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Official demand for US debt: implications for US real rates

Iryna Kaminska and Gabriele Zinna

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Online Appendix

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Official Demand for U.S. Debt: Implications for U.S. Real Rates

by
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The online appendix is organized as follows:

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I. OFFICIAL DEMAND FOR U.S. DEBT

Foreign Officials. The foreign official sector has been playing an important role in the Treasury market since the early 2000s. Foreign central banks tend to buy Treasuries because they are highly liquid assets that provide a reliable store of value and also serve as an insurance against future crises. Exchange rate policy management is another key factor, with a large chunk of emerging market economies' (EMEs) reserves invested in U.S. Treasury securities. Furthermore, certain features of the domestic financial systems of EMEs, especially in Asia, are likely to have played an important role (for a detailed discussion, see ECB, 2006).

As the Treasury International Capital System (TIC) data show, while the U.S. debt market has been growing constantly but slowly since the early 2000s till the crisis, over the period foreign investors' purchases grew at a faster pace (Figure IA.1). In September 2009, non-Americans held more than 50 percent of the all U.S. government notes and bonds outstanding, with foreign official investors holding roughly 37 percent of the total Treasury supply, making them the largest holders of Treasury debt. (Private foreign holdings are usually overstated because TIC data do not capture foreign central banks acquisitions, taking place through a third-country intermediary.)

Of particular interest is that the rapid increase in foreign official holdings of U.S. Treasury bonds coincided with the decline in U.S. long-term interest rates in 2004-2005. The prevailing standard macro-financial literature of the time had difficulties in explaining the decline in rates by relying solely on macroeconomic and financial fundamentals.[37] For this reason, the phenomenon was labeled as a 'conundrum' by Alan Greenspan in 2005; however, he singled out heavy purchases of longer-term Treasury securities by foreign central banks as a possible factor behind the fall in longer-term yields, and quantified its effect in less than 50 basis points.

Federal Reserve. As U.S. policy interest rates reached the ZLB later in 2008, the Fed embarked on the QE program with the aim of reducing long-term interest rates, in order to stimulate economic activity and thus facilitate the recovery from the financial crisis. In particular, to support the QE objectives, the Fed launched several unconventional asset purchase programs, largely focusing on longer-term securities, including government bonds and MBSs.

The first large scale asset purchase program was launched in March 2009 (LSAP1), when the FOMC committed to purchase 300 $billions of longer-term Treasury securities and 850 $billions of agency securities in addition to the 600 $billions of MBSs and agency debt announced earlier on in November 2008. As the recovery lost momentum, in November 2010 the FOMC announced 600 $billions of additional purchases of longer-term Treasury securities to be completed by mid-2011 (LSAP2). To further improve financial market conditions and provide support for the economic recovery, in September 2011, the FOMC started the Maturity Extension Program (MEP), which aimed to increase the average maturity of the Fed's portfolio of Treasury securities without a further expansion of the Fed's balance sheet. Under the MEP, the Fed sold a total of 667 $billions of shorter-term Treasury securities and used the proceeds to purchase longer-term Treasury securities.

The last round of QE was announced in September 2012, when the Fed launched LSAP3. Initially, it consisted of buying only a fixed amount of agency MBS per month but the purchases were soon extended to long-term Treasury bonds. Under LSAP3, the Fed bought 823 $billions of MBSs and 790 $billions of Treasury securities. In December 2013 the Fed announced they would have been tapering back, and the program ended in October 2014. As a consequence of the UMPs undertaken, the Fed's asset holdings, as well as the average maturity of its assets, expanded substantially (Figure IA.2).

On August 10, 2010, the Fed announced its new reinvestment policy and stated that principal payments from agency debt and agency MBS will be reinvested in longer-term Treasury securities. Subsequently, in September 2011 the Federal Reserve switched the reinvestments of principal payments from agency securities into MBS rather than into longer-term Treasury securities. The Fed has continued to roll over maturing Treasury debt up until the end of our sample (December 2016).

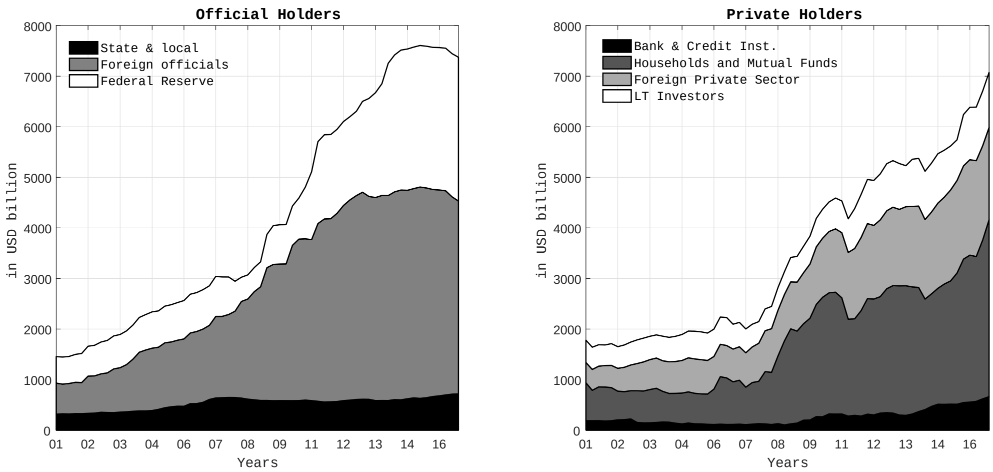

Figure IA.1: Estimated Ownership of U.S. Treasury Securities

Note: The figure shows the holdings of Treasuries, in billions USD, by official and and private holders. Official holders include state and local authorities (including public pension funds), foreign officials, and the Federal Reserve. Private holders are grouped as follows, bank and credit institutions, households and mutual funds, foreign private sector, and long-term investors. Source: U.S. Department of Treasury, Office of Debt Management, and authors' calculations.

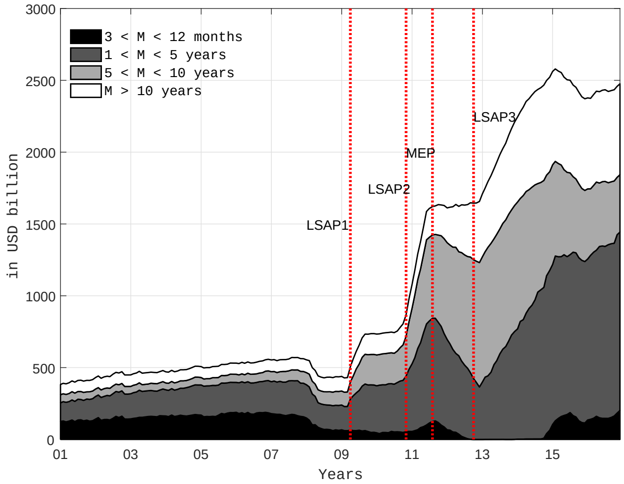

Figure IA.2: Fed's Treasury Holdings by Maturity Buckets

Note: The figure shows the maturity structure of the Fed's Treasury portfolio. The Fed's holdings of Treasuries are displayed in USD billion and in maturity buckets, where M is the maturity. Vertical dotted lines mark the start of the LSAP1, LSAP2, LSAP3 and MEP programs. Source: St. Louis FRED, U.S. Treasury Department, and Bank for International Settlements.

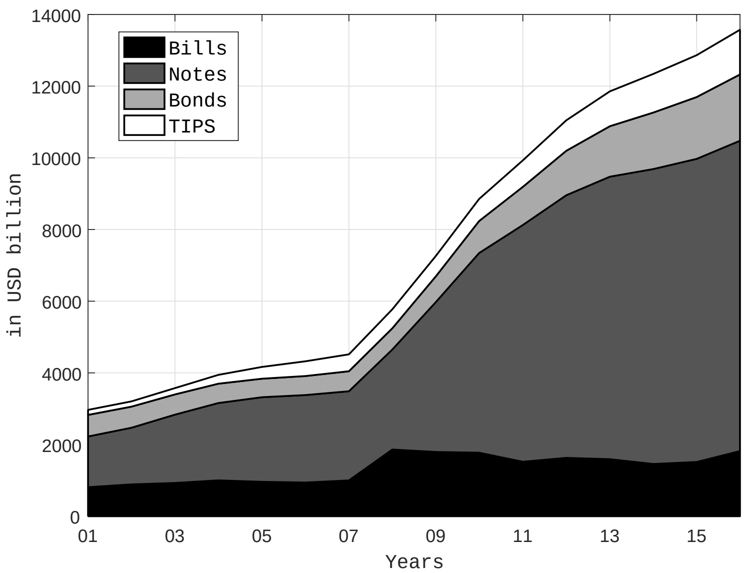

Figure IA.3: U.S. Treasury Debt Outstanding

Note: The figure shows the U.S. Treasury debt outstanding by type of Treasury securities, nominal vs index linked. Nominal bonds are then grouped by maturity into bonds, notes and bills. TIPS denote the Treasuries bonds indexed to inflation. Source: U.S. Treasury Department.

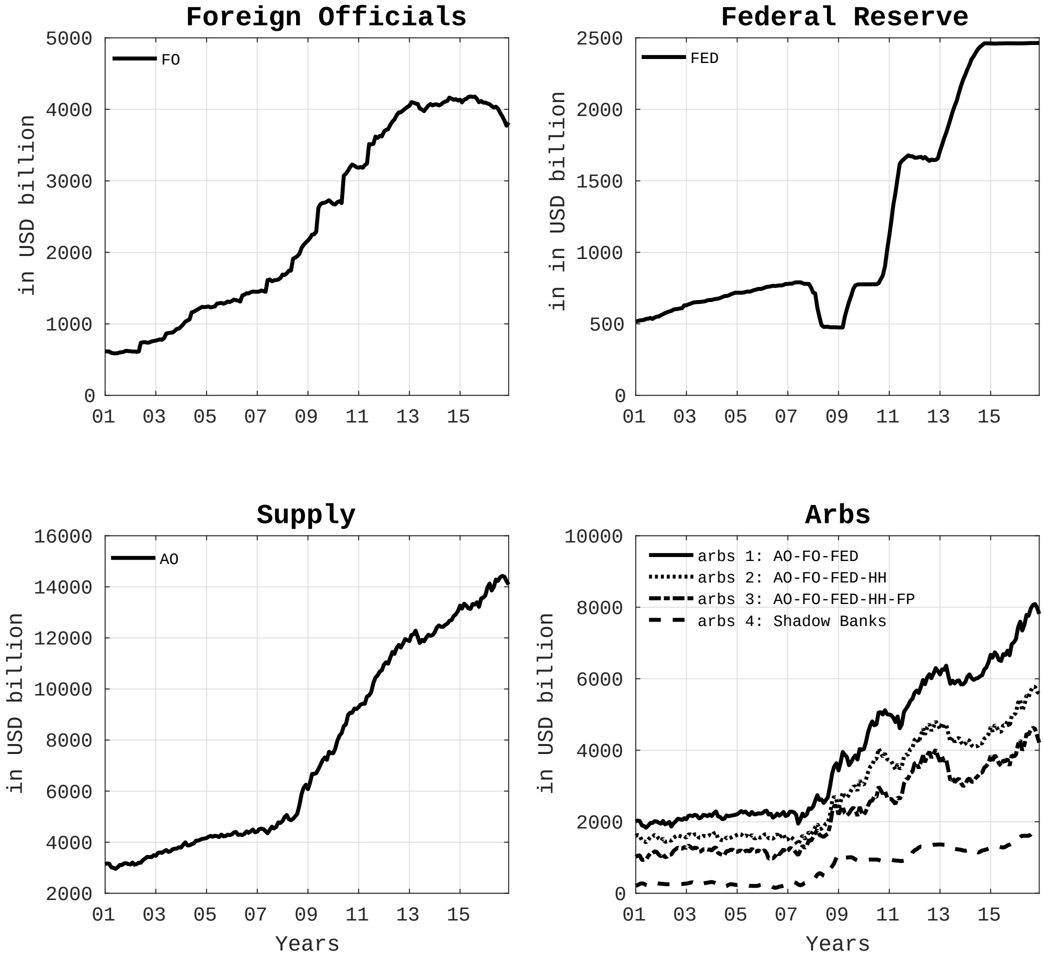

Figure IA.4: Holdings of US Treasury Securities by Type of Investor

Note: This figure presents the raw observable measures of official demand and supply, i.e. in USD billions, as well as the arbitrageurs' holdings of US Treasury securities for different choices of arbitrageurs; Foreign Official holdings of Treasury securities (FO, top left panel); Federal Reserve holdings of medium- to long-term (with maturities greater than, or equal to, five years) Treasury securities (FED, top right panel); the amount of marketable Treasury securities outstanding (AO, bottom left panel); and, four measures of arbs' holdings of Treasury securities (Arbs, bottom right panel). The arbs are defined as follows: arbs 1 is AO-FO-FED; arbs 2 is AO-FO-FED-HH; arbs 3 is AO-FO-FED-HH-FP; and arbs 4: Shadow Banks. Source AO, FED, FO are from Treasury International Capital System and St. Louis Fred, HH (Households), FP (Foreign private) and Shadow Banks from the US Flow of Funds.

II. TWO-FACTOR MODEL

II.1 Derivation

We conjecture equilibrium spot rates to be affine in the risk factors, *i.e.*, the short-rate (r_t) and the excess-supply (β_t) factors, so that the equilibrium bond prices take the following exponential form

$$P_{t,\tau} = e^{-\left[A_r(\tau)r_t + A_\beta(\tau)\beta_t + C(\tau)\right]} \tag{II.1}$$

for three functions $A_r(\tau), A_\beta(\tau), C(\tau)$ that depend on maturity τ. Applying Ito's Lemma to (II.1) and using the dynamics (1) of r_t and (5) of β_t, we find that the instantaneous return on the bond with maturity τ is

$$\frac{dP_{t,\tau}}{P_{t,\tau}} = \mu_{t,\tau}dt - A_r(\tau)\sigma_r dB_{r,t} - A_\beta(\tau)\sigma_\beta dB_{\beta,t}, \tag{II.2}$$

where

$$\begin{aligned} \mu_{t,\tau} \equiv\ & A_r'(\tau)r_t + A_\beta'(\tau)\beta_t + C'(\tau) - A_r(\tau)\kappa_r(\overline{r} - r_t) - A_\beta(\tau)\kappa_\beta(\overline{\beta} - \beta_t) \\ & + \frac{1}{2}A_r(\tau)^2\sigma_r^2 + \frac{1}{2}A_\beta(\tau)^2\sigma_\beta^2 + \rho A_r(\tau)A_\beta(\tau)\sigma_r\sigma_\beta \end{aligned} \tag{II.3}$$

is the instantaneous expected return. Substituting (II.2) into the arbitrageurs' budget constraint (7), we can solve the arbitrageurs' optimization problem.

Next we show how to derive bond risk premiums (or excess returns) of equation (9). Using (II.2), we can write

$$\begin{aligned} dW_t &= \left[W_t r_t - \int_0^T x_{t,\tau}(\mu_{t,\tau} - r_t)d\tau\right]dt \\ & - \left[\int_0^T x_{t,\tau}A_r(\tau)d\tau\right]\sigma_r dB_{r,t} - \left[\int_0^T x_{t,\tau}A_\beta(\tau)d\tau\right]\sigma_\beta dB_{\beta,t}, \end{aligned}$$

and (6) as

$$\begin{aligned} \max_{\{x_{t,\tau}\}_{\tau\in(0,T]}} & - \int_0^T x_{t,\tau}(\mu_{t,\tau} - r_t)d\tau - \frac{a\sigma_r^2}{2}\left[\int_0^T x_{t,\tau}A_r(\tau)d\tau\right]^2 \\ - \frac{a\sigma_\beta^2}{2}\left[\int_0^T x_{t,\tau}A_\beta(\tau)d\tau\right]^2 & - a\sigma_\beta\sigma_r\rho\left[\int_0^T x_{t,\tau}A_r(\tau)d\tau\right]\left[\int_0^T x_{t,\tau}A_\beta(\tau)d\tau\right]. \end{aligned} \tag{II.4}$$

Point-wise maximization of (A4) yields (9).

