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Flexible inflation targeting with active fiscal policy

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Abstract

This paper studies optimal time-consistent monetary policy in a simple New Keynesian model with long-term nominal government debt. Fiscal policy is 'active', so that stabilisation of the government debt stock is a binding constraint on monetary policy. Away from the lower bound on the monetary policy rate, optimal monetary policy cannot fully offset the effects of shocks to the natural rate of interest, reducing welfare. At the lower bound, recessionary shocks increase the real value of government debt, generating the anticipation of higher future inflation to stabilise real debt. Higher inflation expectations reduce real interest rates, mitigating the effects of recessionary shocks. If debt duration is long enough, improved performance at the lower bound may outweigh higher welfare losses in normal times, compared with the case in which fiscal policy is 'passive'.

Key words: Optimal monetary policy, fiscal policy, effective lower bound, government debt.

JEL classification: E52, E58, E62.

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1 Introduction

This paper studies how optimal monetary policy is affected by the nature of fiscal policy behavior. The focus is a case in which fiscal policy is 'active' (Leeper, 1991) so that monetary policy must ensure that the real value of (nominal) government debt is stabilized. This contrasts with the typical assumption adopted in textbook models, in which taxes and spending are passively adjusted to stabilize the real government debt stock in response to changes in financing costs or real debt generated by monetary policy actions. Such 'passive' fiscal behavior ensures that the real government debt stock is stabilized for any path of prices so that the government's intertemporal budget constraint is irrelevant for the monetary policymaker.

The analytical framework is a standard New Keynesian model (Galí, 2008; Woodford, 2003) extended to include long-term nominal government debt. The presence of this debt has no implications for optimal monetary policy under the textbook assumption of passive fiscal policy. However, as noted by Sims (2011) and Cochrane (2018), under active fiscal policy, the presence of long-term government debt allows the price level adjustments required to stabilize government debt to be spread out over time. Long-term debt therefore influences the combinations of output, inflation and real government debt that can be achieved by an optimizing monetary policymaker.

To explore the implications of debt duration for optimal monetary policy, two parameterizations of the model are studied. The baseline parameterization assumes that the duration of government debt is four years, based on data for a selection of OECD countries. An alternative 'long duration' parameterization assumes that the duration of government debt is eight years. 'Active' fiscal policy is captured by an assumption that real taxes are held fixed. While stark, this is an obvious starting point for the analysis of active fiscal policy, since fiscal instruments are independent of the level of real government debt.

Optimal monetary policy is studied using a log-linearized version of the model and a social welfare function based on a quadratic approximation to household utility. Policymakers are constrained to follow time-consistent policies, which is interpreted as an approximation to the way that flexible inflation targeting mandates are pursued in practice.

In the absence of a lower bound on the short-term nominal interest rate, optimal policy is characterized by a standard linear-quadratic optimization problem that can be studied analytically. That analysis reveals several insights.

Under optimal policy the behavior of the output gap and inflation is determined by the elasticity of the government debt stock with respect to previously accumulated debt. In equilibrium, this elasticity increases with the duration of the debt stock. This implies that longer duration debt allows for a slower stabilization of the debt stock following a shock and reduces the scale of fluctuations in the output gap and inflation required to deliver that stabilization. The duration of government debt therefore underpins the extent of the "debt stabilization bias" under time-consistent policy (Leith and Wren-Lewis, 2013; Leeper and Leith, 2016).

Two further results are uncovered from a comparison to the textbook model with passive fiscal policy. First, the so-called "divine coincidence" (Blanchard and Galí, 2007) no longer holds. In the textbook model with passive fiscal policy, the effects of variations in the natural rate of interest on the output gap and inflation are perfectly offset under optimal time-consistent monetary policy. With active fiscal policy, the additional constraint that the government debt stock is stabilized requires the monetary policymaker to allow deviations of output from potential and inflation from target. These fluctuations generate welfare costs.

Conversely, welfare costs generated by cost-push shocks may be smaller under active fiscal policy. A negative cost-push shock that lowers inflation in the near term increases the real value of existing nominal debt. Stabilizing the real debt stock requires future policymakers to deliver higher inflation. That increases expected inflation and makes the trade-off between stabilizing the output gap and inflation more favorable.

In the presence of a lower bound on the short-term interest rate, the non-linearity induced by the lower bound requires the model to be solved by numerical methods. Nonetheless, the analytical results from the linear quadratic analysis also provide intuition for results in this case.

In the presence of a lower bound on the short-term interest rate, active fiscal policy may lead to higher welfare than passive fiscal policy. This result depends on the balance between two effects. Away from the lower bound, welfare losses are larger under active fiscal policy, since the 'divine coincidence' does not hold. However, when constrained by the lower bound, the combination of active fiscal policy and long-duration debt reduces welfare losses. Deflationary shocks that drive the policy rate to the lower bound raise the real value of government debt, requiring future policymakers to generate higher inflation to stabilize the debt stock. This increases inflation expectations, lowering the real interest rate and thereby mitigates the recessionary effects of the shock.

For the baseline parameterization, the improvement in outcomes at the zero bound associated with active fiscal policy is not large enough to offset poorer performance in normal times. However, this result is reversed for the 'long duration' parameterization, so that overall welfare is higher than the level achieved under passive fiscal policy.

These results contribute to a large literature studying the interaction between monetary and fiscal policies, which has become an area of renewed interest following the Global Financial Crisis and the Covid-19 pandemic. Much of that literature studies the role of particular fiscal policy actions (for example Bianchi et al., 2020), the combination of alternative monetary and fiscal policy rules (for example Billi and Walsh, 2021) and jointly optimal monetary and fiscal policies (Matveev, 2018). That literature has also explored the interplay between fiscal policy and unconventional monetary policies such as quantitative easing (Bhattarai et al., 2015, 2019).

The present paper has a narrower focus, examining the implications for optimal interest rate policy under specific assumptions about fiscal policy. Indeed the analysis abstracts from changes in fiscal instruments, assuming that real tax revenue is held fixed, to maintain the focus on monetary policy implications. This type of approach is similar to that of Kumhof et al. (2010), who study simple monetary policy rules under active fiscal policy and Benigno and Woodford (2006), who focus on optimal commitment policies. In contrast, this paper studies time-consistent monetary policy, similarly to Blake and Kirsanova (2012), but assuming that the government finances its activities using long-term debt and that monetary policy is constrained by a lower bound on the short-term nominal interest rate. The interplay between government debt duration, fiscal policy behavior and the lower bound on the short-ter rate drives many of the results in the present paper.

The rest of the paper is structured as follows. Section 2 sets out the model, welfare-based loss function and the baseline and 'long duration' parameterizations. Section 3 presents the analysis of optimal time-consistent policy, assuming that there is no lower bound on the short-term interest rate. Section 4 examines optimal time-consistent policy when the lower bound on the short-term interest rate is accounted for. Section 5 concludes.

2 Model

This section describes the key elements of the model structure. A detailed derivation is provided in Appendix A.

2.1 Households

The representative household maximizes a utility function defined over consumption, c, and hours worked, n, subject to a budget constraint that defines how proceeds from wage and profit income, net of taxes are allocated to short-term and long-term government bonds.

The optimization problem is

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \phi_t \left\{ \frac{c_t^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} - \frac{n_t^{1+\psi}}{1+\psi} \right\}$$

subject to

$$V_t D_t + B_t = (\varrho + \chi V_t) D_{t-1} + R_{t-1} B_{t-1} + W_t n_t - T_t + F_t - P_t c_t \tag{1}$$

where P is the price of consumption, W is the nominal wage, T is a lump sum tax and F represents dividend payments from firms. The household may invest in one period government bonds B or long-term government debt, D. To keep the model close to the textbook New Keynesian benchmark a 'cashless limit' economy, following Woodford (2003), is considered.[1]

Long-term debt is a security that pays a sequence of nominal coupons that decay geometrically at rate $\chi < 1$.[2] The nominal value of a newly issued bond at date t is V_t and such a bond pays a coupon stream of $\varrho, \varrho\chi, \varrho\chi^2, \ldots$ in periods $t+1, t+2, t+3, \ldots$. The importance of the initial coupon $\varrho > 0$ is discussed in Section 2.3.

[1] Early explorations of the monetary policy implications of fiscal policy behavior focused on the implications for money growth and seigniorage (see, for example, Sargent and Wallace, 1981). More recent treatments have abstracted from money as government debt represents the vast majority of outstanding government liabilities in most countries (see, for example, Cochrane, 2018).

[2] A key benefit of this setup is that the value of a bond issued at date $t-j$ is equal to $\chi^j V_t$ so that holdings of all previously issued bonds can be summarized in terms of an equivalent quantity of newly issued bonds, simplifying aggregation.

Preferences are subject to an exogenous shock, ϕ_t which follows the process

$$\ln \phi_t = \rho_\phi \ln \phi_{t-1} + \sigma_\phi \varepsilon_t^\phi \tag{2}$$

where ε_t^ϕ is an iid normally-distributed shock with unit variance.

2.2 Firms

A continuum of monopolistically competitive producers of unit mass, indexed by $j \in (0,1)$, produce differentiated products that form a Dixit-Stiglitz bundle purchased by households. Preferences over differentiated products are given by

$$y_t = \left[\int_0^1 y_{j,t}^{1-\eta_t^{-1}} dj \right]^{\frac{1}{1-\eta_t^{-1}}}$$

where y_j is firm j's output and the elasticity of demand η_t varies over time according to

$$\ln \eta_t - \ln \eta = \rho_\eta \left(\ln \eta_{t-1} - \ln \eta \right) + \sigma_\eta \varepsilon_t^\eta \tag{3}$$

Firms produce using a constant returns production function in the single input (labor):

$$y_{j,t} = A_t n_{j,t}$$

where A_t is an exogenous productivity process that follows:

$$\ln A_t - \ln A = \rho_A \left(\ln A_{t-1} - \ln A \right) + \sigma_A \varepsilon_t^A \tag{4}$$

Firms set prices according to the Calvo (1983) staggered pricing scheme, with a probability $1-\alpha$ of changing price each period. A fixed production subsidy ensures that the steady state is efficient.

2.3 Government debt and fiscal policy

The nominal government flow budget constraint is:

$$B_t + V_t D_t = R_{t-1} B_{t-1} + \left(\varrho + \chi V_t \right) D_{t-1} + G_t - T_t$$

To focus on the case in which the government issues only long-term debt, short-term and long-term debt are assumed to be issued in fixed proportions:

$$B_t = \vartheta D_t$$

for $\vartheta \geq 0$.

The focus will be on the case of purely long-term debt, $\vartheta \to 0$, so that:

$$V_t D_t = (\varrho + \chi V_t)\, D_{t-1} + G_t - T_t = (\varrho + \chi V_t)\, D_{t-1} - S_t$$

where the primary surplus is defined as $S \equiv T - G$.

Denoting real quantities with lower case letters means that the real-valued government budget constraint is :

$$V_t d_t = (\varrho + \chi V_t)\, \pi_t^{-1} d_{t-1} - s_t \tag{5}$$

Appendix A.1 demonstrates that the steady-state value of price of debt, V, is equal to unity if the initial coupon satisfies $\varrho = \beta^{-1} - \chi$. Invoking this assumption allows d to be interpreted as the (real) par value of long-term debt. This is useful for calibration purposes, since most data on government debt stocks are measured at par, rather than market value.

Real government spending, $g_t \, (\equiv G_t/P_t)$, is assumed to evolve according to a simple exogenous process around its long-run steady state level, $g\,(> 0)$:

$$\bar{g}_t = \rho_g \bar{g}_{t-1} + \varepsilon_t^g \tag{6}$$

where $\bar{g}_t \equiv g_t - g$ denotes the linear deviation of spending from steady state.

The baseline assumption for fiscal policy is that lump sum taxes are held fixed at $\tau_t = \tau > g, \forall t$ in real terms (with $\tau_t \equiv T_t/P_t$). This means that the real primary surplus, s is determined entirely by movements in real government spending. While stark, this assumption is a natural starting point for analyzing the implications of 'active' fiscal policy for optimal monetary policy, since the real primary surplus is completely independent of the level of real government debt.[3] The simplicity of the assumed fiscal policy behavior also makes it possible to derive some key results analytically.

Leeper (1991) develops a taxonomy of monetary and fiscal policy configurations that are consistent with stable real government debt and determinate inflation. Leeper labels monetary and fiscal policies as 'passive' or 'active' depending on whether or not they are constrained to respond to the level of (or disturbances to) real government debt. Leeper's taxonomy of policy configurations has shaped much of the subsequent research on monetary and fiscal policy interactions, much of it in the context of simple policy rules for monetary and fiscal policies.[4]

As will be demonstrated below, conditional on these (stark) assumptions for fiscal policy, *optimal* monetary policy will be passive, since stabilization of the real debt stock is a binding constraint on the monetary policymaker.[5]

[3] Billi and Walsh (2021) also adopt this assumption for one of their specifications of active fiscal policy.

[4] Woodford (2001, 2003) distinguishes between 'Ricardian' and 'non-Ricardian' fiscal policies, which correspond to Leeper's passive and active specifications respectively. Though Woodford's terminology is well known, Leeper's active/passive distinction is used for the remainder of this paper.

[5] It is assumed that fiscal and/or monetary policy must ensure *debt stabilization*. This requires the real value of the government debt stock to be stationary, returning to a fixed steady-state level following a disturbance. This is a stricter condition than requiring that the debt stock be sustainable, in part reflecting the difficulty of identifying sustainability from the stochastic properties of government debt. For example, on empirical grounds Bohn (2007, p1846, emphasis added) argues that "A second strategy [to assess debt sustainability] is to consider stronger conditions

2.4 Log-linearized model

Appendix A contains the derivation of the log-linearized approximation of the model around the efficient steady state. The log-linear model equations are:

$$\hat{x}_t = \mathbb{E}_t \hat{x}_{t+1} - \widetilde{\sigma} \left[\hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1} - r_t^* \right] \tag{7}$$

$$\hat{\pi}_t = \kappa \hat{x}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1} + u_t \tag{8}$$

$$\hat{d}_t = \beta^{-1} \left(\hat{d}_{t-1} - \hat{\pi}_t \right) - (1-\chi) \hat{V}_t + \zeta^{-1} \bar{g}_t \tag{9}$$

$$\hat{V}_t = -\hat{R}_t + \chi \beta \mathbb{E}_t \hat{V}_{t+1} \tag{10}$$

where ζ is the steady-state ratio of government debt to output ($\frac{d}{y}$) and the parameters $\widetilde{\sigma}$ and κ satisfy:

$$\widetilde{\sigma} \equiv \sigma (1-g)$$

$$\kappa \equiv \frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \left(\psi + \widetilde{\sigma}^{-1} \right)$$

The natural real interest rate, r^*, and cost-push shock, u_t, are given by:

$$r_t^* = \mathbb{E}_t \left[\frac{1+\psi}{1+\psi\widetilde{\sigma}} \left(\hat{A}_{t+1} - \hat{A}_t \right) - \left(\hat{\phi}_{t+1} - \hat{\phi}_t \right) - \frac{\psi}{1+\psi\widetilde{\sigma}} \left(\bar{g}_{t+1} - \bar{g}_t \right) \right]$$

$$u_t = -\frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \frac{\eta}{\eta-1} \hat{\eta}_t$$

Equations (7) and (8) are the familiar New Keynesian IS and Phillips curves (Galí, 2008; Woodford, 2003). Equation (9) is the government debt accumulation equation. Equation (10) is a log-linearized version of the no-arbitrage condition between long-term and short-term bonds.

When fiscal policy is active, the government budget constraint is a constraint on monetary policy. Variations in the monetary policy instrument (the short-term interest rate $\hat{R}$) influence the evolution of long-term debt via their effects on the price of long-term debt ($\hat{V}$) and inflation. Monetary policy must be set so that the government debt stock is stabilized.

2.5 Welfare-based loss function

Appendix B demonstrates that a loss function based on a second-order approximation to household utility is given by:

$$\mathcal{L}_t = \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \left[\hat{\pi}_{t+i}^2 + \omega \hat{x}_{t+i}^2 \right] \tag{11}$$

on policy, e.g., upper bounds on debt motivated by a limited capacity to service debt. Then *stationarity in levels is the most relevant econometric condition*, and additional restrictions may apply." In common with most of the monetary policy literature, default on government debt is ruled out. Allowing for default gives rise to a richer set of interactions between monetary and fiscal policy (see, for example, Uribe, 2006; Bi et al., 2018).

where

$$\omega = \kappa\eta^{-1}$$

Unsurprisingly, relative to the standard assumption of passive fiscal policy, active fiscal policy affects the constraints upon the monetary policymaker, but not their objectives. Following Vestin (2006), time-consistent pursuit of (11) is interpreted as 'flexible inflation targeting'.

More broadly, the focus on time-consistent optimal policy is motivated by two considerations. First, optimal monetary policy under commitment in this class of models tends to generate an extreme form of time inconsistency.[6] Second, there is substantial narrative evidence that monetary policymakers have doubts over their ability to credibly pre-commit to future policy actions (see, for example, Nakata, 2015).

2.6 A 'textbook' benchmark with passive fiscal policy

A textbook New Keynesian model with passive fiscal policy is a natural benchmark against which to assess the implications of active fiscal policy. In the textbook model, a fiscal policy reaction function ensures that primary surpluses are adjusted to ensure that the trajectory for government debt consistent with (9) is stable. The fiscal solvency condition is satisfied for any path of the price level, regardless of the actions of the monetary policymaker. Since government spending is exogenous, passive fiscal policy requires that lump sum taxes adjust to ensure that the intertemporal government budget constraint is satisfied.

