^\Omega \left[w_{t+1}\right] = (1+\phi_\pi)\, w_t - \phi_\pi w_{t-1} \quad \textit{and} \quad \mathrm{Cov}\left(w_t, u_s\right) = 0 \;\forall\, s, t \tag{8d}$$

$$\lambda = \min\left\{1\,,\ \phi_\pi\right\} \tag{8e}$$

$$\gamma = \begin{cases} \frac{1}{1-\rho} & \textit{if } \phi_\pi < 1 \\ \frac{1}{\phi_\pi - \rho} & \textit{if } \phi_\pi \geq 1 \end{cases} \tag{8f}$$

Proof. See appendix A. □

It is straightforward to show that assuming $\phi_\pi > 1$ renders both the purely backward-looking solution (8c) and the extrinsic bubble (8d) explosive. When combined with a transversality condition, these elements are then eliminated and the solution is identical to (6).

As a preview of later results, however, note that if backward-looking solutions and the extrinsic bubble could be removed without imposing the Blanchard-Kahn conditions, then a unique solution would exist for all $\phi_\pi \geq 0$ and the price level would be *stationary* when $\phi_\pi \in [0, 1)$. This latter point arises because, when substituting (7) forward, the convergent coefficient against p_{t-1} corresponds is the "MOD solution" of McCallum (2007).

2.1 The Cochrane Critique

Although only rarely considered, the plausibility of the Blanchard-Kahn assumptions depends on the economic context of the model being solved. In the present circumstance, Cochrane (2011) argues that neither assumption is valid when solving the New Keynesian model. Against the eigenvalue restriction, he notes that by ensuring that the model is explosive in inflation when off the desired equilibrium path, the Taylor principle cannot be an *ex ante* credible commitment for the central bank to make, since in the event of off-equilibrium inflation, it will retain *ex post* options for bringing inflation in check without deliberately sending the economy into a hyperinflationary spiral. Against the no-bubble condition, he argues that while a transversality assumption may be reasonable for real variables, its imposition on nominal variables is less defensible, as periods of hyperinflation patently do happen.[7]

Building on this rejection of standard solution methods in the New Keynesian framework, Cochrane (2016) has emphasised that, absent the Blanchard-Kahn conditions, one admissible solution under full information is that which is "backward stable" (i.e. non-explosive as $t \to -\infty$). Under this solution, which is necessarily backward-looking, when the interest rate is pegged below its original steady state value forever, inflation does not explode but, instead, falls to accommodate the change – a result he dubs 'neo-Fisherian'. García-Schmidt and Woodford (2015) describe this as a paradox of perfect foresight and propose a deviation from rational expectations – based on iterative, but incomplete revisions of beliefs each period – which avoids it. Gabaix (2016) describes another boundedly-rational variant of the New Keynesian model in which agents pay reduced attention to specific variables when forecasting and, together with an ad hoc assumption about how agents form opinions of trend inflation, obtains results that are Neo-Fisherian in the long run.

[7] Other selection criteria have been proposed (e.g. Evans and Honkapohja, 2001), but these still retain at least one of the assumptions described above, and so remain subject to at least some aspect of Cochrane's critique.

3 The Model

I start from the canonical three equation model of Galí (2008), extended only to deny full information to price-setting firms. It is cashless, and features Ricardian equivalence and lump sum taxes to eliminate any influence of fiscal policy. There is a continuum of firms, indexed $j \in [0,1]$, that supply differentiated goods to a representative household, who values them via a Dixit and Stiglitz (1977) aggregator. The household provides labour to the firms, with decreasing marginal productivity, in a competitive labour market. There is no capital. Firms are subject to Calvo (1983) pricing and information frictions, while the household and the central bank each possess full information. All agents are fully rational and trend inflation is taken to be zero.

Combined with market clearing, the household's Euler equation is:

$$y_t = E_t^{\Omega}[y_{t+1}] - \sigma\left(i_t - \left(E_t^{\Omega}[p_{t+1}] - p_t\right) - x_t\right) \tag{9}$$

$$x_t = \rho x_{t-1} + u_t \tag{10}$$

where y_t is output; p_t is the aggregate price level; i_t is the nominal interest rate; σ is the elasticity of intertemporal substitution; x_t is a persistent demand shock (with $\rho \in (0,1)$ and $u_t \sim N(0, \sigma_u^2)$), implemented here as a shock to the natural rate of interest; and $E_t^{\Omega}[\cdot] = E[\cdot|\Omega_t]$ is the mathematical expectation conditional on all period-t information. The central bank makes use of a contemporaneous Taylor rule:

$$i_t = \phi_y y_t + \phi_\pi (p_t - p_{t-1}) \tag{11}$$

Individual firms have an independent probability, θ, of not being able to update their price in each period, so that the aggregate price level evolves as:

$$p_t = \theta p_{t-1} + (1-\theta) g_t \tag{12}$$

where $g_t \equiv \int_0^1 g_t(j)\, dj$ is the average reset price in period t. Firms' individual reset prices are given by their expectations of their optimal reset prices:

$$g_t(j) = (1-\beta\theta) E_t(j)[p_t + \omega y_t] + (\beta\theta) E_t(j)[g_{t+1}] \tag{13}$$

where β is the household discount factor, ω is a function of the various elasticities of intertemporal substitution, demand, labour supply and marginal cost; and $E_t(j)[\cdot] \equiv E[\cdot|\mathcal{I}_t(j)]$ is firm j's (rational) expectation based on an incomplete information set: $\mathcal{I}_t(j) \subset \Omega_t$. Taking an average of (13) and combining it with (12) then gives the following expression for the price level:

$$\begin{aligned} p_t = \theta p_{t-1} + (1-\theta(1+\beta))\overline{E}_t[p_t] \\ + (\beta\theta)\overline{E}_t[p_{t+1}] + (1-\theta)(1-\beta\theta)\omega\overline{E}_t[y_t] \end{aligned} \tag{14}$$

where $\overline{E}_t[\cdot] \equiv \int_0^1 E_t(j)[\cdot]\,dj$ is the average firm expectation. For reference, note that this may be readily rearranged (using $\pi_t \equiv p_t - p_{t-1}$) to give:

$$\begin{aligned} \pi_t = (1-\theta)\overline{E}_t[\pi_t] + (1-\theta)\left\{\overline{E}_t[p_{t-1}] - p_{t-1}\right\} \\ + (1-\theta)(1-\beta\theta)\omega\overline{E}_t[y_t] \\ + (\beta\theta)\overline{E}_t[\pi_{t+1}] \end{aligned} \tag{15}$$

which is the *Incomplete Information New Keynesian Phillips Curve*, first presented by Nimark (2008), although generalised here to allow for uncertainty about the previous period's price-level. It should be clear that with full information, the term in $\left\{\overline{E}_t[p_{t-1}] - p_{t-1}\right\}$ drops out and expectations around period-t variables become accurate, leading to the canonical *full information NKPC*:

$$\pi_t = \kappa y_t + \beta E_t^{\Omega}[\pi_{t+1}] \quad \text{where} \quad \kappa = \frac{(1-\theta)(1-\beta\theta)}{\theta}\omega \tag{16}$$

3.1 Timing

Unlike in models of full information, where all variables are jointly determined by a Walrasian auctioneer, I suppose that each period proceeds in two stages:

1. In stage one ("overnight"), firms observe their signals and, when able, adjust their prices accordingly, thereby determining inflation.
2. In stage two ("the working day"), the household and monetary authority jointly determine the market-clearing nominal interest rate and nominal wage. The household reveals the quantity demanded from each firm at the given prices, firms discover their current-period marginal costs and produce the goods. The household consumes the goods entirely.

3.2 Firms' information

Firms retain complete information about the trend path for the economy, but have only incomplete and heterogeneous access to information about its deviations from that trend. Each period, each firm (regardless of whether they are free to adjust their price) observes a set of signals about the aggregate economy and uses these to update their beliefs. Note that equation (13) implies that there is strategic complementarity in firms' decision-making, so that each of them will care about not only the real marginal cost they will individually face but also the decisions (and beliefs) of all other firms.

As may already be clear, and will in any case be shown below, the underlying state of the economy includes the exogenous driving process (x_t) and the lagged price level (p_{t-1}). I therefore assume that each firm observes:

$$\boldsymbol{s}_t(j) = \begin{bmatrix} x_t & + v_t^x(j) \\ p_{t-1} & + v_t^p(j) \end{bmatrix} \text{ where } \underbrace{\begin{bmatrix} v_t^x(j) \\ v_t^p(j) \end{bmatrix}}_{\boldsymbol{v}_t(j)} \sim N\left(\mathbf{0}, \sigma_v^2 I_2\right) \tag{17a}$$

so that

$$\mathcal{I}_t(j) = \{\mathcal{I}_{t-1}(j), \boldsymbol{s}_t(j)\} \tag{17b}$$

The idiosyncratic noise, which I assume to be transitory, may be thought of as firms' failure to directly observe a public signal or a misinterpretation of the same (perhaps instead getting only an impression from newspaper coverage); an error of judgement; or as the imperfect applicability of national public signals to the aggregation level most relevant to each firm (e.g. at an industry or sector level). Idiosyncratic noise shocks are taken to be independent of aggregate shocks, so that $\text{Cov}(u_t, v_s^*(j)) = 0 \quad \forall\, t, s, j$ and $* \in \{x, p\}$.

This signal structure has the benefit of nesting *full information* as a special case by setting $\sigma_v^2 = 0$. As is commonly known and was illustrated above in section 2, forward-looking models with rational expectations and full information are indeterminate in general, meaning that additional assumptions are needed to select a solution.

More generally, *common (but incomplete) information* – a setting explored, for example, by Currie, Levine and Pearlman (1986) – can be nested here by supposing that $\mathrm{Cov}\left(\boldsymbol{v}_t\left(i\right),\boldsymbol{v}_t\left(j\right)\right) = \sigma_v^2 \quad \forall i,j,t$. This would add additional dynamics to the full-information model, as past noise shocks would affect current behaviour, but it would not address the question of determinacy. An equivalent multiplicity of solutions still emerges and additional equilibrium-selection assumptions are still required, just as with the full information case. Any criticism that may be made of them under full information applies equally well when information is incomplete and common.

In this paper, I suppose a framework of *dispersed information*, where firms' noise shocks are i.i.d. so that $\mathrm{Cov}\left(\boldsymbol{v}_s\left(i\right),\boldsymbol{v}_t\left(j\right)\right) = 0 \quad \forall i,j,s,t$, and demonstrate that this is sufficient to ensure determinacy without needing to impose the Taylor principle. Note, in particular, that firms do not perfectly observe the past price level. This assumption will prove to be critical in ensuring uniqueness below. This requirement seems, to this author, to be quite a weak assumption, however, given the constantly-evolving nature of official estimates of economic data.[8] It bears emphasising, too, that uniqueness will only require the presence of *any* amount of idiosyncratic noise, no matter how small.

Other information assumptions may, of course, be made. Common noise shocks could be added, for example, to capture the effect of measurement errors by national statistical agencies or 'animal spirits'.[9] Alternatively, the signal regarding the natural rate of interest could be replaced with a similarly noisy signal about the previous period's aggregate output. This might arguably be a more plausible description of information actually used by firms in their pricing decision, but would no longer nest the case of full information. In the language of Baxter, Graham and Wright (2011), the model would then be only asymptotically invertible when $\sigma_v^2 = 0$, rather than instantly invertible.

[8]For example, the Bureau of Economic Analysis conducts both an annual revision of US data, typically focusing on the preceding three years, and a 'comprehensive revision' of data every five years, in which all time periods of published data can be altered (Kornfeld et al., 2008). The latest comprehensive review, conducted in 2013, included changes to national accounts dating to 1929 (McCulla, Holdren and Smith, 2013).

[9]A second way of accommodating general movements in agents' sentiments, as described by Angeletos and La'O (2013), would be to grant firms noisy signals about other firms' signals. In either scenario, these would then be added, alongside the natural rate of interest, to the list of exogenous shocks that firms would need to estimate.

4 Solving the model

To solve the model, I proceed in three stages. I first characterise the purely forward-looking solution under full information when the model is written in terms of the price level. Next, I derive the corresponding forward-looking solution under imperfect common knowledge as a pertubation from its full-information counterpart. Finally, I demonstrate uniqueness by showing that backward-looking solutions and extrinsic bubbles are ruled out, regardless of the parameters of the model.

4.1 The forward-looking solution with full information

Substituting the central bank's decision rule (11) into the Euler equation (9), it is clear that a systematic response to the output gap by the central bank induces the household to discount the future:

$$y_t = \delta E_t^{\Omega}\left[y_{t+1}\right] - \delta\sigma\left(\phi_\pi\left(p_t - p_{t-1}\right) - \left(E_t^{\Omega}\left[p_{t+1}\right] - p_t\right) - x_t\right) \tag{18}$$

where $\delta = 1/(1+\sigma\phi_y)$. Imposing full information on the price-level (14) and combining it with (18), the model may be written compactly as:

$$A_0\boldsymbol{\zeta}_t = A_1 E_t^{\Omega}\left[\boldsymbol{\zeta}_{t+1}\right] + B_1\boldsymbol{\zeta}_{t-1} + C_0 x_t \tag{19}$$

where $\boldsymbol{\zeta}_t = \begin{bmatrix} p_t & y_t \end{bmatrix}'$ and A_0, A_1, B_1 and C_0 are matrices of parameters.[10] The standard approach to solving models like (19) is to stack the variables and to rearrange it so that the forecast variables are on the left-hand side:[11]

$$\begin{bmatrix} E_t^{\Omega}\left[\boldsymbol{\zeta}_{t+1}\right] \\ \boldsymbol{\zeta}_t \end{bmatrix} = \underbrace{\begin{bmatrix} A_1^{-1}A_0 & -A_1^{-1}B_1 \\ I & \mathbf{0} \end{bmatrix}}_{D} \begin{bmatrix} \boldsymbol{\zeta}_t \\ \boldsymbol{\zeta}_{t-1} \end{bmatrix} + \begin{bmatrix} -A_1^{-1}C_0 \\ \mathbf{0} \end{bmatrix} x_t \tag{20}$$

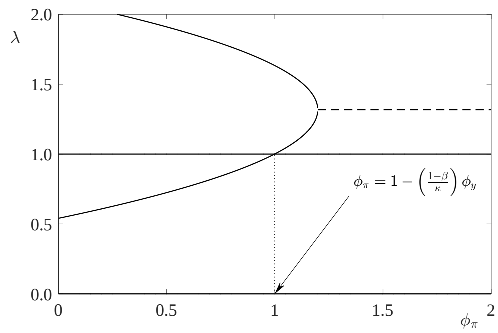
[10] $A_0 = \begin{bmatrix} 1 & -\frac{\kappa}{1+\beta} \\ \sigma\left(\phi_\pi + 1\right) & \frac{1}{\delta} \end{bmatrix}$, $A_1 = \begin{bmatrix} \frac{\beta}{1+\beta} & 0 \\ \sigma & 1 \end{bmatrix}$, $B_1 = \begin{bmatrix} \frac{1}{1+\beta} & 0 \\ \sigma\phi_\pi & 0 \end{bmatrix}$ and $C_0 = \begin{bmatrix} 0 \\ \sigma \end{bmatrix}$.