Then, we derive the factor loadings $A_r(\tau)$ and $A_\beta(\tau)$. By imposing market clearing, so that $x_{t,\tau} = -y_{t,\tau}$, and using equation (2), (II.1) and the definition of $R_{t,\tau}$, we find

$$x_{t,\tau} = \alpha(\tau)\left\{\beta_t\tau - \left[A_r(\tau)r_t + A_\beta(\tau)\beta_t + C(\tau)\right]\right\}. \tag{II.5}$$

Substituting $(\mu_{t,\tau}, \lambda_{r,t}, \lambda_{\beta,t}, x_{t,\tau})$ from (II.3), (10), (11) and (II.5) into (9), we find an affine equation in (r_t, β_t). Setting linear terms in (r_t, β_t) to zero yields

$$A_r'(\tau) + \kappa_r A_r(\tau) - 1 = A_r(\tau)M_{1,1} + A_\beta(\tau)M_{1,2}, \tag{II.6a}$$

$$A_\beta'(\tau) + \kappa_\beta A_r(\tau) = A_r(\tau)M_{2,1} + A_\beta(\tau)M_{2,2}, \tag{II.6b}$$

where the matrix M is given by

$$\begin{aligned}
M_{1,1} &\equiv -a\sigma_r \int_0^T \alpha(\tau)A_r(\tau)\left[\sigma_r A_r(\tau) + \rho\sigma_\beta A_\beta(\tau)\right] d\tau,\\
M_{1,2} &\equiv -a\sigma_\beta \int_0^T \alpha(\tau)A_r(\tau)\left[\rho\sigma_r A_r(\tau) + \sigma_\beta A_\beta(\tau)\right] d\tau,\\
M_{2,1} &\equiv a\sigma_r \int_0^T \alpha(\tau)\left[\tau\theta(\tau) - A_\beta(\tau)\right]\left[\sigma_r A_r(\tau) + \rho\sigma_\beta A_\beta(\tau)\right] d\tau,\\
M_{2,2} &\equiv a\sigma_\beta \int_0^T \alpha(\tau)\left[\tau\theta(\tau) - A_\beta(\tau)\right]\left[\rho\sigma_r A_r(\tau) + \sigma_\beta A_\beta(\tau)\right] d\tau.
\end{aligned}$$

The solution to the system of (II.6a) and (II.6b) is given by equations (II.7) and (II.8):

$$A_r(\tau) = \frac{1 - e^{-\nu_1\tau}}{\nu_1} + \gamma_r\left(\frac{1 - e^{-\nu_2\tau}}{\nu_2} - \frac{1 - e^{-\nu_1\tau}}{\nu_1}\right), \tag{II.7}$$

$$A_\beta(\tau) = \gamma_\beta\left(\frac{1 - e^{-\nu_2\tau}}{\nu_2} - \frac{1 - e^{-\nu_1\tau}}{\nu_1}\right). \tag{II.8}$$

To determine $(\nu_1, \nu_2, \gamma_r, \gamma_\beta)$, we substitute (II.7) and (II.8) into (II.6a) and (II.6b), and identify terms in $\frac{1-e^{-\nu_1\tau}}{\nu_1}$ and $\frac{1-e^{-\nu_2\tau}}{\nu_2}$. This yields

$$(1 - \gamma_r)(\nu_1 - \kappa_r + M_{1,1}) - \gamma_\beta M_{1,2} = 0, \tag{II.9}$$

$$\gamma_r(\nu_2 - \kappa_r + M_{1,1}) + \gamma_\beta M_{1,2} = 0, \tag{II.10}$$

in the case of (II.6a) and

$$\gamma_\beta(\nu_1 - \kappa_\beta + M_{2,2}) - (1 - \gamma_r)M_{2,1} = 0, \tag{II.11}$$

$$-\gamma_\beta(\nu_2 - \kappa_\beta + M_{2,2}) - \gamma_r M_{2,1} = 0, \tag{II.12}$$

in the case of (II.6b). Combining (II.9) and (II.10), we find the equivalent equations

$$\nu_1 + \gamma_r(\nu_2 - \nu_1) - \kappa_r + M_{1,1} = 0, \tag{II.13}$$

$$\gamma_r(1-\gamma_r)(\nu_1 - \nu_2) - \gamma_\beta M_{1,2} = 0, \tag{II.14}$$

and combining (II.11) and (II.12), we find the equivalent equations

$$\gamma_\beta(\nu_1 - \nu_2) - M_{2,1} = 0, \tag{II.15}$$

$$\kappa_\beta - \nu_2 - \gamma_r(\nu_1 - \nu_2) - M_{2,2} = 0. \tag{II.16}$$

Equations (II.13)-(II.16) are a system of four scalar non-linear equations in the unknowns $(\nu_1, \nu_2, \gamma_r, \gamma_\beta)$.

To solve the system of (II.13)-(II.16), we must assume functional forms for $\alpha(\tau), \theta(\tau)$. Many parametrizations are possible. A convenient one that we adopt from now on is $\alpha(\tau) \equiv \alpha e^{-\delta\tau}$ and $\theta(\tau) = 1$ (*i.e.*, the excess-supply factor affects all maturities equally in the absence of arbitrageurs). We also set $\alpha = 1$, which is without loss of generality because α matters only through the product αa.

Next, we show how to determine the function $C(\tau)$. Setting $x_{t,\tau} = -y_{t,\tau}$ in (10) and (11), and using $R_{t,\tau} \equiv -\frac{\log(P_{t,\tau})}{\tau}$, equations (2) and (II.1), we find

$$\begin{aligned}
\lambda_{r,t} \equiv a\sigma_r^2 \int_0^T \alpha(\tau)\left[\beta_t\tau - A_r(\tau)r_t + A_\beta(\tau)\beta_t + C(\tau)\right]A_r(\tau)d\tau \\
+ a\sigma_r\rho\sigma_\beta \int_0^T \alpha(\tau)\left[\beta_t\tau - A_r(\tau)r_t + A_\beta(\tau)\beta_t + C(\tau)\right]A_\beta(\tau)d\tau,
\end{aligned} \tag{II.17}$$

$$\begin{aligned}
\lambda_{\beta,t} \equiv a\sigma_\beta\rho\sigma_r \int_0^T \alpha(\tau)\left[\beta_t\tau - A_r(\tau)r_t + A_\beta(\tau)\beta_t + C(\tau)\right]A_r(\tau)d\tau \\
+ a\sigma_\beta^2 \int_0^T \alpha(\tau)\left[\beta_t\tau - A_r(\tau)r_t + A_\beta(\tau)\beta_t + C(\tau)\right]A_\beta(\tau)d\tau.
\end{aligned} \tag{II.18}$$

Substituting $\mu_{t,\tau}$ from (II.3), $\lambda_{r,t}$ from (II.17), $\lambda_{\beta,t}$ from (II.18), we find

$$\begin{aligned}
&C'(\tau) - \kappa_r\bar{r}A_r(\tau) + \frac{1}{2}\sigma_r^2A_r(\tau)^2 + \frac{1}{2}\sigma_\beta^2A_\beta(\tau)^2 + \rho\sigma_r\sigma_\beta A_r(\tau)A_\beta(\tau) \\
&= a\sigma_rA_r(\tau)\int_0^T \alpha(\tau)\left[\overline{\beta}\tau - C(\tau)\right]\left[\sigma_rA_r(\tau) + \rho\sigma_\beta A_\beta(\tau)\right]d\tau \\
&+ a\sigma_\beta A_\beta(\tau)\int_0^T \alpha(\tau)\left[\overline{\beta}\tau - C(\tau)\right]\left[\rho\sigma_rA_r(\tau) + \sigma_\beta A_\beta(\tau)\right]d\tau.
\end{aligned} \tag{II.19}$$

The solution to (II.19) is given by

$$C(\tau) = z_r \int_0^\tau A_r(u)du + z_\beta \int_0^\tau A_\beta(u)du \\ - \frac{\sigma_r^2}{2}\int_0^\tau A_r(u)^2du - \frac{\sigma_\beta^2}{2}\int_0^\tau A_\beta(u)^2du - \rho\sigma_r\sigma_\beta \int_0^\tau A_r(u)A_\beta(u)du, \tag{II.20}$$

where

$$z_r \equiv \kappa_r\overline{r} - a\sigma_r \int_0^T \alpha(\tau)C(\tau)\left[\sigma_r A_r(\tau) + \rho\sigma_\beta A_\beta(\tau)\right]d\tau, \tag{II.21}$$

$$z_\beta \equiv \kappa_\beta\overline{\beta} - a\sigma_\beta \int_0^T \alpha(\tau)C(\tau)\left[\rho\sigma_r A_r(\tau) + \sigma_\beta A_\beta(\tau)\right]d\tau. \tag{II.22}$$

Substituting $C(\tau)$ into (II.21) and (II.22), we can derive (z_r, z_β) as the solution to a linear system of equations.

II.2 MCMC Algorithm

In this section, we first present the extended state-space framework that accounts for the additional state variable, $l_{t+\Delta}$, which differently from the specification used in the paper is allowed to be measured with error. We then turn to describing the details of the MCMC algorithm.

State-space Representation. The complete state-space representation in discrete time is given by:

State Equations:

$$X_{t+\Delta} = G(P_1) + F(P_1)X_t + u_{t+\Delta} \qquad u_t \sim N(0, Q) \tag{II.23}$$

$$l_{t+\Delta} = \kappa_0 + \kappa_1 l_t + v_{t+\Delta} \qquad v_t \sim N(0, \sigma_l^2) \tag{II.24}$$

Measurement Equations:

$$Y_{t+\Delta} = f(P, X_{t+\Delta}) + \nu l_{t+\Delta} + \varepsilon_{t+\Delta} \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon^2 I) \tag{II.25}$$

$$r^o_{t+\Delta} = r_{t+\Delta} + \epsilon_{1,t+\Delta} \qquad \epsilon_{1,t} \sim N(0, \sigma^2_{1,\epsilon}) \tag{II.26}$$

$$\beta^o_{t+\Delta} = \beta_{t+\Delta} + \epsilon_{2,t+\Delta} \qquad \epsilon_{2,t} \sim N(0, \sigma^2_{2,\epsilon}) \tag{II.27}$$

$$l^o_{t+\Delta} = l_{t+\Delta} + \epsilon_{3,t+\Delta} \qquad \epsilon_{3,t} \sim N(0, \sigma^2_{3,\epsilon}) \tag{II.28}$$

where $r^o_{t+\Delta}$ is the proxy for the one-month real rate, $\beta^o_{t+\Delta} = \theta b'_{t+\Delta}$ is the observed excess-supply factor, where $\theta = [\theta_0, \theta_1, \theta_2, \theta_3]$ and $b_{t+\Delta} = [1, b^{FO}_{t+\Delta}, b^{FED}_{t+\Delta}, b^{SUP}_{t+\Delta}]$, and $l^o_{t+\Delta}$ is the proxy for liquidity. To simplify the notation, we group the parameters of the VV model as $P_1 = (\rho, \sigma_r, \sigma_\beta, \kappa_r, \kappa_\beta, \overline{r}, \overline{\beta})$ and $P_2 = (a, \alpha, \delta)$, and $P = (P_1, P_2)$. The system matrices take the form of

$$G(P_1) = \begin{bmatrix} \kappa_r \overline{r} \Delta \\ \kappa_\beta \overline{\beta} \Delta \end{bmatrix} \text{ and } F(P_1) = \begin{bmatrix} 1 - \kappa_r \Delta & 0 \\ 0 & 1 - \kappa_\beta \Delta \end{bmatrix};$$

the factors' variance-covariance matrix is

$$Q = \Delta \begin{bmatrix} \sigma_r^2 & \rho \sigma_r \sigma_\beta \\ \rho \sigma_r \sigma_\beta & \sigma_\beta^2 \end{bmatrix};$$

and σ_ε^2 is the common variance of the independent and normally distributed measurement errors, $\varepsilon_{t+\Delta}$. The time step Δ at a monthly frequency is 1/12 .

Likelihood Functions. Let us define the liquidity-augmented state vector $\widetilde{X}^T = [r^T, \beta^T, l^T]$, the parameters $\widetilde{P}_1 = [P_1, \kappa]$ and $\widetilde{Q}$ is a 3×3 matrix $[Q, \mathbf{0}; \mathbf{0}', \sigma_l^2]$, whereby $\mathbf{0}$ is a 1×2 vector of zeros. The density of the factors is

$$\pi(\widetilde{X}^T | \widetilde{P}_1) \propto \prod |\widetilde{Q}|^{-1} \exp(-\frac{1}{2} \widetilde{u}_t' \widetilde{Q}^{-1} \widetilde{u}_t)$$

where $\widetilde{u}_t$ are the transition equation errors. Let us denote the data by $\widetilde{Y}^T = [Y_T, r^{*T}, l^{*T}, D^T]$, and $\widetilde{P} = [\widetilde{P}_1, \widetilde{P}_2]$, where $\widetilde{P}_2^T = [P_2, \nu]$ then conditional on a realization of the parameters and latent factors, the likelihood function of the data is

$$L(\widetilde{Y}^T | \widetilde{P}, \Sigma, \widetilde{X}^T) \propto \prod |\Sigma|^{-1} \exp\left(-\frac{1}{2} \widetilde{\varepsilon}_t{}'_t \Sigma^{-1} \widetilde{\varepsilon}_t \right)$$

where the diagonal 8×8 variance-covariance matrix Σ stacks on the main diagonal the pricing error variances σ_ε^2, for each maturity n, and the variances of the additional measurement equations σ_ϵ^2. Similarly, the pricing errors, $\widetilde{\varepsilon}_t = [\varepsilon_t, \epsilon_t]$, are given by equations (II.25)-(II.28).

Finally, the joint posterior distribution of the model parameters and the latent factors is given by

$$\pi(\widetilde{P}, \Sigma, \widetilde{X}^T | \widetilde{Y}^T) \propto L(\widetilde{Y}^T | \widetilde{P}, \Sigma, \widetilde{X}^T) \pi(\widetilde{X}^T | \widetilde{P}_1) \pi(P)$$

which is therefore given by the product of the likelihood of the observations, the density of the factors and the priors of the parameters.

Next, we present the block-wise slice-sampling algorithm within Gibbs sampler that allows us to draw from the full posterior, $\pi(\widetilde{P}, \Sigma, \widetilde{X}^T | \widetilde{Y}^T)$. We approximate the target density by repeatedly simulating from the conditional distributions of each block in turn. If the conditional distributions were known, this algorithm then consists of a series of Gibbs sampler steps. But, in our case, for those parameters which enter the bond pricing function f(·) the conditional distributions are not recognizable, so we replace Gibbs sampler steps with slice-sampling steps.

Drawing Factors. The term structure model is linear and has a Gaussian state-space representation. The measurement and transition equations are linear in the unobserved factors, X^T. And both equations have Gaussian distributed errors. So we use the Carter and Kohn (1994) simulation smoother to obtain a draw from the joint posterior density of the factors. In short, a run of the Kalman filter yields $\pi(\widetilde{P}, \Sigma, \widetilde{X}^T|\widetilde{Y}^T)$ and the predicted and smoothed means and variances of the states, while the simulation smoother provides the updated estimates of the conditional means and variances that fully determine the remaining densities (Kim and Nelson, 1999).

Drawing Parameters: Slice-sampling Steps. Although in the discretized case, VAR parameters have conjugate normal posterior distribution given the factors X^T, in our model the drift parameters and volatilities (P_1) also enter the pricing of yields. Thus, their conditional posteriors are unknown. We therefore draw these parameters from the joint posterior using the slice-sampling method proposed by Neal (2003), and adopted in a number of recent studies (*e.g.*, Li and Zinna, 2014). An alternative method would be the Random-Walk Metropolis (RWM) algorithm (see, *e.g.*, Johannes and Polson, 2004).