In the textbook benchmark model, only the IS curve (7) and Phillips curve (8) are constraints on the monetary policymaker. Indeed, in the absence of a lower bound on the short-term interest rate, the IS curve is not a binding constraint and, in this case, the optimal time-consistent monetary policy delivers the following 'targeting rule' (Galí, 2008; Woodford, 2003):

$$\omega\hat{x}_t + \kappa\hat{\pi}_t = 0 \tag{12}$$

as will be demonstrated below.

Under passive fiscal policy, the precise formulation of the fiscal reaction function for the lump sum tax rate does not affect equilibrium outcomes for the output gap and inflation. However, for the purposes of comparison, it is assumed that the lump sum tax rate is adjusted to hold the stock of government debt constant at its steady-state level at all times: $\hat{d}_t = 0, \forall t$.

2.7 Parameter values

Table 1 shows the baseline parameter values.

The values of parameters governing preferences and technology are similar to those used by Harrison (2017). The parameter values used in that paper implies a non-negligible chance of

[6] In a similar model, Leith and Wren-Lewis (2013) demonstrate that the optimal commitment policy implies that government debt follows a random walk, allowing the policymaker to increase welfare in the near term by inducing permanent movements in debt.

encountering the zero lower bound, when the steady-state level of nominal interest rates is around 3%, as implied by the choice of β.

The parameters governing the persistence of the exogenous shocks (ρ_a, ρ_g, ρ_ϕ) are set equal to the posterior mean estimates of the analogous shocks in Burgess et al. (2013) and Del Negro et al. (2015) as appropriate. Markup shocks are assumed to be white noise ($\rho_\eta = 0$), following Burgess et al. (2013).

	Value	Source/motivation
σ	1	Log utility
β	0.9926	Steady-state annual real interest rate $\approx 3\%$
g	0.2	Sims and Wolff (2013)
ζ	2	Reinhart et al. (2012) (advanced economies, pre-crisis)
η	7.88	Rotemberg and Woodford (1997)
α	0.855	Implies $\kappa \approx 0.05$
ψ	0.55	Smets and Wouters (2007)
ρ_η	0	Burgess et al. (2013)
ρ_g	0.91	Burgess et al. (2013) (ρ_G)
ρ_A	0.96	Del Negro et al. (2015) (ρ_z)
ρ_ϕ	0.71	Burgess et al. (2013) (ρ_B)
χ	0.945 ($\equiv \bar{\chi}$)	'Average duration' variant (see text)
	0.976 ($\equiv \chi_L$)	'Long duration' variant (see text)

Table 1: Baseline parameter values

The parameter χ is important for the present study as it determines the maturity of government debt in the model. Two values for this parameter using the OECD data on the Macaulay duration of domestic government debt shown in Table 2. The baseline calibration sets the duration of government debt to four years, slightly below average from Table 2 (4.2 years). The 'long duration' variant sets χ to deliver a Macaulay duration of eight years, double the baseline duration and broadly in line with the longest duration of government debt in Table 2.[7]

[7] The steady-state Macaulay duration is given by $(1 - \beta\chi)^{-1}$ (see, for example, Chen et al. (2012) and Harrison (2017)). So if the desired Macaulay duration is M years, the required value of χ is given by $\chi = \beta^{-1}\left(1 - (4M)^{-1}\right)$, which incorporates the fact that each time period in the model is one quarter.

Country	Sample	Average	Minimum	Maximum
Austria	2000–2010	5.5	4.1	7
Denmark	2000–2010	4.7	3.7	7.3
Finland	2000–2010	2.8	2.4	3.5
France	2001–2004	4.5	4.3	4.8
Hungary	2000–2010	2.3	1.4	2.8
Italy	2000–2010	4.2	3.4	4.9
Norway	2000–2010	3.0	1.9	3.5
Spain	2000–2010	4.7	3.9	5.2
Sweden	2000–2005	2.8	2.7	3.1
United Kingdom	2000–2010	8.0	6.9	9.0
United States	2000–2010	3.5	3.4	4.0

Table 2: Macaulay duration of domestic government debt, selected countries
Notes: Macaulay duration is measured in years. Data were downloaded from OECD statistics library (https://stats.oecd.org/) on 24 November 2018.

3 Time-consistent policy without a lower bound

This section examines the behavior of the model under time-consistent optimal monetary policy.

The policymaker at date t is treated as a Stackelberg leader with respect to both private agents at date t and policymakers (and private agents) in dates $t+i, i \geq 1$. The equilibrium Markov perfect policy is one in which optimal decisions are a function only of the payoff relevant state variables in the model $\left\{\eta_t, \bar{g}_t, \hat{A}_t, \hat{\phi}_t, \hat{d}_{t-1}\right\}$. The policymaker recognizes that future allocations will satisfy time-invariant policy functions with this property. Current policy decisions affect future outcomes through their impact on the endogenous state variable, which in the context of the present model is the stock of real government debt.

To derive insights that can be studied analytically, the lower bound on the short-term bond rate is ignored. Given the quadratic objective function and fully linear constraints, the Markov perfect policy functions are linear functions of the state variables. Section 4 examines the behavior of the model when the presence of the lower bound on the short-term bond rate is accounted for.

3.1 The optimal policy problem

The policymaker's optimization problem is characterized by the following Lagrangean:

$$\begin{aligned}\widetilde{\mathcal{L}}_t =& \frac{1}{2}\left[\hat{\pi}_t^2 + \omega \hat{x}_t^2\right] - \mu_t^x \left[\hat{x}_t - \mathbb{E}_t \hat{x}_{t+1} + \sigma (1-g)\left(\hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1} - r_t^*\right)\right] - \mu_t^\pi \left[\hat{\pi}_t - \kappa \hat{x}_t - \beta \mathbb{E}_t \hat{\pi}_{t+1} - u_t\right] \\ & - \mu_t^d \left[\hat{d}_t - \beta^{-1}\left(\hat{d}_{t-1} - \hat{\pi}_t\right) + (1-\chi)\hat{V}_t - \zeta^{-1}\bar{g}_t\right] - \mu_t^V \left[\hat{V}_t + \hat{R}_t - \chi\beta \hat{V}_{t+1}\right] + \beta \mathbb{E}_t \widetilde{\mathcal{L}}_{t+1}\end{aligned}$$

The first order conditions for minimization are:

$$0 = \hat{\pi}_t - \mu_t^\pi - \beta^{-1}\mu_t^d \tag{13}$$

$$0 = \omega\hat{x}_t - \mu_t^x + \kappa\mu_t^\pi \tag{14}$$

$$\begin{aligned} 0 = \mu_t^x \left[\frac{\partial \mathbb{E}_t \hat{x}_{t+1}}{\partial \hat{d}_t} + \sigma(1-g)\frac{\partial \mathbb{E}_t \hat{\pi}_{t+1}}{\partial \hat{d}_t} \right] + \beta\mu_t^\pi \frac{\partial \mathbb{E}_t \hat{\pi}_{t+1}}{\partial \hat{d}_t} \\ - \mu_t^d + \chi\beta\mu_t^V \frac{\partial \mathbb{E}_t \hat{V}_{t+1}}{\partial \hat{d}_t} + \beta \frac{\partial \mathbb{E}_t \widetilde{\mathcal{L}}_{t+1}}{\partial \hat{d}_t} \end{aligned} \tag{15}$$

$$0 = -(1-\chi)\mu_t^d - \mu_t^V \tag{16}$$

$$0 = -\sigma(1-g)\mu_t^x - \mu_t^V \tag{17}$$

Derivatives of $\mathbb{E}_t \widetilde{\mathcal{L}}_{t+1}$ can be eliminated by noting that:

$$\frac{\partial \widetilde{\mathcal{L}}_t}{\partial \hat{d}_{t-1}} = \beta^{-1}\mu_t^d \Rightarrow \frac{\partial \mathbb{E}_t \widetilde{\mathcal{L}}_{t+1}}{\partial \hat{d}_t} = \beta^{-1}\mathbb{E}_t \mu_{t+1}^d$$

The linear-quadratic nature of the problem and the focus on Markov-perfect equilibria implies that equilibrium allocations are linear functions of the state variables. This means that:

$$\frac{\partial \mathbb{E}_t Z_{t+1}}{\partial \hat{d}_t} \equiv F_Z$$

for some (fixed) coefficient F_Z for any variable Z.

These observations can be used to write (15) as:

$$\mu_t^d = [F_{\hat{x}} + \sigma(1-g)F_{\hat{\pi}}]\mu_t^x + \beta F_{\hat{\pi}}\mu_t^\pi + \chi\beta F_{\hat{V}}\mu_t^V + \mathbb{E}_t \mu_{t+1}^d \tag{18}$$

A straightforward, but tedious, application of the method of undetermined coefficients can be used to characterize the solutions of the coefficients $\left\{F_{\hat{\pi}}, F_{\hat{x}}, F_{\hat{V}}, F_{\hat{d}}\right\}$. Appendix C contains the details and demonstrates that (conditional on solutions for $F_{\hat{\pi}}$ and $F_{\hat{d}}$):

$$\begin{aligned} F_{\hat{x}} &= \kappa^{-1}F_{\hat{\pi}} - \kappa^{-1}\beta F_{\hat{\pi}}F_{\hat{d}} \\ F_{\hat{V}} &= (1-\chi)^{-1}\beta^{-1} - (1-\chi)^{-1}\beta^{-1}F_{\hat{\pi}} - (1-\chi)^{-1}F_{\hat{d}} \end{aligned}$$

Solving for $F_{\hat{\pi}}$ and $F_{\hat{d}}$ involves solving a coupled system of quadratic equations. The quadratic equation for $F_{\hat{\pi}}$ has a solution (conditional on $F_{\hat{d}}$) given by the following function:

$$F_{\hat{\pi}} = m\left(F_{\hat{d}}\right) \equiv \frac{1+\chi-(1+\beta\chi)F_{\hat{d}}}{\beta\left(\frac{\omega}{\kappa\Xi}\left(1-\beta F_{\hat{d}}\right)+\frac{\kappa}{\Xi}\right)^{-1} + (1-\chi)(\kappa\widetilde{\sigma})^{-1}\left(1-\beta F_{\hat{d}}\right)} \tag{19}$$

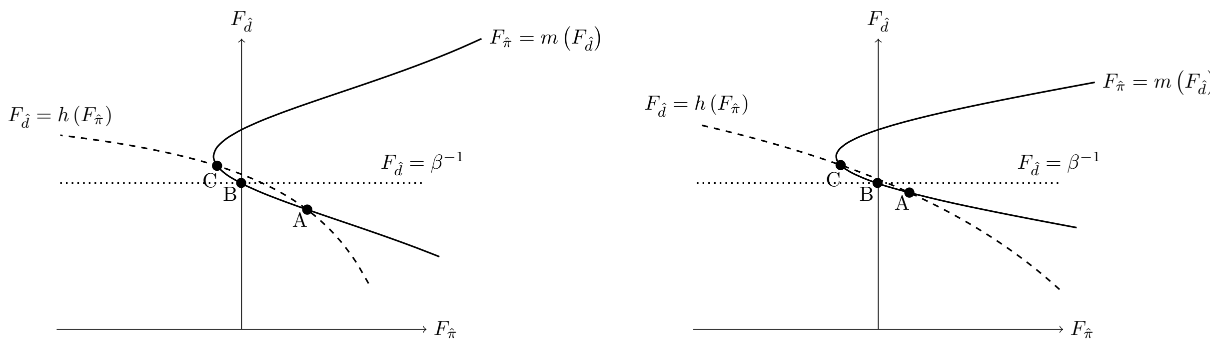
where $\Xi \equiv (1-\chi)\widetilde{\sigma}^{-1} + \kappa\beta^{-1}$.

The quadratic equation for $F_{\hat{d}}$ can be factorized. One solution is shown to be $F_{\hat{d}} = \beta^{-1}$.

Equation (19) then implies that $F_{\hat{\pi}} = \frac{\kappa}{\Xi\beta}\left(1-\beta^{-1}\right) < 0$. The other solution, conditional on $F_{\hat{\pi}}$, satisfies:

$$F_{\hat{d}} \equiv h\left(F_{\hat{\pi}}\right) = \frac{\left(1+\kappa\widetilde{\sigma}\left[\beta\left(1-\chi\right)\right]^{-1}\right)F_{\hat{\pi}} - \kappa\widetilde{\sigma}\left[\beta\left(1-\chi\right)\right]^{-1}}{F_{\hat{\pi}} - \chi\kappa\widetilde{\sigma}\left(1-\chi\right)^{-1}} \tag{20}$$

The model has a unique, stable ('determinate') solution if the debt stock returns to steady state from any initial condition, which requires that $F_{\hat{d}} < 1$. Blake and Kirsanova (2012) demonstrate the the presence of endogenous state variables can generate multiple stable Markov perfect equilibria for time-consistent linear-quadratic optimal policy problems.[8]



(a) Baseline model ($\chi = 0.945$) **(b)** 'Long duration' variant ($\chi = 0.976$)

Figure 1: Solutions for $F_{\hat{\pi}}$ and $F_{\hat{d}}$

Notes: Each panel plots the functions m and h defined by equations (19) and (20) respectively. Panel (a) shows the baseline model and panel (b) shows the variant with long-duration government debt. In each case, point A denotes the solution for the coefficients $F_{\hat{\pi}}$ and $F_{\hat{d}}$ consistent with the unique Markov perfect equilibrium.

Figure 1 provides a graphical analysis of the candidate equilibria for $F_{\hat{\pi}}$ and $F_{\hat{d}}$ for the baseline model (panel (a)) and 'long duration' parameterization (panel (b)). In both cases there are three candidate equilibria, labeled A, B and C. Of these, B and C generate an explosive trajectory for debt (since $F_{\hat{d}} \geq \beta^{-1} > 1$). Point A is the unique stable solution and is the equilibrium used for the experiments in the next subsections.[9]

Comparing panels (a) and (b) reveals that intersection point A in panel (b) lies to the North-West of the corresponding intersection point in panel (a). So the equilibrium trajectory for government debt is more persistent in the long duration variant. The longer-term debt variant also has the property that inflation depends less strongly on previously accumulated debt.

Appendix C demonstrates that the first order conditions can be combined into a targeting rule:

$$\omega\hat{x}_t + \kappa\hat{\pi}_t = \Xi\mu_t^d \tag{21}$$

[8] Indeed, one of their motivating examples adds (one period) government debt to a textbook New Keynesian model. Aside from one-period debt, the model considered by Blake and Kirsanova (2012) differs from the present model in two important respects: taxation has a distortionary effect on labor supply; and the tax rate is adjusted in response to previously accumulated government debt. There is a unique Markov-perfect equilibrium if real taxes are held fixed in the Blake and Kirsanova (2012) model (as in the baseline model considered in the present paper).

[9] Figure 1 provides a 'local' analysis, plotting the m and h functions in the vicinity of the intersection points. Figure 9 in Appendix D expands the range over which the functions are plotted to present a global picture, demonstrating that no other candidates exist.

Equation (21) reveals that the value of the multiplier on the government debt accumulation equation, μ_t^d, affects the optimal achievable combination of the output gap and inflation. If government debt accumulation is not a constraint on monetary policy (as is the case under passive fiscal policy) then $\mu_t^d = 0, \forall t$ and (21) collapses to the targeting rule in the New Keynesian benchmark model, (12), as previously claimed.

3.2 Impulse responses

This section examines the impulse responses of the model to shocks, comparing the baseline ('average duration') model to the textbook New Keynesian benchmark and to the 'long duration' variant.