[11] The shock x_t may also be added to the stacked variables so that the driving process is i.i.d., but this would simply add ρ to the list of eigenvalues of D.

and then to proceed as per Blanchard and Kahn (1980).[12] It is straightforward to show that, in this instance, D has four distinct eigenvalues:

$$\lambda \in \left\{0, 1, \frac{\beta+\delta+\kappa\sigma\delta}{2\beta\delta} \pm \frac{\sqrt{(\beta+\delta+\kappa\sigma\delta)^2 - 4\beta\delta\left(1+\kappa\sigma\delta\phi_\pi\right)}}{2\beta\delta}\right\} \tag{21a}$$

These are plotted below in figure 1.[13] Note, in particular, that $\frac{\beta+\delta+\kappa\sigma\delta}{2\beta\delta} > 1$ and that the lower of the two quadratic solutions crosses $\lambda = 1$ when $\phi_\pi = 1 - \left(\frac{1-\beta}{\kappa}\right)\phi_y$ (the Taylor threshold). When $\phi_\pi > 1 - \left(\frac{1-\beta}{\kappa}\right)\phi_y$, the number of eigenvalues outside the unit circle matches the number of forecast variables, thus ensuring that any backward-looking solution will be explosive.



Note: The chart plots eigenvalues of the basic NK model when solved under full information (λ) as a function of the central bank's marginal response to inflation (ϕ_π). The dashed line represents the real component of two complex solutions. Structural parameters are $\{\beta, \phi_y, \sigma, \kappa\} = \left\{0.994, \frac{0.5}{4}, 1, 0.5\right\}$.

Figure 1: Eigenvalues of the New Keynesian model

As is usually the case in such models, the coefficients against lagged variables in the solution is given by the lowest eigenvalues of the system:

Proposition 2. *The purely forward-looking solution to the price level in* (19) *and* (10) *under full information is:*

$$p_t = \lambda\, p_{t-1} + \gamma\, x_t \tag{22a}$$

[12]If A_1 were not invertible, the generalized Schur form could be used, as per Klein (2000).

[13]The quadratic roots are complex when $\phi_\pi > \left(\frac{(1+\beta+\kappa\sigma)^2-4\beta}{4\beta\kappa\sigma}\right) - \left(\frac{1-\beta-\kappa\sigma}{2\kappa}\right)\phi_y + \left(\frac{\beta\sigma}{4\kappa}\right)\phi_y^2$.

where

$$\lambda = \min\left\{1, \frac{\beta+\delta+\kappa\sigma\delta}{2\beta\delta} - \sqrt{\left(\frac{\beta+\delta+\kappa\sigma\delta}{2\beta\delta}\right)^2 - \left(\frac{1+\kappa\sigma\delta\phi_\pi}{\beta\delta}\right)}\right\} \tag{22b}$$

$$\gamma = \frac{\kappa\sigma\delta}{(1-\delta\rho)\left(1+\kappa\sigma+\beta\left(1-\rho-\lambda\right)\right) - \kappa\sigma\frac{(1-\delta)(1-\delta\phi_\pi)}{(1-\delta\lambda)}} \tag{22c}$$

Proof. See appendix B.[14] □

When $\phi_\pi > 1 - \left(\frac{1-\beta}{\kappa}\right)\phi_y$, the full information, purely forward-looking solution to the New Keynesian model has a unit root in prices (as illustrated by Galí, 2008), but when ϕ_π is below the Taylor threshold, the purely forward-looking solution features a stationary price level. Lest readers be concerned with this stationarity, it bears noting that when x_t is sufficiently persistent, only this solution will produce a finite solution to γ.

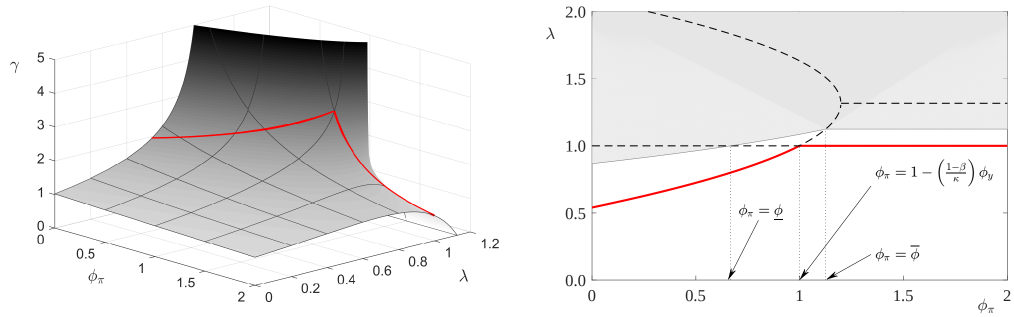
Corollary 1. *A solution for λ **other than** that specified in (22b) would be economically plausible (in the sense that γ is positive and finite) only when $\phi_\pi \in \left(\underline{\phi_\pi}, \overline{\phi_\pi}\right)$, where $\underline{\phi_\pi} = 1-(1-\rho)\left(1+\frac{1-\beta\rho}{\sigma\kappa}\right)-\left(\frac{1-\beta\rho}{\kappa}\right)\phi_y$ and $\overline{\phi_\pi} = 1+\sigma\phi_y$. Furthermore, this interval vanishes as $\phi_y \to 0$ and $\rho \to 1$.*

This point is illustrated in figure 2. Note that the region $\phi_\pi < \underline{\phi_\pi}$ with $\lambda = 1$ is the non-convergence region highlighted by Cho and McCallum (2015).

4.2 The forward-looking solution under imperfect common knowledge

With firms making use of heterogeneous information sets, it becomes necessary to consider the hierarchy of their (average) expectations. Let the 0$^{\text{th}}$-order expectation of a variable be the variable itself; the 1$^{\text{st}}$-order expectation be

[14] Correspondingly, $y_t = \underbrace{\left(\frac{\sigma\left(\phi_\pi - (1+\phi_\pi)\lambda + \lambda^2\right)}{1+\sigma\phi_y - \lambda}\right)}_{\omega} p_{t-1} + \left(\frac{\omega\gamma+\sigma(1-\gamma(1+\phi_\pi-\lambda-\rho))}{1+\sigma\phi_y-\rho}\right) x_t$.



$$p_t = \lambda p_{t-1} + \gamma x_t$$

Note: The left-hand chart plots values for γ that would emerge if λ were a choice variable, while the right-hand chart plots the solution for λ (the solid red line in both charts is the correct solution). The grey shaded region in the right-hand chart covers values of λ for which γ is not positive and finite: that is, such that a positive demand shock would fail to induce higher prices. The lower threshold is $\underline{\phi} = 1 - (1-\rho)\left(1 + \frac{1-\beta\rho}{\sigma\kappa}\right) - \left(\frac{1-\beta\rho}{\kappa}\right)\phi_y$. The higher threshold is $\overline{\phi} = 1 + \sigma\phi_y$. Parameters are $\{\beta, \phi_y, \sigma, \kappa, \rho\} = \left\{0.994, \frac{0.5}{4}, 1, 0.5, 0.8\right\}$.

Figure 2: Economic plausibility of the New Keynesian model

firms' average expectation about the variable; the 2^{nd}-order expectation be firms' average expectation about the 1^{st}-order expectation, and so on:

$$x_{t|t}^{(0)} \equiv x_t \tag{23a}$$

$$x_{t|t}^{(k)} \equiv \overline{E}_t\left[x_{t|t}^{(k-1)}\right] \ \forall k \geq 1 \tag{23b}$$

with $p_{t-1|t}^{(k)}$ similarly defined. The state of the model will be the (4×1) vector

$$Z_t \equiv \begin{bmatrix} x_t & \widetilde{x}_{t|t} & p_{t-1} & \widetilde{p}_{t-1|t} \end{bmatrix}' \tag{24a}$$

where $\widetilde{x}_{t|t}$ and $\widetilde{p}_{t-1|t}$ are weighted averages of firms' higher-order expectations regarding x_t and p_{t-1}:

$$\widetilde{x}_{t|t} \equiv (1-\varphi)\sum_{k=1}^{\infty}\varphi^{k-1}x_{t|t}^{(k)} \tag{24b}$$

$$\widetilde{p}_{t-1|t} \equiv (1-\varphi)\sum_{k=1}^{\infty}\varphi^{k-1}p_{t-1|t}^{(k)} \tag{24c}$$

for some $\varphi \in (-1, 1)$.

It is also possible, of course, to define an infinite-dimension state vector including every higher order expectation ($X_t \equiv \left[x_t \quad p_{t-1} \quad \overline{E}_t \left[X_t\right]'\right]'$), in which case the model may be solved according to Nimark (2011). Until recently the literature has generally held that a solution could *only* be expressed in terms of X_t when agents are forward-looking and observe endogenous signals. However, Huo and Takayama (2016) have demonstrated that a finite-state representation must exist, provided that agents do not observe endogenous signals contemporaneously. I show here that the finite-state representation may still be used when the endogenous signals are observed with a lag.

Proposition 3. *For the New Keynesian model with prices set under imperfect common knowledge, the purely forward-looking solution is of the form:*

$$
\begin{aligned}
Z_t &= A Z_{t-1} + B u_t && \text{(25a)} \\
p_t &= \boldsymbol{\alpha}' Z_t \\
&= \theta\, p_{t-1} + (\lambda - \theta)\, \widetilde{p}_{t-1|t} + \gamma\, \widetilde{x}_{t|t} && \text{(25b)}
\end{aligned}
$$

Furthermore, (25) *equals the corresponding solution under full information* (22) *when* $\sigma_v^2 = 0$*, and approaches it smoothly as* $\sigma_v^2 \to 0$.

Proof. See appendix C for detail, although I outline the bulk of the proof here.

Obtaining a single competitive equilibrium condition

Substituting (18) forward, I obtain:[15]

$$
\begin{aligned}
y_t = \sigma\delta\,(1-\delta\rho)^{-1} &\quad x_t \\
+\, \sigma\delta\,\phi_\pi &\quad p_{t-1} \\
-\, \sigma\delta\,(1 - \phi_\pi\delta + \phi_\pi) &\quad p_t \\
+\, \sigma\delta\,(1-\delta\phi_\pi)\,(1-\delta) &\quad \sum_{s=0}^{\infty} \delta^s E_t^\Omega\left[p_{t+s+1}\right]
\end{aligned} \tag{26}
$$

[15] A limiting term of $\lim_{s\to\infty} \delta^s E_t^\Omega\left[y_{t+s+1}\right]$ has been implicitly set to zero in (26). Since transversality is satisfied *by definition* in purely forward-looking solutions and I later demonstrate the inadmissibility of backward-looking solutions, its absence here is innocuous.

Substituting (26) into (14) then gives the model's equilibrium condition:

$$
\begin{aligned}
p_t = b_p \overline{E}_t [x_t] + \theta p_{t-1} &+ \zeta_{-1} \overline{E}_t [p_{t-1}] \\
&+ \zeta_0 \ \overline{E}_t [p_t] \\
&+ \beta\theta \ \overline{E}_t [p_{t+1}] \\
&+ \zeta_{1+} \overline{E}_t \left[(1-\delta) \sum_{s=0}^{\infty} \delta^s p_{t+s+1} \right]
\end{aligned} \tag{27a}
$$

This gives the current log deviation of the price level from its steady-state path in terms of the previous period's log deviation; firms' average expectation of the current value of the underlying shock process; and firms' average expectations of the past, current and all future price levels (note that p_{t+1} appears in both of the bottom two lines). The compound parameters are given by:

$$b_p = \theta\kappa\sigma\delta (1-\delta\rho)^{-1} \tag{27b}$$

$$\zeta_{-1} = \theta\kappa\sigma\delta\phi_\pi \tag{27c}$$

$$\zeta_0 = 1 - \theta (1+\beta) - \theta\kappa\sigma\delta (1 - \phi_\pi\delta + \phi_\pi) \tag{27d}$$

$$\zeta_{1+} = \theta\kappa\sigma\delta (1 - \phi_\pi\delta) \tag{27e}$$

Although perhaps unusual, (27) is a perfectly valid statement of the equilibrium condition underlying Galí (2008), extended here only to accomodate incomplete information among price-setting firms. Note that the term on the final line of (27a) is a *weighted* average of all future price deviations. When $\phi_y > 0$ it is skewed in favour of the near-term, while when $\phi_y = 0$ it is a simple average. Since trend inflation is assumed to be zero, it follows that $\lim_{\phi_y \to 0} (1-\delta) \sum_{s=0}^{\infty} \delta^s p_{t+s+1} = \lim_{s\to\infty} p_{t+s}$. This will be non-zero for any $x_t \neq 0$ if prices exhibit a unit root, as in the standard solution to the NK model.

Finding the solution

Firms' expectations

Without full information, individual firms must form expectations about the current state of the economy (Z_t). Since firms' signals may be written as $\boldsymbol{s}_t (j) = NZ_t + \sigma_v^2 I_2$, the model is in state-space form and the Bayes-rational estimator is the Kalman filter:

$$E_t(j)[Z_t] = E_{t-1}(j)[Z_t] + M_t\left\{\boldsymbol{s}_t(j) - E_{t-1}(j)[\boldsymbol{s}_t(j)]\right\} \tag{28}$$

where M_t is the (4×2) Kalman gain, common to all firms as their problems are symmetric. Defining $V_{t|t-1} \equiv Var\left(Z_t - E_{t-1}(j)[Z_t]\right)$ as the variance of firms' prior expectation errors, then for a given law of motion, the optimal filter converges to a time-invariant $M \equiv \begin{bmatrix}\boldsymbol{m}_x & \boldsymbol{m}_p\end{bmatrix}$ that satisfies:[16]

$$M = VN'\left(NVN' + \sigma_v^2 I_2\right)^{-1} \tag{29a}$$

$$V = A\left(V - VN'\left(NVN' + \sigma_v^2 I_2\right)^{-1} NV\right)A' + \sigma_u^2 BB' \tag{29b}$$

Reduced-form coefficients and the law of motion

Simple inspection of the equilibrium condition (27) is sufficient to note that $\boldsymbol{\alpha}' = \begin{bmatrix}0 & \alpha_2 & \theta & \alpha_4\end{bmatrix}$. Next, note that it must be the case that (i) $\widetilde{x}_{t|t} = x_t$ and $\widetilde{p}_{t-1|t} = p_{t-1}$ under full information; and (ii) $\widetilde{x}_{t|t} \to x_t$ and $\widetilde{p}_{t-1|t} \to p_{t-1}$ as $\sigma_v^2 \to 0$ by the optimality of the Kalman filter. It therefore follows that $\boldsymbol{\alpha}$ must be consistent with the solution under full information (22), so that:

$$\boldsymbol{\alpha}' = \begin{bmatrix}0 & \gamma & \theta & \lambda - \theta\end{bmatrix} \tag{30}$$

The process for deriving the law of motion (25a) is identical to that in Woodford (2003). Conditional on a corresponding solution under full information (λ, γ) and a value for φ, I show in the appendix that the result here is:

$$A = \begin{bmatrix} \rho & 0 & 0 & 0 \\ \rho\boldsymbol{\varphi}_x'\boldsymbol{m}_x & \rho\left(1 - \boldsymbol{\varphi}_x'\boldsymbol{m}_x\right) & \theta\boldsymbol{\varphi}_x'\boldsymbol{m}_p & -\theta\boldsymbol{\varphi}_x'\boldsymbol{m}_p \\ 0 & \gamma & \theta & \lambda - \theta \\ \rho\boldsymbol{\varphi}_p'\boldsymbol{m}_x & \gamma - \rho\boldsymbol{\varphi}_p'\boldsymbol{m}_x & \theta\boldsymbol{\varphi}_p'\boldsymbol{m}_p & \lambda - \theta\boldsymbol{\varphi}_p'\boldsymbol{m}_p \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ \boldsymbol{\varphi}_x'\boldsymbol{m}_x \\ 0 \\ 0 \end{bmatrix} \tag{31a}$$

where

$$\boldsymbol{\varphi}_x' = \begin{bmatrix} (1-\varphi) & \varphi & 0 & 0 \end{bmatrix} \tag{31b}$$

$$\boldsymbol{\varphi}_p' = \begin{bmatrix} 0 & 0 & (1-\varphi) & \varphi \end{bmatrix} \tag{31c}$$

[16]For a derivation, see Hamilton (1994).