The conditional distributions of the (P_2) parameters are also unknown, however, they do not enter the density of the states. The estimation of aα and δ is similar in spirit to that of market price of risk parameters in traditional no-arbitrage models. (Note that the arbitrageurs' risk aversion, a, and excess demand elasticity, α, are not separately identified, so we estimate their product $a\alpha$.) These parameters are notably difficult to estimate because they only enter the measurement equation (bond pricing). We again use a slice-sampling method to draw these parameters from the posterior density.

Drawing Parameters: Gibbs-sampling Steps. It is rather straightforward to draw the remaining parameters as we can recur to simple Gibbs steps. Take for example the variance of the bond pricing errors. Conditional on the other parameters, $\widetilde{P}$, the factors and the observed yields, we get the measurement errors, ε_t. And, because we assume a common variance for all the maturities, we implicitly pool the n vectors of residuals into a single series. Thus, the inverse Gamma distribution becomes the natural prior for the variance, σ_ε^2. We also use the inverse Gamma to draw the volatilities of the remaining measurement equations.

We get the excess-supply factor loadings, θ, and the variance of the demand measurement errors, $\sigma_{\epsilon,2}^2$, conditional on the factors and the observed demand and supply variables. Essentially, conditional on a draw of β_t, this consists of estimating a linear regression model, $\beta_t = \theta b_t' + \epsilon_{2,t+\Delta}$, where β_t is the dependent variable and b_t are the regressors. This would involve two simple Gibbs steps. We would first draw the θ parameters from $\pi(\theta|\beta^T, B^T)$, which is the Gaussian distribution with mean $\left(\beta^{T\prime}\beta^T\right)^{-1}\left(\beta^{T\prime}B^T\right)$. Then, conditionally on θ, the factors and demand/supply variables, we get the measurement errors $\epsilon_{2,t}$ and draw the variance of the demand measurement errors, $\sigma_{\epsilon,2}^2$. However, as explained in the main text, to account for the serial correlation in the residuals, $\epsilon_{2,t}$, we employ the artificial 'sandwich' posterior method for Bayesian inference proposed by Müller (2013) to obtain θ in the first step. We implement this method following Miranda-Agrippino and Ricco (2017); specifically, we use the artificial Gaussian posterior centered at the MLE but with a HAC-corrected covariance matrix to draw the parameters. Once a draw of the θs is obtained, it is then possible to draw the measurement error volatility from the inverse Gamma distribution, as explained above.

We draw the liquidity parameters ν using a Gibbs step. Indeed, conditional on the $\widetilde{P}$ parameters and the states, which drive the bond pricing in the VV model, we obtain the pricing error $Y_t^\tau - f(r_t, \beta_t, P; \tau)$. Thus, conditional on σ_ε^2, the ν_τ parameter can be drawn simply as the coefficient in a linear regression model with dependent variable $Y_t^\tau - f(r_t, \beta_t, P; \tau)$ and independent variable l_t.

Priors. We set the priors such that they are proper but only little informative. The volatilities, in the Gibbs step, generally take inverse Gamma densities. The rest of the parameters have normal or, in a few cases, truncated normal distributions. For example, we impose arbitrageurs risk aversion αa, the elasticity δ, the mean reversion parameters to be positive (to insure factors' stationarity). These constraints can be easily imposed using the slice-sampling method.

Implementations Details. We perform 20,000 replications, of which the first 10,000 are 'burned' to insure convergence of the chain to the ergodic distribution. Note that the slice sampling method requires far less draws than the Metropolis-Hasting as for each draw the density is evaluated many times. We save 1 every 10 draws of the last 10,000 replications of the Markov chain to limit the autocorrelation of the draws.

Convergence Check. In order to check the convergence of the Markov chain we use two convergence diagnostics: the numerical standard error (NSE), and the convergence diagnostic (CD) of Geweke (1992)[38]. The NSE is a widely used measure of the approximation error. A good estimate of NSE has to compensate for the correlation in the draws (Koop, 2003). The second diagnostic, CD, relies on the idea that an estimate of the parameter based on the first half of the draws must be essentially the same to an estimate based on the last half. If this is not the case, then either the number of replications is too small, or the effect of the starting value has not vanished.

Three-factor Model. We do not provide a detailed description of the algorithm given that the main steps are essentially unchanged. Also note that, to account for the additional state and avoid over fitting, given that the model is now essentially a four-factor model including liquidity, we add the seven-year rate to the term structure of observed real rates.

II.3 Additional Tables and Figures

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II.3.1 Miscellaneous

Table A1: *Yield Summary Statistics and PC analysis*

Panel A: Summary Statistics						
		2-yr	5-yr	10-yr	15-yr	20-yr
Mean		0.09	0.87	1.45	1.73	1.85
Std.Dev.		1.44	1.31	1.11	0.97	0.89
AC(1)		0.96	0.96	0.97	0.97	0.96
Min		-2.12	-1.69	-0.78	-0.18	0.14
Max		5.01	3.62	3.57	3.59	3.61
Panel B: PC Analysis						
	EV	2-yr	5-yr	10-yr	15-yr	20-yr
PC1	92.3	0.6	0.5	0.4	0.3	0.3
PC2	7.2	-0.7	0.0	0.3	0.4	0.5
PC3	0.5	0.4	-0.7	-0.3	0.2	0.5

Note: The table reports summary statistics of real rates, and the principal component analysis. Panel A, *Summary Statistics*, presents mean, standard deviation, autocorrelation, maximum and minimum values of the real yields for the 2-, 5-, 10-, 15- and 20-year maturities for the estimation period from January 2001 to December 2016. The 2-year real rate was not available prior to Jan-2004; for this reason, the principal component (PC) analysis is performed over the January 2004 to December 2016 period. Panel B, *PC Analysis*, reports the proportion of the total variance explained by each PC in percent, EV, and the PC loadings for the selected maturities.

Table A2: *QE Price Impact by Arbs' Type*

	Panel I: **arbs1 (AO-FO-FED)**					Panel II: **arbs2 (AO-FO-FED-HH)**				
	Panel I.A: FED					*Panel II.A: FED*				
	std	LSAP1	LSAP2	MEP	LSAP3	std	LSAP1	LSAP2	MEP	LSAP3
2yr	-33.0	-19.6	-45.1	-18.0	-14.9	-32.3	-22.5	-44.6	-17.0	-14.6
5yr	-60.6	-36.0	-82.7	-33.0	-27.3	-59.4	-41.3	-81.8	-31.2	-26.8
10yr	-76.6	-45.5	-104.5	-41.7	-34.5	-75.0	-52.2	-103.4	-39.5	-33.8
15yr	-76.9	-45.7	-104.9	-41.9	-34.6	-75.4	-52.5	-103.9	-39.7	-34.0
20yr	-72.1	-42.8	-98.3	-39.2	-32.4	-70.6	-49.2	-97.4	-37.2	-31.8
	Panel I.B: AO					*Panel II.B: AO*				
	std	LSAP1	LSAP2	MEP	LSAP3	std	LSAP1	LSAP2	MEP	LSAP3
2yr	19.5	23.7	38.4	-9.2	23.9	23.4	49.0	37.9	-10.9	27.3
5yr	35.8	43.5	70.5	-16.8	44.0	42.9	90.1	69.7	-20.0	50.1
10yr	45.2	54.9	89.0	-21.2	55.5	54.3	113.8	88.1	-25.3	63.3
15yr	45.4	55.2	89.4	-21.3	55.8	54.5	114.3	88.5	-25.5	63.6
20yr	42.5	51.7	83.8	-20.0	52.3	51.1	107.1	82.9	-23.8	59.6
	Panel III: **arbs3 (AO-FO-FED-HH-FP)**					Panel IV: **arbs4 (Shadow Banks)**				
	Panel III.A: FED					*Panel IV.A: FED*				
	std	LSAP1	LSAP2	MEP	LSAP3	std	LSAP1	LSAP2	MEP	LSAP3
2yr	-31.6	-18.0	-41.0	-14.2	-24.5	-23.5	-17.0	-36.1	-1.0	-25.5
5yr	-58.0	-33.1	-75.2	-26.0	-44.9	-43.1	-31.3	-66.4	-1.8	-46.9
10yr	-73.3	-41.8	-95.1	-32.8	-56.8	-54.6	-39.6	-84.0	-2.2	-59.3
15yr	-73.6	-42.0	-95.5	-33.0	-57.0	-54.9	-39.8	-84.4	-2.3	-59.6
20yr	-69.0	-39.3	-89.5	-30.9	-53.4	-51.5	-37.3	-79.2	-2.1	-55.9
	Panel III.B: AO					*Panel IV.B: AO*				
	std	LSAP1	LSAP2	MEP	LSAP3	std	LSAP1	LSAP2	MEP	LSAP3
2yr	26.7	24.9	30.6	-10.7	43.8	59.0	22.5	10.7	-29.6	31.3
5yr	49.0	45.8	56.3	-19.6	80.4	108.5	41.3	19.7	-54.5	57.5
10yr	61.9	57.8	71.1	-24.8	101.6	137.3	52.3	25.0	-68.9	72.7
15yr	62.2	58.1	71.4	-24.9	102.0	138.1	52.5	25.1	-69.3	73.1
20yr	58.3	54.5	66.9	-23.3	95.6	129.4	49.3	23.5	-65.0	68.5

Note: The table reports the price impact of the Fed (Panels A: FED) and of supply (Panels B: AO) on the term structure of U.S. real rates for the selected maturities scaling demand and supply variables using several definitions of arbitrageurs (arbs). The arbs are defined as follows: arbs 1 is AO-FO-FED (our baseline as in Table 4), (Panel I); arbs 2 is AO-FO-FED-HH (Panel II); arbs 3 is AO-FO-FED-HH-FP (Panel III); and arbs 4: Shadow Banks (Panel IV). Specifically, AO is the amount of Treasury securities outstanding; FO the holdings of foreign officials; FED the holdings of the Fed; HH the holdings of domestic households; FP the holdings of foreign private investors; Shadow Banks as defined in Andolfatto and Spewak (2018). The column *std* shows the price impact of one-standard deviation change of the variable at hand. As for the sample periods, *LSAP1* is the first stage of the Fed asset purchase program, from March 2009 to November 2009; *LSAP2* is the second stage of the program, from November 2010 to June 2011; *LSAP3* is the third stage of program, from October 2012 to October 2014; and, *MEP* is the maturity extension program, from September 2011 to June 2012. The price impacts are quantified using equation (16) and are reported in basis points.

Table A3: *Price Impact: Demand and Supply Variables Scaled by U.S. GDP*

	Model I: Avg. TIPS Fitting Errors					Model II: On/off-the-Run Spread			
FO	2001-16	pre-QE	QE	post-QE	FO	2001-16	pre-QE	QE	post-QE
2yr	-70.8	-48.2	-38.3	15.7	2yr	-81.6	-55.6	-44.2	18.1
10yr	-164.4	-112.0	-89.0	36.6	10yr	-181.1	-123.3	-98.0	40.3
20yr	-154.6	-105.3	-83.7	34.4	20yr	-159.3	-108.5	-86.2	35.4
FED	2001-16	pre-QE	QE	post-QE	FED	2001-16	pre-QE	QE	post-QE
2yr	-15.5	-0.6	-23.9	9.1	2yr	-17.8	-0.7	-27.5	10.5
10yr	-35.9	-1.5	-55.5	21.1	10yr	-39.4	-1.6	-61.0	23.2
20yr	-33.7	-1.4	-52.2	19.8	20yr	-34.7	-1.4	-53.6	20.4
AO	2001-16	pre-QE	QE	post-QE	AO	2001-16	pre-QE	QE	post-QE
2yr	7.5	2.6	4.4	0.6	2yr	22.4	7.6	13.1	1.6
10yr	17.5	5.9	10.3	1.3	10yr	49.6	16.9	29.1	3.6
20yr	16.5	5.6	9.7	1.2	20yr	43.6	14.8	25.6	3.2
Total	2001-16	pre-QE	QE	post-QE	Total	2001-16	pre-QE	QE	post-QE
2yr	-78.7	-46.3	-57.8	25.4	2yr	-77.0	-48.7	-58.5	30.2
10yr	-182.8	-107.5	-134.2	58.9	10yr	-170.9	-108.1	-129.8	67.1
20yr	-171.8	-101.1	-126.2	55.4	20yr	-150.3	-95.1	-114.2	59.0

Note: The table reports the price impact of Foreign Officials (FO), of the Fed (FED), of supply (AO) and the combined impact of FO, FED and AO (Total) on the term structure of U.S. real rates for the selected maturities, whereby the demand and supply variables are standardized by U.S. nominal GDP. As for the sample periods, *2001-16* is the entire sample, from January 2001 to December 2016; *pre-QE* is the period prior to the start of QE, from January 2001 to March 2009; *QE* is the period of QE, from March 2009 to October 2014; *post-QE* is the period after the end of QE, from October 2014 to December 2016. The price impacts are quantified using equation (16) and are reported in basis points. Model I uses the average TIPS fitting errors to control for (il)liquidity, whereas Model II the on/off-the-run spread.

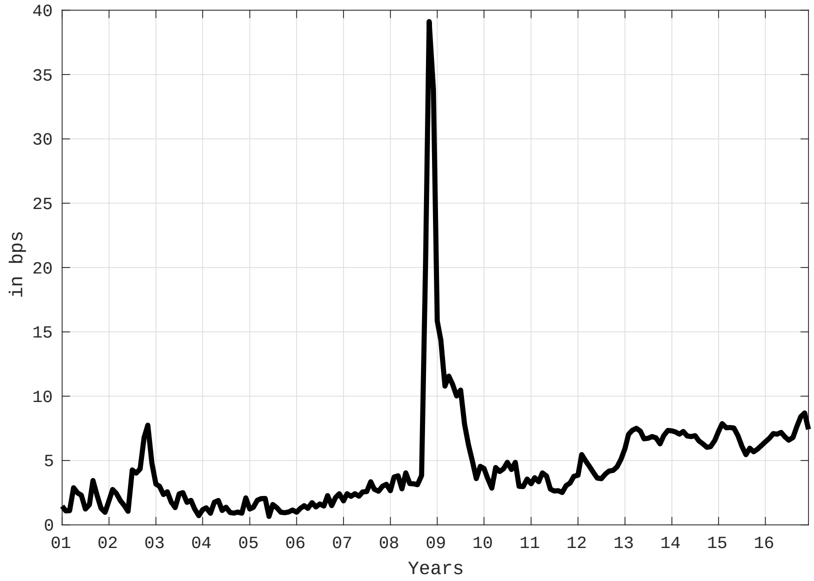


Figure IA.5: Average TIPS Curve Fitting Errors

Note: The figure shows a measure of (il)liquidity conditions in the TIPS market, *i.e.*, the average mean fitting errors from the Svensson TIPS yield curve. This measure is constructed as in D'Amico, Kim and Wei (2018). Source: Bloomberg, authors' calculations.

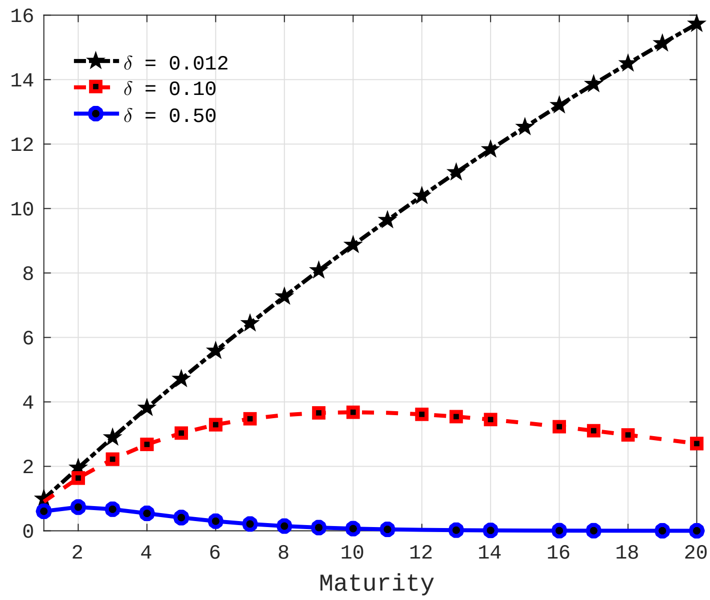


Figure IA.6: Impact Curve Robustness

Note: The figure shows the $\alpha(\tau) = \tau\alpha e^{-\delta\tau}$ function for different values of δ for $\alpha = 1$. The value of δ=0.012 results from the model estimation. Maturity is denoted in years.