The responses of the par value of debt, $\hat{d}$ and the long-term bond rate, $\hat{\mathcal{R}}$, are shown. The long-term bond rate is computed as the the yield to maturity:

$$\hat{\mathcal{R}}_t = \chi\beta\mathbb{E}_t\hat{\mathcal{R}}_{t+1} + (1-\chi\beta)\,\hat{R}_t$$

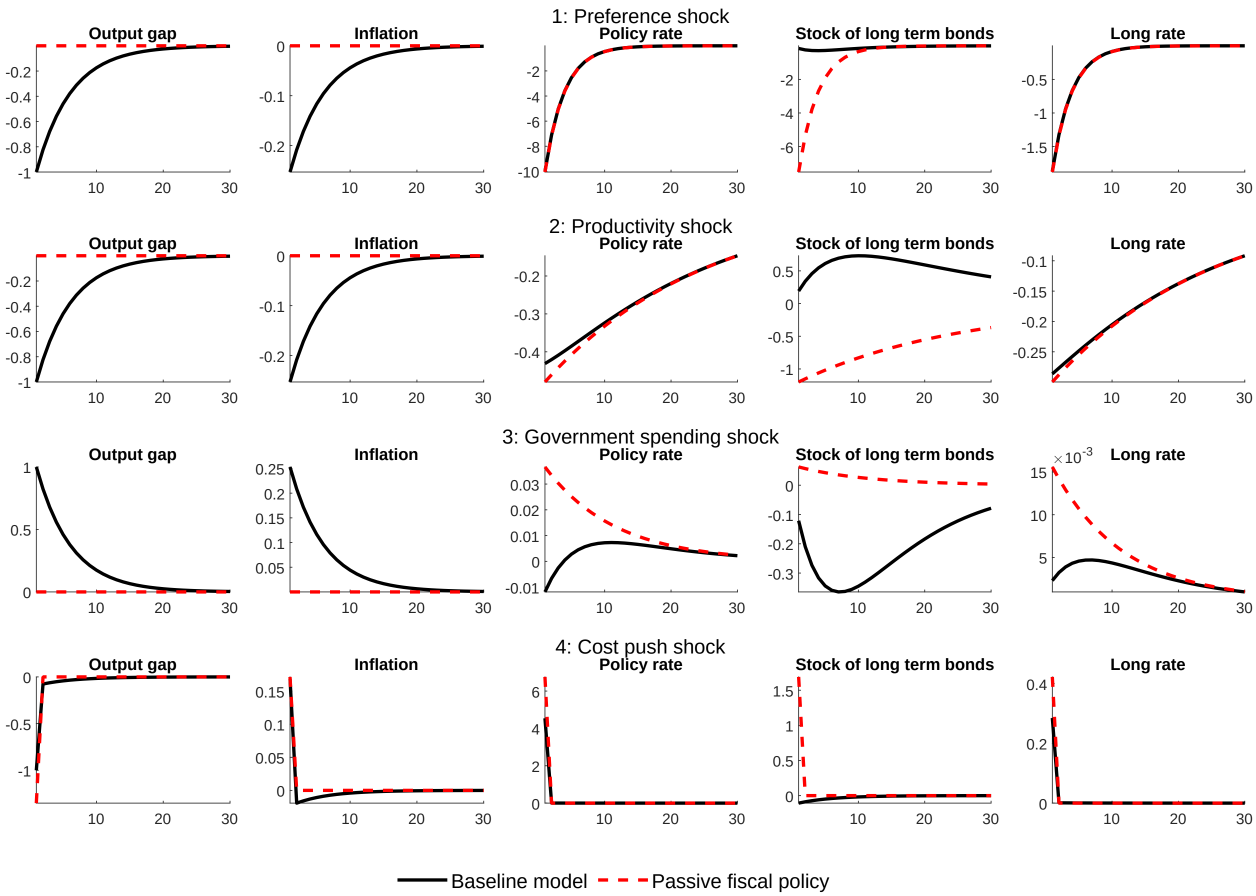


Figure 2: Responses to shocks: baseline model and textbook model with passive fiscal policy

Notes: Impulse responses to shocks to the baseline model (solid black lines) and the 'textbook' variant with passive fiscal policy described in Section 2.6 (dashed red lines). The scale of all shocks is normalized to deliver a 1% response of the output gap in the baseline variant. Policy rate and long rate plotted in annualized units. All variables are shown in percentage point deviations from steady state.

Figure 2 plots responses of the model to a positive innovation in each of the shocks (solid black lines) alongside the responses of the textbook New Keynesian model with passive fiscal policy (red dashed lines). To aid the comparison, the scale of each shock is chosen so that it has an identical impact effect on the output gap in the baseline model (1 percentage point in absolute terms).

Two key results emerge from Figure 2.

First, the so-called "divine coincidence" (Blanchard and Galí, 2007) disappears when fiscal policy is active. The divine coincidence result refers to the fact that optimal time-consistent policy in the textbook model with passive fiscal policy (dashed red lines) achieves complete stability of both the output gap and inflation in response to government spending, preference and productivity shocks (rows 1–3). So, under passive fiscal policy, the relevant targeting criterion (12) can be achieved with $x_t = \pi_t = 0, \forall t$. This is feasible because these shocks affect the economy solely through their effects on the natural real interest rate, r^* and time-consistent policy tracks movements in r_t^* with the short-term nominal interest rate $\hat{R}_t$, delivering a zero output gap and (hence) zero inflation.[10]

However, this policy response is not feasible when fiscal policy is active. Tracking exogenous movements in r^* with the nominal interest rate would not stabilize the government debt stock.[11] A necessary condition for debt stabilization is that the interest rate responds to the debt stock. Because the nominal interest rate must respond to the debt stock, it cannot fully insulate the economy from the effects of exogenous changes in r^* and so costly fluctuations in the output gap and inflation cannot be avoided.

The second key result is that the dynamic responses of inflation and the output gap to government spending, preference and productivity shocks are perfectly correlated (rows 1–3). In all cases, the responses of inflation and the output gap satisfy the targeting criterion (21). Appendix C.1 shows that, *in the absence of cost push shocks*, this targeting criterion can be combined with the first order condition for government debt (18) and the IS curve (7) to deliver a second order difference equation for inflation. Appendix C.1 further demonstrates that the solution to that difference equation implies that inflation and the output gap follow AR(1) processes given by:

$$\hat{\pi}_{t+1} = F_{\hat{d}} \hat{\pi}_t$$
$$\hat{x}_{t+1} = F_{\hat{d}} \hat{x}_t$$

for $t \geq 1$.

This result means that both inflation and the output gap follow identical (to scale) AR(1) processes in response to preference, productivity and government spending shocks. In Figure 2, the initial response of the output gap in period 1 is equal to unity in all cases, by virtue of the normalization assumption. The fact that the autoregressive parameter is equal to $F_{\hat{d}}$ is an important

[10]See Galí (2008) for a complete analysis of optimal monetary policy in the textbook model.

[11]An informal proof by contradiction is as follows. Suppose that tracking exogenous movements in r^* with $\hat{R}$ does stabilize inflation so that $\hat{\pi}_t = 0, \forall t$. Note now that equation (10) implies that the value of long-term debt will be a function of the exogenous fluctuations in r^* (since $\hat{R}_t = r_t^*$). Inspecting (9) reveals than an exogenous impulse to $\hat{V}_t$ with $\hat{\pi}_t = 0, \forall t$ generates an explosive trajectory for $\hat{d}_t$ given that $\beta^{-1} > 1$. So full stabilization of inflation by tracking r^* with the policy rate does not also ensure that the debt stock returns to steady state.

result, discussed below.

Figure 3 compares responses from the baseline model (solid black lines) with the 'long duration' variant (red black lines).

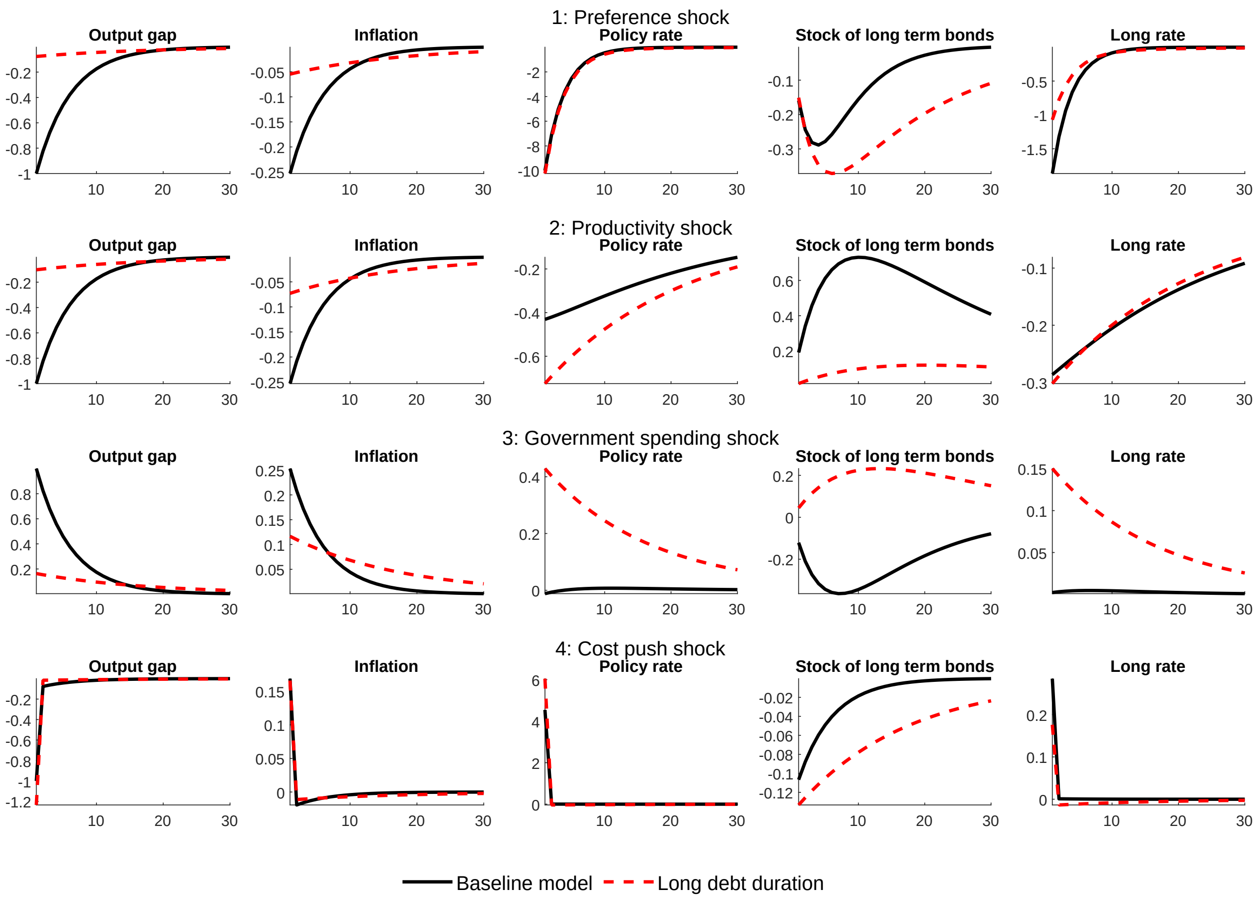


Figure 3: Responses to shocks in baseline model and 'long duration' variant

Notes: Impulse responses to shocks to the baseline model (solid black lines) and variant with long debt duration (dashed red lines). The scale of all shocks is normalized to deliver a 1% response of the output gap in the baseline variant. Policy rate and long rate plotted in annualized units. All variables are shown in percentage point deviations from steady state.

A key result is that the output gap and inflation are better stabilized in response to preference, productivity and government spending shocks for the long duration variant (rows 1–3). The responses of the output gap and inflation are also more persistent in this variant, compared with the baseline calibration. A corollary of the smaller responses of inflation and the output gap is that the policy rate must adjust by more in response to each of these shocks.

The greater persistence of the output gap and inflation responses in the long-duration variant follows from the result that these variables both follow AR(1) processes with parameter $F_{\hat{d}}$. As shown in Figure 1, $F_{\hat{d}}$ is larger for the long duration variant.

These results relate to previous findings that optimal time-consistent policy may exhibit a "debt stabilization bias". Leith and Wren-Lewis (2013) study jointly optimal monetary and fiscal policy in a similar model.[12] In their model, optimal time consistent policy rapidly returns the government debt stock back to steady state following a shock. That in turn requires large movements in output and inflation to achieve the required change in real debt values. Leeper et al. (2019) study the responses of a non-linear model with optimal monetary and fiscal policy for alternative assumptions about government debt duration. They also find that longer duration debt dampens the responses of macroeconomic variables to shocks under optimal policy.

The presence of long-term debt reduces the degree to which large and immediate movements in inflation will reduce the real value of government debt. Instead, it is possible to stabilize the debt stock through smaller but more persistent movements in inflation (which may nonetheless have a sizable effect on the market value of debt). The presence of government debt with a longer duration allows even smaller, and even more persistent, changes in inflation to be used to bring the debt stock back to steady state following a shock.

3.3 Welfare implications

Figure 4 examines the welfare implications of the shocks across model variants. In each panel an approximation to the square root of the loss $\mathcal{L}_t$ is plotted for each model variant.[13] The square root transformation facilitates comparison of the model variants, but obviously understates the true welfare differences between them.

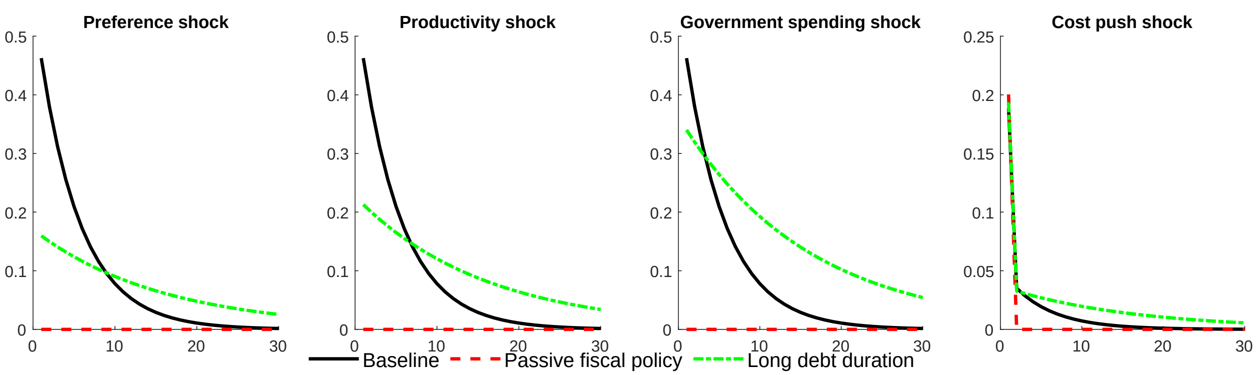


Figure 4: Normalized losses for each model variant

Notes: Normalized losses are defined as $\sqrt{\hat{\mathcal{L}}_t}$, where $\hat{\mathcal{L}}_t \equiv \mathbb{E}_t \sum_{i=0}^{H} \beta^i \left[\hat{\pi}_{t+i}^2 + \omega \hat{x}_{t+i}^2\right]$, a finite-horizon approximation to the loss defined in (11). The horizon H is set to 200.

As expected, the divine coincidence result for the textbook model with passive fiscal policy implies that there are no welfare losses from preference, productivity or government spending shocks in that variant (red dashed lines). For these shocks, losses are, initially, much smaller for the long duration variant (green dash-dot lines) compared with the baseline model (solid black lines). As the shocks dissipate, however, losses are larger for the long duration variant.

These results are consistent with the observation from Figure 3 that inflation and the output

[12]The main differences are the inclusion of distortionary taxation and the assumption that the government finances its activities using one-period debt.

[13]The approximation is to compute losses over a finite horizon, H: $\hat{\mathcal{L}}_t \equiv \mathbb{E}_t \sum_{i=0}^{H} \beta^i \left[\hat{\pi}_{t+i}^2 + \omega \hat{x}_{t+i}^2\right]$. Results are shown for $H = 200$, but are not sensitive to this assumption.

gap exhibit muted and persistent responses to preference, productivity and government spending shocks in the long duration variant. The presence of longer duration debt increases the extent to which the monetary policymaker is able to smooth welfare losses across time. The existence of longer duration debt means that bond prices ($\hat{V}$) can be materially affected by relatively small, but very persistent movements in inflation. The optimal monetary policy exploits this mechanism to mitigate welfare losses in the near term, at the expense of larger losses in the longer term.

For cost-push shocks, the welfare ranking across model variants is, initially, reversed. On impact, losses are greatest for the textbook model with passive fiscal policy and the long duration variant generates larger losses than the baseline model. Because the cost-push shock has no persistence, losses under passive fiscal policy are zero from period 2 onwards.[14]

The smaller initial losses with active fiscal policy can be understood by observing that inflation is below target from period 2 onwards in these variants (Figures 2 and 3). Relative to the textbook model with passive fiscal policy, therefore, the Phillips curve trade-off in the *first* period is improved, because inflation expectations are lower. This allows the policymaker to achieve a less costly mix of inflation and the output gap in period 1. From period 2 onwards, losses are higher than under passive fiscal policy because the requirement that monetary policy stabilizes the government debt stock requires a persistent deviation of inflation and the output gap. The net effect on the present value loss, $\mathcal{L}_1$ depends on whether the gains from the improved trade-off in period 1 outweigh the future losses.

Figure 5 shows outcomes in period 1 for the baseline model and the textbook model with passive fiscal policy. The solid black lines show the baseline model and gray dashed lines show the textbook model. In both cases, the inflation-output gap trade-off is determined by the intersection of an upward-sloping Phillips curve (8) and a downward-sloping optimal policy criterion. The optimal policy criterion under passive fiscal policy is given by (12). Appendix C.1 demonstrates that the targeting criterion in the baseline model can be written as:

$$\omega \hat{x}_t + \left(\kappa - \beta \Xi F_{\hat{\pi}} \Omega^{-1}\right) \hat{\pi}_t = \Omega^{-1} \mathbb{E}_t \left(\omega \hat{x}_{t+1} + \kappa \hat{\pi}_{t+1}\right) \tag{22}$$

where $\Omega = \chi + F_{\hat{\pi}} \left(1 - (1-\beta) \frac{1-\chi}{\kappa \tilde{\sigma}}\right) > 0$.

As noted above, the optimal response in the baseline model implies that inflation is negative from period 2 onward. The spike in inflation in period 1 reduces the real value of debt and negative inflation thereafter helps to stabilize real debt. The negative inflation in period 2 implies that the Phillips curve in the baseline model lies below the Phillips curve under passive fiscal policy.[15]

Other things equal, a downward shift in the Phillips curve allows the monetary policymaker to achieve a better outcome (a smaller increase in inflation and a smaller reduction in the output gap). The optimal combination of the output gap and inflation that is chosen, however, depends on the

[14] The textbook model has no endogenous state variables, so the absence of any disturbance from period 2 onward allows complete stabilization of the output gap and inflation (Galí, 2008, Figure 5.1).

[15] The upward-sloping black line in Figure 5 lies below the upward-sloping gray dashed line. Inflation expectations under passive fiscal policy are zero ($\hat{\pi}_2 = 0$).