The equilibrium degree of strategic complementarity

The coefficient φ is the degree of strategic complementarity in firms' price-setting decisions after taking account of demand and the entire expected future path of prices. To obtain it, note that:

$$\overline{E}_t\left[p_{t+q}\right] = \boldsymbol{\alpha}' A^{q-1}\boldsymbol{e}_3\overline{E}_t\left[p_t\right] + \boldsymbol{\alpha}' A^{q-1}J_3 A\overline{E}_t\left[Z_t\right] \tag{32}$$

where $\boldsymbol{e}_3$ is a column vector of zeros with a one in the third position, and J_3 is the identity matrix modified to put a zero in the third position of the lead diagonal. Substituting (32) into the competitive equilibrium condition (27) and gathering like terms then gives:[17]

$$p_t = \theta p_{t-1} + \boldsymbol{d}'\overline{E}_t\left[Z_t\right] + \varphi\overline{E}_t\left[p_t\right] \tag{33a}$$

where

$$\varphi = \zeta_0 + \beta\theta\boldsymbol{\alpha}'\boldsymbol{e}_3 + \zeta_{1+}\boldsymbol{\alpha}'\left((1-\delta)\sum_{q=0}^{\infty}(\delta A)^q\right)\boldsymbol{e}_3 \tag{33b}$$

Bringing everything together

We then have that, conditional on a particular forward-looking solution under full information (λ, γ), the law of motion is a function of the Kalman gain and the strategic complementarity $(A = f(M, \varphi))$; the Kalman gain is a function of the law of motion $(M = g(A))$; and the strategic complementarity is a function of the law of motion $(\varphi = h(A))$. The solution is then the fixed point of equations (29) , (31a) and (33b): $A = f(g(A), h(A))$. □

4.3 Uniqueness

Since the purely forward-looking solution under full information is unique among fundamental solutions (proposition 2) and the forward-looking solution under incomplete common knowledge is a purtubation from that full-information solution (proposition 3), all that remains is to demonstrate that

[17] Although not needed to calculate the solution, $\boldsymbol{d}' = \begin{bmatrix} b_p & 0 & \zeta_{-1} & 0 \end{bmatrix} + \beta\theta\boldsymbol{\alpha}' J_3 A + \zeta_{1+}\boldsymbol{\alpha}'\left((1-\delta)\sum_{q=0}^{\infty}(\delta A)^q\right)J_3 A$.

backward-looking solutions and extrinsic bubbles may be rejected under imperfect common knowledge.

It should be clear that since $\widetilde{x}_{t|t}$ and $\widetilde{p}_{t-1|t}$ are weighted averages of firms' entire hierarchy of expectations, there exists an infinite-state representation as a counterpart to (25). It should also be clear that since Z_t follows a vector AR(1) process, so too must the constituent higher-order beliefs. Let this alternative representation be given by:

$$p_t = \boldsymbol{\psi}' X_t \quad \text{where} \quad X_t \equiv \begin{bmatrix} x_t & p_{t-1} & \overline{E}_t \left[X_t\right]' \end{bmatrix}' \tag{34a}$$

$$X_t = F X_{t-1} + G u_t \tag{34b}$$

The full set of potential solutions, including those with some backward-looking component and an extrinsic shock, can then be written as:

$$p_t = \boldsymbol{\mu}' X_t + \xi q_{t-1} + w_t \quad \text{where} \quad \xi \in \mathbb{R} \tag{35a}$$

$$q_t \equiv \boldsymbol{a}\left(L\right)' X_t + b\left(L\right) p_t \tag{35b}$$

$$\text{Cov}\left(w_t, u_s\right) = 0 \; \forall \, s, t \tag{35c}$$

for some polynomial functions $\boldsymbol{a}\left(L\right)$ and $b\left(L\right)$ and scalar ξ.

Proposition 4. *For the New Keynesian model with prices set under imperfect common knowledge, the solution (25) is unique, with $\xi = w_t = 0 \; \forall \, t$ in (35).*

Proof. See appendix D. □

To help with intuition, I here rule out the following specific candidate solution:

$$p_t = \boldsymbol{\mu}' X_t + d p_{t-1} \tag{36}$$

This represents candidate solutions in which additional (if $d > 0$) weight is given to the lagged price level over and above θ. To begin, step (36) forward and take the period-t average expectation to get:

$$\overline{E}_t \left[p_{t+s}\right] = \boldsymbol{\mu}' \left(\sum_{q=0}^{s} d^q F^{s-q} \right) \overline{E}_t \left[X_t\right] + d^{s+1} \overline{E}_t \left[p_{t-1}\right] \quad \forall \, s \geq 0 \tag{37}$$

Next, define T as the selection matrix such that $TX_t = \overline{E}_t[X_t]$ (shifting the vector up two places). Substituting (37) into the equilibrium condition (27),[18] making use of T and gathering like terms, it is straightforward to show that a candidate of the form of (36) can therefore only be a solution if:

$$\boldsymbol{\mu}' = \begin{bmatrix} 0 & \theta & b_p & \zeta_{-1} & \mathbf{0}_{1\times\infty} \end{bmatrix} + \boldsymbol{\mu}' \left(\zeta_0 I + \beta\theta\left(F + dI\right) + \zeta_{1^+}\left(1-\delta\right)\sum_{s=0}^{\infty}\left(\sum_{q=0}^{s+1} d^q F^{s+1-q}\right)\right) T \tag{38a}$$

$$dp_{t-1} = d\left(\zeta_0 + d\beta\theta + d\zeta_{1^+}\left(1-\delta\right)\sum_{s=0}^{\infty}\left(d\delta\right)^s\right)\overline{E}_t\left[p_{t-1}\right] \tag{38b}$$

If $d=0$, (38a) reduces to the solution for $\boldsymbol{\psi}$ given in the appendix. Turning (38b) around and defining $\chi = \left(\zeta_0 + d\beta\theta + d\zeta_{1^+}\left(1-\delta\right)\sum_{s=0}^{\infty}\left(d\delta\right)^s\right)^{-1}$, gives

$$\overline{E}_t\left[p_{t-1}\right] = \chi\, p_{t-1} \tag{39}$$

which must hold for (36) to be valid. But (39) is inconsistent with rational expectations. To see this, consider an individual firm's filter regarding p_{t-1}:

$$E_t\left(j\right)\left[p_{t-1}\right] = E_{t-1}\left(j\right)\left[p_{t-1}\right] + K_t\left\{\boldsymbol{s}_t\left(j\right) - E_{t-1}\left(j\right)\left[\boldsymbol{s}_t\left(j\right)\right]\right\} \tag{40}$$

for some projection matrix K_t. Taking the average of this and splitting out the firm's two signals gives:

$$\overline{E}_t\left[p_{t-1}\right] = \overline{E}_{t-1}\left[p_{t-1}\right] + \rho\, K_{x,t}\left\{x_{t-1} - \overline{E}_{t-1}\left[x_{t-1}\right]\right\} + K_{x,t}\, u_t + K_{p,t}\left\{p_{t-1} - \overline{E}_{t-1}\left[p_{t-1}\right]\right\} \tag{41}$$

Since u_t is unforecastable, p_{t-1} cannot be a function of it. A necessary condition for (39) to hold is therefore that $K_{x,t}=0$. But since shocks are persistent ($\rho > 0$), this can only hold if (i) firms are not rational, which we rule out by assumption; (ii) firms have *no* information about the state ($\sigma_v^2 = \infty$); or (iii) firms have *full* information about the state ($\sigma_v^2 = 0$).

[18] It may already be clear at this point that the result is established, as the only term in p_{t-1} (as distinct from $\overline{E}_t\left[p_{t-1}\right]$) that remains on the right-hand side of (27) has a coefficient of θ, meaning that d must be zero unless firms know p_{t-1} with certainty.

Identical logic applies to any lagged variable. In short, backward-looking solutions require co-ordination between firms, and co-ordination requires common knowledge. So long as firms' signals contain *any* amount of idiosyncratic noise, so that they can never perfectly agree on past values of state variables, co-ordination is not possible and backward-looking solutions are eliminated.

5 Some (testable) implications

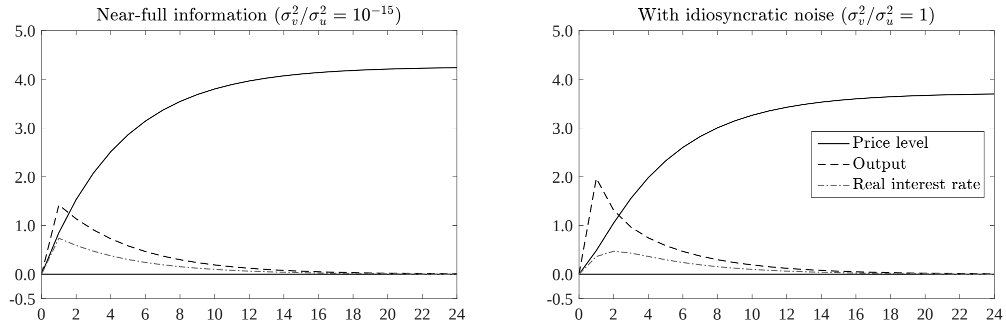
The ability to identify a unique solution to an otherwise-standard New Keynesian model when the central bank does not satisfy the Taylor principle has a variety of implications for how the model may be interpreted. I explore some of the most striking here, emphasising in advance that all are conditional on the model at hand, including the assumed common knowledge trend in prices.

5.1 Impulse responses

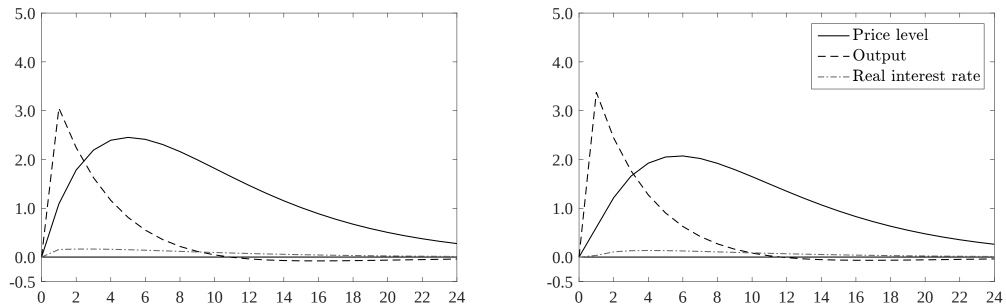
As a point of context for the corollaries listed below, figure 3 first provides impulse responses for the price level, output and the ex ante real interest rate following a positive shock to demand for different central bank designs and different levels of idiosyncratic noise, holding the following structural parameters as fixed: $\{\beta, \sigma, \theta, \omega, \rho\} = \{0.994, 1, 0.7, 0.994, 0.8\}$. The left-hand panels plot those under near-full information, with $\sigma_v^2 = 10^{-15}$, while the right-hand panels plot those under idiosyncratically noisy information, with $\sigma_v^2 = 1$.

The top row implements a standard Taylor-type rule, with $\phi_\pi = 1.5$ and $\phi_y = 0.1$. The top-left panel therefore reproduces the results of the textbook New Keynesian model. The top-right panel plots responses when firms' signals have material amounts of idiosyncratic noise.[19] Even under the optimal signal extraction process, firms' beliefs are slow to update and prices consequently deviate by less than they do under full information. The reduced price response subsequently induces a larger response in output.

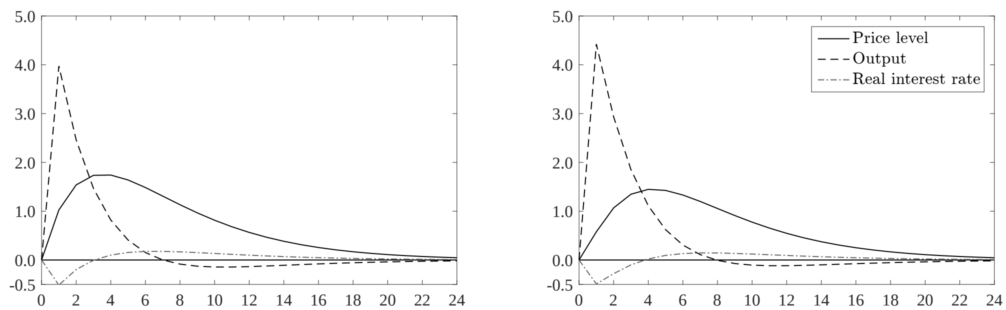
[19] It is therefore similar to Nimark (2008), albeit without firms having perfect knowledge of the lagged price level.



(a) Standard Taylor rule ($\phi_\pi = 1.5$ and $\phi_y = 0.1$): $\lambda = 1.00 \quad \gamma = 0.85$



(b) Subdued rule ($\phi_\pi = 0.5$ and $\phi_y = 0.1$): $\lambda = 0.84 \quad \gamma = 1.09$



(c) State-invariant rule ($\phi_\pi = 0$ and $\phi_y = 0$): $\lambda = 0.70 \quad \gamma = 1.03$

Note: The charts plot impulse response functions (IRFs) for the price level, output and the ex ante real interest rate following a positive shock to demand, when solutions for the price level under full information are: $p_t = \lambda p_{t-1} + \gamma x_t$. The left-hand panels impose near-full information ($\sigma_v^2 = 10^{-15}$), while in the right-hand panels firms' signals are subject to idiosyncratic noise ($\sigma_v^2 = 1$). Other parameters are $\{\beta, \sigma, \theta, \omega, \rho\} = \{0.994, 1, 0.7, 0.994, 0.8\}$.

Figure 3: Impulse responses following a demand shock

The middle row depicts the unique solutions (again, under near-full and dispersed information) when the central bank's marginal response to inflation is more subdued, at only 0.5 instead of 1.5. Since this coefficient is below the Taylor threshold, the aggregate price level itself becomes stationary, with inflation initially rising above trend and then falling below trend. The weaker price effect induces a larger movement in output on impact, but the sustained period of below-trend inflation later causes a small contraction. Despite the central bank's decision rule, the real interest rate remains positive throughout.

The bottom two panels show the unique solutions when the central bank does not respond to the state of the economy at all, instead keeping the nominal interest rate pegged at its steady-state level. The price response is both smaller and less persistent, causing the response of output to be substantially larger again. With no movement in the nominal interest rate, the real rate is initially negative as the household anticipates the subsequent price increases. Once the price level peaks and inflation falls below trend, however, the real interest rate becomes, and remains, positive thereafter.