II.3.2 2-Factor Model with No Adjustment for Liquidity

Table A4: *QE Price Impact: Two-Factor Model Without Liquidity*

	Panel I: FED				
	std	LSAP1	LSAP2	MEP	LSAP3
2yr	-34.3	-20.3	-46.7	-18.6	-15.4
5yr	-61.5	-36.5	-83.9	-33.5	-27.7
10yr	-75.6	-44.9	-103.1	-41.1	-34.0
15yr	-74.3	-44.1	-101.4	-40.5	-33.5
20yr	-68.6	-40.7	-93.6	-37.3	-30.9
	Panel II: AO				
	std	LSAP1	LSAP2	MEP	LSAP3
2yr	18.0	21.9	35.6	-8.5	22.2
5yr	32.4	39.4	63.9	-15.2	39.8
10yr	39.8	48.4	78.4	-18.7	48.9
15yr	39.1	47.6	77.2	-18.4	48.1
20yr	36.1	43.9	71.2	-17.0	44.4

Note: The table reports the price impact of the Fed (Panel I: FED) and of supply (Panel II: AO) on the term structure of U.S. real rates for the selected maturities, using the two-factor VV model without controlling for liquidity. The column *std* shows the price impact of one-standard deviation change of the variable at hand. As for the sample periods, *LSAP1* is the first stage of the Fed asset purchase program, from March 2009 to November 2009; *LSAP2* is the second stage of the program, from November 2010 to June 2011; *LSAP3* is the third stage of program, from October 2012 to October 2014; and, *MEP* is the maturity extension program, from September 2011 to June 2012. The price impacts are quantified using equation (16) and are reported in basis points.

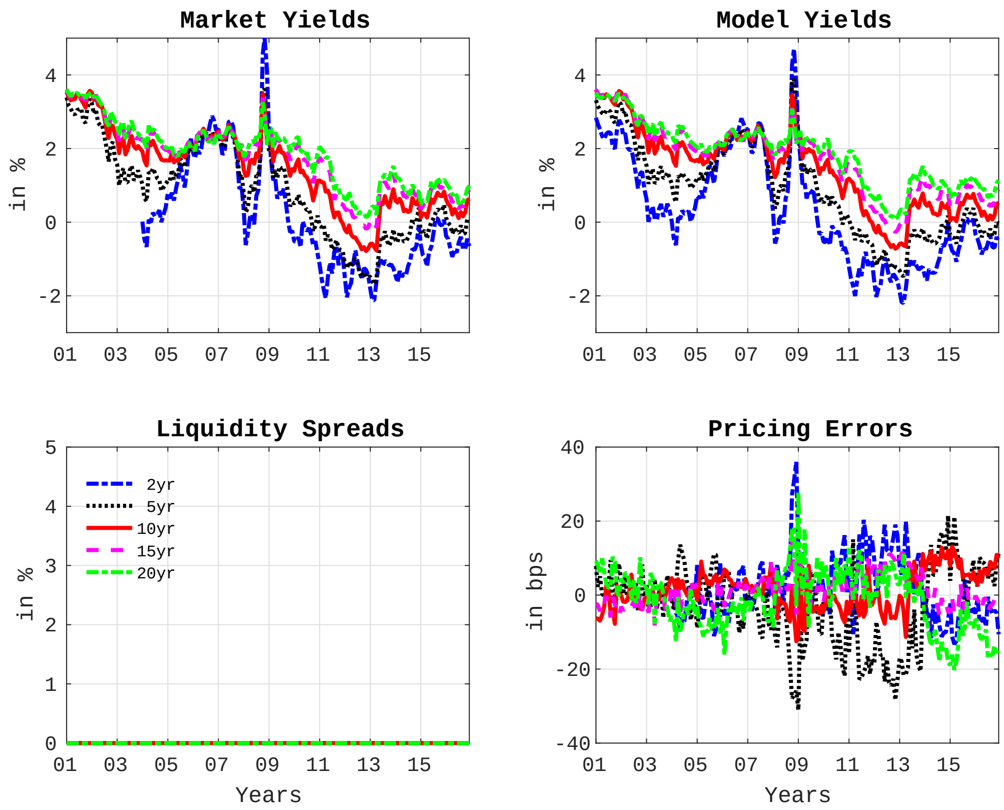


Figure IA.7: Decomposing Real Rates: Model Without Liquidity

Note: The figure shows the observed term structure of the monthly real rates for the 2-, 5-, 10-, 15- and 20-year maturities in the top left panel, *Market Yields*; the model real rates that result from the two-factor VV model in the top right panel, *Model Yields*; the liquidity spreads are zero as we do not control for illiquidity, in the bottom left panel, *Liquidity Spreads*; and, the pricing errors obtained as market yields minus model yields. Model implied rates result from the Bayesian estimation of the model presented in Section 4.3, by fixing at 0 the liquidity loadings, νs. The sample period ranges from January 2001 to December 2016, but the 2-year rate is available from January 2004.

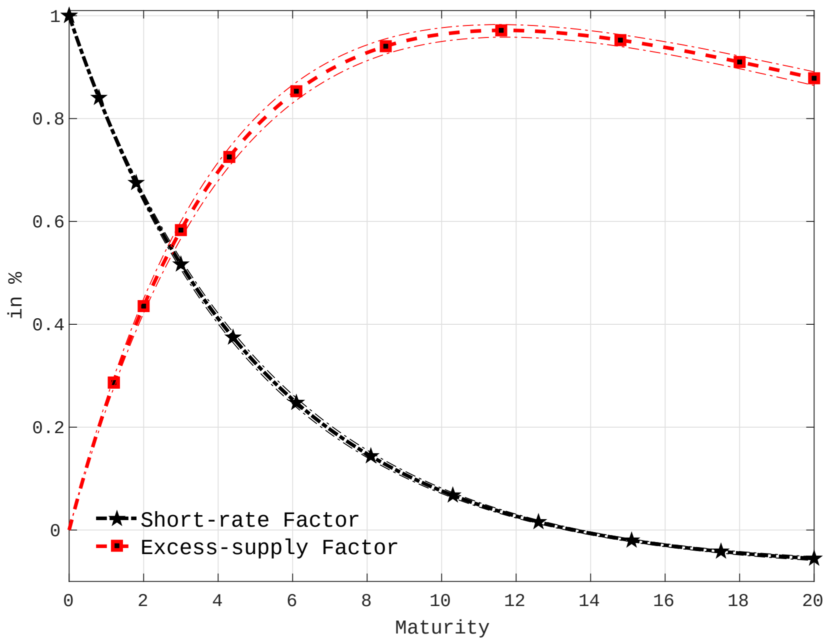


Figure IA.8: Factor Loadings: Model Without Liquidity

Note: This figure shows the effect of a 1% rise in the r_t and β_t factors on the term structure of spot real rates for maturities from 0 to 20 years resulting from the two-factor VV model, where liquidity is not controlled for. Dotted lines denote the 68% credible intervals.

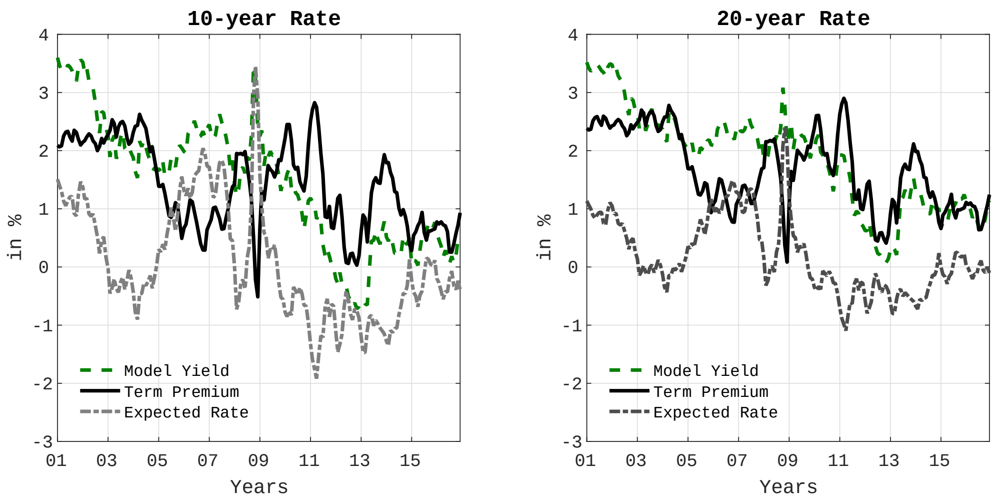


Figure IA.9: Model-Implied Real Rate, Expected Rate and Term Premium: Model Without Liquidity

Note: The figure shows the decomposition of the 10- and 20-year model implied real rates into the term premium and the average expected short rate over the 10- and 20-year horizons, respectively, resulting from the alternative specification of the two-factor VV model that does not control for liquidity.

II.3.3 Two-Factor Model with Off/On-the-Run Spread

Table A5: *QE Price Impact: Two-Factor Model with Off/On-the-Run Spread*

	Panel I: FED				
	std	LSAP1	LSAP2	MEP	LSAP3
2yr	-32.6	-19.4	-44.5	-17.8	-14.7
5yr	-59.1	-35.1	-80.6	-32.2	-26.6
10yr	-72.4	-43.0	-98.8	-39.4	-32.6
15yr	-70.4	-41.8	-96.0	-38.3	-31.7
20yr	-63.8	-37.9	-87.0	-34.7	-28.7

	Panel II: AO				
	std	LSAP1	LSAP2	MEP	LSAP3
2yr	20.9	25.4	41.2	-9.8	25.7
5yr	37.9	46.1	74.7	-17.8	46.6
10yr	46.4	56.5	91.5	-21.8	57.1
15yr	45.1	54.8	88.9	-21.2	55.4
20yr	40.9	49.7	80.6	-19.2	50.3

Note: The table reports the price impact of the Fed (Panel I: FED) and of supply (Panel II: AO) on the term structure of U.S. real rates for the selected maturities, resulting from the alternative specification based on the two-factor VV model that uses the off/on-the-run spread to control for illiquidity (see Figure IA.10). The column *std* shows the price impact of one-standard deviation change of the variable at hand. As for the sample periods, *LSAP1* is the first stage of the Fed asset purchase program, from March 2009 to November 2009; *LSAP2* is the second stage of the program, from November 2010 to June 2011; *LSAP3* is the third stage of program, from October 2012 to October 2014; and, *MEP* is the maturity extension program, from September 2011 to June 2012. The price impacts are quantified using equation (16) and are reported in basis points.

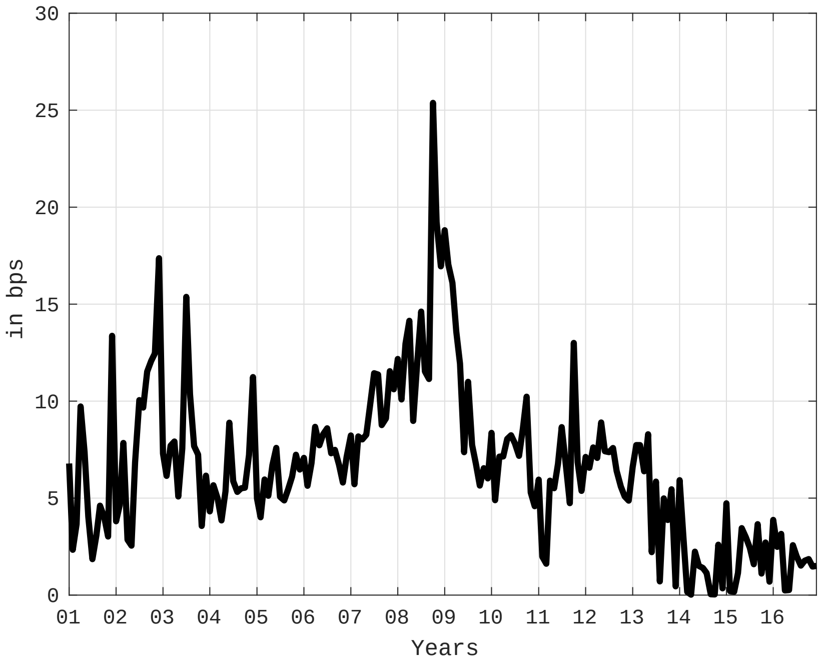


Figure IA.10: Off/On-the-Run Spread

Note: The figure shows an alternative measure of (il)liquidity conditions in the Treasury market, *i.e.*, the off/on-the-run spread. The off/on-the-run spread is calculated as the difference between the 10-year yield obtained from the yield curve (computed by the authors) using the off-the-run US Treasury bonds and the US on-the-run 10-year Treasury yield. Our estimate is somewhat less smooth than other estimates found in the literature, in particular it displays a less pronounced downward trend prior to the crisis. Source: U.S. Treasury, authors' calculations.

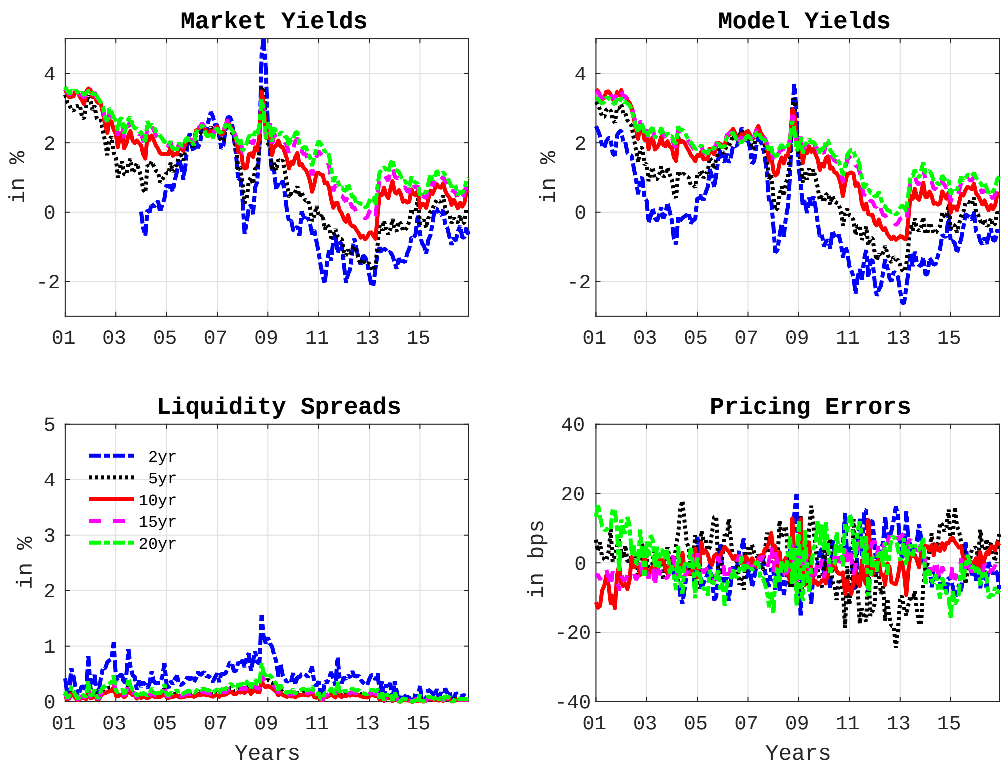


Figure IA.11: Decomposing Real Rates: Model with Off/On-the-Run Spread

Note: The figure shows the observed term structure of the monthly real rates for the 2-, 5-, 10-, 15- and 20-year maturities in the top left panel, *Market Yields*; the model real rates which result from the 2-factor VV model in the top right panel, *Model Yields*; the liquidity spreads, which result from using the off/on-the-run spread as alternative measure of liquidity, in the bottom left panel, *Liquidity Spreads*; and, the pricing errors obtained as market yields minus model yields and liquidity spreads. Model implied rates and liquidity spreads result from the Bayesian estimation of the model presented in Section 4.3. The sample period ranges from January 2001 to December 2016, but the 2-year rate is available from January 2004.