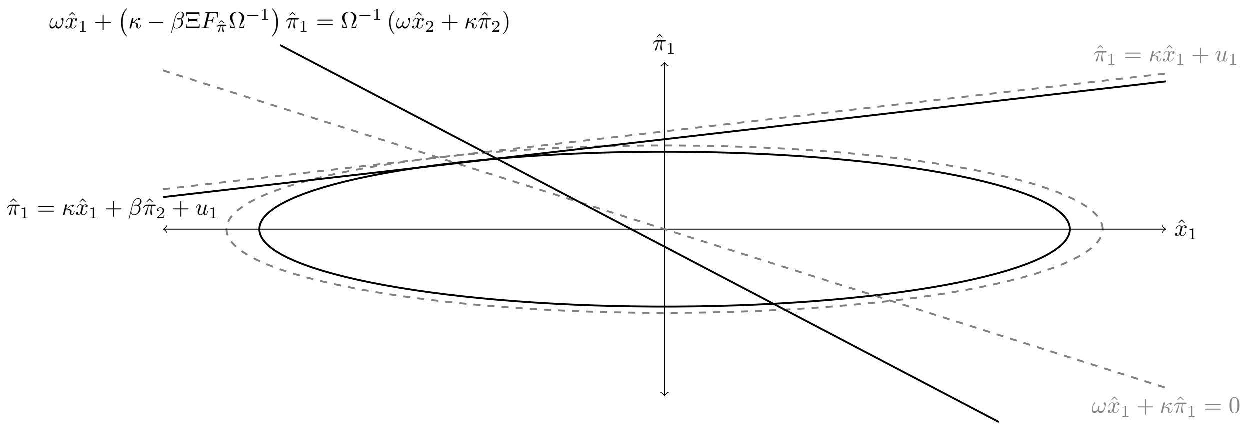


Figure 5: Optimal responses to cost-push shock in period 1

Notes: The diagram shows the optimal policy decisions when a cost-push shock arrives in period 1, for the baseline model (solid black lines) and the textbook model with passive fiscal policy described in Section 2.6 (dashed gray lines). The upward-sloping lines are the Phillips curve, (8), conditional on expected outcomes in period 2. The downward-sloping lines are the optimal trade-off criteria: equation (22) for the baseline model and equation (12) for the passive fiscal policy variant. The ellipses are iso-loss lines, tracing out combinations of the output gap ($\hat{x}$) and inflation ($\hat{\pi}$) that deliver the same value of the period 1 loss, $L_1 \equiv \hat{\pi}_1^2 + \omega\hat{x}_1^2$.

trade-off criterion (22). Relative to passive fiscal policy, the trade-off criterion in the baseline model features a downward shift and an clockwise tilt. The downward shift reflects the fact that the right hand side of (22) is negative.[16] The tilt occurs because $\left(\kappa - \beta\Xi F_{\hat{\pi}}\Omega^{-1}\right) < \kappa$. The resulting optimal combination of the output gap and inflation features a similar inflation rate to the New Keynesian case, but a noticeably smaller negative output gap.

The ellipses in Figure 5 are iso-loss curves, showing combinations of the output gap and inflation in period 1 that satisfy $\hat{\pi}_1^2 + \omega\hat{x}_1^2 = L$. The ellipse for the baseline model (solid black line) lies within the ellipse for the model with passive fiscal policy (dashed gray), so that the losses incurred *in period 1* are lower in the baseline model. While per-period losses are larger from period 2 onwards, the gain in period 1 is sufficient for the discounted loss $\mathcal{L}_1$ to be lower in the baseline model than under passive fiscal policy.

4 Time-consistent monetary policy at the lower bound

The analysis in this section accounts for the existence of a lower bound on the short-term bond rate. To do so, the model is solved using projection methods. To reduce the number of state variables (and hence the dimensionality of the problem), the variances of government spending shocks and productivity shocks are set to zero.[17]

[16] The output gap and inflation in period 2 are both negative.

[17] One motivation for ignoring government spending shocks is that the precise nature of the effect of government spending on debt is determined by the particular (extreme) assumption that real lump sum taxes are held fixed. A motivation for ignoring productivity shocks is that productivity and preference shocks both influence the model only through their effects on the natural rate of interest, r^*. If these two shocks had identical persistence ($\rho_A = \rho_\phi$), then their effects are identical, up to scale.

These simplifications imply that the natural real interest rate, r^* can be treated as a 'primitive' disturbance, which is assumed to follow a first-order autoregressive process:

$$r_t^* = \rho_r r_{t-1}^* + \sigma_r \varepsilon_t^r \tag{23}$$

Similar, the cost push shock process is given by:

$$u_t = \rho_u u_{t-1} + \sigma_u \varepsilon_t^u \tag{24}$$

Both ε^r and ε^u are i.i.d., normally distributed and have unit variance. Values for the shock parameters are shown in Table 3.

	Description	Value
ρ_r	Natural rate persistence	0.85
$100 \times \sigma_r$	Natural rate shock standard deviation	0.225
ρ_u	Cost-push shock persistence	0
$100 \times \sigma_u$	Cost-push shock standard deviation	0.135

Table 3: Shock process parameters

Again, the parameter values are set with reference to the assumptions in Harrison (2017). Relative to that paper, the variance of the disturbance to r^* is slightly smaller.[18] The assumed persistence of the r^* process is close to the average persistence of the shock processes (productivity, preference and government spending) used in Section 3.

4.1 Optimal policy problem and solution

The policymaker's optimization problem is the same as considered in Section 3.1, with the addition of a constraint on the short-term bond rate:

$$\hat{R}_t \geq \ln \beta \tag{25}$$

where the lower bound on the nominal interest rate is assumed to be zero.[19]

The first order conditions (13)–(16) are unchanged. The first order condition for the short-term nominal rate becomes

$$0 = -\widetilde{\sigma}\mu_t^x - \mu_t^V - \mu_t^Z$$

where μ_t^Z is the Lagrange multiplier on the constraint (25). The fact that the bound on the policy rate binds occasionally gives rise to a contemporary slackness condition reported in Appendix E.1.

[18] One difference between the model specification used here and that in Harrison (2017) is that the effective slope of the IS curve is equal to $\widetilde{\sigma} = \sigma(1-g) < \sigma$. This means that the effect of policy rate changes on aggregate demand is smaller, making the stabilization problem more challenging in the presence of the zero lower bound.

[19] Variables in the model are measured relative to steady state, so $\ln \beta$ measures the log difference between the steady state gross nominal interest rate $R = \beta^{-1}$ and a gross nominal rate of $R = 1$ (corresponding to a net nominal interest rate of zero).

The model is solved using projection methods. To reduce the dimensionality of the state space, the stochastic processes (23) and (24) are approximated using finite state Markov processes with transition matrices derived using the Rouwenhorst (1995) method.[20] The model is solved using policy function iteration. This approach is facilitated by using a good initial 'guess' for the policy functions, which is provided by the piecewise linear solution algorithm for time-consistent optimal policy subject to instrument constraints in Harrison and Waldron (2021). A detailed description of the solution approach is provided in Appendix E.

4.2 Outcomes at the lower bound

The behavior of the model at the lower bound is explored using a recessionary scenario. In period 0, the model is assumed to be at its deterministic steady state. In period 1, the natural real interest rate is initialized at a negative value (-4% on an annualized basis) and is assumed to follow the process (23) (with $\varepsilon_t^r = 0, t = 1, \dots$). The values of the cost-push state are set to zero throughout the simulation ($u_t = 0, t = 1, \dots$). Conditional on the initial value of the natural rate, r_1^*, the exogenous states $\{u_t, r_t^*\}$ follow their most likely paths.[21] However, in each periods outcomes for the endogenous variables account for the *risk* that future shocks arrive, including those that would prolong the time spent at the lower bound.

Figure 6 shows the effects of the recessionary scenario. The solid black lines show the baseline model, the red dashed lines show the variant with long duration debt and the gray lines show the textbook model with passive fiscal policy. The dotted line in the top left panel shows the trajectory of the natural real interest rate, r^*.

Relative to the textbook model with passive fiscal policy (gray lines), active fiscal policy reduces the scale of the recession and allows the short-term policy rate to lift off from the zero bound earlier. The deflationary effect of the recession increases the real value of government debt. Other things equal, this increases inflation expectations, as higher future inflation will be required to stabilize the real debt stock. This mechanism reduces expected real interest rates, stimulating spending and supporting inflation. In turn, that mitigates the recessionary effects of the fall in the natural real interest rate.

Comparing the results from the baseline and 'long duration' variants reveals that longer duration debt is associated with a (slightly) smaller recession and a later liftoff from the zero bound. The results in Section 3.2 reveal that, away from the zero bound, optimal policy is able to better stabilize the output gap and inflation with long duration debt. However, achieving this improved stabilization performance requires larger movements in the policy rate.

This implies that, with long duration debt, the zero bound is a more binding constraint on the setting of the policy rate required to deliver smaller output and inflation responses to a recessionary

[20] Kopecky and Suen (2010) demonstrate that this approach generates accurate approximations to autoregressive process with high persistence. In the context of the present model, this reduces the computational burden of analyzing the case in which shocks to the natural rate of interest r^* are very persistent.

[21] Simulations are constructed using linear interpolation of the policy functions. Linear interpolation is also applied to the natural rate, r^*, so that the trajectory for r^* satisfies $r_t^* = \rho_r^{t-1} r_1^*$.

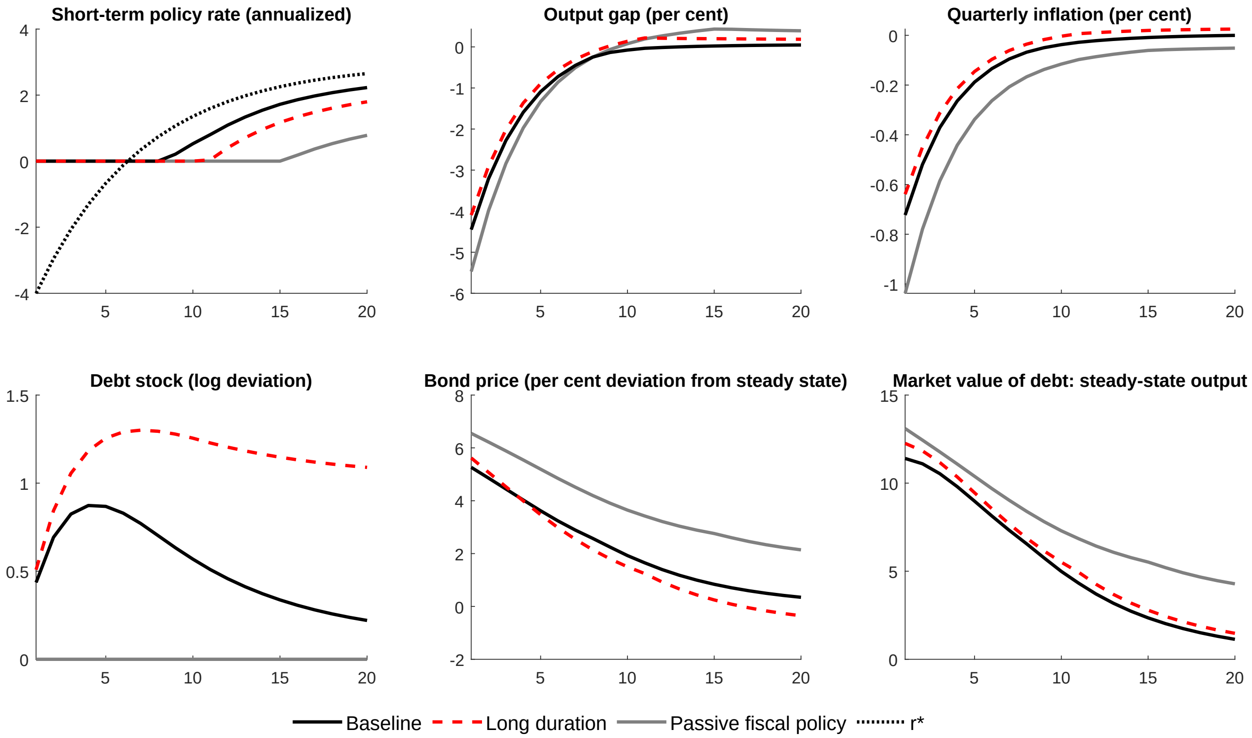


Figure 6: A recessionary scenario that causes the ZLB to bind

Notes: The panels show outcomes from simulations of the baseline model (solid black lines), the variant with long-duration debt (dashed red lines) and the textbook model with passive fiscal policy described in Section 2.6 (gray lines). In each case, the simulation is constructed from the policy functions solved by projection methods. The initial value of the natural rate state r_1^* (plus the deterministic steady-state interest rate) is set to -4% on an annualized basis. Thereafter r^* follows the process (23), with shocks set to their most likely value of zero, $\varepsilon_t^r = 0, t = 2, \ldots$. The cost-push state is set equal to its most likely value, $u_t = 0, t = 1, \ldots$. The initial debt stock is at the deterministic steady state: $\hat{d}_0 = 0$.

shock. For the particular shock examined in Figure 6, the net effect is a slight improvement in output and inflation stabilization. However, this improvement requires the policy rate to remain at the zero bound for an additional two quarters compared with the baseline model.

These effects are also apparent in the simulated distributions of key variables, for which Table 4 provides a summary.[22]

The results for the textbook model with passive fiscal policy demonstrate the familiar "deflation bias" (Eggertsson, 2006): under time-consistent policy, the zero lower bound induces a downward skew in the distributions for inflation and the output gap, both of which have a negative mean. The mean of the policy rate is below the deterministic steady-state value of 3%, as the downward skew in the distribution of expected inflation dominates the positive effect on the mean from truncation of the distribution at zero.

In contrast, the average policy rate is at or above the deterministic steady-state value of 3% for the variants of the model with active fiscal policy. In these cases, the truncation effect of the zero bound is dominant, pushing up on the mean policy rate. One driver of this result is that the policy rate is more variable when fiscal policy is active, particularly for the long duration debt variant

[22]Each model variant was simulated for 260,000 periods, with the first 10,000 periods discarded.

	Active fiscal policy				Passive fiscal policy	
	Baseline		Long duration		'Textbook model'	
	Mean	Std dev	Mean	Std dev	Mean	Std dev
Quarterly inflation, %	0.01	0.17	0.04	0.14	-0.08	0.15
Output gap, %	0.00	0.80	0.01	0.66	-0.01	0.68
Annualized policy rate, %	3.0	2.9	3.1	3.4	2.7	3.5
Debt stock, % deviation from SS	0.18	0.23	1.3	0.31	0	0
Loss per period $\left(\hat{\pi}_t^2 + \omega \hat{x}_t^2\right)$	0.034	–	0.023	–	0.032	–
When at ZLB:	0.057	–	0.023	–	0.059	–
When not at ZLB:	0.026	–	0.023	–	0.010	–
ZLB incidence, %	24		32		45	

Table 4: Summary statistics from alternative model variants

(see Section 3.2 and Table 4). Higher variability of the policy rate increases the truncation effect, other things equal.

Another driver of this result is the distribution of debt under active fiscal policy. Table 4 shows that average debt is above the deterministic steady-state for both the baseline and long duration model variants. This is because recessionary shocks that cause the lower bound to bind generate increases in the debt stock via the debt deflation mechanism described above (Figure 6). As there is no upper bound on the policy rate, the debt deflation mechanism does not operate in reverse for large expansionary shocks and the debt distribution shifts to the right (relative to a symmetric distribution around the deterministic steady state).

The implications of initial debt levels for the responses to an expansionary shock are explored in Figure 7. The top panel shows the responses of the baseline model to a scenario in which the initial level of (annualized) r^* is 1pp above steady state under two assumptions for the initial stock of debt (d). The solid black lines show a case in which the initial stock of debt is equal to the mean of the stochastic distribution (from table 4). The dashed red lines show the case in which the initial debt level is equal to the deterministic steady state. The bottom panel repeats this experiment for the variant of the model with long duration debt.

Figure 7 shows that the expansionary scenario generates a larger positive output gap and more inflation when the initial level of debt is at its stochastic mean, compared to the case in which the initial debt level is at the deterministic steady state. When debt is relatively high, additional inflation is required to stabilize the real debt stock. As a result, the rightward shift in the distribution of debt generates a rightward shift in the distributions of the output gap and inflation. Indeed, average inflation and output gaps are slightly *positive* under active fiscal policy.[23] Unsurprisingly, the rightward shift in these distributions is particularly evident for the long duration variant, for which debt is higher on average.

The results so far indicate that, relative to the textbook model with passive fiscal policy, active fiscal policy generates a rightward shift in the distributions of the output gap and inflation. As

[23] The mean output gap for the baseline model is slightly positive but rounds to zero to two decimal places, as reported in Table 4.