5.2 Central bank design determines persistence in the price level

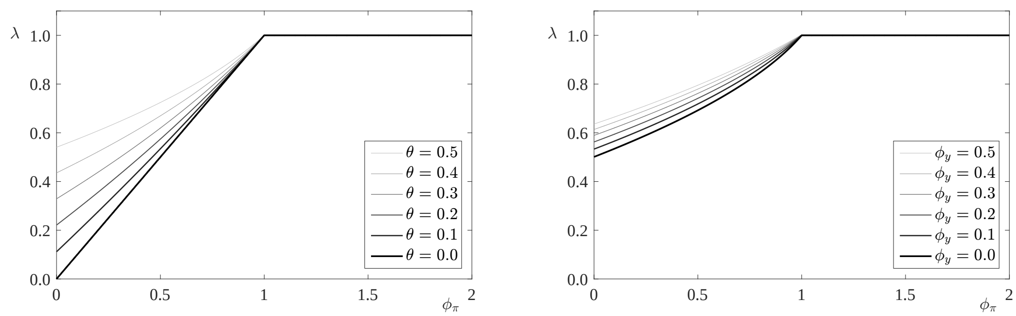
Corollary 2. *When the central bank chooses to satisfy the Taylor principle, the price level exhibits a unit root. When the central bank declines to satisfy the Taylor principle, the price level is stationary, with persistence strictly increasing in the coefficients of the central bank's decision rule:*

$$\frac{\partial \lambda}{\partial \phi_\pi} = \kappa\sigma\delta\left(\left(\beta+\delta+\kappa\sigma\delta\right)^2 - 4\beta\delta\left(1+\kappa\sigma\delta\phi_\pi\right)\right)^{-\frac{1}{2}} > 0 \quad (42a)$$

$$\frac{\partial \lambda}{\partial \phi_y} = \frac{1}{2}\sigma\left(1+\delta\left(2-\beta\right)\left(\left(\beta+\delta+\kappa\sigma\delta\right)^2 - 4\beta\delta\left(1+\kappa\sigma\delta\phi_\pi\right)\right)^{-\frac{1}{2}}\right) > 0 \quad (42b)$$

Figure 4 plots the solutions for λ as a function of ϕ_π while varying ϕ_y and θ. The positive slope when below the Taylor threshold may be understood by referring to the model's competitive equilibrium condition (27). Increasing ϕ_π lowers the weight that firms place on their beliefs about current and future prices, but increases the coefficient on beliefs about the lagged price level.

(a) Varying θ (b) Varying ϕ_y



Note: Both charts plot the intrinsic persistence of the price level (λ) as a function of the central bank's marginal response to inflation (ϕ_π). On the left, the marginal response to output is held fixed at $\phi_y = \frac{0.5}{4}$. On the right, the Calvo parameter is $\theta = 0.5$. Other parameters are $\{\beta, \sigma, \kappa\} = \{0.994, 1, 0.5\}$.

Figure 4: Persistence of the price level under different parameter choices

Figure 4a highlights a curious oddity that has long applied to the canonical solution to the New Keynesian model. When the central bank satisfies the Taylor principle, so that $\lambda = 1$ and the forward-looking full-information solution is $\pi_t = \gamma x_t$, changing the stickiness of firms' prices (θ) does not alter the persistence of the model following a shock, only the magnitude of its effect. When the central bank does not satisfy the Taylor principle, however, increasing θ does achieve the intuitively anticipated result of increasing the model's endogenous persistence.

Figure 4b shows, curiously, that the persistence of the price level is increasing in ϕ_y if the central bank does not satisfy the Taylor principle. Inspection of equation (33b) helps to explain this result. Setting ϕ_π to 0 and treating $(1-\delta)\sum_{q=0}^{\infty}(\delta A)^q$ as being roughly constant as δ varies, we see that increasing ϕ_y lowers δ and therefore serves to increase firms' strategic complementarity.

5.3 The monetary authority does not *need* to respond to cyclical deviations

Corollary 3. *Provided that $\sigma_v > 0$, a unique and stable solution exists when the nominal interest rate remains pegged at its steady-state value ($\phi_y = \phi_\pi =$*

0*), with the following corresponding full-information coefficients:*

$$\lambda^{peg} = \frac{(\beta + 1 + \kappa\sigma) - \sqrt{(\beta + 1 + \kappa\sigma)^2 - 4\beta}}{2\beta} \quad \xrightarrow{\theta \to 0} \quad 0 \tag{43a}$$

$$\gamma^{peg} = \left(\frac{1}{1-\rho}\right)\left(\frac{\kappa\sigma}{1 + \beta(1 - \rho - \lambda) + \kappa\sigma}\right) \quad \xrightarrow{\theta \to 0} \quad \frac{1}{1-\rho} \tag{43b}$$

This result stands in partial contrast to the indeterminacy result of Sargent and Wallace (1975), although it bears emphasising that the peg here is restricted to the steady-state level of the interest rate. Note that under flexible prices ($\theta = 0$), these become simply $\lambda = 0$ and $\gamma = \frac{1}{1-\rho}$. This makes sense, as with an interest rate peg ($i_t = 0$) the household's Euler equation (9) becomes:

$$y_t = E_t[y_{t+1}] + \sigma\{E_t[p_{t+1}] - p_t + x_t\} \tag{44}$$

Under full information and price flexibility, current and expected future prices adjust to fully offset x_t, keeping the term in braces equal to zero so that output never deviates from trend.

5.4 The real interest rate still responds

It is commonly suggested that the purpose of the Taylor principle is to ensure that the real interest rate moves in the same direction as prices (inflation). However, this is not necessary when the price level is stationary. Following a positive demand shock that initially raises prices, the period of below-trend inflation that occurs to bring the price level back to trend will also raise the real interest rate, even if the nominal rate remains fixed.

Corollary 4. *Suppose that $\phi_y = 0$. Then under full information:*

- *The ex ante real interest rate is given by:*

$$r_t = (1 + \phi_\pi - \rho - \lambda)\gamma x_t + (1 - \lambda)(\lambda - \phi_\pi) p_{t-1} \tag{45a}$$

- *The impulse response function (IRF) of the real interest rate is given by:*

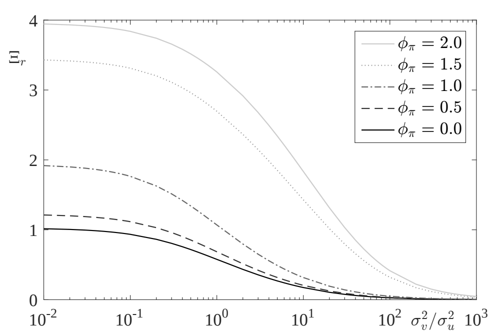
$$\frac{\partial r_{t+s}}{\partial u_t} = \gamma\left((1 + \phi_\pi - \rho - \lambda)\rho^s + (1 - \lambda)(\lambda - \phi_\pi)\sum_{q=0}^{s-1}\lambda^{s-1-q}\rho^q\right) \tag{45b}$$

- *The sum of all current and future IRF values is given by:*

$$\Xi_r \equiv \sum_{s=0}^{\infty} \frac{\partial r_{t+s}}{\partial u_t} = \begin{cases} \gamma & > 0 \quad \textit{if } \phi_\pi \leq 1 \\ \gamma \left(\frac{\phi_\pi - \rho}{1-\rho}\right) & > 0 \quad \textit{if } \phi_\pi > 1 \end{cases} \tag{45c}$$

When the Taylor principle is satisfied, the impulse response simplifies to $\frac{\partial r_{t+s}}{\partial u_t} = \gamma\left(\phi_\pi - \rho\right)\rho^s$, which is always positive. When the Taylor principle is not satisfied, the real rate will be negative on impact if $1 + \phi_\pi - \rho - \lambda < 0$.[20] Even then, however, it eventually turns positive and the absolute sum of later periods exceeds that of early periods so that the total effect is positive.

Under idiosyncratically noisy information, the sum of real interest rates is lower (as the dampened response of prices means that inflation deviations are smaller), but remains strictly positive. Figure 5 illustrates this point, plotting Ξ_r for various values of ϕ_π as the amount of idiosyncratic noise varies. Although not shown, setting $\phi_y > 0$ raises Ξ_r in all cases.



Note: The chart plots the sum of all current and future deviations of the real interest rate from trend caused by a positive demand shock ($\Xi_r \equiv \sum_{s=0}^{\infty} \frac{\partial r_{t+s}}{\partial u_t}$) as a function of the level of idiosyncratic noise faced by price-setting firms (σ_v^2/σ_u^2) for various values of the central bank's marginal response to inflation (ϕ_π). Other parameters are $\{\beta, \phi_y, \sigma, \theta, \omega, \rho\} = \{0.994, 0, 1, 0.7, 0.994, 0.8\}$.

Figure 5: The total effect of a demand shock on the real interest rate

[20] This occurs if prices are sufficiently sticky.

5.5 Inflation stability can still occur with 'passive' monetary policy

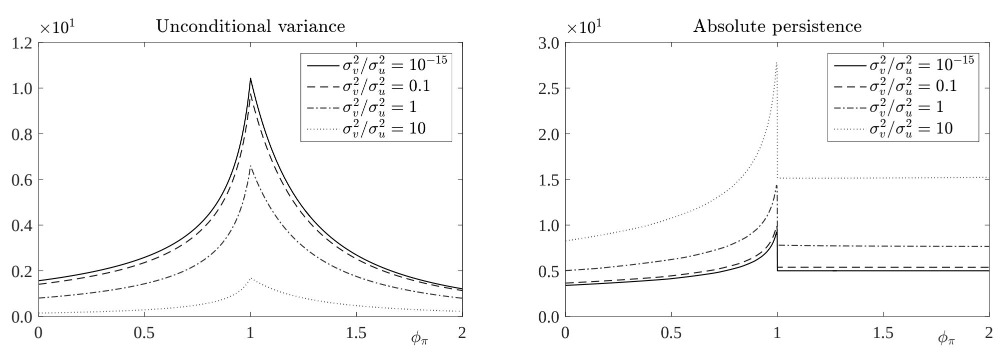
Corollary 5. *In an economy with only demand shocks, the unconditional variance of inflation is:*

$$\mathrm{Var}\left(\pi_t\right) = 2\left(\frac{1-\lambda}{1-\lambda^2}\right)\left(\frac{1}{1-\rho\lambda}\right)\left(\frac{1-\rho}{1-\rho^2}\right)\gamma^2\sigma_u^2 \tag{46}$$

under full information and strictly falls as σ_v^2 rises.

When varying ϕ_π, (46) peaks at the Taylor threshold. When the Taylor principle is satisfied, inflation volatility is decreasing in ϕ_π. In this case, $\lambda = 1$ so that (46) simplifies to $\mathrm{Var}\left(\pi_t\right) = \left(\gamma^2 / \left(1-\rho^2\right)\right)\sigma_u^2 = \gamma^2 \,\mathrm{Var}\left(x_t\right)$. An increase in ϕ_π lowers γ and, thus, lowers inflation volatility.

When the Taylor principle is not satisfied, inflation volatility is increasing in ϕ_π. Since $\lambda \in (0,1)$, (46) is increasing in λ (and, hence, if γ were held fixed, in ϕ_π). When ϕ_π increases, this second effect dominates changes in γ, leading to higher volatility.



Note: The charts plot the unconditional variance and absolute persistence $\left(\left(\sum_{s=0}^{\infty}\left|\frac{\partial \pi_{t+s}}{\partial u_t}\right|\right) / \left(\frac{\partial \pi_t}{\partial u_t}\right)\right)$ of deviations of inflation from trend as a functions of the central bank's marginal response to inflation (ϕ_π). Other parameters are $\{\phi_y, \beta, \sigma, \theta, \omega, \rho\} = \{0, 0.994, 1, 0.7, 0.994, 0.8\}$.

Figure 6: Unconditional volatility and persistence of inflation

Figure 6 illustrates this point, plotting the unconditional variance of inflation and its absolute persistence — the sum of absolute deviations of inflation from trend, divided by the on-impact deviation: $\left(\sum_{s=0}^{\infty}\left|\frac{\partial \pi_{t+s}}{\partial u_t}\right|\right) / \left(\frac{\partial \pi_t}{\partial u_t}\right)$ — while holding $\phi_y = 0$. When the Taylor principle is satisfied, this ratio is the same regardless of the particular value of ϕ_π. Under full information, inflation is simply a multiple of x_t so that the ratio is given simply by $\frac{1}{1-\rho} = 5$ in this calibration. As idiosyncratic noise becomes larger, persistence increases but unconditional variance decreases.

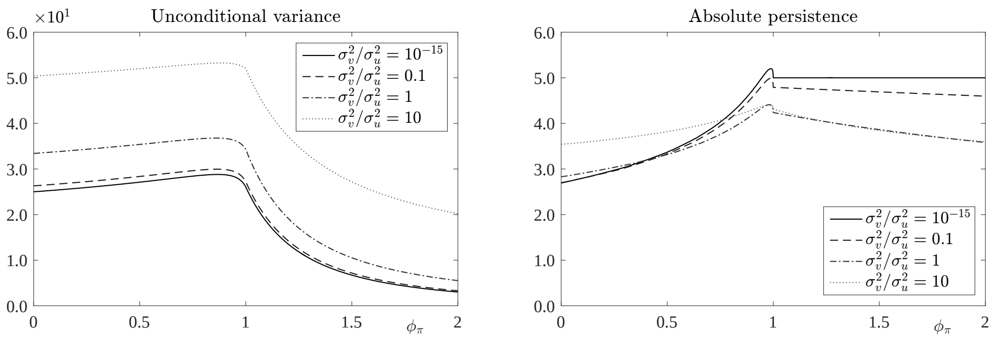
When the Taylor principle is not satisfied, the absolute persistence of inflation is a convexly increasing function of ϕ_π (note that since the price level is stationary in this region, the regular sum of current and future deviations would be zero). It reaches a peak at the Taylor threshold and, as the solution switches to the different root for λ, the persistence then steps down to the constant values discussed above.

5.6 Deviations of output from trend are more persistent with 'active' monetary policy, but output volatility is nevertheless lower

Corollary 6. *The return of output to its trend following a demand shock is more rapid when the central bank does not satisfy the Taylor principle.*

This result follows from the stationarity of the price level. With a positive demand shock the price level initially rises above, but subsequently falls back to, trend. During the initial period, output rises above its trend. But in the latter period, since the household anticipates that inflation will be below trend, it correspondingly lowers demand more quickly than it would if prices remained above their initial level (as they do when the central bank satisfies the Taylor principle). Indeed, as seen in the impulse responses shown in figure 3, when ϕ_π is below the Taylor threshold, output overshoots slightly so that after a positive demand shock it ultimately returns to trend from below. The same argument applies in reverse following a negative shock.

The more rapid return of output to trend generally produces lower output persistence, although not always. Figure 7 plots the unconditional variance of output and its absolute persistence while holding $\phi_y = 0$.