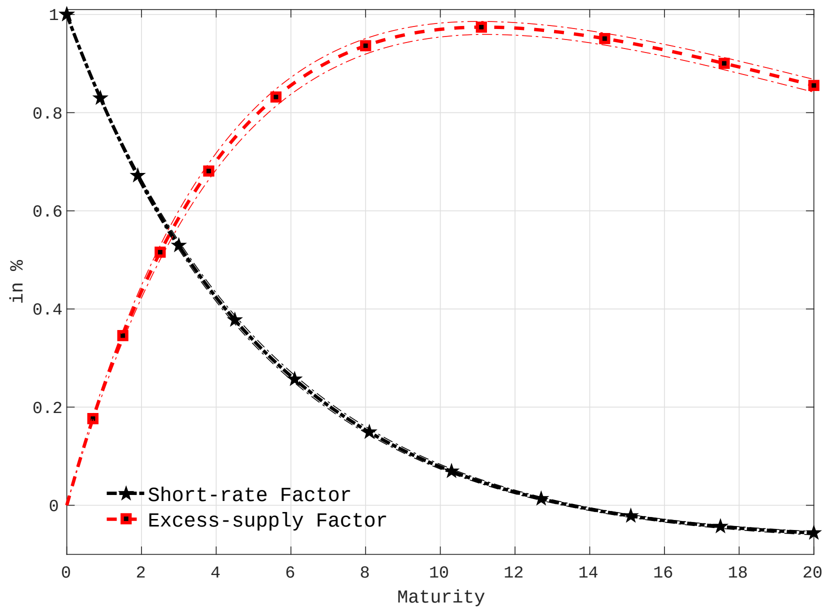


Figure IA.12: Factor Loadings: Model with Off/On-the-Run Spread

Note: This figure shows the effect of a 1% rise in the r_t and β_t factors on the term structure of spot real rates for maturities from 0 to 20 years in a model which uses the off/on-the-run spread as alternative measure of liquidity. Dotted lines denote the 68% credible intervals.

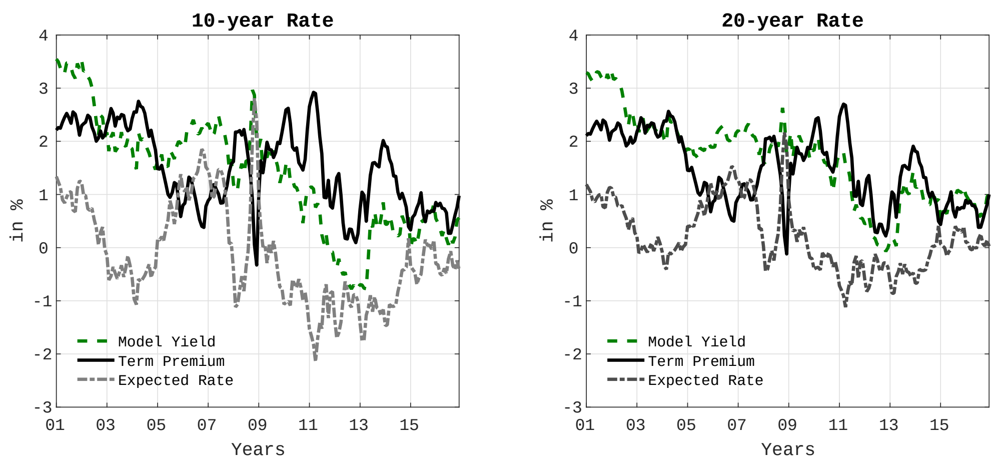
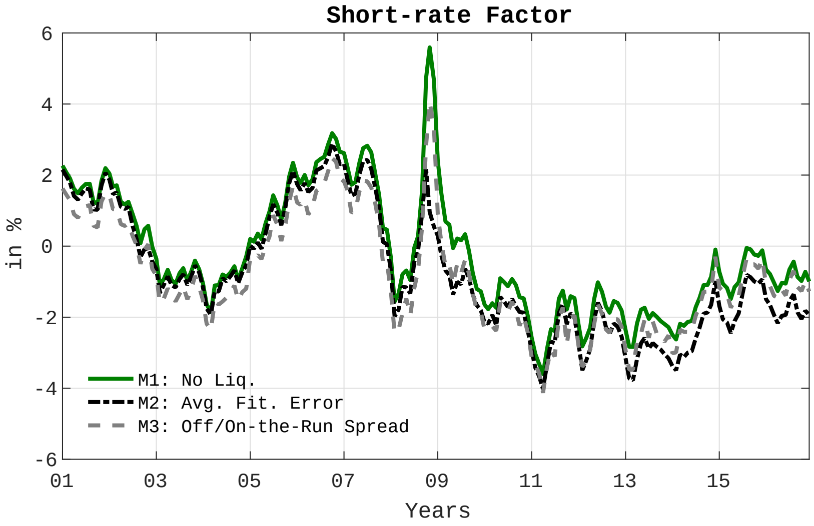


Figure IA.13: Model-Implied Real Rate, Expected Rate and Term Premium: Model with Off/On-the-Run Spread

Note: The figure shows the decomposition of the 10- and 20-year model implied real rates into the term premium and the average expected short rate over the 10- and 20-year horizons, respectively, resulting from the alternative model that uses off/on-the-run spread as measure of illiquidity. The model rate is the component of the real rate which is obtained from the 2-factor VV model, and thus does not include the liquidity spread component.



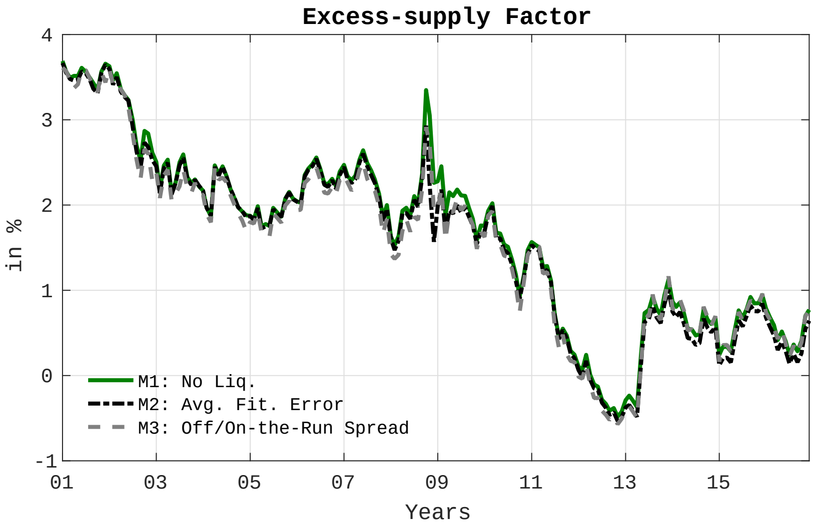


Figure IA.14: Robustness of Factor Estimates to Liquidity Effects

Note: The figure shows the factors estimated from three alternative models. In *M1: No Liq.*, the r_t and β_t factors are obtained from the estimation of the 2-factor VV model, whereby the real rate for maturity τ is given by $Y_t^\tau = f(P;\tau)[r_t;\beta_t] + \epsilon_t$. In *M2: Avg Fit. Error*, the r_t and β_t factors are obtained from the estimation of the 2-factor VV model augmented with the (il)liquidity spread, whereby the real rate for maturity τ is given by $Y_t^\tau = f(P,;\tau)[r_t;\beta_t] + \nu_\tau l^Q + \epsilon_t$, where $f(\cdot)$ is the real rate pricing based on the 2-factor VV model, l^Q is the observable liquidity factor, *i.e.*, the Average Fitted Errors, and ν_τ is the estimated liquidity loading for the τ-year rate. In *M3: Off/On-the-Run Spread*, the (il)liquidity measure used is the on/off the run spread.

III. THREE-FACTOR MODEL

III.1 Derivation

The short rate follows the process

$$dr_t = \kappa_r \left(\overline{r} + \eta_t - r_t\right) dt + \sigma_r dB_{r,t}, \tag{III.1}$$

which reverts to the stochastic mean $\overline{r} + \eta_t$. The term η_t has zero mean and follows the Ornstein-Uhlenbeck process:

$$d\eta_t = -\kappa_\eta \eta_t dt + \sigma_\eta dB_{\eta,t}. \tag{III.2}$$

Instantaneous correlation between $B_{r,t}$ and $B_{\eta,t}$ is $\rho_{r,\eta}$.

As in the two-factor model, the *excess demand* for the bond with maturity τ is assumed to be a linear function of the bond's yield $R_{t,\tau}$:

$$y_{t,\tau} = \alpha\left(\tau\right) \tau \left(R_{t,\tau} - \beta_t\right), \tag{III.3}$$

The aggregate excess-supply factor β_t follows the Ornstein-Uhlenbeck process

$$d\beta_t = \kappa_\beta \left(\overline{\beta} - \beta_t\right) dt + \sigma_\beta dB_{\beta,t}. \tag{III.4}$$

Instantaneous correlation between $B_{r,t}$ and $B_{\beta,t}$ is $\rho_{r,\beta}$, and between $B_{\eta,t}$ and $B_{\beta,t}$ is $\rho_{\eta,\beta}$.

We assume that arbitrageurs' investment strategy follows a mean-variance portfolio optimization, such that the arbitrageurs' optimization problem is given by

$$\max_{\{x_{t,\tau}\}_{\tau\in(0,T]}} \left[E_t(dW_t) - \frac{a}{2} Var_t(dW_t)\right], \tag{III.5}$$

with a denoting arbitrageurs' risk-aversion coefficient, $x_{t,\tau}$ denoting their dollar investment in the bond with maturity τ, and W_t denoting arbitrageurs time-t wealth. Arbitrageurs' budget constraint is assumed to be:

$$dW_t = \left(W_t - \int_0^T x_{t,\tau}\right) r_t dt + \int_0^T x_{t,\tau} \frac{dP_{t,\tau}}{P_{t,\tau}}, \tag{III.6}$$

where $P_{t,\tau}$ is the time-t price of the bond with maturity τ that pays \$1 at time $t+\tau$.

We conjecture equilibrium spot rates that are affine in the risk factors $(r_t,\, \eta_t,\, \beta_t)$, *i.e.*,

$$P_{t,\tau} = e^{-\left[A_r(\tau)r_t + A_\eta(\tau)\eta_t + A_\beta(\tau)\beta_t + C(\tau)\right]} \tag{III.7}$$

for four functions $A_r(\tau)$,$A_\eta(\tau)$, $A_\beta(\tau)$ and $C(\tau)$ that depend on maturity τ.

Applying Ito's Lemma to (III.7) and using the dynamics (III.1) of r_t, (III.2) of η_t and (III.4) of β_t, we find that the instantaneous return on the bond with maturity τ is

$$\frac{dP_{t,\tau}}{P_{t,\tau}} = \mu_{t,\tau} dt - A_r(\tau)\sigma_r dB_{r,t} - A_\eta(\tau)\sigma_\eta dB_{\eta,t} - A_\beta(\tau)\sigma_\beta dB_{\beta,t}, \tag{III.8}$$

where

$$\mu_{t,\tau} \equiv A_r'(\tau) r_t + A_\eta'(\tau)\eta_t + A_\beta'(\tau)\beta_t + C'(\tau)
- A_r(\tau)\kappa_r(\overline{r} + \eta_t - r_t) + A_\eta(\tau)\kappa_\eta\eta_t + A_\beta(\tau)\kappa_\beta\beta_t
+ \frac{1}{2}A_r(\tau)^2\sigma_r^2 + \frac{1}{2}A_\eta(\tau)^2\sigma_\eta^2 + \frac{1}{2}A_\beta(\tau)^2\sigma_\beta^2
+ \rho_{r,\eta}A_r(\tau)A_\eta(\tau)\sigma_r\sigma_\eta + \rho_{r,\beta}A_r(\tau)A_\beta(\tau)\sigma_r\sigma_\beta + \rho_{\eta,\beta}A_\eta(\tau)A_\beta(\tau)\sigma_\eta\sigma_\beta \tag{III.9}$$

is the instantaneous expected return. Substituting (III.8) into the arbitrageurs' budget constraint (III.6), we find that the arbitrageurs' first-order condition is

$$\mu_{t,\tau} - r_t = A_r(\tau)\lambda_{r,t} + A_\eta(\tau)\lambda_{\eta,t} + A_\beta(\tau)\lambda_{\beta,t}, \tag{III.10}$$

where

$$\lambda_{r,t} = a\sigma_r \int_0^T x_{t,\tau}\left[\sigma_r A_r(\tau) + \rho_{r,\eta}\sigma_\eta A_\eta(\tau) + \rho_{r,\beta}\sigma_\beta A_\beta(\tau)\right] d\tau, \tag{III.11}$$

$$\lambda_{\eta,t} = a\sigma_\eta \int_0^T x_{t,\tau}\left[\sigma_\eta A_\eta(\tau) + \rho_{r,\eta}\sigma_r A_r(\tau) + \rho_{\eta,\beta}\sigma_\beta A_\beta(\tau)\right] d\tau, \tag{III.12}$$

$$\lambda_{\beta,t} = a\sigma_\beta \int_0^T x_{t,\tau}\left[\sigma_\beta A_\beta(\tau) + \rho_{r,\beta}\sigma_r A_r(\tau) + \rho\eta, \beta\sigma_\eta A_\eta(\tau)\right] d\tau. \tag{III.13}$$

Market clearing implies that $x_{t,\tau} = -y_{t,\tau}$. Combining with III.3, $\beta_{t,\tau} = \theta(\tau)\beta_t$, III.7 and the definition of $R_{t,\tau}$, we obtain

$$x_{t,\tau} = \alpha(\tau)\{[\theta(\tau)\beta_t]\tau - [A_r(\tau)r_t + A_r(\tau)\eta_t + A_\beta(\tau)\beta_t + C(\tau)]\} \tag{III.14}$$