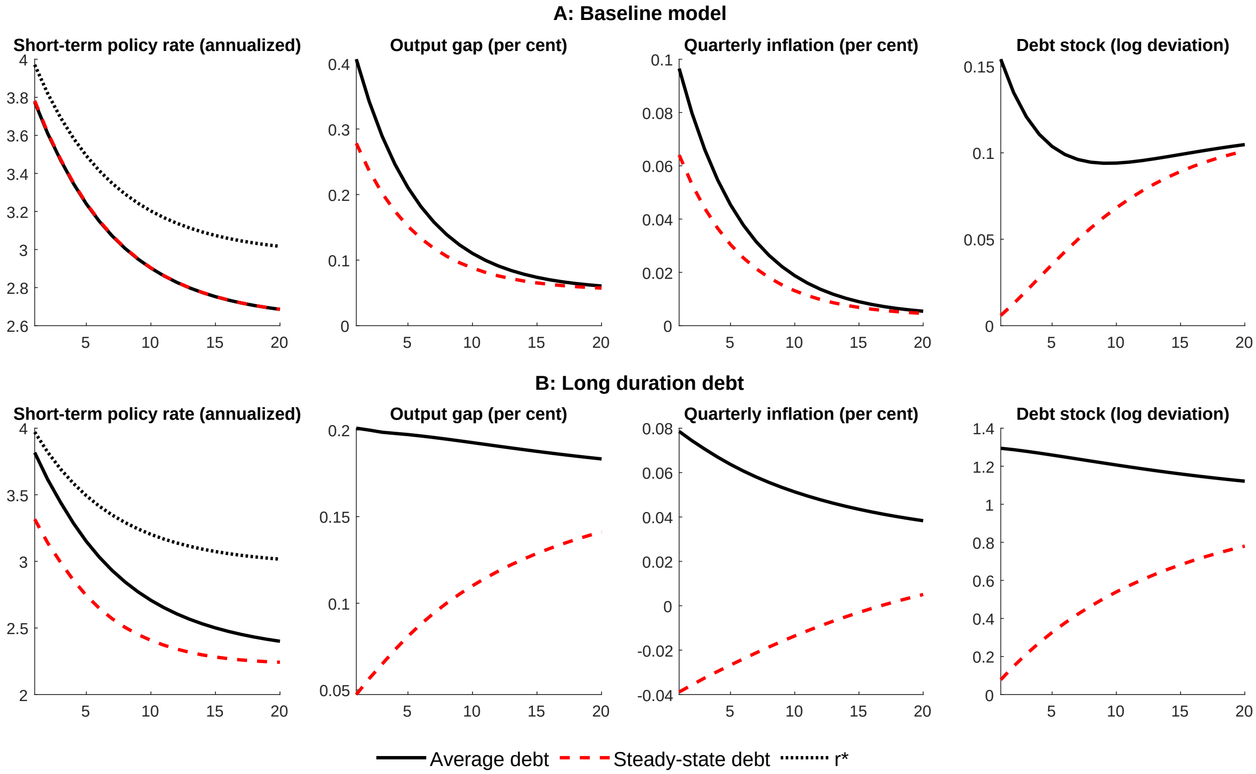


Figure 7: An expansionary shock with for different assumptions about initial debt

Notes: The top panels (A) show outcomes from simulations of the baseline model. The lower panels (B) show outcomes from simulations of the variant with long-duration debt. In all cases, the simulation is constructed from the policy functions solved by projection methods. The initial value of the natural rate state r_1^* is set to 1% above the deterministic steady-state (on an annualized basis). Thereafter r^* follows the process (23), with shocks set to their most likely value of zero, $\varepsilon_t^r = 0, t = 2, \ldots$. The cost-push state is set equal to its most likely value, $u_t = 0, t = 1, \ldots$. The dashed red lines show the case in which the initial value of the debt stock is set to the deterministic steady state, $\hat{d}_0 = 0$. The solid black lines show the case in which the initial value of the debt stock is equal to the mean of the stochastic distribution reported in Table 4.

noted in the discussion of Figure 6, this increases inflation expectations and hence mitigates the effects of recessionary shocks when the policy rate is constrained at the lower bound. On the other hand, the analysis in Section 3.2 revealed that, absent the zero bound, welfare would be higher in the textbook model with passive fiscal policy.

Which of these effects dominates?

The policy rate is at the zero bound 24% of the time in the baseline model, compared with 45% under passive fiscal policy (Table 4). Conditional on being at the lower bound, losses are slightly lower on average. However, these performance improvements are outweighed by higher losses when policy is not constrained by the zero bound. Relative to passive fiscal policy, average losses are therefore slightly higher in the baseline model.

For the variant of the model with long duration debt, the policy rate is at the zero bound around a third of the time, midway between the baseline model and passive fiscal policy. Conditional on being at the zero bound, losses are considerably lower than the other variants. This performance improvement is sufficient to compensate for higher losses (compared with passive fiscal policy) away

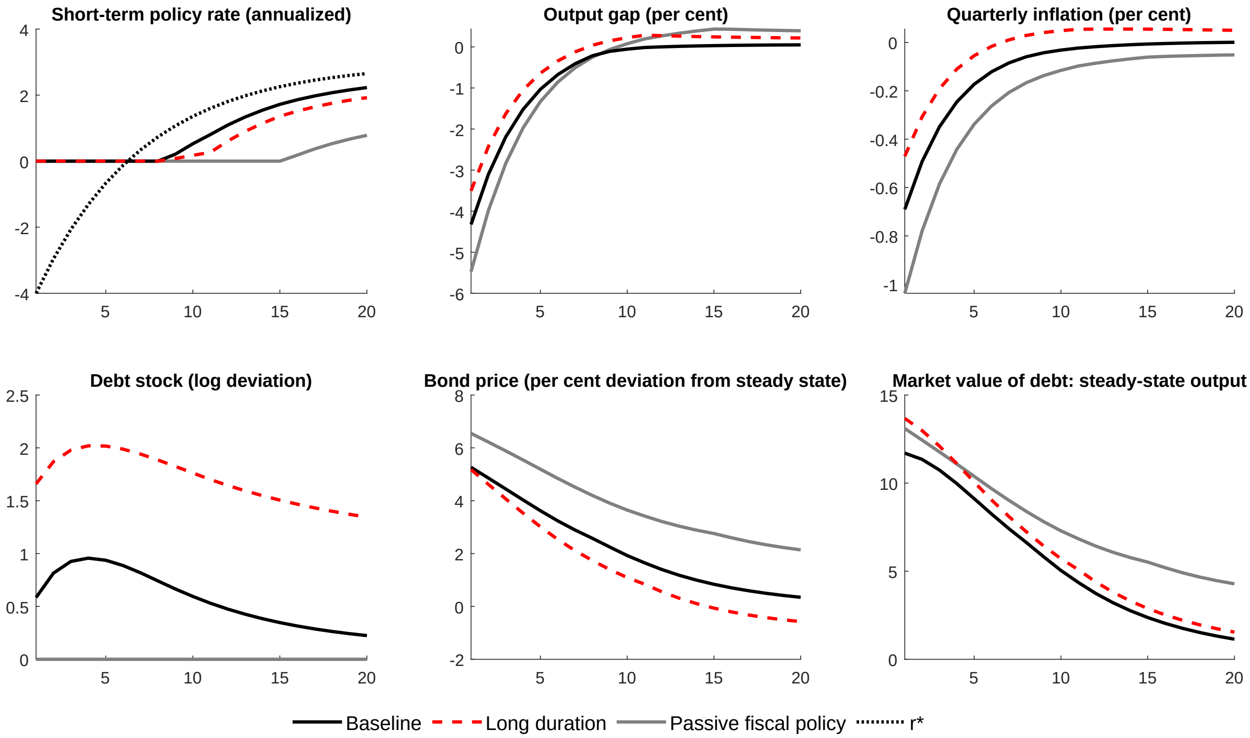


Figure 8: Recessionary scenario under average debt levels

Notes: The panels show outcomes from simulations of the baseline model (solid black lines), the variant with long-duration debt (dashed red lines) and the textbook model with passive fiscal policy described in Section 2.6 (gray lines). In each case, the simulation is constructed from the policy functions solved by projection methods. The initial value of the natural rate state r_1^* (plus the deterministic steady-state interest rate) is set to -4% on an annualized basis. Thereafter r^* follows the process (23), with shocks set to their most likely value of zero, $\varepsilon_t^r = 0, t = 2, \ldots$. The cost-push state is set equal to its most likely value, $u_t = 0, t = 1, \ldots$. The initial value of the debt stock is set equal to the mean of the stochastic distribution reported in Table 4.

from the zero bound.

The material welfare improvements at the zero bound for the long duration debt variant are not evident from Figure 6. Once again, the distribution of government debt matters. Figure 8 repeats the experiment from Figure 6, but under the assumption that initial debt stocks are equal to the mean of the stochastic distribution. This reveals greater performance improvements in the model with long-duration debt. Higher average debt generates higher inflation expectations, reducing real interest rates and stimulating aggregate demand. In this case, liftoff from the zero bound occurs at the same time as the baseline model, though the policy path remains slightly lower after liftoff.

5 Conclusion

This paper studies the behavior of a simple model with long-term government debt, time consistent monetary policy and active fiscal policy. Active fiscal policy may support inflation expectations at the lower bound, as agents expect higher future inflation to be used to reduce the real value of debt accumulated during the recession. This effect may be sufficient to improve welfare relative to the textbook case in which fiscal policy is passive.

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A The log-linear model

This appendix derives a log-linear representation of the model. For variable X_t, $\hat{X}_t \equiv \ln X_t - \ln X$ defines the log-deviation of X_t from its steady-state value, X. A useful feature of the derivation is that the steady-state level of output is normalized to unity, as described in Appendix A.4.

A.1 Households

The first-order conditions for the optimization problem are:

$$\phi_t c_t^{-\frac{1}{\sigma}} = \mu_t P_t \tag{26}$$

$$\phi_t n_t^{\psi} = W_t \mu_t \tag{27}$$

$$0 = -\mu_t + \beta R_t \mathbb{E}_t \mu_{t+1} \tag{28}$$

$$0 = -V_t \mu_t + \beta \mathbb{E}_t \left(\varrho + \chi V_{t+1}\right) \mu_{t+1} \tag{29}$$

where μ is the Lagrange multiplier on the nominal budget constraint (1).

Let the real Lagrange multiplier be defined as:

$$\Lambda_t \equiv P_t \mu_t$$

and real short bond holdings and long-term debt as

$$b_t \equiv \frac{B_t}{P_t}$$

$$d_t \equiv \frac{D_t}{P_t}$$

The first order conditions for short-term and long-term bond holdings, (28) and (29) can be written in terms of real-valued variables as:

$$0 = -\Lambda_t + \beta R_t \mathbb{E}_t \Lambda_{t+1} \pi_{t+1}^{-1} \tag{30}$$

$$0 = -\Lambda_t V_t + \beta \mathbb{E}_t \left(\varrho + \chi V_{t+1}\right) \Lambda_{t+1} \pi_{t+1}^{-1} \tag{31}$$

Combining these equations gives:

$$R_t V_t \mathbb{E}_t \Lambda_{t+1} \pi_{t+1}^{-1} = \mathbb{E}_t \left(\varrho + \chi V_{t+1}\right) \Lambda_{t+1} \pi_{t+1}^{-1} \tag{32}$$

In steady state this implies that:

$$V = \varrho \left(R - \chi\right)^{-1} = \varrho \left(\beta^{-1} - \chi\right)^{-1}$$

Setting $\varrho = \beta^{-1} - \chi$ therefore implies that $V = 1$, that is, the steady-state price of debt is unity. Adopting this assumption means that the real debt stock d can be treated as the real *par* value of debt.

Log-linearizing (32) gives:

$$RV\left(\hat{R}_t + \hat{V}_t\right) = \chi V \hat{V}_{t+1}$$

which implies that:

$$\hat{V}_t = -\hat{R}_t + \chi\beta\hat{V}_{t+1} \tag{33}$$

which uses the fact that $R = \beta^{-1}$ in a zero inflation steady state.

Combining (26) and (30) creates an Euler equation for consumption:

$$\phi_t c_t^{-\frac{1}{\sigma}} = \beta R_t \mathbb{E}_t \phi_{t+1} c_{t+1}^{-\frac{1}{\sigma}} \pi_{t+1}^{-1}$$

which can be log-linearized to give:

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \sigma\left[\hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1}\right] - \sigma\mathbb{E}_t\left(\hat{\phi}_{t+1} - \hat{\phi}_t\right) \tag{34}$$

The first order conditions for labor supply (27) and consumption (26) can be combined and log-linearized to give

$$\psi\hat{n}_t = \hat{w}_t - \sigma^{-1}\hat{c}_t \tag{35}$$

A.2 Firms

The real profit of producer j is:

$$\frac{(1+\Gamma) P_{j,t}}{P_t} y_{j,t} - w_t n_{j,t} = \left((1+\Gamma)\frac{P_{j,t}}{P_t} - \frac{w_t}{A_t}\right)\left(\frac{P_{j,t}}{P_t}\right)^{-\eta_t} y_t$$

where $\Gamma > 0$ is the subsidy that ensures that the steady-state level of output is efficient.

The objective function for a producer that is able to reset prices is:

$$\max \mathbb{E}_t \sum_{k=t}^{\infty} \Lambda_k (\beta\alpha)^{k-t}\left((1+\Gamma)\frac{P_{j,t}}{P_k} - \frac{w_k}{A_k}\right)\left(\frac{P_{j,t}}{P_k}\right)^{-\eta_t} y_k$$

where Λ represents the household's stochastic discount factor and $0 \le \alpha < 1$ is the probability that the producer is *not* allowed to reset its price each period.

Familiar manipulations (see, for example, Harrison, 2017) deliver a log-linearized inflation equation:

$$\hat{\pi}_t = \frac{(1-\beta\alpha)(1-\alpha)}{\alpha}\left(\hat{w}_t - \hat{A}_t - \frac{\eta}{\eta-1}\hat{\eta}_t\right) + \beta\mathbb{E}_t\hat{\pi}_{t+1} \tag{36}$$

A.3 Government

Log-linearizing (5) gives:

$$\hat{V}_t + \hat{d}_t = \frac{\varrho + \chi V}{V}\left(\hat{d}_{t-1} - \pi_t\right) + \chi\hat{V}_t - \frac{s_t - s}{Vd}$$

Since the steady-state level of output is normalized to unity, $Vd = \zeta$ where ζ is the steady state ratio of government debt to output. Moreover, in the steady state, $\frac{\varrho+\chi V}{V} = R = \beta^{-1}$.

These observations imply that:

$$\hat{d}_t = \beta^{-1}\left(\hat{d}_{t-1} - \hat{\pi}_t\right) - (1-\chi)\,\hat{V}_t - \zeta^{-1}\bar{s}_t \tag{37}$$

where $\bar{s}_t \equiv s_t - s$ is the linear deviation of the surplus from steady state.

The real surplus is given by:

$$s_t = \tau - g_t$$

since taxes are held fixed ($\tau_t = \tau, \forall t$).

In deviations from steady state, this implies that:

$$\bar{s}_t = -\bar{g}_t \tag{38}$$

Using this result in (37) gives

$$\hat{d}_t = \beta^{-1}\left(\hat{d}_{t-1} - \hat{\pi}_t\right) - (1-\chi)\,\hat{V}_t + \zeta^{-1}\bar{g}_t \tag{39}$$

A.4 Market clearing and the efficient allocation

Without loss of generality, the steady-state level of productivity, A is chosen to ensure that the steady-state level of output y is equal to unity. This implies that the steady-state level of government spending g represents that fraction of output consumed by the government (so $0 \leq g < 1$).

Aggregate output satisfies

$$A_t n_t = \mathcal{D}_t y_t \tag{40}$$

where

$$\mathcal{D}_t \equiv \int_0^1 \left(\frac{P_{jt}}{P_t}\right)^{-\eta} \mathrm{d}j \tag{41}$$

is a measure of price dispersion.

Goods market clearing requires:

$$c_t + g_t = y_t$$

where $\mathcal{D}_t$ is a *second order* price dispersion term (see, for example, Galí, 2008).

To a log-linear approximation, this is:

$$(1-g)\,\hat{c}_t + \bar{g}_t = \hat{y}_t \tag{42}$$

The subsidy required to make the steady state efficient is:

$$\Gamma = \frac{\eta}{\eta - 1}$$

In a flexible price equilibrium with no distortion from monopolistic competition, the real wage will equal the marginal product of labor, which is equal to A_t. So the efficient allocations, denoted with an asterisk, can be found from the labor supply relation (35):

$$\psi \hat{n}_t^* = \hat{A}_t - \sigma^{-1} \hat{c}_t^*$$

Imposing market clearing, $(1-g)\, c_t^* + \bar{g}_t = \hat{A}_t + n_t^* = y_t^*$ implies that potential output is given by:

$$\hat{y}_t^* = \frac{\sigma (1-g)(1+\psi)}{1+\psi\sigma(1-g)} \hat{A}_t + \frac{1}{1+\psi\sigma(1-g)} \bar{g}_t$$

A.5 The 'gap' representation

The Phillips curve and Euler equation can be written in terms of the output gap, defined as the deviation between output and the efficient level of output.

Substituting the labor supply equation (35) into the log-linearized pricing equation (36) gives:

$$\begin{aligned}
\hat{\pi}_t &= \frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \left(\psi \hat{n}_t + \sigma^{-1} \hat{c}_t - \hat{A}_t \right) + \beta \mathbb{E}_t \hat{\pi}_{t+1} \\
&\quad - \frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \frac{\eta}{\eta-1} \hat{\eta}_t \\
&= \frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \left(\left(\psi + \sigma^{-1} (1-g)^{-1} \right) \hat{y}_t - (1+\psi) \hat{A}_t - \sigma^{-1} (1-g)^{-1} \bar{g}_t \right) \\
&\quad + \beta \mathbb{E}_t \hat{\pi}_{t+1} - \frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \frac{\eta}{\eta-1} \hat{\eta}_t \\
&= \frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \left(\psi + \sigma^{-1} (1-g)^{-1} \right) \hat{x}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1} + u_t \\
&\quad + \frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \left[\left(\psi + \sigma^{-1} (1-g)^{-1} \right) \hat{y}_t^* - (1+\psi) \hat{A}_t - \sigma^{-1} (1-g)^{-1} \bar{g}_t \right]
\end{aligned}$$

where the second line uses market clearing and the third line uses the definition of the output gap $\hat{y}_t - \hat{y}_t^* \equiv \hat{x}_t$ and defines the cost push shock, u, as:

$$u_t \equiv -\frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \frac{\eta}{\eta-1} \hat{\eta}_t$$

Notice that the final term in brackets on the final line is given by:

$$\begin{aligned}
&\left(\psi + \sigma^{-1} (1-g)^{-1} \right) \hat{y}_t^* - (1+\psi) \hat{A}_t - \sigma^{-1} (1-g)^{-1} \bar{g}_t \\
&= \sigma^{-1} (1-g)^{-1} \left[(1+\psi\sigma(1-g)) \hat{y}_t^* - (1+\psi) \sigma (1-g) \hat{A}_t - \bar{g}_t \right] \\
&= 0
\end{aligned}$$

The Phillips curve can therefore be written as:

$$\hat{\pi}_t = \kappa \hat{x}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1} + u_t \tag{43}$$

where

$$\kappa \equiv \frac{(1-\beta\alpha)(1-\alpha)}{\alpha}\left(\psi+\sigma^{-1}(1-g)^{-1}\right)$$

The Euler equation for consumption (34) can be written as:

$$\begin{aligned}(1-g)^{-1}\hat{y}_t-(1-g)^{-1}\bar{g}_t= &\ (1-g)^{-1}\mathbb{E}_t\left[\hat{y}_{t+1}-\bar{g}_{t+1}\right]-\sigma\left[\hat{R}_t-\mathbb{E}_t\hat{\pi}_{t+1}\right]\\ &-\sigma\mathbb{E}_t\left(\hat{\phi}_{t+1}-\hat{\phi}_t\right)\end{aligned}$$

which incorporates the market clearing condition for output.