Note: The charts plot the unconditional variance and absolute persistence $\left(\left(\sum_{s=0}^{\infty}\left|\frac{\partial y_{t+s}}{\partial u_t}\right|\right)/\left(\frac{\partial y_t}{\partial u_t}\right)\right)$ of deviations of output from trend as a functions of the central bank's marginal response to inflation (ϕ_π). Other parameters are $\{\phi_y, \beta, \sigma, \theta, \omega, \rho\} = \{0, 0.994, 1, 0.7, 0.994, 0.8\}$.

Figure 7: Unconditional volatility and persistence of output

When the Taylor principle is satisfied and firms have full information, this ratio is the same regardless of the particular value of ϕ_π. Since, in this case, output is simply a multiple of x_t, the ratio is given simply by $\frac{1}{1-\rho} = 5$ in this calibration. As idiosyncratic noise is introduced, however, the absolute persistence becomes a decreasing function of ϕ_π.

When the Taylor principle is not satisfied, persistence is generally lower than for when it holds. Persistence is slightly higher when close to, but below, the Taylor threshold, but then falls as ϕ_π falls towards zero. Despite this, unconditional volatility is generally higher when the principle is not satisfied, the on-impact response being large enough to offset the fall in persistence.

6 Conclusion

This paper makes a simple point, but one with striking implications. When price-setting firms are subject to idiosyncratic noise in their information sets about both current and past deviations of the economy from its trend, the solution is unique (ruling out sunspots) and features nominal stability, regardless of the responsiveness of the central bank.

Standard solutions to the New Keynesian model are nested when the Taylor principle is satisfied and the noise faced by firms is taken to zero. But when the Taylor principle is not satisfied, including when the nominal interest rate is simply pegged to its steady-state level, a unique and stable solution still emerges, and features stationarity in the aggregate price level, provided that firms face at least some heterogeneous uncertainty. In all cases, as is typical in such models, the information friction represents a real rigidity, with persistence following a shock increasing in the amount of noise faced by firms.

It is important to emphasise that the model, as implemented, is log-linearised around a deterministic steady state. This imposes an assumption that although firms do not share common knowledge about the actual price level, they do agree on its underlying trend. In effect, this amounts to an assumption that while firms' expectations about near-term inflation remain dispersed, their beliefs about long-run inflation are perfectly anchored. Conditional on this assumption, nominal stability around that trend need not require a systematic central bank response to the state of the economy. Although currently linearised around a zero-inflation trend, this would presumably also address the (in)determinacy concerns of Ascari and Ropele (2009) in the presence of positive trend inflation.

The determinacy obtained under an interest rate peg is striking, but ultimately perfectly intuitive. The peg applied above is to the steady-state value for the nominal interest rate (which, with trend inflation at zero, is just the steady state real interest rate, here $1/\beta$). So long as the natural interest rate returns to this value, and firms *know* that it will return, then the logic of Wicksell (1898) remains intact. If the interest rate were indefinitely pegged to a different value, however, it would represent a change *of* trend. Dynamics would then depend on if, and how, agents' beliefs shift between a Wicksellian world view (mistakenly assuming no change in trend) and a Fisherian one (where they accept it). The 'backwards-stable' criteria of Cochrane (2016) – which necessarily focusses on backward-looking solutions to the cyclical component of the model – is ruled out when firms face imperfect common knowledge, but the broader point of Cochrane's neo-Fisherian question remains.

This paper makes no comment on how agents might arrive at a consensus about the steady state of the economy. If, for example, systematic policy is necessary to ensure that long-run inflation expectations remain well anchored then that would be in addition to the results discussed above. Nevertheless, it bears noting that when the central bank's response to inflation is less than one, the full, non-linear model features a unique, globally stable steady-state equilibrium even after allowing for the possibility of a lower bound on interest rates (albeit one with cyclical indeterminacy under full information).[21] This suggests that a learning model of the steady state, combined with the approach described here for solutions around a given steady state, may prove fruitful in both addressing the neo-Fisherian question and removing the deflationary trap emphasised by Benhabib, Schmitt-Grohe and Uribe (2001).[22]

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[21] That is, the economy would be *structurally determinate* (in the absence of shocks), but *cyclically indeterminate* (when shocks are added) if there were full information.

[22] To my knowledge, the only work to date combining uncertainty about both the model the state is Graham (2011), who finds rapid convergence to model-consistent expectations.

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Appendix

A Proof of proposition 1

I start with solutions based only on past, current and expected future values of the fundamental shock. The purely backward-looking solution is found simply by imposing perfect foresight on (7), giving:

$$p_t^{(B)} = (1+\phi_\pi)p_{t-1} - \phi_\pi p_{t-2} - x_{t-1} \tag{A.1}$$

To obtain the purely forward-looking solution, (7) must be substituted forward. Following Cho and Moreno (2011), define $m_1 = \frac{1}{1+\phi_\pi}$, $\lambda_1 = \frac{\phi_\pi}{1+\phi_\pi}$, and $\gamma_1 = \frac{1}{1+\phi_\pi}$, so that (7) may be written as $p_t = m_1 E_t^\Omega [p_{t+1}] + \lambda_1 p_{t-1} + \gamma_1 x_t$. Stepping this forward and taking the period-t expectation gives $E_t^\Omega [p_{t+1}] = m_1 E_t^\Omega [p_{t+2}] + \lambda_1 p_t + \rho\gamma_1 x_t$. Combining the two then yields $p_t = m_2 E_t^\Omega [p_{t+2}] + \lambda_2 p_{t-1} + \gamma_2 x_t$ where $m_2 = (1-m_1\lambda_1)^{-1} m_1 m_1$, $\lambda_2 = (1-m_1\lambda_1)^{-1}\lambda_1$ and $\gamma_2 = (1-m_1\lambda_1)^{-1}(\gamma_1 + m_1\gamma_1\rho)$. Repeating the process then gives:

$$p_t = m_s E_t^\Omega [p_{t+s}] + \lambda_s p_{t-1} + \gamma_s x_t \tag{A.2a}$$

where

$$m_s = \left(\frac{1}{1-m_1\lambda_{s-1}}\right) m_1 m_{s-1} = \frac{m_{s-1}}{1+\phi_\pi - \lambda_{s-1}} \tag{A.2b}$$

$$\lambda_s = \left(\frac{1}{1-m_1\lambda_{s-1}}\right) \lambda_1 = \frac{\phi_\pi}{1+\phi_\pi - \lambda_{s-1}} \tag{A.2c}$$

$$\gamma_s = \left(\frac{1}{1-m_1\lambda_{s-1}}\right) \left(\gamma_1 + m_1\gamma_{s-1}\rho\right) = \frac{1+\rho\gamma_{s-1}}{1+\phi_\pi - \lambda_{s-1}} \tag{A.2d}$$

In the limit, therefore, we have that:

$$p_t = \lambda p_{t-1} + \gamma x_t + \lim_{s\to\infty} m_s E_t^\Omega [p_{t+s}] \tag{A.3}$$

where $\lambda = \lim_{s\to\infty} \lambda_s$ and $\gamma = \lim_{s\to\infty} \gamma_s$.

Next define the function $f(\lambda) = \phi_\pi / (1+\phi_\pi - \lambda)$. It is clear that $\{\phi, 1\}$ are the two solutions to the quadratic $\lambda = f(\lambda)$. Since (i) $f'(\lambda) > 0$; (ii)

$f''(\lambda) > 0$; (iii) $\exists\,\lambda$ s.t. $f(\lambda) < 1$; and (iv) $\lambda_1 = \frac{\phi_\pi}{1+\phi_\pi} < \min\{\phi, 1\}$, it follows that $\lambda_s \xrightarrow{s\to\infty} \min\{\phi, 1\}$ from below. Consequently, when $\lambda = \phi_\pi$, $\gamma = \frac{1}{1-\rho}$ and when $\lambda = 1$ (that is, $\phi_\pi \geq 1$), $\gamma = \frac{1}{\phi_\pi - \rho}$.

Note that (A.3) is simply a restatement of (7) when substituted foward, so all possible solutions, whether foward- or backward-looking, must satisfy (A.3). Since $p_t^{(F)} = \lambda p_{t-1} + \gamma x_t$ is the (candidate) purely forward-looking solution, the term $b_t \equiv \lim_{s\to\infty} m_s E_t^{\Omega}[p_{t+s}]$ therefore represents the possibility of backward-looking solutions.

With $\lambda = \min\{\phi_\pi, 1\}$, it follows that:

$$m_s = \begin{cases} \left(\frac{1}{1+\phi_\pi}\right) & \text{if } \phi_\pi < 1 \\ \left(\frac{1}{1+\phi_\pi}\right)\left(\frac{1}{\phi_\pi}\right)^{s-1} & \text{if } \phi_\pi \geq 1 \end{cases} \tag{A.4}$$

for large s. Forecasts for p_{t+s} under the purely forward-looking solution are given by:

$$\begin{aligned} E_t^{\Omega}[p_{t+s}]^{(F)} &= \lambda^{s+1} p_{t-1} + \gamma\left(\sum_{q=0}^{s} \lambda^{s-q}\rho^q\right) x_t \\ &= \lambda^{s+1}\left[p_{t-1} + \gamma\left(\frac{1-\left(\frac{\rho}{\lambda}\right)^{s+1}}{\lambda - \rho}\right) x_t\right] \\ &= \begin{cases} \phi_\pi^{s+1}\left[p_{t-1} + \left(\frac{1-\left(\frac{\rho}{\phi_\pi}\right)^{s+1}}{(\phi_\pi-\rho)(1-\rho)}\right) x_t\right] & \text{if } \phi_\pi < 1 \\ p_{t-1} + \left(\frac{1-\rho^{s+1}}{(\phi_\pi-\rho)(1-\rho)}\right) x_t & \text{if } \phi_\pi \geq 1 \end{cases} \end{aligned} \tag{A.5}$$

while under the purely backward-looking solution:

$$E_t^{\Omega}\left[p_{t+1}\right]^{(B)} = (1+\phi_\pi)\left\{(1+\phi_\pi)p_{t-1} - \phi_\pi p_{t-2} - x_{t-1}\right\} - \phi_\pi p_{t-1} - x_t$$
$$= \left(1+\phi_\pi+\phi_\pi^2\right)p_{t-1} - \phi_\pi\left(1+\phi_\pi\right)p_{t-2} - \left(1+\phi_\pi\right)x_{t-1} - x_t$$

$$E_t^{\Omega}\left[p_{t+2}\right]^{(B)} = \left(1+\phi_\pi+\phi_\pi^2\right)\left\{(1+\phi_\pi)p_{t-1} - \phi_\pi p_{t-2} - x_{t-1}\right\} - \phi_\pi\left(1+\phi_\pi\right)p_{t-1} - \left(1+\phi_\pi\right)x_t - x_{t+1}$$
$$= \left(1+\phi_\pi+\phi_\pi^2+\phi_\pi^3\right)p_{t-1} - \phi_\pi\left(1+\phi_\pi+\phi_\pi^2\right)p_{t-2} - \left(1+\phi_\pi+\phi_\pi^2\right)x_{t-1} - \left(1+\phi_\pi-\rho\right)x_t$$

$$E_t^{\Omega}\left[p_{t+s}\right]^{(B)} = \left(\sum_{q=0}^{s+1}\phi_\pi^q\right)p_{t-1} - \phi_\pi\left(\sum_{q=0}^{s}\phi_\pi^q\right)p_{t-2} - \left(\sum_{q=0}^{s}\phi_\pi^q\right)x_{t-1} - \left(\sum_{q=0}^{s-1}\rho^q\left(\sum_{k=0}^{s-1-q}\phi_\pi^k\right)\right)x_t$$

Adding and subtracting $\phi_\pi\left(\sum_{q=0}^{s-1}\phi_\pi\right)p_{t-1}$ on the right-hand side gives

$$\begin{aligned}E_t^{\Omega}\left[p_{t+s}\right]^{(B)} = &\left(\sum_{q=0}^{s}\phi_\pi^q\right)\Big\{\left(1+\phi_\pi\right)p_{t-1} - \phi_\pi p_{t-2} - x_{t-1}\Big\}\\ &- \phi_\pi\left(\sum_{q=0}^{s-1}\phi_\pi^q\right)p_{t-1} - \left(\sum_{q=0}^{s-1}\rho^q\left(\sum_{k=0}^{s-1-q}\phi_\pi^k\right)\right)x_t\end{aligned}$$

or, rearranging the final term,

$$\begin{aligned}E_t^{\Omega}\left[p_{t+s}\right]^{(B)} = &\left(\sum_{q=0}^{s}\phi_\pi^q\right)\Big\{\left(1+\phi_\pi\right)p_{t-1} - \phi_\pi p_{t-2} - x_{t-1}\Big\}\\ &- \phi_\pi\left(\sum_{q=0}^{s-1}\phi_\pi^q\right)p_{t-1} - \left(\sum_{q=0}^{s-1}\phi_\pi^q\left(\sum_{k=0}^{s-1-q}\rho^k\right)\right)x_t\end{aligned} \tag{A.6}$$

which applies regardless of the value of ϕ_π. Expanding b_t gives:

$$\begin{aligned}b_t &\equiv \lim_{s\to\infty} m_s E_t^{\Omega}\left[p_{t+s}\right]\\ &= \lim_{s\to\infty} m_s\left(\left(1-\alpha\right)E_t^{\Omega}\left[p_{t+s}\right]^{(F)} + \alpha E_t^{\Omega}\left[p_{t+s}\right]^{(B)}\right)\\ &= \left(1-\alpha\right)\lim_{s\to\infty} m_s E_t^{\Omega}\left[p_{t+s}\right]^{(F)} + \alpha\lim_{s\to\infty} m_s E_t^{\Omega}\left[p_{t+s}\right]^{(B)}\end{aligned} \tag{A.7}$$

for some $\alpha \in \mathbb{R}$. Looking first at $\phi_\pi < 1$, substituting in (A.4), (A.5) and (A.6) then produces:

$$b_t = \left(\frac{1}{1+\phi_\pi}\right) \lim_{s\to\infty} \left\{ \begin{array}{l} (1-\alpha)\,\phi_\pi^{s+1} \left[p_{t-1} + \left(\frac{1-\left(\frac{\rho}{\phi_\pi}\right)^{s+1}}{(\phi_\pi-\rho)(1-\rho)}\right) x_t\right] \\ +\alpha \left[\left(\sum_{q=0}^{s} \phi_\pi^q\right) p_t^{(B)} - \phi_\pi \left(\sum_{q=0}^{s-1} \phi_\pi^q\right) p_{t-1} - \left(\sum_{q=0}^{s-1} \phi_\pi^q \left(\sum_{k=0}^{s-1-q} \rho^k\right)\right) x_t\right] \end{array} \right\}$$

$$= \left(\frac{1}{1+\phi_\pi}\right) \alpha \left[\left(\frac{1}{1-\phi_\pi}\right) p_t^{(B)} - \phi_\pi \left(\frac{1}{1-\phi_\pi}\right) p_{t-1} - \left(\frac{1}{1-\phi_\pi}\right)\left(\frac{1}{1-\rho}\right) x_t\right]$$

$$= \left(\frac{1}{1+\phi_\pi}\right)\left(\frac{1}{1-\phi_\pi}\right) \alpha \left[p_t^{(B)} - p_t^{(F)}\right] \tag{A.8}$$