Substituting ($\mu_{t,\tau}$, $\lambda_{\eta,t}$, $\lambda_{\beta,t}$, $x_{t,\tau}$, $\lambda_{r,t}$) from III.9 and III.11-III.14, we find an affine equation in (r_t, η_t, β_t). Setting linear terms to zero yields

$$A_r'(\tau) + \kappa_r A_r(\tau) - 1 = A_r(\tau)M_{1,1} + A_\eta M_{1,2} + A_\beta M_{1,3} \tag{III.15}$$

$$A_\eta'(\tau) - \kappa_r A_r(\tau) + \kappa_\eta A_\eta(\tau) = A_r(\tau)M_{2,1} + A_\eta M_{2,2} + A_\beta M_{2,3} \tag{III.16}$$

$$A_\beta'(\tau) + \kappa_\beta A_\beta(\tau) = A_r(\tau)M_{3,1} + A_\eta M_{3,2} + A_\beta M_{3,3} \tag{III.17}$$

where

$$M_{1,1} = -a\sigma_r \int_0^T \alpha(\tau)A_r(\tau)\left[\sigma_r A_r(\tau) + \rho_{r,\eta}\sigma_\eta A_\eta(\tau) + \rho_{r,\beta}\sigma_\beta A_\beta(\tau)\right] d\tau, \quad \text{(III.18)}$$

$$M_{1,2} = -a\sigma_\eta \int_0^T \alpha(\tau)A_r(\tau)\left[\sigma_r \rho_{r,\eta} A_r(\tau) + \sigma_\eta A_\eta(\tau) + \rho_{\eta,\beta}\sigma_\beta A_\beta(\tau)\right] d\tau, \quad \text{(III.19)}$$

$$M_{1,3} = -a\sigma_\beta \int_0^T \alpha(\tau)A_r(\tau)\left[\sigma_r \rho_{r,\beta} A_r(\tau) + \rho_{\eta,\eta}\sigma_\eta A_\eta(\tau) + \sigma_\beta A_\beta(\tau)\right] d\tau, \quad \text{(III.20)}$$

$$M_{2,1} = -a\sigma_r \int_0^T \alpha(\tau)A_\eta(\tau)\left[\sigma_r A_r(\tau) + \rho_{r,\eta}\sigma_\eta A_\eta(\tau) + \rho_{r,\beta}\sigma_\beta A_\beta(\tau)\right] d\tau, \quad \text{(III.21)}$$

$$M_{2,2} = -a\sigma_\eta \int_0^T \alpha(\tau)A_\eta(\tau)\left[\sigma_r \rho_{r,\eta} A_r(\tau) + \sigma_\eta A_\eta(\tau) + \rho_{\eta,\beta}\sigma_\beta A_\beta(\tau)\right] d\tau, \quad \text{(III.22)}$$

$$M_{2,3} = -a\sigma_\beta \int_0^T \alpha(\tau)A_\eta(\tau)\left[\sigma_r \rho_{r,\beta} A_r(\tau) + \rho_{\eta,\eta}\sigma_\eta A_\eta(\tau) + \sigma_\beta A_\beta(\tau)\right] d\tau, \quad \text{(III.23)}$$

$$M_{3,1} = -a\sigma_r \int_0^T \alpha(\tau)\hat{A}_\beta(\tau)\left[\sigma_r A_r(\tau) + \rho_{r,\eta}\sigma_\eta A_\eta(\tau) + \rho_{r,\beta}\sigma_\beta A_\beta(\tau)\right] d\tau, \quad \text{(III.24)}$$

$$M_{3,2} = -a\sigma_\eta \int_0^T \alpha(\tau)\hat{A}_\beta(\tau)\left[\sigma_r \rho_{r,\eta} A_r(\tau) + \sigma_\eta A_\eta(\tau) + \rho_{\eta,\beta}\sigma_\beta A_\beta(\tau)\right] d\tau, \quad \text{(III.25)}$$

$$M_{3,3} = -a\sigma_\beta \int_0^T \alpha(\tau)\hat{A}_\beta(\tau)\left[\sigma_r \rho_{r,\beta} A_r(\tau) + \rho_{\eta,\eta}\sigma_\eta A_\eta(\tau) + \sigma_\beta A_\beta(\tau)\right] d\tau, \quad \text{(III.26)}$$

and $\hat{A}_\beta(\tau) = [\tau\theta(\tau) - A_\beta(\tau)]$. Equations (III.15-III.17) constitute a system of three linear differential equations in $A_r(\tau)$, $A_\eta(\tau)$ and $A_\beta(\tau)$, in which the coefficients of $A_r(\tau)$, $A_\eta(\tau)$ and $A_\beta(\tau)$ depend on integral involving these functions. The solution to the system (III.15-III.17) is given by

$$A_r(\tau) = \frac{1-e^{-\nu_1\tau}}{\nu_1} + \gamma_{r,1}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right) + \gamma_{r,2}\left(\frac{1-e^{-\nu_3\tau}}{\nu_3} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right), \quad \text{(III.27)}$$

$$A_\eta(\tau) = \gamma_{\eta,1}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right) + \gamma_{\eta,2}\left(\frac{1-e^{-\nu_3\tau}}{\nu_3} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right), \quad \text{(III.28)}$$

$$A_\beta(\tau) = \gamma_{\beta,1}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right) + \gamma_{\beta,2}\left(\frac{1-e^{-\nu_3\tau}}{\nu_3} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right), \quad \text{(III.29)}$$

where (ν_1,ν_2,ν_3,$\gamma_{r,1}$,$\gamma_{r,2}$,$\gamma_{\eta,1}$,$\gamma_{\eta,2}$,$\gamma_{\beta,1}$,$\gamma_{\beta,2}$) are scalars. To determine these scalars, we substitute (III.27)-(III.29) into (III.15-III.17), and identify terms in $\frac{1-e^{-\nu_k\tau}}{\nu_k}$, for k=1,2,3. This yields a system of nine scalar non-linear equations in the unknowns (ν_1,ν_2,ν_3,$\gamma_{r,1}$,$\gamma_{r,2}$, $\gamma_{\eta,1}$,$\gamma_{\eta,2}$,$\gamma_{\beta,1}$,$\gamma_{\beta,2}$). Thus, by including the central tendency process we move from a system of four to one of nine equations. To solve the resulting system, as in the case of the two-factor model, we need to assume the functional forms of $\alpha(\tau)$ and $\theta(\tau)$; we adopt the same specifications as in the two-factor case.

Finally, the function $C(\tau)$ is determined by

$$C'(\tau) - \kappa_r \overline{r} A_r(\tau) + \frac{1}{2}\sigma_r^2 A_r(\tau)^2 + \frac{1}{2}\sigma_\eta^2 A_\eta(\tau)^2 + \frac{1}{2}\sigma_\beta^2 A_\beta(\tau)^2$$
$$+ \rho_{r,\eta}\sigma_r\sigma_\eta A_r(\tau)A_\eta(\tau) + \rho_{r,\beta}\sigma_r\sigma_\beta A_r(\tau)A_\beta(\tau) + \rho_{\eta,\beta}\sigma_\eta\sigma_\beta A_\eta(\tau)A_\beta(\tau)$$
$$= a\sigma_r A_r(\tau) \int_0^T \alpha(\tau) \left[\overline{\beta}\tau - C(\tau)\right] \left[\sigma_r A_r(\tau) + \rho_{r,\eta}\sigma_\eta A_\eta(\tau) + \rho_{r,\beta}\sigma_\beta A_\beta(\tau)\right] d\tau$$
$$+ a\sigma_\eta A_\eta(\tau) \int_0^T \alpha(\tau) \left[\overline{\beta}\tau - C(\tau)\right] \left[\rho_{r,\eta}\sigma_r A_r(\tau) + \sigma_\eta A_\eta(\tau) + \rho_{\eta,\beta}\sigma_\beta A_\beta(\tau)\right] d\tau$$
$$+ a\sigma_\beta A_\beta(\tau) \int_0^T \alpha(\tau) \left[\overline{\beta}\tau - C(\tau)\right] \left[\rho_{r,\beta}\sigma_r A_r(\tau) + \rho_{\eta,\beta}\sigma_\eta A_\eta(\tau) + \sigma_\beta A_\beta(\tau)\right] d\tau. \quad \text{(III.30)}$$

The solution to (III.30) is

$$\begin{aligned} C(\tau) = {} & z_r \int_0^\tau A_r(u)du + z_\eta \int_0^\tau A_\eta(u)du + z_\beta \int_0^\tau A_\beta(u)du \\ & - \frac{\sigma_r^2}{2} \int_0^\tau A_r(u)^2 du - \frac{\sigma_\eta^2}{2} \int_0^\tau A_\eta(u)^2 du - \frac{\sigma_\beta^2}{2} \int_0^\tau A_\beta(u)^2 du \\ & - \rho_{r,\eta}\sigma_r\sigma_\eta \int_0^\tau A_r(u)A_\eta(u)du - \rho_{r,\beta}\sigma_r\sigma_\beta \int_0^\tau A_r(u)A_\beta(u)du \\ & - \rho_{\eta,\beta}\sigma_\eta\sigma_\beta \int_0^\tau A_\eta(u)A_\beta(u)du, \end{aligned} \quad \text{(III.31)}$$

where

$$z_r \equiv \kappa_r \overline{r} - a\sigma_r \int_0^T \alpha(\tau) \left[C(\tau)\right] \left[\sigma_r A_r(\tau) + \rho_{r,\eta}\sigma_\eta A_\eta(\tau) + \rho_{r,\beta}\sigma_\beta A_\beta(\tau)\right] d\tau, \quad \text{(III.32)}$$
$$z_\eta \equiv -a\sigma_\eta \int_0^T \alpha(\tau) \left[C(\tau)\right] \left[\rho_{r,\eta}\sigma_r A_r(\tau) + \sigma_\eta A_\eta(\tau) + \rho_{\eta,\beta}\sigma_\beta A_\beta(\tau)\right] d\tau, \quad \text{(III.33)}$$
$$z_\beta \equiv \kappa_\beta \overline{\beta} - a\sigma_\beta \int_0^T \alpha(\tau) \left[C(\tau)\right] \left[\rho_{r,\beta}\sigma_r A_r(\tau) + \rho_{\eta,\beta}\sigma_\eta A_\eta(\tau) + \sigma_\beta A_\beta(\tau)\right] d\tau. \quad \text{(III.34)}$$

Substituting $C(\tau)$ from (III.31) into (III.32)-(III.34), we can derive z_r, z_η, and z_β as the solution to a linear system of equations.

III.2 Additional Tables and Figures

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  - Table A7: Impact of Fed Asset Purchase Policies on Real Rates: Three-factor Model with Average Fitting Errors
  - Figure IA.15: Decomposing Real Rates: Three-factor Model with Average Fitting Errors
  - Figure IA.16: Estimated r_t, η_t and β_t Factors: Three-factor Model with Average Fitting Errors
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III.2.1 Model with Average Fitting Errors

Table A6: *Parameter Estimates; Three Factor Model with Average Fitting Errors*

	PANEL A: Model Parameters						PANEL B: Liquidity Parameters				
	mean	lb	ub	nse	CD		mean	lb	ub	nse	CD
κ_r	0.89	0.83	0.94	0.001	0.278	ν_{2yr}	9.37	8.28	10.45	0.01	4.32
$\overline{r}$	0.10	-0.24	0.38	0.003	9.677	ν_{5yr}	5.35	4.43	6.20	0.01	5.11
κ_η	0.11	0.10	0.11	0.000	1.380	ν_{7yr}	4.20	3.34	4.94	0.01	5.14
κ_β	0.95	0.91	0.99	0.000	0.258	ν_{10yr}	2.98	2.19	3.66	0.01	4.97
$\overline{\beta}$	1.12	0.98	1.30	0.002	6.424	ν_{15yr}	1.79	1.04	2.40	0.01	4.72
σ_r	1.99	1.98	2.00	0.000	0.979	ν_{20yr}	1.37	0.65	1.95	0.01	4.53
σ_η	0.99	0.98	1.00	0.000	1.100						
σ_β	0.96	0.92	1.01	0.000	3.491		Pricing Errors (in bps)				
$\rho_{r,\eta}$	1.60	-4.27	7.67	0.059	1.067		mean	lb	ub	nse	CD
$\rho_{r,\beta}$	46.15	42.74	48.99	0.029	0.047	σ_ε	4.08	3.89	4.28	0.00	0.13
$\rho_{\eta,\beta}$	1.79	1.57	2.05	0.002	2.580	$\sigma_{1,\epsilon}$	164.88	155.75	173.91	0.09	0.67
$a\alpha$	48.43	47.20	49.72	0.016	0.250						
δ	0.017	0.011	0.023	0.000	3.563						

Note: The table presents posterior means, 68% credible intervals, numerical standard errors (nse), and the absolute value of the convergence diagnostic (CD), as in Geweke (1992), for the estimated structural parameters of the the three-factor VV model, the liquidity parameters (ν), and the measurement error volatilities of the yields (σ_ε) and short-rate proxy ($\sigma_{1,\epsilon}$). The parameters $\overline{r}$ and $\overline{\beta}$, as well as the factor volatilities and correlations are reported in percent, and σ_ε and $\sigma_{1,\epsilon}$ in basis points. The model is estimated over the period ranging from Jan-2001 to Dec-2016 using real rates for the 2-, 5-, 7-, 10-, 15- and 20-year maturities, the average TIPS fitting errors (Figure IA.5) to proxy for (il)liquidity, and the baseline observable demand and supply factors presented in Section 4.

Table A7: *Impact of Fed Asset Purchase Policies on Real Rates: Three-factor Model with Average TIPS Fitting Errors*

Panel I: FED					
	std	LSAP1	LSAP2	MEP	LSAP3
2yr	-51.0	-30.3	-69.5	-27.7	-22.9
5yr	-72.1	-42.8	-98.3	-39.2	-32.4
7yr	-76.1	-45.2	-103.8	-41.4	-34.2
10yr	-77.5	-46.0	-105.8	-42.2	-34.9
15yr	-75.9	-45.1	-103.5	-41.3	-34.1
20yr	-72.6	-43.1	-99.1	-39.5	-32.7

Panel II: AO					
	std	LSAP1	LSAP2	MEP	LSAP3
2yr	29.8	36.3	58.8	-14.0	36.7
5yr	42.2	51.3	83.2	-19.8	51.9
7yr	44.5	54.1	87.7	-20.9	54.7
10yr	45.4	55.2	89.4	-21.3	55.8
15yr	44.4	54.0	87.5	-20.9	54.6
20yr	42.5	51.7	83.8	-20.0	52.3

Note: The table reports the price impact of the Fed (Panel I: FED) and of supply (Panel II: AO) on the term structure of U.S. real rates for the selected maturities, using the three-factor version of the VV model using, in addition to the term structure of real rates, the following data: the average TIPS fitting errors (Figure IA.5) to proxy for (il)liquidity; the proxy of the short real rate, and observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t. The column *std* shows the price impact of one-standard deviation change of the variable at hand. As for the sample periods, *LSAP1* is the first stage of the Fed asset purchase program, from March 2009 to November 2009; *LSAP2* is the second stage of the program, from November 2010 to June 2011; *LSAP3* is the third stage of program, from October 2012 to October 2014; and, *MEP* is the maturity extension program, from September 2011 to June 2012. The price impacts are quantified using equation (16) and are reported in basis points.

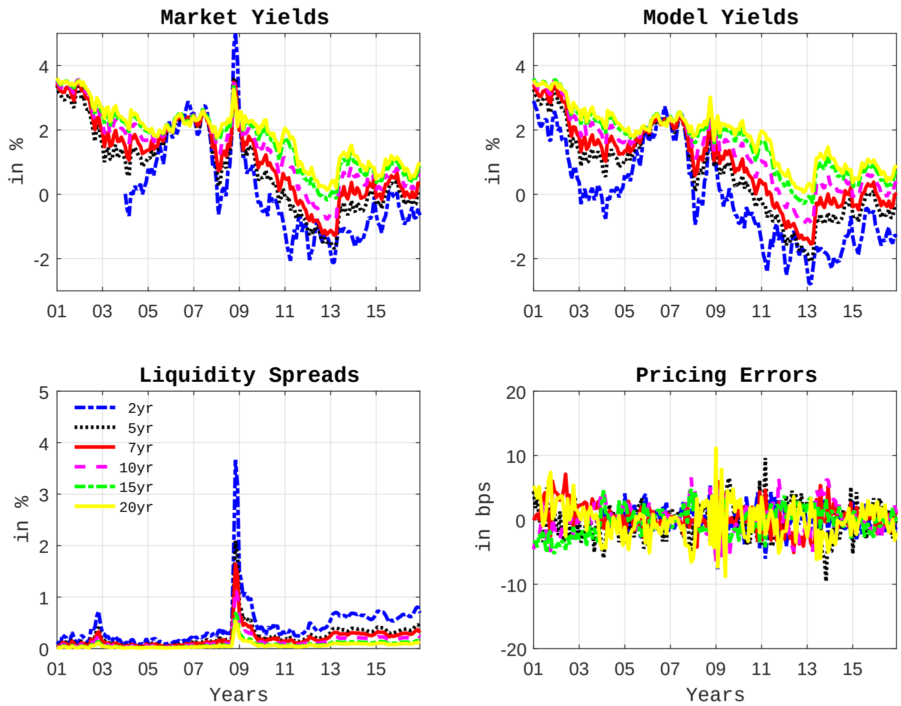

Figure IA.15: Decomposing Real Rates: Three-factor Model with Average Fitting Errors

Note: The figure shows the observed term structure of the real rates for the 2-, 5-, 7-, 10-, 15- and 20-year maturities in the top left panel, *Market Yields*; the model real rates which result from the 3-factor VV model in the top right panel, *Model Yields*; the liquidity spreads, which result from using the average fitting errors as proxy for (il)liquidity, in the bottom left panel, *Liquidity Spreads*; and, the pricing errors obtained as market yields minus model yields and liquidity spreads. Model implied rates and liquidity spreads result from the Bayesian estimation of the three-factor version of the VV model using, in addition to the term structure of real rates, the following data: the average fitting errors (Figure IA.5) to proxy for (il)liquidity; the proxy of the short real rate, and observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t. The sample period ranges from January 2001 to December 2016, but the 2-year rate is available from January 2004.