Rearranging gives:

$$\hat{y}_t=\mathbb{E}_t\hat{y}_{t+1}-\sigma(1-g)\left[\hat{R}_t-\mathbb{E}_t\hat{\pi}_{t+1}\right]+\mathbb{E}_t\left[\bar{g}_t-\bar{g}_{t+1}-\sigma(1-g)\left(\hat{\phi}_{t+1}-\hat{\phi}_t\right)\right]$$

This implies that:

$$\begin{aligned}\hat{y}_t-\hat{y}_t^*+\hat{y}_t^*= &\ \mathbb{E}_t\left(\hat{y}_{t+1}-y_{t+1}^*+y_{t+1}^*\right)-\sigma(1-g)\left[\hat{R}_t-\mathbb{E}_t\hat{\pi}_{t+1}\right]\\ &+\mathbb{E}_t\left[\bar{g}_t-\bar{g}_{t+1}-\sigma(1-g)\left(\hat{\phi}_{t+1}-\hat{\phi}_t\right)\right]\end{aligned}$$

or

$$\hat{x}_t=\mathbb{E}_t\hat{x}_{t+1}-\sigma(1-g)\left[\hat{R}_t-\mathbb{E}_t\hat{\pi}_{t+1}-r_t^*\right] \tag{44}$$

where the efficient rate of interest r^* satisfies

$$\begin{aligned}r_t^* &= \mathbb{E}_t\left[\sigma^{-1}(1-g)^{-1}\left(y_{t+1}^*-y_t^*+\widetilde{g}_t-\widetilde{g}_{t+1}\right)-\left(\hat{\phi}_{t+1}-\hat{\phi}_t\right)\right]\\ &= \mathbb{E}_t\sigma^{-1}(1-g)^{-1}\left(\begin{array}{c}\frac{\sigma(1-g)(1+\psi)}{1+\psi\sigma(1-g)}\hat{A}_{t+1}+\frac{1}{1+\psi\sigma(1-g)}\bar{g}_{t+1}\\ -y_t^*+\bar{g}_t-\bar{g}_{t+1}\end{array}\right)\\ &\quad-\mathbb{E}_t\left(\hat{\phi}_{t+1}-\hat{\phi}_t\right)\\ &= \mathbb{E}_t\left[-\left(\hat{\phi}_{t+1}-\hat{\phi}_t\right)+\frac{1+\psi}{1+\psi\sigma(1-g)}\left(\hat{A}_{t+1}-\hat{A}_t\right)\right]\\ &\quad-\frac{\psi}{1+\psi\sigma(1-g)}\mathbb{E}_t\left(\bar{g}_{t+1}-\bar{g}_t\right)\end{aligned}$$

B The utility-based loss function

Ignoring constants, the period utility function is:

$$U_t=\phi_t\left[\frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}-\frac{n_t^{1+\psi}}{1+\psi}\right]$$

Markup shocks are ignored (by setting $\eta_t=\eta,\forall t$) to simplify notation. Since cost push shocks are independent of policy this does not affect the derivation.

Approximating the utility from consumption to second order gives:

$$\phi_t \frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \approx c^{1-\frac{1}{\sigma}} \left(\frac{c_t - c}{c} \right) - \frac{1}{2\sigma} c^{1-\frac{1}{\sigma}} \left(\frac{c_t - c}{c} \right)^2 + c^{1-\frac{1}{\sigma}} \frac{c_t - c}{c} \frac{\phi_t - \phi}{\phi} + t.i.p. \tag{45}$$

where $t.i.p.$ denotes 'terms independent of policy' (that is, functions of exogenous disturbances) and the fact that $\phi = 1$ in steady state is used to simplify the first two terms.

Using the second order approximation for the percentage changes in consumption implies that:

$$\frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \approx c^{1-\frac{1}{\sigma}} \left(\hat{c}_t + \frac{1}{2} \left(1 - \sigma^{-1}\right) \hat{c}_t^2 + \hat{c}_t \hat{\phi}_t \right) + h.o.t.$$

where $h.o.t.$ are 'higher order terms'.

The sub-utility function for labor supply is:

$$\begin{aligned}
\frac{\phi_t n_t^{1+\psi}}{1+\psi} \approx & \frac{n^{1+\psi}}{1+\psi} + n^{1+\psi} \frac{n_t - n}{n} + \frac{\psi n^{1+\psi}}{2} \left(\frac{n_t - n}{n} \right)^2 + \frac{n^{1+\psi}}{1+\psi} \frac{\phi_t - \phi}{\phi} \\
& + n^{1+\psi} \frac{n_t - n}{n} \frac{\phi_t - \phi}{\phi} \\
\approx & n^{1+\psi} \frac{n_t - n}{n} + \frac{\psi n^{1+\psi}}{2} \left(\frac{n_t - n}{n} \right)^2 + n^{1+\psi} \frac{n_t - n}{n} \frac{\phi_t - \phi}{\phi} + t.i.p.
\end{aligned}$$

Using the mapping from percentage changes to log-deviations, to second order, implies that:

$$\frac{\phi_t n_t^{1+\psi}}{1+\psi} \approx n^{1+\psi} \left[\hat{n}_t + \frac{(1+\psi)}{2} \hat{n}_t^2 + \hat{n}_t \hat{\phi}_t \right] + h.o.t.$$

A second order approximation to the aggregate production function (40) is:

$$\hat{y}_t + \frac{1}{2} \hat{y}_t^2 = \hat{n}_t + \frac{1}{2} \hat{n}_t^2 + \hat{A}_t \hat{n}_t - \hat{\mathcal{D}}_t + t.i.p.$$

which uses the fact that $\hat{\mathcal{D}}_t$ is a second-order term.

This implies that:

$$\frac{\phi_t n_t^{1+\psi}}{1+\psi} \approx n^{1+\psi} \left[\hat{y}_t + \frac{(1+\psi)}{2} \hat{y}_t^2 - (1+\psi) \hat{y}_t \hat{A}_t + \hat{y}_t \hat{\phi}_t - \hat{\mathcal{D}}_t \right] + h.o.t. + t.i.p.$$

The second-order approximation to the utility function is therefore

$$\begin{aligned}
U_t \approx & c^{1-\frac{1}{\sigma}} \left(\hat{c}_t + \frac{1}{2} \left(1 - \sigma^{-1}\right) \hat{c}_t^2 + \hat{c}_t \hat{\phi}_t \right) \\
& - n^{1+\psi} \left[\hat{y}_t + \frac{(1+\psi)}{2} \hat{y}_t^2 - (1+\psi) \hat{y}_t \hat{A}_t + \hat{y}_t \hat{\phi}_t - \hat{\mathcal{D}}_t \right]
\end{aligned}$$

The steady-state labor supply relationship is

$$n^{\psi} = wc^{-1/\sigma} = Ac^{-1/\sigma}$$

Steady-state market clearing is

$$c + g = y = An$$

since steady-state price dispersion is $\mathcal{D} = 1$.

This implies that

$$n^{1+\psi} = (1-g)^{-1} c^{1-\frac{1}{\sigma}}$$

so that the utility function can be written as

$$U_t \approx c^{1-\frac{1}{\sigma}} \left[\begin{array}{c} \hat{c}_t + \frac{1}{2}\left(1-\sigma^{-1}\right)\hat{c}_t^2 + \hat{c}_t\hat{\phi}_t - (1-g)^{-1}\hat{y}_t - \frac{(1+\psi)}{2(1-g)}\hat{y}_t^2 \\ +\frac{(1+\psi)}{(1-g)}\hat{y}_t\hat{A}_t - (1-g)^{-1}\hat{y}_t\hat{\phi}_t - (1-g)^{-1}\hat{\mathcal{D}}_t \end{array} \right]$$

The goods market clearing condition is:

$$c_t = y_t - g_t$$

A second order approximation to the goods market clearing condition is:

$$(1-g)\hat{c}_t + \frac{1-g}{2}\hat{c}_t^2 = \hat{y}_t + \frac{1}{2}\hat{y}_t^2 + t.i.p.$$

These results can be used to write the approximation to utility in terms of output deviations and price dispersion only.

First substitute for

$$\hat{c}_t = -\frac{1}{2}\hat{c}_t^2 + \frac{\hat{y}_t}{1-g} + \frac{1}{2(1-g)}\hat{y}_t^2$$

to give:

$$\begin{aligned} U_t &\approx c^{1-\frac{1}{\sigma}} \left[\begin{array}{c} -\frac{1}{2}\hat{c}_t^2 + \frac{\hat{y}_t}{1-g} + \frac{1}{2(1-g)}\hat{y}_t^2 + \frac{1}{2}\left(1-\sigma^{-1}\right)\hat{c}_t^2 + \hat{c}_t\hat{\phi}_t \\ -(1-g)^{-1}\hat{y}_t - \frac{(1+\psi)}{2(1-g)}\hat{y}_t^2 + \frac{(1+\psi)}{(1-g)}\hat{y}_t\hat{A}_t \\ -(1-g)^{-1}\hat{y}_t\hat{\phi}_t - (1-g)^{-1}\hat{\mathcal{D}}_t \end{array} \right] \\ &\approx \frac{c^{1-\frac{1}{\sigma}}}{1-g}\left[-\hat{\mathcal{D}}_t - \frac{1-g}{2\sigma}\hat{c}_t^2 - \frac{\psi}{2}\hat{y}_t^2 + (1-g)\hat{c}_t\hat{\phi}_t + (1+\psi)\hat{y}_t\hat{A}_t - \hat{y}_t\hat{\phi}_t\right] \end{aligned}$$

where the second line collects common terms.

Substituting for $\hat{c}_t^2$ gives:

$$U_t \approx \frac{c^{1-\frac{1}{\sigma}}}{1-g}\left[-\hat{\mathcal{D}}_t - \frac{\psi + \widetilde{\sigma}^{-1}}{2}\hat{y}_t^2 + \frac{1}{\widetilde{\sigma}}\hat{y}_t\bar{g}_t + (1-g)\hat{c}_t\hat{\phi}_t + (1+\psi)\hat{y}_t\hat{A}_t - \hat{y}_t\hat{\phi}_t\right]$$

where $\widetilde{\sigma} \equiv (1-g)\sigma$ as in the main text.

Noting that

$$(1-g)\,\hat{c}_t\hat{\phi}_t - \hat{y}_t\hat{\phi}_t = \hat{\phi}_t\left[(1-g)\,\hat{c}_t - \hat{y}_t\right] = \hat{\phi}_t\left[(1-g)\,\hat{c}_t - (1-g)\,\hat{c}_t - \bar{g}_t\right] = -\hat{\phi}_t\bar{g}_t$$

which is independent of policy gives:

$$U_t \approx \frac{c^{1-\frac{1}{\sigma}}}{1-g}\left[-\hat{\mathcal{D}}_t - \frac{\psi+\widetilde{\sigma}^{-1}}{2}\hat{y}_t^2 + \frac{1}{\widetilde{\sigma}}\hat{y}_t\bar{g}_t + (1+\psi)\,\hat{y}_t\hat{A}_t\right]$$

The terms in $\hat{y}_t$ can be written as:

$$\begin{aligned}
-\frac{\psi+\widetilde{\sigma}^{-1}}{2}\hat{y}_t^2 + \frac{1}{\widetilde{\sigma}}\hat{y}_t\bar{g}_t + (1+\psi)\,\hat{y}_t\hat{A}_t &= -\frac{\psi+\widetilde{\sigma}^{-1}}{2}\left[\hat{y}_t^2 - 2\hat{y}_t\left(\begin{array}{c}\frac{1}{1+\psi\widetilde{\sigma}}\bar{g}_t\\ +\frac{\widetilde{\sigma}(1+\psi)}{1+\psi\widetilde{\sigma}}\hat{A}_t\end{array}\right)\right]\\
&= -\frac{\psi+\widetilde{\sigma}^{-1}}{2}\left[\hat{y}_t^2 - 2\hat{y}_t\hat{y}_t^*\right]\\
&= -\frac{\psi+\widetilde{\sigma}^{-1}}{2}\left[\hat{y}_t^2 - 2\hat{y}_t\hat{y}_t^* + (\hat{y}_t^*)^2 - (\hat{y}_t^*)^2\right]\\
&= -\frac{\psi+\widetilde{\sigma}^{-1}}{2}(\hat{y}_t - \hat{y}_t^*)^2 + \frac{\psi+\widetilde{\sigma}^{-1}}{2}(\hat{y}_t^*)^2\\
&= -\frac{\psi+\widetilde{\sigma}^{-1}}{2}x_t^2 + t.i.p.
\end{aligned}$$

Define the discounted loss function to be minimized as:

$$\mathcal{L} = -2\,(1-g)\,c^{\frac{1}{\sigma}-1}\sum_{t=0}^{\infty}\beta^t U_t = \sum_{t=0}^{\infty}\beta^t\left[2\hat{\mathcal{D}}_t + \left(\psi+\widetilde{\sigma}^{-1}\right)\hat{x}_t^2\right]$$

The analysis of the price dispersion term $\hat{\mathcal{D}}_t$ is standard (see, for example, Galí, 2008; Harrison, 2017) so that the loss function can be written as:

$$\mathcal{L} = \sum_{t=0}^{\infty}\beta^t\left[\frac{\alpha\eta}{(1-\alpha\beta)(1-\alpha)}\hat{\pi}_t^2 + \left(\psi+\widetilde{\sigma}^{-1}\right)\hat{x}_t^2\right]$$

Normalizing the coefficient on inflation to unity implies that the loss function is:

$$\mathcal{L} = \sum_{t=0}^{\infty}\beta^t\left[\hat{\pi}_t^2 + \frac{(1-\alpha\beta)(1-\alpha)}{\alpha\eta}\left(\psi+\widetilde{\sigma}^{-1}\right)\hat{x}_t^2\right] = \sum_{t=0}^{\infty}\beta^t\left[\hat{\pi}_t^2 + \frac{\kappa}{\eta}\hat{x}_t^2\right]$$

C Time-consistent linear-quadratic policy

This appendix focuses on solving for the coefficients that describe the dependence of endogenous variables on the debt stock (that is, $F_{\hat{\pi}}$,$F_{\hat{x}}$,$F_{\hat{V}}$,$F_{\hat{d}}$). The starting point is the first order conditions derived in Section 3.1 of the main text.