While, for $\phi_\pi > 1$, substituting in (A.4), (A.5) and (A.6) gives:

$$b_t = \left(\frac{1}{1+\phi_\pi}\right) \lim_{s\to\infty} \left\{ \begin{array}{l} (1-\alpha)\left(\frac{1}{\phi_\pi}\right)^{s+1} \left[p_{t-1} + \left(\frac{1-\rho^{s+1}}{(\phi_\pi-\rho)(1-\rho)}\right) x_t\right] \\ +\alpha \left(\frac{1}{\phi_\pi}\right)^{s+1} \left[\left(\sum_{q=0}^{s} \phi_\pi^q\right) p_t^{(B)} - \phi_\pi \left(\sum_{q=0}^{s-1} \phi_\pi^q\right) p_{t-1} - \left(\sum_{q=0}^{s-1} \phi_\pi^q \left(\sum_{k=0}^{s-1-q} \rho^k\right)\right) x_t\right] \end{array} \right\}$$

$$= \left(\frac{1}{1+\phi_\pi}\right) \alpha \left[\left(\frac{1}{\phi_\pi-1}\right) p_t^{(B)} - \phi_\pi \left(\frac{1}{\phi_\pi-1}\right) p_{t-1} - \left(\frac{1}{\phi_\pi}\right)^2 \lim_{s\to\infty} \left(\sum_{q=0}^{s-1} \left(\frac{1}{\phi_\pi}\right)^q \left(\sum_{k=0}^{s-1-q} \left(\frac{\rho}{\phi_\pi}\right)^k\right)\right) x_t\right]$$

$$= \left(\frac{1}{1+\phi_\pi}\right) \alpha \left[\left(\frac{1}{\phi_\pi-1}\right) p_t^{(B)} - \phi_\pi \left(\frac{1}{\phi_\pi-1}\right) p_{t-1} - \left(\frac{1}{\phi_\pi-1}\right)\left(\frac{1}{\phi_\pi-\rho}\right) x_t\right]$$

$$= \left(\frac{1}{1+\phi_\pi}\right)\left(\frac{1}{\phi_\pi-1}\right) \alpha \left[p_t^{(B)} - p_t^{(F)}\right] \tag{A.9}$$

so that, combining (A.3), (A.7), (A.8) and (A.9), I obtain:

$$p_t = (1-\xi)\, p_t^{(F)} + \xi\, p_t^{(B)} \tag{A.10a}$$

$$\xi = \left\{ \begin{array}{ll} \left(\frac{1}{1+\phi_\pi}\right)\left(\frac{1}{1-\phi_\pi}\right)\alpha & \text{if } \phi_\pi < 1 \\ \left(\frac{1}{1+\phi_\pi}\right)\left(\frac{1}{\phi_\pi-1}\right)\alpha & \text{if } \phi_\pi \geq 1 \end{array} \right. \tag{A.10b}$$

for any $\alpha \in \mathbb{R}$. Finally, note that any extrinsic stochastic process that matches the functional form of (7) without the structural shock may also be added, so that we arrive at:

$$p_t = (1-\xi)\, p_t^{(F)} + \xi\, p_t^{(B)} + w_t \quad \text{where } \xi \in \mathbb{R} \tag{A.11a}$$

$$E_t^{\Omega}\left[w_{t+1}\right] = (1-\phi_\pi)\, w_t - \phi_\pi w_{t-1} \quad \text{where } Cov\left(w_t, u_s\right) = 0\ \forall\, s, t \tag{A.11b}$$

B Proof of proposition 2

To find values for γ and λ, note that under full information the competitive equilibrium condition (27) simplifies to:

$$p_t = \left(\frac{1}{1-\zeta_0}\right)\left(b_p x_t + (\theta + \zeta_{-1})\, p_{t-1} + \beta\theta E_t^{\Omega}[p_{t+1}] + \zeta_{1+} E_t^{\Omega}\left[(1-\delta)\sum_{s=0}^{\infty}\delta^s p_{t+s+1}\right]\right) \tag{B.1}$$

While firms' expectation of future prices must be formed as:

$$E_t^{\Omega}[p_{t+1}] = \gamma(\rho+\lambda)\, x_t + \lambda^2 p_{t-1} \tag{B.2a}$$
$$E_t^{\Omega}[p_{t+2}] = \gamma\left(\rho^2+\lambda\rho+\lambda^2\right) x_t + \lambda^3 p_{t-1} \tag{B.2b}$$
$$E_t^{\Omega}[p_{t+3}] = \gamma\left(\rho^3+\lambda\rho^2+\lambda^2\rho+\lambda^3\right) x_t + \lambda^4 p_{t-1} \tag{B.2c}$$
$$\vdots$$

Substituting (B.2) into (B.1) then gives

$$p_t = \left(\frac{1}{1-\zeta_0}\right)\left(\begin{array}{l} \quad (b_p + \beta\theta\gamma(\rho+\lambda))\, x_t \\ + (\theta + \zeta_{-1} + \beta\theta\lambda^2)\, p_{t-1} \\ + \zeta_{1+}(1-\delta)\displaystyle\sum_{s=0}^{\infty}\delta^s\left(\gamma\left(\sum_{q=0}^{s+1}\rho^q\lambda^{s+1-q}\right) x_t + \lambda^{s+2} p_{t-1}\right)\end{array}\right) \tag{B.3}$$

Gathering like terms, it follows that

$$\gamma = \left(\frac{1}{1-\zeta_0}\right)\left(b_p + \beta\theta c(\rho+\lambda) + \gamma\,\zeta_{1+}(1-\delta)\sum_{s=0}^{\infty}\delta^s\left(\sum_{q=0}^{s+1}\rho^q\lambda^{s+1-q}\right)\right) \tag{B.4a}$$

$$\lambda = \left(\frac{1}{1-\zeta_0}\right)\left(\theta + \zeta_{-1} + \beta\theta\lambda^2 + \lambda^2\zeta_{1+}\left(\frac{1-\delta}{1-\delta\lambda}\right)\right) \tag{B.4b}$$

The coefficient γ

Starting with the expression for γ, note that (B.4a) may be rewritten as

$$\gamma = \frac{b_p}{\xi} \quad \text{where} \quad \xi = 1 - \zeta_0 - \beta\theta(\rho+\lambda) - \zeta_{1+}(1-\delta)\sum_{s=0}^{\infty}\delta^s\left(\sum_{q=0}^{s+1}\rho^q\lambda^{s+1-q}\right) \tag{B.5a}$$

The expression for ξ can then be re-expressed as:

$$\begin{aligned}\xi &= 1-\zeta_0-\beta\theta\left(\rho+\lambda\right)-\zeta_{1^+}\left(1-\delta\right)\sum_{s=0}^{\infty}\delta^s\lambda^{s+1}\left(\sum_{q=0}^{s+1}\left(\frac{\rho}{\lambda}\right)^q\right)\\ &= 1-\zeta_0-\beta\theta\left(\rho+\lambda\right)-\zeta_{1^+}\left(1-\delta\right)\sum_{s=0}^{\infty}\delta^s\lambda^{s+1}\left(\frac{1-\left(\frac{\rho}{d}\right)^{s+2}}{1-\frac{\rho}{\lambda}}\right)\\ &= 1-\zeta_0-\beta\theta\left(\rho+\lambda\right)-\zeta_{1^+}\left(1-\delta\right)\left(\frac{\lambda}{1-\frac{\rho}{\lambda}}\right)\sum_{s=0}^{\infty}\left(\delta\lambda\right)^s\left(1-\left(\frac{\rho}{\lambda}\right)^{s+2}\right)\\ &= 1-\zeta_0-\beta\theta\left(\rho+\lambda\right)-\zeta_{1^+}\left(1-\delta\right)\left(\frac{\lambda}{1-\frac{\rho}{\lambda}}\right)\left(\frac{1}{1-\delta\lambda}-\left(\frac{\rho}{\lambda}\right)^2\frac{1}{1-\delta\rho}\right)\end{aligned} \tag{B.6}$$

where the final equality requires that $\delta\lambda < 1$. For values of $\lambda \geq \frac{1}{\delta}$, the sum $\sum_{s=0}^{\infty}\left(\delta\lambda\right)^s$ will explode, leading to $c = 0$ (that is, non-existence of a solution).[23] The expression (B.6) simplifies further as

$$\begin{aligned}\xi &= 1-\zeta_0-\beta\theta\left(\rho+\lambda\right)-\zeta_{1^+}\left(1-\delta\right)\left(\frac{1}{\lambda-\rho}\right)\left(\frac{\lambda^2}{1-\delta\lambda}-\frac{\rho^2}{1-\delta\rho}\right)\\ &= 1-\zeta_0-\beta\theta\left(\rho+\lambda\right)-\zeta_{1^+}\left(1-\delta\right)\left(\frac{\lambda+\rho-\delta\rho\lambda}{\left(1-\delta\lambda\right)\left(1-\delta\rho\right)}\right)\end{aligned} \tag{B.7}$$

Expanding ζ_0 and ζ_{1^+}, this then becomes

$$\begin{aligned}\xi &= \theta\left(1+\beta\right)+\theta\kappa\sigma\delta\left(\phi_\pi+1-\phi_\pi\delta\right)\\ &-\beta\theta\left(\rho+\lambda\right)\\ &-\theta\kappa\sigma\delta\left(1-\phi_\pi\delta\right)\left(1-\delta\right)\left(\frac{\lambda+\rho-\delta\rho\lambda}{\left(1-\delta\lambda\right)\left(1-\delta\rho\right)}\right)\end{aligned} \tag{B.8}$$

or, after some straightforward manipulation,

$$\xi = \theta+\beta\theta\left(1-\rho-\lambda\right)+\theta\kappa\sigma\left(1-\frac{\left(1-\delta\right)}{\left(1-\delta\lambda\right)}\frac{\left(1-\delta\phi_\pi\right)}{\left(1-\delta\rho\right)}\right) \tag{B.9}$$

[23] Note that since $\rho \in (0,1)$ and $\delta \in (0,1]$, it must be the case that $\delta\rho < 1$. Also note that the third equality does not require that $\frac{\rho}{\lambda} < 1$ in order to write $\left(\frac{1-\left(\frac{\rho}{\lambda}\right)^{s+2}}{1-\frac{\rho}{\lambda}}\right)$, as the latter is simplifying a finite (rather than infinite) sum.

The coefficient λ

Next looking at the expression for λ, we can rewrite (B.4b) as

$$\left\{\beta\theta\delta\right\}\lambda^3-\left\{\beta\theta+\left(1-\zeta_0\right)\delta+\zeta_{2+}\right\}\lambda^2+\left\{1-\zeta_0+\left(\theta+\zeta_{-1}\right)\delta\right\}\lambda-\left\{\theta+\zeta_{-1}\right\}=0 \tag{B.10}$$

Expanding the latter three compound parameters, we have

$$\left\{\beta\theta+\left(1-\zeta_0\right)\delta+\zeta_{2+}\right\}=\beta\theta+\delta\theta\left(1+\beta\right)+\theta\kappa\sigma\delta \tag{B.11a}$$
$$\left\{1-\zeta_0+\left(\theta+\zeta_{-1}\right)\delta\right\}=\beta\theta+\theta\left(1+\delta\right)+\theta\kappa\sigma\delta\left(1+\phi_\pi\right) \tag{B.11b}$$
$$\left\{\theta+\zeta_{-1}\right\}=\theta+\theta\kappa\sigma\delta\phi_\pi \tag{B.11c}$$

It is easy to confirm that (B.10) has a root of $\lambda=1$:

$$\left\{\beta\theta\delta\right\}(1)^3-\left\{\beta\theta+\left(1-\zeta_0\right)\delta+\zeta_{2+}\right\}(1)^2+\left\{1-\zeta_0+\left(\theta+\zeta_{-1}\right)\delta\right\}(1)-\left\{\theta+\zeta_{-1}\right\}=0 \tag{B.12}$$

Given this, (B.10) may be rewritten as:

$$\left(\lambda-1\right)\left(\left\{\beta\theta\delta\right\}\lambda^2-\left\{\beta\theta+\delta\theta+\theta\kappa\sigma\delta\right\}\lambda+\left\{\theta+\theta\kappa\sigma\delta\phi_\pi\right\}\right)=0 \tag{B.13}$$

from which the other two roots may be readily obtained as

$$\lambda=\frac{\beta+\delta+\kappa\sigma\delta}{2\beta\delta}\pm\frac{\sqrt{\left(\beta+\delta+\kappa\sigma\delta\right)^2-4\beta\delta\left(1+\kappa\sigma\delta\phi_\pi\right)}}{2\beta\delta} \tag{B.14}$$

These are the non-zero eigenvalues of the system highlighted in the main text. To see that the solution is the lower envelope of these, start from equation (19) in the main text. Cho and Moreno (2011) show that substituting this expression forward gives:

$$\boldsymbol{\zeta}_t=M_kE_t^\Omega\left[\boldsymbol{\zeta}_{t+k}\right]+\Lambda_k\boldsymbol{\zeta}_{t-1}+\Gamma_kx_t \tag{B.15a}$$

where $M_1=A$, $\Lambda_1=B$, $\Gamma_1=C$ and, for $k\geq 2$,

$$M_k=\left(I-A\Lambda_{k-1}\right)^{-1}AM_{k-1} \tag{B.15b}$$
$$\Lambda_k=\left(I-A\Lambda_{k-1}\right)^{-1}B \tag{B.15c}$$
$$\Gamma_k=\left(I-A\Lambda_{k-1}\right)^{-1}\left(C+A\Gamma_{k-1}\rho\right) \tag{B.15d}$$

so that, in the limit,

$$\boldsymbol{\zeta}_t = \Lambda\boldsymbol{\zeta}_{t-1} + \Gamma x_t + \lim_{k\to\infty} M_k E_t^{\Omega}\left[\boldsymbol{\zeta}_{t+k}\right] \tag{B.16}$$

where $\Lambda = \lim_{k\to\infty}\Lambda_k$ and $\Gamma = \lim_{k\to\infty}\Gamma_k$ and under the purely forward-looking solution the limiting expectation term (which accomodates backward-looking solutions) is zero. Since the eigenvalues of D are all distinct, the model must have a dominant solvent (S_1) and a minimal solvent (S_2), where

$$\min\left\{|\lambda| : \lambda \in \lambda\left(S_1\right)\right\} > \max\left\{|\lambda| : \lambda \in \lambda\left(S_2\right)\right\} \tag{B.17}$$

When S_1 and S_2 exist (as they do here), Rendahl (2017) proves that the sequence (B.15c) must converge to S_2, provided that $\Lambda_1 \neq S_1$. But since we have $\Lambda_1 = B$, the proof is established. Given the simplicity of the basic NK model, it is also straightforward here to confirm convergence to the minimal solution numerically.