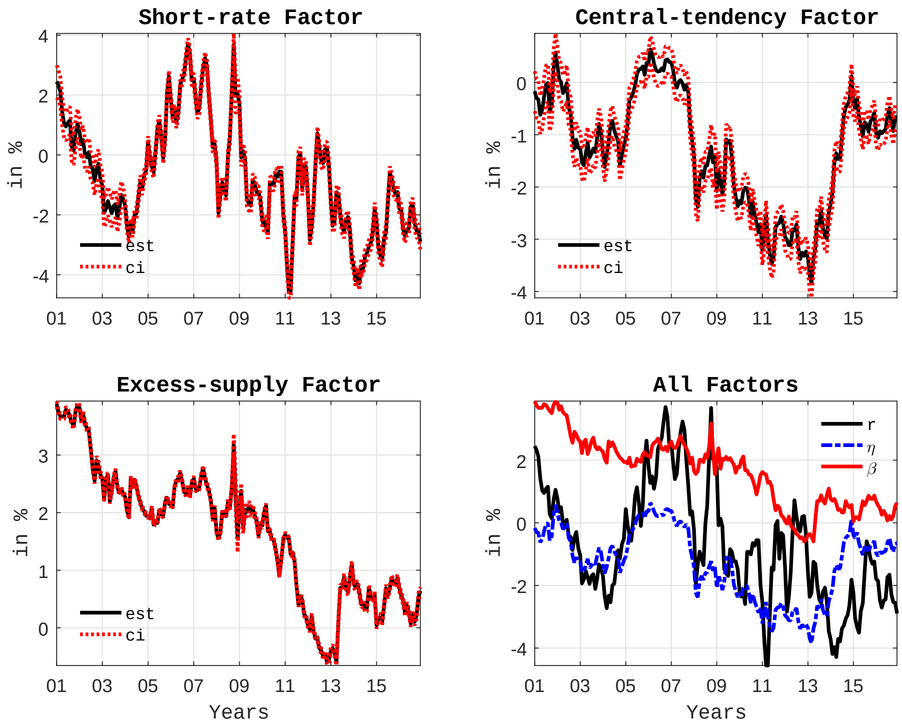

Figure IA.16: Estimated r_t, η_t and β_t Factors: Three-factor Model with Average Fitting Errors

Note: Smoothed factors with 68% credible intervals and the corresponding observable factors. Top left panel plots the short-term real rate in black, r_t; Top right panel plots the central tendency factor, η_t; bottom left panel plots the the excess-supply factor, β_t. The loadings are constructed using the parameters obtained from the Bayesian estimation of the three-factor version of the VV model using, in addition to the term structure of real rates, the following data: the average fitting errors (Figure IA.5) to proxy for (il)liquidity; the proxy of the short real rate, and observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t.

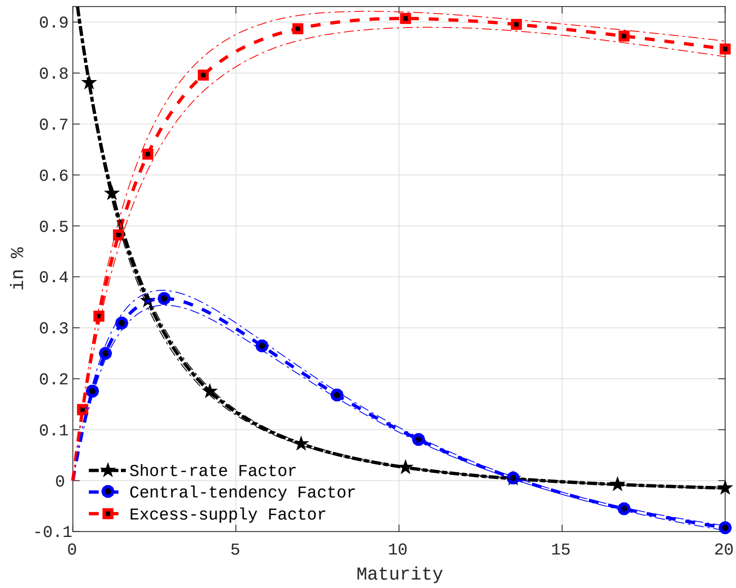

Figure IA.17: Factor Loadings: Three-factor Model with Average Fitting Errors

Note: This figure shows the effect of a 1% rise in the r_t, η_t and β_t factors on the term structure of spot real rates for maturities from 0 to 20 years. Dotted lines denote the 68% credible intervals. The loadings are constructed using the parameters obtained from the Bayesian estimation of the three-factor version of the VV model using, in addition to the term structure of real rates, the following data: the average fitting errors (Figure IA.5) to proxy for (il)liquidity; the proxy of the short real rate, and observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t.

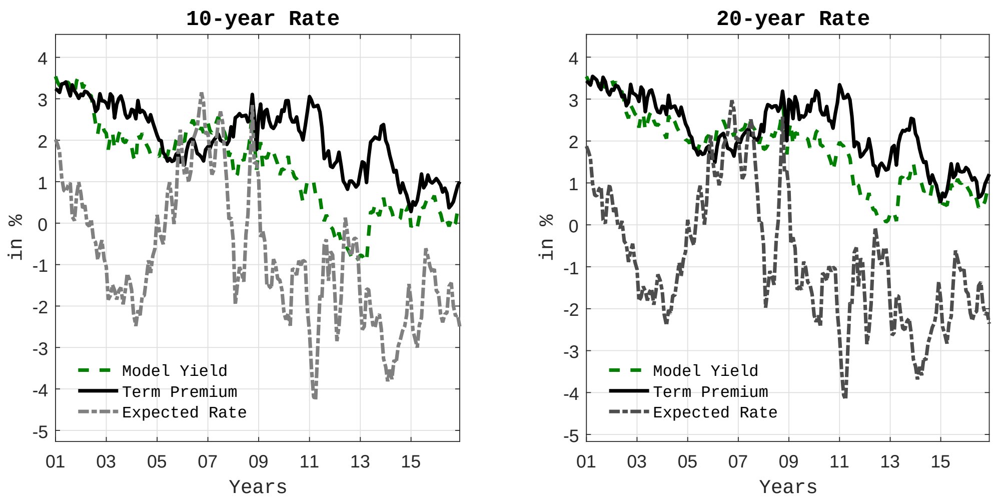

Figure IA.18: Model-Implied Real Rate, Expected Rate and Term Premium: Three-factor Model with Average Fitting Errors

Note: The figure shows the decomposition of the 10- and 20-year model implied real rates into the term premium and the average expected short rate over the 10- and 20-year horizons, respectively. The model rate is the component of the real rate which is obtained from the 3-factor VV model, and thus does not include the liquidity spread component. The decomposition is obtained using the output from the Bayesian estimation of the three-factor version of the VV model using, in addition to the term structure of real rates, the following data: the average fitting errors (Figure IA.5) to proxy for (il)liquidity; the proxy of the short real rate, and observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t.

III.2.2 Model with No Liquidity Adjustment

Table A8: *Impact of Fed Asset Purchase Policies on Real Rates: 3-Factor Model Without Liquidity*

	Panel I: Fed				
	std	LSAP1	LSAP2	MEP	LSAP3
2yr	-52.5	-31.2	-71.6	-28.6	-23.6
5yr	-73.7	-43.7	-100.5	-40.1	-33.2
7yr	-76.7	-45.6	-104.6	-41.7	-34.5
10yr	-76.8	-45.6	-104.7	-41.8	-34.5
15yr	-73.7	-43.8	-100.5	-40.1	-33.2
20yr	-70.0	-41.6	-95.5	-38.1	-31.5
	Panel II: AO				
	std	LSAP1	LSAP2	MEP	LSAP3
2yr	28.2	34.4	55.7	-13.3	34.7
5yr	39.6	48.2	78.1	-18.6	48.7
7yr	41.3	50.2	81.4	-19.4	50.8
10yr	41.3	50.3	81.4	-19.4	50.8
15yr	39.7	48.2	78.2	-18.7	48.8
20yr	37.7	45.8	74.3	-17.7	46.3

Note: The table reports the price impact of the Fed (Panel I: FED) and of supply (Panel II: AO) on the term structure of U.S. real rates for the selected maturities, using the three-factor version of the VV model using, in addition to the term structure of real rates, the following data: the proxy of the short real rate, and observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t. The column *std* shows the price impact of one-standard deviation change of the variable at hand. As for the sample periods, *LSAP1* is the first stage of the Fed asset purchase program, from March 2009 to November 2009; *LSAP2* is the second stage of the program, from November 2010 to June 2011; *LSAP3* is the third stage of program, from October 2012 to October 2014; and, *MEP* is the maturity extension program, from September 2011 to June 2012. The price impacts are quantified using equation (16) and are reported in basis points.

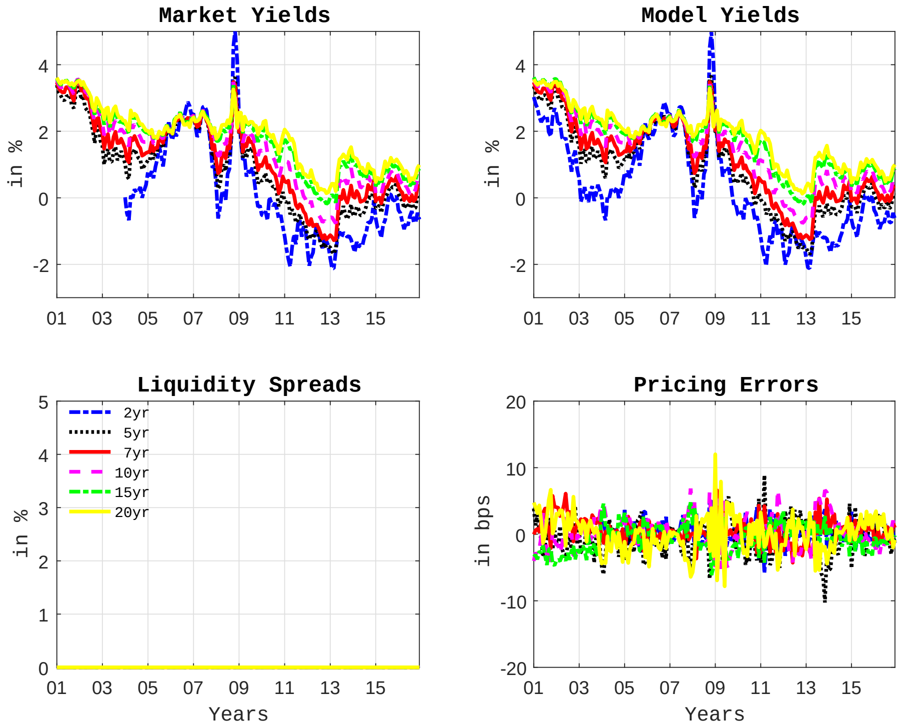

Figure IA.19: Decomposing Real Rates: Three-factor Model Without Liquidity

Note: The figure shows the observed term structure of the monthly real rates for the 2-, 5-, 7-, 10-, 15- and 20-year maturities in the top left panel, *Market Yields*; the model real rates which result from the 3-factor VV model in the top right panel, *Model Yields*; the liquidity spreads are zero as liquidity is not controlled for, in the bottom left panel, *Liquidity Spreads*; and, the pricing errors obtained as market yields minus model yields and liquidity spreads. Model implied rates result from the Bayesian estimation of the three-factor version of the VV model using, in addition to the term structure of real rates, the following data: the proxy of the short real rate; and, the observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t and parameters, and by fixing at 0 the liquidity loadings, νs. The sample period ranges from January 2001 to December 2016, but the 2-year rate is available from January 2004.

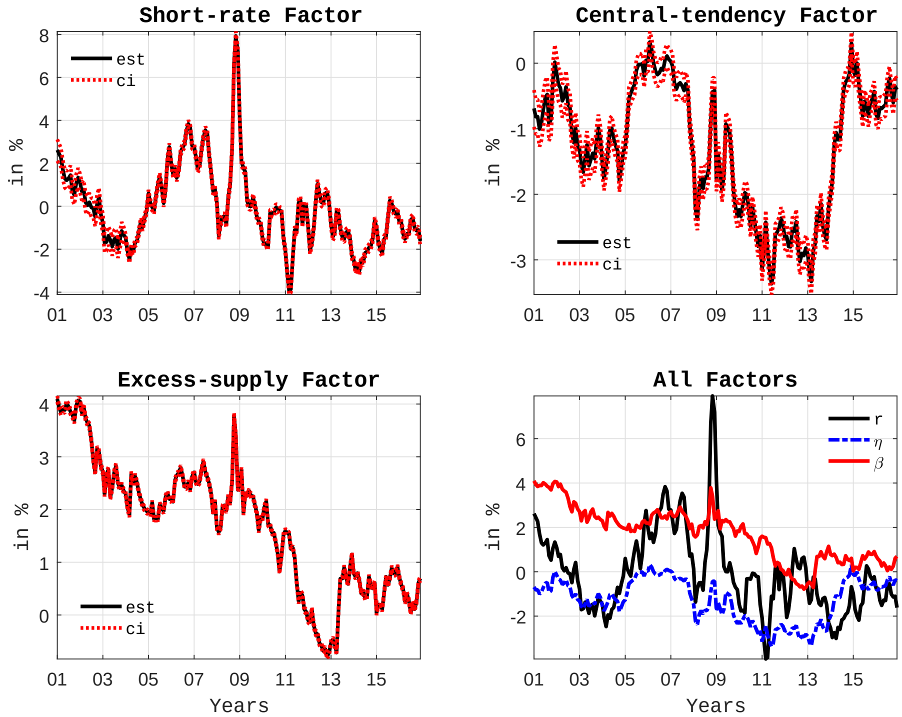

Figure IA.20: Estimated r_t, η_t and β_t Factors: Three-factor Model Without Liquidity

Note: Smoothed factors with 68% credible intervals and the corresponding observable factors. Top left panel plots the short-term real rate in black, r_t; Top right panel plots the central tendency factor, η_t; bottom left panel plots the the excess-supply factor, β_t. The factors result from the Bayesian estimation of the three-factor version of the VV model using, in addition to the term structure of real rates, the following data: the proxy of the short real rate; and, the observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t and parameters, and by fixing at 0 the liquidity loadings, νs. The sample period ranges from January 2001 to December 2016, but the 2-year rate is available from January 2004.