Note first that (16) and (17) imply that:

$$\mu_t^x = (1-\chi)\,\widetilde{\sigma}^{-1}\mu_t^d \tag{46}$$

and

$$\mu_t^V = -(1-\chi)\,\mu_t^d \tag{47}$$

Using the preceding results in the equation for μ_t^d gives:

$$\begin{aligned}\mu_t^d = &\ [F_{\hat{x}} + \widetilde{\sigma} F_{\hat{\pi}}](1-\chi)\,\widetilde{\sigma}^{-1}\mu_t^d + \beta F_{\hat{\pi}}\mu_t^{\pi} - \chi\beta F_{\hat{V}}(1-\chi)\,\mu_t^d + \mathbb{E}_t\mu_{t+1}^d \\ = &\ [F_{\hat{x}} + \widetilde{\sigma} F_{\hat{\pi}}](1-\chi)\,\widetilde{\sigma}^{-1}\mu_t^d + \beta F_{\hat{\pi}}\left(\hat{\pi}_t - \beta^{-1}\mu_t^d\right) - \chi\beta F_{\hat{V}}(1-\chi)\,\mu_t^d \\ &+ \mathbb{E}_t\mu_{t+1}^d\end{aligned}$$

Collecting terms and rearranging gives:

$$\left[1 + \chi\beta(1-\chi)\,F_{\hat{V}} + \chi F_{\pi} - (1-\chi)\,\widetilde{\sigma}^{-1}F_{\hat{x}}\right]\mu_t^d = \beta F_{\hat{\pi}}\hat{\pi}_t + \mathbb{E}_t\mu_{t+1}^d \tag{48}$$

Combining (13) and (14) gives:

$$0 = \omega\hat{x}_t - \mu_t^x + \kappa\hat{\pi}_t - \kappa\beta^{-1}\mu_t^d$$

which implies that:

$$\omega\hat{x}_t + \kappa\hat{\pi}_t = \left((1-\chi)\,\widetilde{\sigma}^{-1} + \kappa\beta^{-1}\right)\mu_t^d \tag{49}$$

or

$$\mu_t^d = \frac{\omega}{\Xi}\hat{x}_t + \frac{\kappa}{\Xi}\hat{\pi}_t \tag{50}$$

where

$$\Xi \equiv (1-\chi)\,\widetilde{\sigma}^{-1} + \kappa\beta^{-1} > 0 \tag{51}$$

Then:

$$\begin{aligned}&\left[1 + \chi\beta(1-\chi)\,F_{\hat{V}} + \chi F_{\hat{\pi}} - (1-\chi)\,\widetilde{\sigma}^{-1}F_{\hat{x}}\right]\left(\frac{\omega}{\Xi}\hat{x}_t + \frac{\kappa}{\Xi}\hat{\pi}_t\right) \\ &= \beta F_{\hat{\pi}}\hat{\pi}_t + \mathbb{E}_t\left(\frac{\omega}{\Xi}\hat{x}_{t+1} + \frac{\kappa}{\Xi}\hat{\pi}_{t+1}\right)\end{aligned} \tag{52}$$

To solve for the 'F' coefficients, ignore exogenous terms and substitute out for expectations (so, for example, $\mathbb{E}_t x_{t+1} = F_{\hat{x}} d_t = F_{\hat{x}} F_{\hat{d}} d_{t-1}$):

$$\begin{aligned}\beta F_{\hat{\pi}}F_{\hat{\pi}} + \left(\frac{\omega}{\Xi}F_{\hat{x}}F_{\hat{d}} + \frac{\kappa}{\Xi}F_{\hat{\pi}}F_{\hat{d}}\right) = &\left[1 + \chi\beta(1-\chi)\,F_{\hat{V}} + \chi F_{\hat{\pi}} - (1-\chi)\,\widetilde{\sigma}^{-1}F_{\hat{x}}\right] \\ &\times\left(\frac{\omega}{\Xi}F_{\hat{x}} + \frac{\kappa}{\Xi}F_{\hat{\pi}}\right)\end{aligned}$$

which implies that

$$\left[1+\chi\beta\left(1-\chi\right)F_{\hat{V}}+\chi F_{\hat{\pi}}-\left(1-\chi\right)\widetilde{\sigma}^{-1}F_{\hat{x}}-F_{\hat{d}}\right]\left(\frac{\omega}{\Xi}F_{\hat{x}}+\frac{\kappa}{\Xi}F_{\hat{\pi}}\right)=\beta F_{\hat{\pi}}^{2} \tag{53}$$

Now apply the same approach for the structural model equations, which gives the following.

$$\begin{aligned}\hat{R}_t &= -\hat{V}_t+\chi\beta F_{\hat{V}}\hat{d}_t \\ \hat{x}_t &= F_{\hat{x}}\hat{d}_t-\widetilde{\sigma}\hat{R}_t+\widetilde{\sigma}F_{\hat{\pi}}\hat{d}_t\end{aligned} \tag{54}$$

which implies that:

$$\hat{x}_t=F_{\hat{x}}\hat{d}_t+\widetilde{\sigma}\hat{V}_t-\widetilde{\sigma}\chi\beta F_{\hat{V}}\hat{d}_t+\widetilde{\sigma}F_{\hat{\pi}}\hat{d}_t$$

and hence:

$$F_{\hat{x}}=F_{\hat{x}}F_{\hat{d}}+\widetilde{\sigma}F_{\hat{V}}-\widetilde{\sigma}\chi\beta F_{\hat{V}}F_{\hat{d}}+\widetilde{\sigma}F_{\hat{\pi}}F_{\hat{d}} \tag{55}$$

The Phillips curve implies that:

$$\hat{\pi}_t=\kappa\hat{x}_t+\beta F_{\hat{\pi}}\hat{d}_t$$

so that:

$$F_{\hat{\pi}}=\kappa F_{\hat{x}}+\beta F_{\hat{\pi}}F_{\hat{d}} \tag{56}$$

The government debt accumulation equation gives:

$$\hat{d}_t=\beta^{-1}\hat{d}_{t-1}-\beta^{-1}\hat{\pi}_t-\left(1-\chi\right)\hat{V}_t$$

which implies that:

$$F_{\hat{d}}=\beta^{-1}-\beta^{-1}F_{\hat{\pi}}-\left(1-\chi\right)F_{\hat{V}} \tag{57}$$

The preceding steps have delivered four equations in four unknowns, repeated here for convenience:

$$\begin{aligned}\beta F_{\hat{\pi}}^{2} &= \left[1+\chi\beta\left(1-\chi\right)F_{\hat{V}}+\chi F_{\hat{\pi}}-\left(1-\chi\right)\widetilde{\sigma}^{-1}F_{\hat{x}}-F_{\hat{d}}\right] \\ &\quad\times\left(\frac{\omega}{\Xi}F_{\hat{x}}+\frac{\kappa}{\Xi}F_{\hat{\pi}}\right)\end{aligned} \tag{58}$$

$$F_{\hat{x}}=F_{\hat{x}}F_{\hat{d}}+\widetilde{\sigma}F_{\hat{V}}-\widetilde{\sigma}\chi\beta F_{\hat{V}}F_{\hat{d}}+\widetilde{\sigma}F_{\hat{\pi}}F_{\hat{d}} \tag{59}$$

$$F_{\hat{\pi}}=\kappa F_{\hat{x}}+\beta F_{\hat{\pi}}F_{\hat{d}} \tag{60}$$

$$F_{\hat{d}}=\beta^{-1}-\beta^{-1}F_{\hat{\pi}}-\left(1-\chi\right)F_{\hat{V}} \tag{61}$$

The final two equations can be used to express $F_{\hat{x}}$ and $F_{\hat{V}}$ as functions of $F_{\hat{\pi}}$ and $F_{\hat{d}}$:

$$F_{\hat{x}}=\kappa^{-1}F_{\hat{\pi}}-\kappa^{-1}\beta F_{\hat{\pi}}F_{\hat{d}} \tag{62}$$

$$F_{\hat{V}}=\left(1-\chi\right)^{-1}\beta^{-1}-\left(1-\chi\right)^{-1}\beta^{-1}F_{\hat{\pi}}-\left(1-\chi\right)^{-1}F_{\hat{d}} \tag{63}$$

Using these expressions in the first two equations gives:

$$\begin{aligned}\beta F_{\hat{\pi}}^2 = & \left[\begin{array}{c} 1+\chi\beta\left(1-\chi\right)\left(1-\chi\right)^{-1}\left(\beta^{-1}-\beta^{-1}F_{\hat{\pi}}-F_{\hat{d}}\right)+\chi F_{\hat{\pi}} \\ -\left(1-\chi\right)\widetilde{\sigma}^{-1}\left(\kappa^{-1}F_{\hat{\pi}}-\kappa^{-1}\beta F_{\hat{\pi}}F_{\hat{d}}\right)-F_{\hat{d}} \end{array}\right] \\ & \times\left(\frac{\omega}{\Xi}\left(\kappa^{-1}F_{\hat{\pi}}-\kappa^{-1}\beta F_{\hat{\pi}}F_{\hat{d}}\right)+\frac{\kappa}{\Xi}F_{\hat{\pi}}\right) \\ = & \left[1+\chi-\chi\beta F_{\hat{d}}-\left(1-\chi\right)\left(\kappa\widetilde{\sigma}\right)^{-1}\left(1-\beta F_{\hat{d}}\right)F_{\hat{\pi}}-F_{\hat{d}}\right] \\ & \times\left(\frac{\omega}{\Xi\kappa}\left(1-\beta F_{\hat{d}}\right)+\frac{\kappa}{\Xi}\right)F_{\hat{\pi}}\end{aligned}$$

and

$$\begin{aligned}\kappa^{-1}F_{\hat{\pi}}\left(1-\beta F_{\hat{d}}\right) = & \kappa^{-1}F_{\hat{\pi}}\left(1-\beta F_{\hat{d}}\right)F_{\hat{d}}+\widetilde{\sigma}F_{\hat{\pi}}F_{\hat{d}} \\ & +\widetilde{\sigma}\left(\left(1-\chi\right)^{-1}\beta^{-1}-\left(1-\chi\right)^{-1}\beta^{-1}F_{\hat{\pi}}-\left(1-\chi\right)^{-1}F_{\hat{d}}\right) \\ & \times\left(1-\beta\chi F_{\hat{d}}\right)\end{aligned}$$

The second equation implies a quadratic equation for $F_{\hat{d}}$ conditional on a solution (or conjecture) for $F_{\hat{\pi}}$. The first equation can be used to solve for $F_{\hat{\pi}}$ conditional on a solution (or conjecture) for $F_{\hat{d}}$.

Specifically, conditional on a solution for $F_{\hat{d}}$, $F_{\hat{\pi}}$ satisfies:

$$F_{\hat{\pi}} = m\left(F_{\hat{d}}\right) \equiv \frac{1+\chi-\left(1+\beta\chi\right)F_{\hat{d}}}{\beta\left(\frac{\omega}{\kappa\Xi}\left(1-\beta F_{\hat{d}}\right)+\frac{\kappa}{\Xi}\right)^{-1}+\left(1-\chi\right)\left(\kappa\widetilde{\sigma}\right)^{-1}\left(1-\beta F_{\hat{d}}\right)} \tag{64}$$

The quadratic equation for $F_{\hat{d}}$ is given by:

$$\begin{aligned}0 = & \left[\frac{\beta\chi\kappa\widetilde{\sigma}}{1-\chi}-\beta F_{\hat{\pi}}\right]F_{\hat{d}}^2+\left[F_{\hat{\pi}}\left(1+\beta+\kappa\widetilde{\sigma}\right)-\frac{\kappa\widetilde{\sigma}}{1-\chi}-\frac{\chi\kappa\widetilde{\sigma}}{\left(1-\chi\right)}\left(1-F_{\hat{\pi}}\right)\right]F_{\hat{d}} \\ & +\left[\frac{\kappa\widetilde{\sigma}}{\beta\left(1-\chi\right)}\left(1-F_{\hat{\pi}}\right)-F_{\hat{\pi}}\right]\end{aligned}$$

which can be rearranged to give:

$$\begin{aligned}0 = & \left[\frac{\chi\kappa\widetilde{\sigma}}{1-\chi}-F_{\hat{\pi}}\right]\beta F_{\hat{d}}^2+\left[F_{\hat{\pi}}-\frac{\chi\kappa\widetilde{\sigma}}{\left(1-\chi\right)}\right]F_{\hat{d}} \\ & +\left[F_{\hat{\pi}}\left(\beta+\kappa\widetilde{\sigma}\right)-\frac{\kappa\widetilde{\sigma}}{1-\chi}+\frac{\chi\kappa\widetilde{\sigma}}{\left(1-\chi\right)}F_{\hat{\pi}}\right]F_{\hat{d}} \\ & +\left[\frac{\kappa\widetilde{\sigma}}{\beta\left(1-\chi\right)}\left(1-F_{\hat{\pi}}\right)-F_{\hat{\pi}}\right]\end{aligned}$$

The third term can be written as:

$$\begin{aligned}
&\left[F_{\hat{\pi}}\left(\beta+\kappa\widetilde{\sigma}\right)-\frac{\kappa\widetilde{\sigma}}{1-\chi}+\frac{\chi\kappa\widetilde{\sigma}}{(1-\chi)}F_{\hat{\pi}}\right]F_{\hat{d}}\\
&=\left[\beta F_{\hat{\pi}}+\frac{(1-\chi)\,\kappa\widetilde{\sigma}}{1-\chi}F_{\hat{\pi}}-\frac{\kappa\widetilde{\sigma}}{1-\chi}+\frac{\chi\kappa\widetilde{\sigma}}{(1-\chi)}F_{\hat{\pi}}\right]F_{\hat{d}}\\
&=\left[\beta F_{\hat{\pi}}+\frac{\kappa\widetilde{\sigma}}{1-\chi}\left(F_{\hat{\pi}}-1\right)\right]F_{\hat{d}}\\
&=-\left[\frac{\kappa\widetilde{\sigma}}{1-\chi}\left(1-F_{\hat{\pi}}\right)-F_{\hat{\pi}}\right]\beta F_{\hat{d}}
\end{aligned}\tag{65}$$

This implies that the quadratic equation can be factorized as:

$$\begin{aligned}
0=&\left[F_{\hat{\pi}}-\frac{\chi\kappa\widetilde{\sigma}}{1-\chi}\right]F_{\hat{d}}\left(1-\beta F_{\hat{d}}\right)+\left[\frac{\kappa\widetilde{\sigma}}{\beta\left(1-\chi\right)}\left(1-F_{\hat{\pi}}\right)-F_{\hat{\pi}}\right]\left(1-\beta F_{\hat{d}}\right)\\
=&\left(1-\beta F_{\hat{d}}\right)\left[\left(F_{\hat{\pi}}-\frac{\chi\kappa\widetilde{\sigma}}{1-\chi}\right)F_{\hat{d}}+\frac{\kappa\widetilde{\sigma}}{\beta\left(1-\chi\right)}\left(1-F_{\hat{\pi}}\right)-F_{\hat{\pi}}\right]
\end{aligned}$$

So one solution is $F_{\hat{d}}=\beta^{-1}$. That implies that $F_{\hat{\pi}}=m\left(\beta^{-1}\right)=\frac{\kappa}{\Xi\beta}\left(1-\beta^{-1}\right)<0$.
The other solution for $F_{\hat{d}}$ satisfies:

$$F_{\hat{d}}\equiv h\left(F_{\hat{\pi}}\right)=\frac{\left(1+\kappa\widetilde{\sigma}\left[\beta\left(1-\chi\right)\right]^{-1}\right)F_{\hat{\pi}}-\kappa\widetilde{\sigma}\left[\beta\left(1-\chi\right)\right]^{-1}}{F_{\hat{\pi}}-\chi\kappa\widetilde{\sigma}\left(1-\chi\right)^{-1}}\tag{66}$$

Note that h can be written as:

$$h\left(\cdot\right)=1+\frac{\kappa\widetilde{\sigma}\left[\beta\left(1-\chi\right)\right]^{-1}\left(F_{\hat{\pi}}-1+\beta\chi\right)}{F_{\hat{\pi}}-\chi\kappa\widetilde{\sigma}\left(1-\chi\right)^{-1}}$$

The previous results can be used to re-write the targeting rule in terms of the output gap and inflation.

C.1 Model properties under time-consistent policy

The coefficient in brackets on the left side of (52) is:

$$\begin{aligned}
\Omega\equiv\;&1+\chi\beta\left(1-\chi\right)F_{\hat{V}}+\chi F_{\hat{\pi}}-\left(1-\chi\right)\widetilde{\sigma}^{-1}F_{\hat{x}}\\
=\;&1+\chi\left(1-F_{\hat{\pi}}-\beta F_{\hat{d}}\right)+\chi F_{\hat{\pi}}-\left(1-\chi\right)\widetilde{\sigma}^{-1}\left(\kappa^{-1}F_{\hat{\pi}}-\kappa^{-1}\beta F_{\hat{\pi}}F_{\hat{d}}\right)\\
=\;&1+\chi-\chi\beta F_{\hat{d}}-\frac{1-\chi}{\kappa\widetilde{\sigma}}F_{\hat{\pi}}+\frac{1-\chi}{\kappa\widetilde{\sigma}}\beta F_{\hat{\pi}}F_{\hat{d}}
\end{aligned}$$

Rearranging (66) reveals that:

$$F_{\hat{d}}F_{\hat{\pi}}=\frac{\chi\kappa\widetilde{\sigma}}{1-\chi}F_{\hat{d}}+\left(1+\frac{\kappa\widetilde{\sigma}}{\beta\left(1-\chi\right)}\right)F_{\hat{\pi}}-\frac{\kappa\widetilde{\sigma}}{\beta\left(1-\chi\right)}$$

so that

$$\frac{1-\chi}{\kappa\widetilde{\sigma}}\beta F_{\hat{\pi}}F_{\hat{d}}=\chi\beta F_{\hat{d}}+\left(1+\beta\frac{1-\chi}{\kappa\widetilde{\sigma}}\right)F_{\hat{\pi}}-1$$

and plugging this into the equation for Ω gives:

$$\Omega=\chi+F_{\hat{\pi}}\left(1-(1-\beta)\,\frac{1-\chi}{\kappa\widetilde{\sigma}}\right)$$

This allows (48) to be written as

$$\mu_t^d=\beta F_{\hat{\pi}}\Omega^{-1}\hat{\pi}_t+\Omega^{-1}\mathbb{E}_t\mu_{t+1}^d$$

and (52) to be written as:

$$\Omega\left(\omega\hat{x}_t+\kappa\hat{\pi}_t\right)=\beta\Xi F_{\hat{\pi}}\hat{\pi}_t+\mathbb{E}_t\left(\omega\hat{x}_{t+1}+\kappa\hat{\pi}_{t+1}\right)$$

or

$$\omega\hat{x}_t+\left(\kappa-\beta\Xi F_{\hat{\pi}}\Omega^{-1}\right)\hat{\pi}_t=\Omega^{-1}\mathbb{E}_t\left(\omega\hat{x}_{t+1}+\kappa\hat{\pi}_{t+1}\right)$$

When cost-push shocks are zero, the Phillips curve implies that the output gap satisfies:

$$\hat{x}_t=\kappa^{-1}\left(\hat{\pi}_t-\beta\mathbb{E}_t\hat{\pi}_{t+1}\right)$$

Which implies that the targeting rule can be written as:

$$\Omega\left(\omega\kappa^{-1}\left(\hat{\pi}_t-\beta\mathbb{E}_t\hat{\pi}_{t+1}\right)+\kappa\hat{\pi}_t\right)=\beta\Xi F_{\hat{\pi}}\hat{\pi}_t+\mathbb{E}_t\left(\omega\kappa^{-1}\left(\hat{\pi}_{t+1}-\beta\mathbb{E}_t\hat{\pi}_{t+2}\right)+\kappa\hat{\pi}_{t+1}\right)$$

which uses the law of iterated conditional expectations.