C Proof of proposition 3

Recall that the candidate solution is of the form:

$$Z_t \equiv \begin{bmatrix} x_t & \widetilde{x}_{t|t} & p_{t-1} & \widetilde{p}_{t-1|t} \end{bmatrix}' \tag{C.1a}$$

$$Z_t = \underbrace{\begin{bmatrix} \rho & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & \alpha_2 & \theta & \alpha_4 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{A} Z_{t-1} + \underbrace{\begin{bmatrix} 1 \\ b_2 \\ 0 \\ b_4 \end{bmatrix}}_{B} u_t \tag{C.1b}$$

$$p_t = \underbrace{\begin{bmatrix} 0 & \alpha_2 & \theta & \alpha_4 \end{bmatrix}}_{\boldsymbol{\alpha}'} Z_t \tag{C.1c}$$

where I have filled in some elements of A, B and $\boldsymbol{\alpha}$ directly from the given law of motion for x_t and the equilibrium condition. Given this solution, firms' signal vectors are expressible as:

$$\boldsymbol{s}_t\left(i\right) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{N} Z_t + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{O} \boldsymbol{v}_t\left(i\right) \tag{C.2}$$

C.1 The reduced-form expression for p_t

Since $\widetilde{x}_{t|t}$ and $\widetilde{p}_{t-1|t}$ are weighted averages of firms' higher-order average expectations, and firms' signals are simply noisy signals of x_t and p_{t-1}, it must be the case that

$$\begin{bmatrix} \widetilde{x}_{t|t} \\ \widetilde{p}_{t-1|t} \end{bmatrix} \xrightarrow{\sigma_v^2 \to 0} \begin{bmatrix} x_t \\ p_{t-1} \end{bmatrix} \tag{C.3}$$

The parameters $\boldsymbol{\alpha}$ must therefore be consistent with the fundamental solution under full information, which is of the form:

$$p_t = \lambda p_{t-1} + \gamma x_t \tag{C.4}$$

The elements of $\boldsymbol{\alpha}$

For the model under ICK to be consistent with the forward solution under full information, it immediately follows that, for given values of λ and γ,

$$\alpha_2 = \gamma \tag{C.5a}$$

$$\alpha_4 = \lambda - \theta \tag{C.5b}$$

C.2 Determining the law of motion

The law of motion for x_t is given and the law of motion for p_{t-1} will come from the solution for $\boldsymbol{\alpha}$ below, so I here focus on those for $\widetilde{x}_{t|t}$ and $\widetilde{p}_{t-1|t}$. First, note that given their definitions, we can write:

$$\widetilde{x}_{t|t} = \underbrace{\begin{bmatrix} (1-\varphi) & \varphi & 0 & 0 \end{bmatrix}}_{\boldsymbol{\varphi}_x'} \overline{E}_t \left[Z_t \right] \tag{C.6a}$$

$$\widetilde{p}_{t-1|t} = \underbrace{\begin{bmatrix} 0 & 0 & (1-\varphi) & \varphi \end{bmatrix}}_{\boldsymbol{\varphi}_p'} \overline{E}_t \left[Z_t \right] \tag{C.6b}$$

or, rearranging these,

$$\overline{E}_t \left[\widetilde{x}_{t|t} \right] = \frac{1}{\varphi} \left(\widetilde{x}_{t|t} - (1-\varphi) \overline{E}_t \left[x_t \right] \right) \tag{C.7a}$$

$$\overline{E}_t \left[\widetilde{p}_{t-1|t} \right] = \frac{1}{\varphi} \left(\widetilde{p}_{t-1|t} - (1-\varphi) \overline{E}_t \left[p_{t-1} \right] \right) \tag{C.7b}$$

Next, write agents' Kalman filter for Z_t:

$$E_t(i)[Z_t] = E_{t-1}(i)[Z_t] + M\left\{\boldsymbol{s}_t(i) - E_{t-1}(i)[\boldsymbol{s}_t(i)]\right\} \tag{C.8}$$

where $M = \begin{bmatrix}\boldsymbol{m}_x & \boldsymbol{m}_p\end{bmatrix}$ is a (4×2) Kalman gain matrix to be determined. Expanding this out and taking the average gives:

$$\overline{E}_t[Z_t] = A\overline{E}_{t-1}[Z_{t-1}] + M\left\{N(AZ_{t-1} + Bu_t) - NA\overline{E}_{t-1}[Z_{t-1}]\right\} \tag{C.9}$$

Gathering like terms and then substituting this into (C.6) then gives:

$$\widetilde{x}_{t|t} = \boldsymbol{\varphi}_x'\left((I - MN)A\overline{E}_{t-1}[Z_{t-1}] + MNAZ_{t-1} + MNBu_t\right) \tag{C.10a}$$

$$\widetilde{p}_{t-1|t} = \boldsymbol{\varphi}_p'\left((I - MN)A\overline{E}_{t-1}[Z_{t-1}] + MNAZ_{t-1} + MNBu_t\right) \tag{C.10b}$$

Note that NA and NB are given by:

$$NA = \begin{bmatrix}\rho & 0 & 0 & 0\\ 0 & \alpha_2 & \theta & \alpha_4\end{bmatrix} \qquad NB = \begin{bmatrix}1\\0\end{bmatrix} \tag{C.11}$$

The law of motion for $\widetilde{x}_{t|t}$

Stepping (C.7) back one period, we can expand (C.10a) to read:

$$\widetilde{x}_{t|t} = (\boldsymbol{\varphi}_x'A - \boldsymbol{\varphi}_x'MNA)\begin{bmatrix}\overline{E}_{t-1}[x_{t-1}]\\ \frac{1}{\varphi}\left(\widetilde{x}_{t-1|t-1} - (1-\varphi)\overline{E}_{t-1}[x_{t-1}]\right)\\ \overline{E}_{t-1}[p_{t-2}]\\ \frac{1}{\varphi}\left(\widetilde{p}_{t-2|t-1} - (1-\varphi)\overline{E}_{t-1}[p_{t-2}]\right)\end{bmatrix}$$
$$+ \boldsymbol{\varphi}_x'MNAZ_{t-1} + \boldsymbol{\varphi}_x'MNBu_t \tag{C.12}$$

Expanding $\boldsymbol{\varphi}'_x A$ and NA and NB, and then gathering like terms, this gives:

$$\begin{aligned}
\widetilde{x}_{t|t} &= \left\{\boldsymbol{\varphi}'_x \boldsymbol{m}_x \rho\right\} x_{t-1} \\
&\quad + \left\{a_{22} - \boldsymbol{\varphi}'_x \boldsymbol{m}_p \frac{\alpha_2}{\varphi} + \boldsymbol{\varphi}'_x \boldsymbol{m}_p \alpha_2\right\} \widetilde{x}_{t-1|t-1} \\
&\quad + \left\{\boldsymbol{\varphi}'_x \boldsymbol{m}_p \theta\right\} p_{t-2} \\
&\quad + \left\{a_{24} - \boldsymbol{\varphi}'_x \boldsymbol{m}_p \frac{\alpha_4}{\varphi} + \boldsymbol{\varphi}'_x \boldsymbol{m}_p \alpha_4\right\} \widetilde{p}_{t-2|t-1} \\
&\quad + \left\{(1-\varphi)\rho + a_{21}\varphi - a_{22}(1-\varphi) - \boldsymbol{\varphi}'_x \boldsymbol{m}_x \rho + \boldsymbol{\varphi}'_x \boldsymbol{m}_p \frac{\alpha_2}{\varphi}(1-\varphi)\right\} \overline{E}_{t-1}\left[x_{t-1}\right] \\
&\quad + \left\{a_{23}\varphi - a_{24}(1-\varphi) - \boldsymbol{\varphi}'_x \boldsymbol{m}_p \theta + \boldsymbol{\varphi}'_x \boldsymbol{m}_p \frac{\alpha_4}{\varphi}(1-\varphi)\right\} \overline{E}_{t-1}\left[p_{t-2}\right] \\
&\quad + \boldsymbol{\varphi}'_x \boldsymbol{m}_x u_t
\end{aligned} \tag{C.13}$$

This will fit the proposed solution if

$$a_{21} = \rho \boldsymbol{\varphi}'_x \boldsymbol{m}_x \tag{C.14a}$$

$$a_{22} = a_{22} + \boldsymbol{\varphi}'_x \boldsymbol{m}_p \left(\frac{1-\varphi}{\varphi}\right) \alpha_2 \tag{C.14b}$$

$$a_{23} = \theta \boldsymbol{\varphi}'_x \boldsymbol{m}_p \tag{C.14c}$$

$$a_{24} = a_{24} + \boldsymbol{\varphi}'_x \boldsymbol{m}_p \left(\frac{1-\varphi}{\varphi}\right) \alpha_4 \tag{C.14d}$$

$$0 = (1-\varphi)\rho + a_{21}\varphi - a_{22}(1-\varphi) - \boldsymbol{\varphi}'_x \boldsymbol{m}_x \rho + \boldsymbol{\varphi}'_x \boldsymbol{m}_p \frac{\alpha_2}{\varphi}(1-\varphi) \tag{C.14e}$$

$$0 = a_{23}\varphi - a_{24}(1-\varphi) - \boldsymbol{\varphi}'_x \boldsymbol{m}_p \theta + \boldsymbol{\varphi}'_x \boldsymbol{m}_p \frac{\alpha_4}{\varphi}(1-\varphi) \tag{C.14f}$$

$$b_2 = \boldsymbol{\varphi}'_x \boldsymbol{m}_x \tag{C.14g}$$

Combining (C.14a), (C.14b) and (C.14e) then gives

$$a_{22} = \rho\left(1 - \boldsymbol{\varphi}'_x \boldsymbol{m}_x\right) \tag{C.15}$$

While combining (C.14c), (C.14d) and (C.14f) gives

$$a_{24} = -\theta \boldsymbol{\varphi}'_x \boldsymbol{m}_p \tag{C.16}$$

The law of motion for $\widetilde{p}_{t-1|t}$

Stepping (C.7) back one period, we can expand (C.10b) to read:

$$\widetilde{p}_{t-1|t} = \left(\boldsymbol{\varphi}_p' A - \boldsymbol{\varphi}_p' MNA\right) \begin{bmatrix} \overline{E}_{t-1}\left[x_{t-1}\right] \\ \frac{1}{\varphi}\left(\widetilde{x}_{t-1|t-1} - (1-\varphi)\,\overline{E}_{t-1}\left[x_{t-1}\right]\right) \\ \overline{E}_{t-1}\left[p_{t-2}\right] \\ \frac{1}{\varphi}\left(\widetilde{p}_{t-2|t-1} - (1-\varphi)\,\overline{E}_{t-1}\left[p_{t-2}\right]\right) \end{bmatrix} + \boldsymbol{\varphi}_p' MNAZ_{t-1} + \boldsymbol{\varphi}_p' MNBu_t \tag{C.17}$$

Expanding $\boldsymbol{\varphi}_x' A$ and NA and NB, and then gathering like terms, this gives:

$$\begin{aligned}\widetilde{p}_{t-1|t} = &\left\{\boldsymbol{\varphi}_p' \boldsymbol{m}_x \rho\right\} x_{t-1} \\ &+ \left\{\left(\frac{\alpha_2(1-\varphi)+a_{42}\varphi}{\varphi}\right) - \boldsymbol{\varphi}_p' \boldsymbol{m}_p \frac{\gamma_2}{\varphi} + \boldsymbol{\varphi}_p' \boldsymbol{m}_p \alpha_2\right\} \widetilde{x}_{t-1|t-1} \\ &+ \left\{\boldsymbol{\varphi}_p' \boldsymbol{m}_p \theta\right\} p_{t-2} \\ &+ \left\{\left(\frac{\alpha_4(1-\varphi)+a_{44}\varphi}{\varphi}\right) - \boldsymbol{\varphi}_p' \boldsymbol{m}_p \frac{\gamma_4}{\varphi} + \boldsymbol{\varphi}_p' \boldsymbol{m}_p \alpha_4\right\} \widetilde{p}_{t-2|t-1} \\ &+ \left\{a_{41}\varphi - \left(\frac{\alpha_2(1-\varphi)+a_{42}\varphi}{\varphi}\right)(1-\varphi) - \boldsymbol{\varphi}_p' \boldsymbol{m}_x \rho + \boldsymbol{\varphi}_p' \boldsymbol{m}_p \frac{\alpha_2}{\varphi}(1-\varphi)\right\} \overline{E}_{t-1}\left[x_{t-1}\right] \\ &+ \left\{\theta(1-\varphi) + a_{43}\varphi - \left(\frac{\alpha_4(1-\varphi)+a_{44}\varphi}{\varphi}\right)(1-\varphi) - \boldsymbol{\varphi}_p' \boldsymbol{m}_p \theta + \boldsymbol{\varphi}_p' \boldsymbol{m}_p \frac{\alpha_4}{\varphi}(1-\varphi)\right\} \overline{E}_{t-1}\end{aligned} \tag{C.18}$$

This will fit the proposed solution if

$$a_{41} = \rho\boldsymbol{\varphi}_p'\boldsymbol{m}_x \tag{C.19a}$$

$$a_{42} = \left(\frac{\gamma_2\left(1-\varphi\right)+a_{42}\varphi}{\varphi}\right) - \boldsymbol{\varphi}_p'\boldsymbol{m}_p\frac{\gamma_2}{\varphi} + \boldsymbol{\varphi}_p'\boldsymbol{m}_p\alpha_2 \tag{C.19b}$$

$$a_{43} = \theta\boldsymbol{\varphi}_p'\boldsymbol{m}_p \tag{C.19c}$$

$$a_{44} = \left(\frac{\gamma_4\left(1-\varphi\right)+a_{44}\varphi}{\varphi}\right) - \boldsymbol{\varphi}_p'\boldsymbol{m}_p\frac{\gamma_4}{\varphi} + \boldsymbol{\varphi}_p'\boldsymbol{m}_p\alpha_4 \tag{C.19d}$$

$$0 = a_{41}\varphi - \left(\frac{\alpha_2\left(1-\varphi\right)+a_{42}\phi}{\varphi}\right)\left(1-\varphi\right) - \boldsymbol{\varphi}_p'\boldsymbol{m}_x\rho + \boldsymbol{\varphi}_p'\boldsymbol{m}_p\frac{\alpha_2}{\varphi}\left(1-\varphi\right) \tag{C.19e}$$

$$0 = \theta\left(1-\varphi\right) + a_{43}\varphi - \left(\frac{\alpha_4\left(1-\varphi\right)+a_{44}\varphi}{\varphi}\right)\left(1-\varphi\right) - \boldsymbol{\varphi}_p'\boldsymbol{m}_p\theta + \boldsymbol{\varphi}_p'\boldsymbol{m}_p\frac{\alpha_4}{\varphi}\left(1-\varphi\right) \tag{C.19f}$$

$$b_4 = 0 \tag{C.19g}$$

Combining (C.19a), (C.19b) and (C.19e) then gives

$$a_{42} = \alpha_2 - \rho\boldsymbol{\varphi}_p'\boldsymbol{m}_x \tag{C.20}$$

While combining (C.19c), (C.19d) and (C.19f) gives

$$a_{44} = \alpha_4 + \theta\left(1 - \boldsymbol{\varphi}_p'\boldsymbol{m}_p\right) \tag{C.21}$$

The overall law of motion

For given values of φ, M and $\boldsymbol{\alpha}$, the law of motion is therefore given by:

$$Z_t = \underbrace{\begin{bmatrix} \rho & 0 & 0 & 0 \\ \rho\boldsymbol{\varphi}_x'\boldsymbol{m}_x & \rho\left(1-\boldsymbol{\varphi}_x'\boldsymbol{m}_x\right) & \theta\boldsymbol{\varphi}_x'\boldsymbol{m}_p & -\theta\boldsymbol{\varphi}_x'\boldsymbol{m}_p \\ 0 & \gamma & \theta & \lambda-\theta \\ \rho\boldsymbol{\varphi}_p'\boldsymbol{m}_x & \gamma-\rho\boldsymbol{\varphi}_p'\boldsymbol{m}_x & \theta\boldsymbol{\varphi}_p'\boldsymbol{m}_p & \lambda-\theta\boldsymbol{\varphi}_p'\boldsymbol{m}_p \end{bmatrix}}_{A} Z_{t-1} + \underbrace{\begin{bmatrix} 1 \\ \boldsymbol{\varphi}_x'\boldsymbol{m}_x \\ 0 \\ 0 \end{bmatrix}}_{B} u_t \tag{C.22}$$

C.3 Optimal Kalman gains

Given the state space representation of

$$Z_t = AZ_{t-1} + Bu_t \tag{C.23a}$$

$$\boldsymbol{s}_t(i) = NZ_t + O\boldsymbol{v}_t(i) \tag{C.23b}$$

the determination of the Kalman filter for firm i's estimation of Z_t is entirely standard:

$$M_t = V_{t|t-1}N'\left(NV_{t|t-1}N' + \sigma_v^2 OO'\right)^{-1} \tag{C.24a}$$

$$V_{t|t-1} = A\left(V_{t-1|t-2} - V_{t-1|t-2}N'\left(NV_{t-1|t-2}N' + \sigma_v^2 OO'\right)^{-1} NV_{t-1|t-2}\right)A' + \sigma_u^2 BB' \tag{C.24b}$$

where M_t is the Kalman gain and $V_{t|t-1} \equiv Var\left(Z_t - E_{t-1}(i)[Z_t]\right)$ is the variance of firms' prior expectation errors (common to all firms as their problems are symmetric).