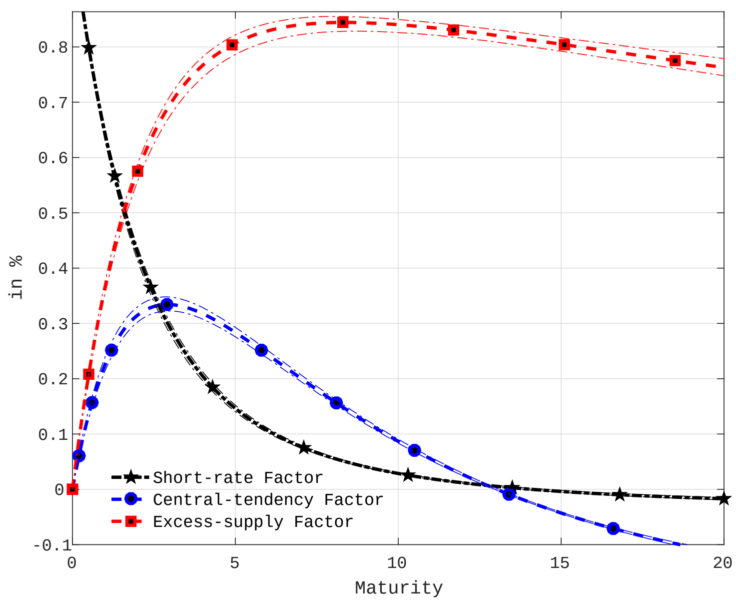

Figure IA.21: Factor Loadings: Three-factor Model Without Liquidity

Note: This figure shows the effect of a 1% rise in the r_t, η_t and β_t factors on the term structure of spot real rates for maturities from 0 to 20 years in a model where liquidity is not controlled for. Dotted lines denote the 68% credible intervals. The loadings are constructed using the parameters obtained from the Bayesian estimation of the three-factor version of the VV model using, in addition to the term structure of real rates, the following data: the proxy of the short real rate; and, the observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t and parameters, and by fixing at 0 the liquidity loadings, νs. The sample period ranges from January 2001 to December 2016, but the 2-year rate is available from January 2004.

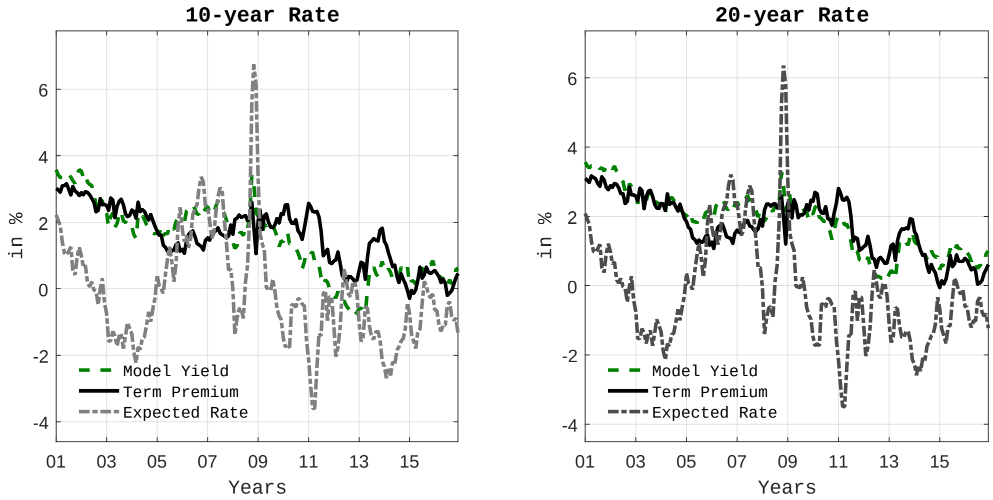

Figure IA.22: Model-Implied Real Rate, Expected Rate and Term Premium: Three-factor Model Without Liquidity

Note: The figure shows the decomposition of the 10- and 20-year model implied real rates into the term premium and the average expected short rate over the 10- and 20-year horizons, respectively, resulting from the alternative model that does not control for liquidity. The model rate is the component of the real rate which is obtained from the 3-factor VV model. The decomposition is obtained using the output the Bayesian estimation of the three-factor version of the VV model using, in addition to the term structure of real rates, the following data: the proxy of the short real rate; and, the observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t and parameters, and by fixing at 0 the liquidity loadings, νs. The sample period ranges from January 2001 to December 2016, but the 2-year rate is available from January 2004.

III.2.3 Model with Off/On-the-Run Spread

Table A9: *Impact of Fed Asset Purchase Policies on Real Rates: Three-factor Model with Off/On-the-Run Spread*

	Panel I: Fed				
	std	LSAP1	LSAP2	MEP	LSAP3
2yr	-47.8	-28.4	-65.2	-26.0	-21.5
5yr	-67.3	-39.9	-91.7	-36.6	-30.3
7yr	-70.3	-41.7	-95.8	-38.2	-31.6
10yr	-70.6	-41.9	-96.2	-38.4	-31.8
15yr	-67.8	-40.3	-92.5	-36.9	-30.5
20yr	-64.1	-38.1	-87.5	-34.9	-28.9
	Panel II: AO				
	std	LSAP1	LSAP2	MEP	LSAP3
2yr	31.9	38.8	62.9	-15.0	39.2
5yr	44.9	54.6	88.5	-21.1	55.2
7yr	46.9	57.1	92.5	-22.1	57.7
10yr	47.1	57.3	92.9	-22.1	57.9
15yr	45.2	55.0	89.2	-21.3	55.6
20yr	42.8	52.1	84.4	-20.1	52.6

Note: The table reports the price impact of the Fed (Panel I: FED) and of supply (Panel II: AO) on the term structure of U.S. real rates for the selected maturities, using the three-factor version of the VV model using, in addition to the term structure of real rates, the following data: the off/on-the-run spread (Figure IA.10) to proxy for (il)liquidity; the proxy of the short real rate, and observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t. The column *std* shows the price impact of one-standard deviation change of the variable at hand. As for the sample periods, *LSAP1* is the first stage of the Fed asset purchase program, from March 2009 to November 2009; *LSAP2* is the second stage of the program, from November 2010 to June 2011; *LSAP3* is the third stage of program, from October 2012 to October 2014; and, *MEP* is the maturity extension program, from September 2011 to June 2012. The price impacts are quantified using equation (16) and are reported in basis points.

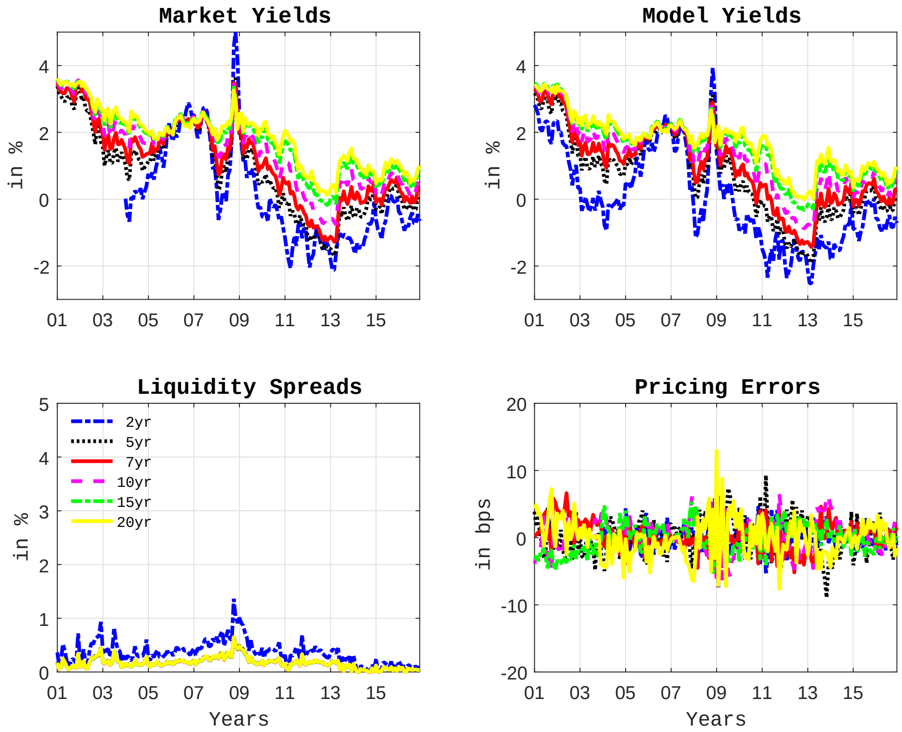

Figure IA.23: Decomposing Real Rates: Three-factor Model with Off/On-the-Run Spread

Note: The figure shows the observed term structure of the real rates for the 2-, 5-, 7-, 10-, 15- and 20-year maturities in the top left panel, *Market Yields*; the model real rates which result from the 3-factor VV model in the top right panel, *Model Yields*; the liquidity spreads, which result from using the off/on-the-run spread as proxy for (il)liquidity, in the bottom left panel, *Liquidity Spreads*; and, the pricing errors obtained as market yields minus model yields and liquidity spreads. Model implied rates and liquidity spreads result from the Bayesian estimation of the three-factor version of the VV model using, in addition to the term structure of real rates, the following data: the average fitting errors (Figure IA.5) to proxy for (il)liquidity; the proxy of the short real rate, and observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t. The sample period ranges from January 2001 to December 2016, but the 2-year rate is available from January 2004.

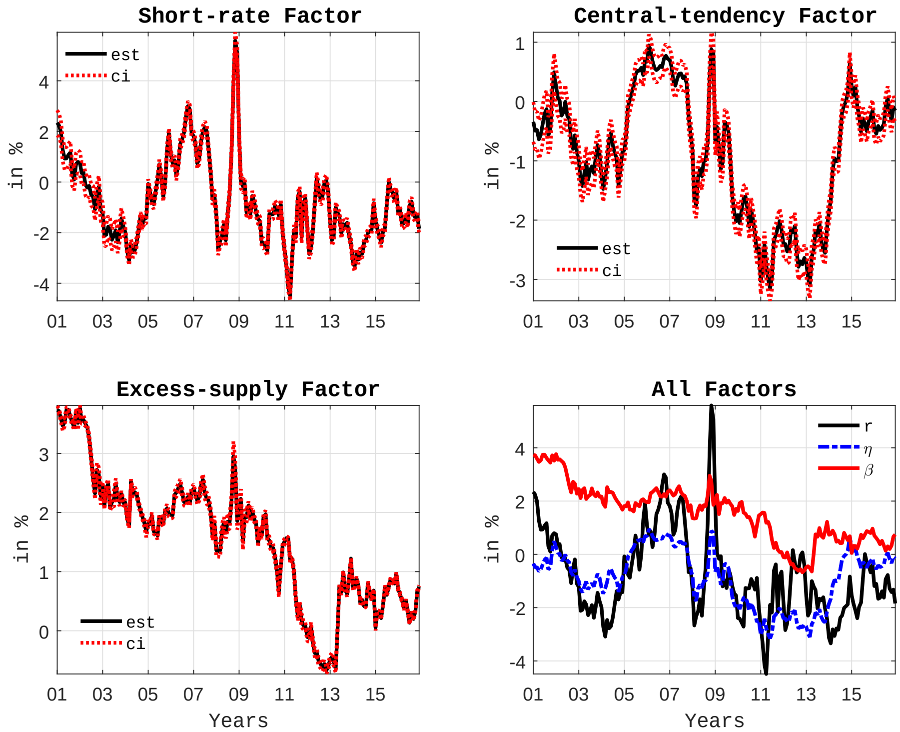

Figure IA.24: Estimated r_t, η_t and β_t Factors: Three-factor Model with Off/On-the-Run Spread

Note: Smoothed factors with 68% credible intervals and the corresponding observable factors. Top left panel plots the short-term real rate in black, r_t; Top right panel plots the central tendency factor, η_t; bottom left panel plots the the excess-supply factor, β_t. The factors result from the Bayesian estimation of the three-factor version of the VV model using, in addition to the term structure of real rates, the following data: the off/on-the-run spread (Figure IA.10) to proxy for (il)liquidity; the proxy of the short real rate, and observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t. The sample period ranges from Jan-2001 to Dec-2016.

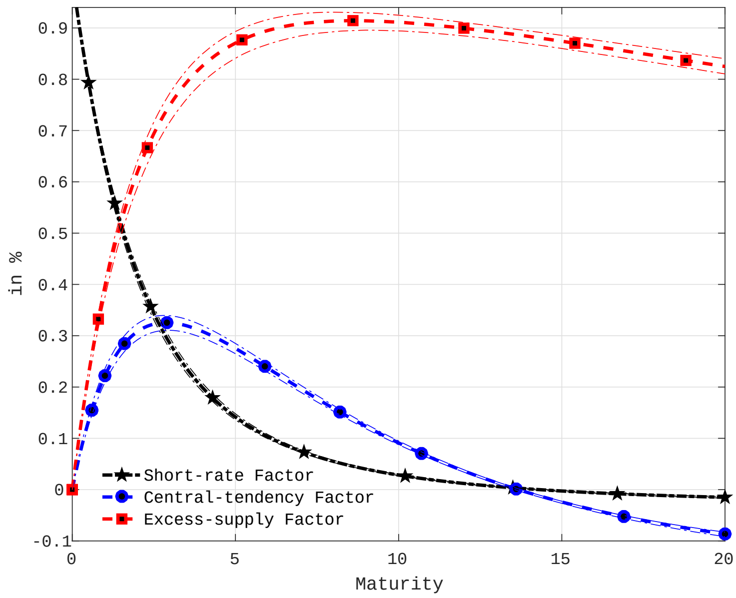

Figure IA.25: Factor Loadings: Three-factor Model with Off/On-the-Run Spread

Note: This figure shows the effect of a 1% rise in the r_t, η_t and β_t factors on the term structure of spot real rates for maturities from 0 to 20 years. Dotted lines denote the 68% credible intervals. The loadings are constructed using the parameters obtained from the Bayesian estimation of the three-factor version of the VV model using, in addition to the real rates, the off/on-the-run spread (Figure IA.10) to proxy for (il)liquidity, a proxy of the short real rate, and observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t.

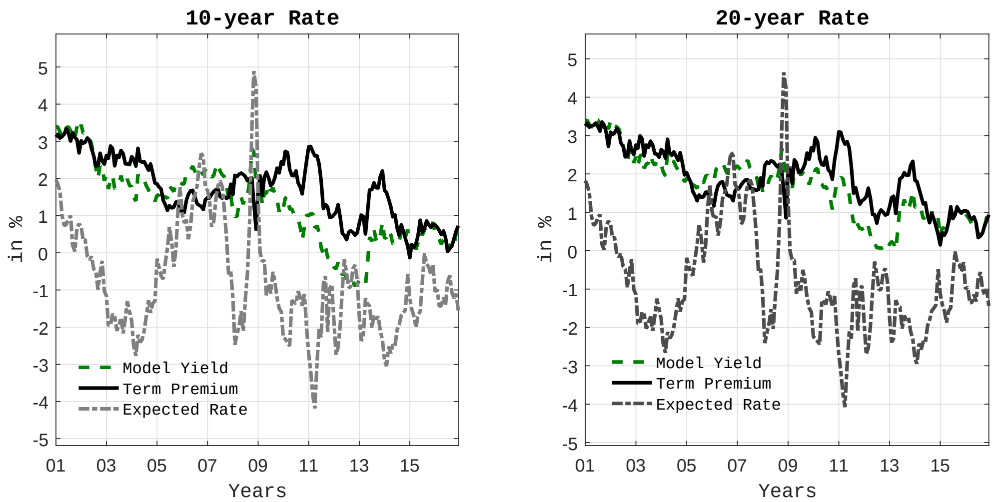

Figure IA.26: Model-Implied Real Rate, Expected Rate and Term Premium: Three-factor Model with Off/On-the-Run Spread

Note: The figure shows the decomposition of the 10- and 20-year model implied real rates into the term premium and the average expected short rate over the 10- and 20-year horizons, respectively. The model rate is the component of the real rate which is obtained from the 3-factor VV model, and thus does not include the liquidity spread component. The decomposition is obtained using the output the Bayesian estimation of the three-factor version of the VV model using, in addition to the real rates, the off/on-the-run spread (Figure IA.10) to proxy for (il)liquidity, a proxy of the short real rate, and observable measures of official demand and supply (Figure 1). The Bayesian algorithm extends that presented in Section 4.3, in order to account for the additional state vector η_t. The sample period ranges from Jan-2001 to Dec-2016.