C.4 Finding φ

The next step is to find φ (the weight used in constructing $\widetilde{x}_{t|t}$ and $\widetilde{p}_{t-1|t}$). We have that

$$p_t = \boldsymbol{\alpha}'Z_t = \theta p_{t-1} + (\lambda - \theta)\,\widetilde{p}_{t-1|t} + \gamma\,\widetilde{x}_{t|t} \tag{C.25}$$

Given A and B, firms' average expectation of the next-period price level is therefore given by:

$$\begin{aligned}\overline{E}_t[p_{t+1}] = \boldsymbol{\alpha}'\overline{E}_t[Z_{t+1}] = \boldsymbol{\alpha}'\overline{E}_t\begin{bmatrix} x_{t+1} \\ \widetilde{x}_{t+1|t+1} \\ p_t \\ \widetilde{p}_{t|t+1}\end{bmatrix} &= \boldsymbol{\alpha}'\overline{E}_t\begin{bmatrix} \boldsymbol{a}_1'Z_t \\ \boldsymbol{a}_2'Z_t \\ p_t \\ \boldsymbol{a}_4'Z_t\end{bmatrix} \\ &= \boldsymbol{\alpha}'\boldsymbol{e}_3\overline{E}_t[p_t] + \boldsymbol{\alpha}'J_3A\overline{E}_t[Z_t]\end{aligned} \tag{C.26}$$

where $\boldsymbol{e}_3$ is a column vector of zeros with a one in the third position, and J_3 is the identity matrix modified to put a zero in the third position of the lead diagonal. For two periods ahead, we have:

$$\overline{E}_t\left[p_{t+2}\right] = \boldsymbol{\alpha}'\overline{E}_t\left[Z_{t+2}\right] = \boldsymbol{\alpha}'A\overline{E}_t\left[Z_{t+1}\right] = \boldsymbol{\alpha}'A\overline{E}_t\begin{bmatrix}\boldsymbol{a}_1'Z_t\\ \boldsymbol{a}_2'Z_t\\ p_t\\ \boldsymbol{a}_4'Z_t\end{bmatrix}$$
$$= \boldsymbol{\alpha}'A\boldsymbol{e}_3\overline{E}_t\left[p_t\right] + \boldsymbol{\alpha}'AJ_3A\overline{E}_t\left[Z_t\right] \tag{C.27}$$

Continuing this process, it should be clear that

$$\overline{E}_t\left[p_{t+q}\right] = \boldsymbol{\alpha}'A^{q-1}\boldsymbol{e}_3\overline{E}_t\left[p_t\right] + \boldsymbol{\alpha}'A^{q-1}J_3A\overline{E}_t\left[Z_t\right] \tag{C.28}$$

Substituting (C.28) into the competitive equilibrium condition (27) then gives

$$\begin{aligned} p_t = \; & \theta\, p_{t-1} \\ & + \begin{bmatrix} b_p & 0 & \zeta_{-1} & 0 \end{bmatrix} \overline{E}_t\left[Z_t\right] \\ & + \zeta_0\, \overline{E}_t\left[p_t\right] \\ & + \beta\theta\, \left(\boldsymbol{\alpha}'\boldsymbol{e}_3\overline{E}_t\left[p_t\right] + \boldsymbol{\alpha}'J_3A\overline{E}_t\left[Z_t\right]\right) \\ & + \zeta_{1+}\, (1-\delta)\sum_{q=1}^{\infty}\delta^{q-1}\left(\boldsymbol{\alpha}'A^{q-1}\boldsymbol{e}_3\overline{E}_t\left[p_t\right] + \boldsymbol{\alpha}'A^{q-1}J_3A\overline{E}_t\left[Z_t\right]\right) \end{aligned} \tag{C.29}$$

Or, gathering like terms,

$$p_t = \theta p_{t-1} + \boldsymbol{d}'\overline{E}_t\left[Z_t\right] + \varphi\overline{E}_t\left[p_t\right] \tag{C.30a}$$

where

$$\boldsymbol{d}' = \begin{bmatrix} b_p & 0 & \zeta_{-1} & 0 \end{bmatrix} + \beta\theta\boldsymbol{\alpha}'J_3A + \zeta_{1+}\boldsymbol{\alpha}'\left((1-\delta)\sum_{q=0}^{\infty}(\delta A)^q\right)J_3A \tag{C.30b}$$

$$\varphi = \zeta_0 + \beta\theta\boldsymbol{\alpha}'\boldsymbol{e}_3 + \zeta_{1+}\boldsymbol{\alpha}'\left((1-\delta)\sum_{q=0}^{\infty}(\delta A)^q\right)\boldsymbol{e}_3 \tag{C.30c}$$

The coefficient φ is the *equilibrium degree of strategic complementarity* in firms' price-setting decisions (that is, after taking account of demand and the entire expected future path of prices). Expanding the compound parameters $\boldsymbol{\alpha}'\boldsymbol{e}_3$, ζ_0 and ζ_{1+}, equation (C.30c) may then be rewritten as:

$$\varphi = (1-\theta)(1-\beta\theta)\left(1-\sigma\omega\delta\phi_\pi - \sigma\omega\delta(1-\phi_\pi\delta)\left(1-\boldsymbol{\alpha}'\left((1-\delta)\sum_{q=0}^{\infty}\left(\delta A(\varphi)\right)^q\right)\boldsymbol{e}_3\right)\right) \tag{C.31}$$

where I have emphasised that the transition matrix A is itself a function of φ.

D Proof of proposition 4

When expressed in terms of the infinite set of higher-order beliefs, the solution must be of the form:

$$p_t = \boldsymbol{\psi}' X_t \quad \text{where} \quad X_t \equiv \underset{(\infty\times 1)}{\begin{bmatrix} x_t \\ p_{t-1} \\ \overline{E}_t[X_t] \end{bmatrix}} \tag{D.1a}$$

$$X_t = F X_{t-1} + G u_t \tag{D.1b}$$

The full set of potential solutions, including those with some backward-looking component, can therefore be written in the following form:

$$p_t = \boldsymbol{\mu}' X_t + \xi q_{t-1} \quad \text{where} \quad q_t = \boldsymbol{a}(L)' X_t + b(L) p_t \tag{D.2}$$

where ξ and q_t are scalars.

The infinite-state representation

Before demonstrating that it must be the case that $\xi = 0$, I first describe, for reference, how the infinite-state representation may be solved. First, note that firms' signals are expressible as:

$$\boldsymbol{s}_t(j) = \underbrace{\begin{bmatrix} 1 & 0 & \mathbf{0}_{1\times\infty} \\ 0 & 1 & \mathbf{0}_{1\times\infty} \end{bmatrix}}_{\Lambda} X_t + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{O} \boldsymbol{v}_t(j) \tag{D.3}$$

Conditional on $X_t = FX_{t-1} + Gu_t$ as the law of motion, firms' Kalman filters will be standard:

$$K_t = V_{t|t-1}\Lambda' \left(\Lambda V_{t|t-1}\Lambda' + \sigma_v^2 OO'\right)^{-1} \tag{D.4a}$$

$$V_{t|t-1} = F\left(V_{t-1|t-2} - V_{t-1|t-2}\Lambda' \left(\Lambda V_{t-1|t-2}\Lambda' + \sigma_v^2 OO'\right)^{-1} \Lambda V_{t-1|t-2}\right) F' + \sigma_u^2 GG' \tag{D.4b}$$

where $K_t \to K$ is the $(\infty \times 2)$ Kalman gain, common to all firms as their problems are symmetric. Firms' average expectation of X_t will therefore update as:

$$\overline{E}_t [X_t] = F\overline{E}_t [X_{t-1}] + K\Lambda \left(FX_{t-1} + Gu_t - F\overline{E}_{t-1} [X_{t-1}]\right) \tag{D.5}$$

Next, define T as the selection matrix such that $TX_t = \overline{E}_t [X_t]$ (shifting the vector up two places). Making use of T in (D.5) makes clear that the law of motion is confirmed, for a given $\boldsymbol{\psi}$, with F and G given implicitly by:

$$F = \begin{bmatrix} \begin{bmatrix}\rho & \mathbf{0}_{1\times\infty}\end{bmatrix} \\ \boldsymbol{\psi}' \\ FT + K\Lambda F(I - T) \end{bmatrix} \qquad G = \begin{bmatrix} 1 \\ 0 \\ K\Lambda G \end{bmatrix} \tag{D.6}$$

Finally, $\boldsymbol{\psi}$ can be obtained by the method of undetermined coefficients. Plugging the solution into the equilibrium condition (27) gives:

$$\begin{aligned}\boldsymbol{\psi}' X_t = b_p S_x T X_t + \theta S_p X_t + \zeta_{-1} S_p T X_t & \\ + \zeta_0 \ \boldsymbol{\psi}' T X_t & \\ + \beta\theta \ \boldsymbol{\psi}' F T X_t & \\ + \zeta_{1+} \boldsymbol{\psi}' \left((1-\delta)\sum_{s=0}^{\infty} (\delta F)^s\right) FTX_t & \end{aligned} \tag{D.7}$$

where S_x and S_p are the selection matrices such that $S_x X_t = x_t$ and $S_p X_t = p_{t-1}$, so that

$$\boldsymbol{\psi}' = \begin{bmatrix} 0 & \theta & b_p & \zeta_{-1} & \mathbf{0}_{1\times\infty}\end{bmatrix} (I - HT)^{-1} \tag{D.8a}$$

$$H = \zeta_0 I + \beta\theta F + \zeta_{1+} (1-\delta)(I - \delta F)^{-1} F \tag{D.8b}$$

provided that the inverses exists. The solution is then the fixed point of (D.4), (D.6) and (D.8), which is found, in practice, by truncating the full state to only include the first k^* higher orders.[24]

Ruling out backward-looking solutions

Step (35) forward and note that:

$$\overline{E}_t\left[p_{t+s}\right]=\boldsymbol{\mu}' F^s\,\overline{E}_t\left[X_t\right]+\xi\,\overline{E}_t\left[q_{t+s-1}\right]\quad\forall s\geq 0 \tag{D.9}$$

Substituting (D.9) into the equilibrium condition (27) and making use of T then gives:

$$\begin{aligned}p_t=&\left\{\begin{bmatrix}0 & \theta & b_p & \zeta_{-1} & \mathbf{0}_{1\times\infty}\end{bmatrix}+\boldsymbol{\mu}'\left(\zeta_0 I+\beta\theta F+\zeta_{1^+}F\left(1-\delta\right)\sum_{s=0}^{\infty}\left(\delta F\right)^s\right)T\right\}X_t\\&+\xi\left(\zeta_0\overline{E}_t\left[q_{t-1}\right]+\beta\theta\overline{E}_t\left[q_t\right]+\zeta_{1^+}\left(1-\delta\right)\sum_{s=0}^{\infty}\delta^s\overline{E}_t\left[q_{t+s}\right]\right)\end{aligned} \tag{D.10}$$

A candidate of the form of (35) can therefore only be a solution if

$$\boldsymbol{\mu}'=\begin{bmatrix}0 & \theta & b_p & \zeta_{-1} & \mathbf{0}_{1\times\infty}\end{bmatrix}+\boldsymbol{\mu}'\left(\zeta_0 I+\beta\theta F+\zeta_{1^+}F\left(1-\delta\right)\sum_{s=0}^{\infty}\left(\delta F\right)^s\right)T \tag{D.11a}$$

$$q_{t-1}=\overline{E}_t\left[Q_t\right] \tag{D.11b}$$

where $Q_t\equiv\zeta_0 q_{t-1}+\beta\theta q_t+\zeta_{1^+}\left(1-\delta\right)\sum_{s=0}^{\infty}\delta^s q_{t+s}$. If (D.11b) holds, then solutions of the form (35) can be true for any value of ξ on the real line, but if it does not hold, then it follows that $\xi=0$ is the only solution. To see that (D.11b) cannot hold, consider an individual firm's filter regarding Q_t:

$$E_t\left(j\right)\left[Q_t\right]=E_{t-1}\left(j\right)\left[Q_t\right]+K_t\left\{\boldsymbol{s}_t\left(j\right)-E_{t-1}\left(j\right)\left[\boldsymbol{s}_t\left(j\right)\right]\right\} \tag{D.12}$$

for some projection matrix K_t. Taking the average of this and splitting out the firm's two signals gives

$$\begin{aligned}\overline{E}_t\left[Q_t\right]=&\,\overline{E}_{t-1}\left[Q_t\right]+\rho\,K_{x,t}\left\{x_{t-1}-\overline{E}_{t-1}\left[x_{t-1}\right]\right\}+K_{x,t}\,u_t\\&+\;K_{p,t}\left\{p_{t-1}-\overline{E}_{t-1}\left[p_{t-1}\right]\right\}\end{aligned} \tag{D.13}$$

[24] See Nimark (2011).

Since u_t is unforecastable, q_{t-1} cannot be a function of it. A necessary condition for (D.11b) to hold is therefore that $K_{x,t} = 0$. But since shocks are persistent ($\rho > 0$), this can only hold if (i) firms are not rational, which we rule out by assumption; (ii) firms have *no* information about the state ($\sigma_v^2 = \infty$); or (iii) firms have *full* information about the state ($\sigma_v^2 = 0$).

Rejecting extrinsic bubbles. Ruling out extrinsic bubbles relies on three points:

1. The equilibrium condition (27) implies that w_t itself cannot appear in the solution, but firms' higher-order average expectations of it ($\overline{E}_t^{(k)}[w_t]$) can.
2. Firms only learn about w_t indirectly by observing the (lagged) price level (17a).
3. Firms' information sets are heterogeneous rather than common.

The first two together imply that expectations of current and future values of w_t can only be a function of noise shocks in firms' signals regarding the price level. The third provides that the law of large numbers may be applied so that the average noise shock in any period is zero.