Collecting terms and rearranging gives a second order difference equation for inflation:

$$\left[\Omega\left(\frac{\omega}{\kappa}+\kappa\right)-\beta\Xi F_{\pi}\right]\hat{\pi}_t-\left[\frac{\omega}{\kappa}\left(1+\beta\Omega\right)+\kappa\right]\mathbb{E}_t\hat{\pi}_{t+1}+\frac{\beta\omega}{\kappa}\mathbb{E}_t\hat{\pi}_{t+2}=0$$

For an impulse response, in which an unexpected shock is revealed in period 1 and no additional information arrives thereafter, the path for inflation must satisfy the difference equation. That is

$$\left[\Omega\left(\frac{\omega}{\kappa}+\kappa\right)-\beta\Xi F_{\pi}\right]\hat{\pi}_t-\left[\frac{\omega}{\kappa}\left(1+\beta\Omega\right)+\kappa\right]\hat{\pi}_{t+1}+\frac{\beta\omega}{\kappa}\hat{\pi}_{t+2}=0$$

where the expectation operator has been removed.

We seek a solution of the form $\pi_{t+1}=G\pi_t$ which implies

$$\left(\left[\Omega\left(\frac{\omega}{\kappa}+\kappa\right)-\beta\Xi F_{\pi}\right]-\left[\frac{\omega}{\kappa}\left(1+\beta\Omega\right)+\kappa\right]G+\frac{\beta\omega}{\kappa}G^2\right)\hat{\pi}_t=0$$

The roots of the characteristic polynominal satisfy:

$$G = \frac{\frac{\omega}{\kappa}(1+\beta\Omega)+\kappa \quad \pm \quad \left(\left[\frac{\omega}{\kappa}(1+\beta\Omega)+\kappa\right]^2 - 4\left[\Omega\left(\frac{\omega}{\kappa}+\kappa\right)-\beta\Xi F_{\hat{\pi}}\right]\frac{\beta\omega}{\kappa}\right)^{\frac{1}{2}}}{2\frac{\beta\omega}{\kappa}} \tag{67}$$

For the parameter values used in both the baseline and long-duration variants, the two solutions for G are real. In both cases, the larger root exceeds 1 and the smaller root is less than 1.

The preceding analysis has shown that, in the absence of cost push shocks, the path for inflation satisfies:

$$\hat{\pi}_{t+1} = G\hat{\pi}_t, \qquad t \geq 1$$

The Phillips curve implies that:

$$\hat{x}_{t+1} = \kappa^{-1}(\hat{\pi}_{t+1} - \beta\hat{\pi}_{t+2}) = \kappa^{-1}(1-\beta G)\hat{\pi}_{t+1}$$

and substituting this into (50) evaluated at $t+1$ gives:

$$\mu^d_{t+1} = \left[\frac{\omega}{\kappa\Xi}(1-\beta G) + \frac{\kappa}{\Xi}\right]\hat{\pi}_{t+1}$$

Also note that, for $t > 1$:

$$\begin{aligned} \hat{x}_{t+2} =& \kappa^{-1}(1-\beta G)\hat{\pi}_{t+2} = \kappa^{-1}(1-\beta G)G\hat{\pi}_{t+1} = G\hat{x}_{t+1} \\ \mu^d_{t+2} =& \left[\frac{\omega}{\kappa\Xi}(1-\beta G)+\frac{\kappa}{\Xi}\right]\hat{\pi}_{t+2} = \left[\frac{\omega}{\kappa\Xi}(1-\beta G)+\frac{\kappa}{\Xi}\right]G\hat{\pi}_{t+1} = G\mu^d_{t+1} \end{aligned}$$

The preceding results can be used in (48) evaluated at $t+1$ to give:

$$\begin{aligned} \left[\begin{array}{c} 1+\chi\beta(1-\chi)F_{\hat{V}} \\ +\chi F_\pi - (1-\chi)\widetilde{\sigma}^{-1}F_{\hat{x}} \end{array}\right]\mu^d_{t+1} &= \beta F_{\hat{\pi}}\hat{\pi}_{t+1} + \mu^d_{t+2} \\ &= \beta F_{\hat{\pi}}\left[\frac{\omega}{\kappa\Xi}(1-\beta G)+\frac{\kappa}{\Xi}\right]^{-1}\mu^d_{t+1} + G\mu^d_{t+1} \end{aligned}$$

which implies that the coefficients satisfy:

$$1+\chi\beta(1-\chi)F_{\hat{V}} + \chi F_\pi - (1-\chi)\widetilde{\sigma}^{-1}F_{\hat{x}} = \beta F_{\hat{\pi}}\left[\frac{\omega}{\kappa\Xi}(1-\beta G)+\frac{\kappa}{\Xi}\right]^{-1} + G$$

Using the solutions for $F_{\hat{x}}$ and $F_{\hat{V}}$ ((62) and (63)) in the left hand side and collecting terms gives:

$$1+\chi\left(1-\beta F_{\hat{d}}\right) - \frac{1-\chi}{\widetilde{\sigma}\kappa}F_{\hat{\pi}}\left(1-\beta F_{\hat{d}}\right) = \beta F_{\hat{\pi}}\left[\frac{\omega}{\kappa\Xi}(1-\beta G)+\frac{\kappa}{\Xi}\right]^{-1} + G$$

which implies that:

$$F_{\hat{\pi}} = \frac{1+\chi\left(1-\beta F_{\hat{d}}\right) - G}{\beta\left[\frac{\omega}{\kappa\Xi}(1-\beta G)+\frac{\kappa}{\Xi}\right]^{-1} + \frac{1-\chi}{\widetilde{\sigma}\kappa}\left(1-\beta F_{\hat{d}}\right)}$$

Equating this to the solution for F_π in (64) gives:

$$\frac{1+\chi\left(1-\beta F_{\hat{d}}\right)-G}{\beta\left[\frac{\omega}{\kappa\Xi}\left(1-\beta G\right)+\frac{\kappa}{\Xi}\right]^{-1}+\frac{1-\chi}{\widetilde{\sigma}\kappa}\left(1-\beta F_{\hat{d}}\right)}$$
$$=\frac{1+\chi-\left(1+\beta\chi\right)F_{\hat{d}}}{\beta\left(\frac{\omega}{\kappa\Xi}\left(1-\beta F_{\hat{d}}\right)+\frac{\kappa}{\Xi}\right)^{-1}+\left(1-\chi\right)\left(\kappa\widetilde{\sigma}\right)^{-1}\left(1-\beta F_{\hat{d}}\right)}$$

which reveals that

$$G=F_{\hat{d}}$$

D Global analysis of stable roots under time-consistent policy

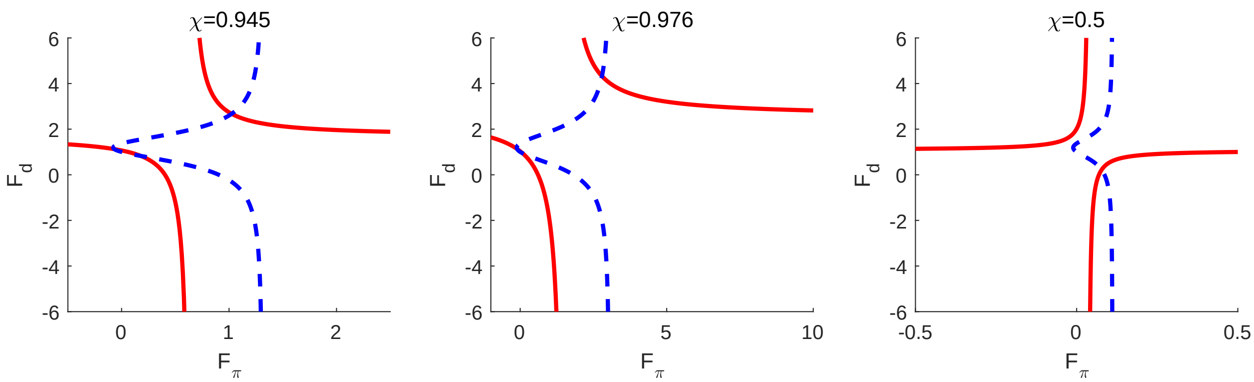

Figure 9: Global analysis of $F_{\hat{\pi}}$ and $F_{\hat{d}}$

Notes: Each panel plots the functions m and h defined by equations (19) (dashed blue line) and (20) (solid red line) respectively. Each panel examines a variant of the model for alternative values f χ: the baseline model ($\chi = 0.945$), the long-duration debt variant ($\chi = 0.976$) and a variant with very short debt duration ($\chi = 0.5$).

Figure 9 presents an analysis of the h and m functions (solid red and dashed blue lines respectively) over a broader range for $F_{\hat{\pi}}$ and $F_{\hat{d}}$ than considered in the main text. This demonstrates that, conditional on the values of the other model parameters, the stable Markov perfect equilibria are unique for the baseline and long duration values of χ as well as for a case in which the duration of bonds is very short ($\chi = 0.5$). In the very short duration case, $F_{\hat{d}} \approx 0$ which suggests that the 'debt stabilization bias' is significant in this case, consistent with the results of Leith and Wren-Lewis (2013) who analyze a model with one period debt ($\chi = 0$).

E Solution of the model accounting for the lower bound

This appendix details the solution of the model when the presence of the zero lower bound is accounted for. The algorithm is presented in subsection E.6, with preceding subsections defining notation and deriving key ingredients required for the solution.

E.1 First order conditions

The constrained loss minimization problem is:

$$\begin{aligned}
&\min \frac{1}{2}\left[\hat{\pi}_t^2+\omega \hat{x}_t^2\right] \\
&-\mu_t^x\left[\hat{x}_t-\mathbb{E}_t \hat{x}_{t+1}+\sigma(1-g)\left(\hat{R}_t-\mathbb{E}_t \hat{\pi}_{t+1}-r_t^*\right)\right] \\
&-\mu_t^\pi\left[\hat{\pi}_t-\kappa \hat{x}_t-\beta \mathbb{E}_t \hat{\pi}_{t+1}-u_t\right] \\
&-\mu_t^d\left[\hat{d}_t-\beta^{-1}\left(\hat{d}_{t-1}-\hat{\pi}_t\right)+(1-\chi) \hat{V}_t-\zeta^{-1} \bar{g}_t\right] \\
&-\mu_t^V\left[\hat{V}_t+\hat{R}_t-\chi \beta \hat{V}_{t+1}\right] \\
&-\mu_t^Z\left[\hat{R}_t-\beta^{-1}+1\right] \\
&+\beta \mathbb{E}_t \widetilde{\mathcal{L}}_{t+1}
\end{aligned}$$

where μ^Z is the Lagrange multiplier on the zero bound constraint.

The first order conditions are:

$$0=\hat{\pi}_t-\mu_t^\pi-\beta^{-1} \mu_t^d \tag{68}$$

$$0=\omega \hat{x}_t-\mu_t^x+\kappa \mu_t^\pi \tag{69}$$

$$\begin{aligned}
0=&\,\mu_t^x\left[\frac{\partial \mathbb{E}_t \hat{x}_{t+1}}{\partial \hat{d}_t}+\sigma(1-g) \frac{\partial \mathbb{E}_t \hat{\pi}_{t+1}}{\partial \hat{d}_t}\right]+\beta \mu_t^\pi \frac{\partial \mathbb{E}_t \hat{\pi}_{t+1}}{\partial \hat{d}_t} \\
&-\mu_t^d+\chi \beta \mu_t^V \frac{\partial \mathbb{E}_t \hat{V}_{t+1}}{\partial \hat{d}_t}+\beta \frac{\partial \mathbb{E}_t \widetilde{\mathcal{L}}_{t+1}}{\partial \hat{d}_t}
\end{aligned} \tag{70}$$

$$0=-(1-\chi) \mu_t^d-\mu_t^V \tag{71}$$

$$0=-\sigma(1-g) \mu_t^x-\mu_t^V-\mu_t^Z \tag{72}$$

$$0=\mu_t^Z\left(\hat{R}_t-\beta^{-1}+1\right) \tag{73}$$

where (73) is the contemporary slackness condition.

Applying the envelope condition implies that (70) can be written as:

$$\begin{aligned}
0=&\,\mu_t^x\left[\frac{\partial \mathbb{E}_t \hat{x}_{t+1}}{\partial \hat{d}_t}+\sigma(1-g) \frac{\partial \mathbb{E}_t \hat{\pi}_{t+1}}{\partial \hat{d}_t}\right]+\beta \mu_t^\pi \frac{\partial \mathbb{E}_t \hat{\pi}_{t+1}}{\partial \hat{d}_t} \\
&-\mu_t^d+\chi \beta \mu_t^V \frac{\partial \mathbb{E}_t \hat{V}_{t+1}}{\partial \hat{d}_t}+\mathbb{E}_t \mu_{t+1}^d
\end{aligned}$$

The policy function iteration technique has the following basic structure (the full algorithm is described below). First, outcomes for each element of the state space are solved, conditional on guesses for expectations and the derivatives of expectations with respect to debt. Then these outcomes are used to form a guess for the policy functions. Those guesses are then used to update the estimates of expectations and the derivatives of those expectations with respect to government debt. This process continues until the policy functions converge.

E.2 Conditional solutions

To simplify notation, the following conventions are adopted. Time subscripts are removed, with a prime used to denote outcomes in the following period. Then For variable z, let partial derivatives be represented as:

$$\mathcal{D}_z \equiv \frac{\partial \mathbb{E} z'}{\partial \hat{d}} \tag{74}$$

Consider first the solution under the assumption that the lower bound does not bind. We can stack the equations characterizing the equilibrium to give:

$$\underbrace{\begin{bmatrix} 0 & 1 & \widetilde{\sigma} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta^{-1} & 0 & 0 & 1-\chi & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -\beta^{-1} & -1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 & \kappa & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \beta\mathcal{D}_{\hat{\pi}} & \mathcal{D}_{\hat{x}} + \widetilde{\sigma}\mathcal{D}_{\hat{\pi}} & \chi\beta\mathcal{D}_{\hat{V}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1-\chi) & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\widetilde{\sigma} & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \hat{\pi} \\ \hat{x} \\ \hat{R} \\ \hat{V} \\ \hat{d}' \\ \mu^d \\ \mu^\pi \\ \mu^x \\ \mu^V \\ \mu^Z \end{bmatrix}}_{z}$$

$$= \underbrace{\begin{bmatrix} \mathbb{E}\hat{x}' + \widetilde{\sigma}\mathbb{E}\hat{\pi}' + \widetilde{\sigma} r^* \\ \beta\mathbb{E}\hat{\pi}' + u \\ \beta^{-1}\hat{d} \\ \chi\beta\mathbb{E}\hat{V}' \\ 0 \\ 0 \\ -\mathbb{E}\mu^{d\prime} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{C} \tag{75}$$

which can be solved for the vector of endogenous variables and Lagrange multipliers as:

$$z = M^{-1}C$$

In the case in which the zero bound does bind we have:

$$
\underbrace{\begin{bmatrix}
0 & 1 & \widetilde{\sigma} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta^{-1} & 0 & 0 & 1-\chi & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -\beta^{-1} & -1 & 0 & 0 & 0 \\
0 & \omega & 0 & 0 & 0 & 0 & \kappa & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & \beta\mathcal{D}_{\hat{\pi}} & \mathcal{D}_{\hat{x}} + \widetilde{\sigma}\mathcal{D}_{\hat{\pi}} & \chi\beta\mathcal{D}_{\hat{V}} & 0 \\
0 & 0 & 0 & 0 & 0 & -(1-\chi) & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\widetilde{\sigma} & -1 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}}_{\widetilde{M}}
\underbrace{\begin{bmatrix}
\hat{\pi} \\ \hat{x} \\ \hat{R} \\ \hat{V} \\ \hat{d}' \\ \mu^{d} \\ \mu^{\pi} \\ \mu^{x} \\ \mu^{V} \\ \mu^{Z}
\end{bmatrix}}_{z}
$$

$$
= \underbrace{\begin{bmatrix}
\mathbb{E}\hat{x}' + \widetilde{\sigma}\mathbb{E}\hat{\pi}' + \widetilde{\sigma}r^* \\
\beta\mathbb{E}\hat{\pi}' + u \\
\beta^{-1}\hat{d} \\
\chi\beta\mathbb{E}\hat{V}' \\
0 \\
0 \\
-\mathbb{E}\mu^{d\prime} \\
0 \\
0 \\
1-\beta^{-1}
\end{bmatrix}}_{\widetilde{C}} \tag{76}
$$

which can be solved for the vector of endogenous variables and Lagrange multipliers as:

$$z = \widetilde{M}^{-1}\widetilde{C}$$

Note that the differences between M and $\widetilde{M}$ and between C and $\widetilde{C}$ are isolated to the bottom row of each matrix.

E.3 State space and policy functions: notation

The description of the algorithm can be simplified by introducing some notation for the key objects that will be solved for.

The vector of endogenous variables are denoted by z, defined implicitly above, but explicitly here:

$$z \equiv \begin{bmatrix} \hat{\pi} \\ \hat{x} \\ \hat{R} \\ \hat{V} \\ \hat{d}' \\ \mu^d \\ \mu^\pi \\ \mu^x \\ \mu^V \\ \mu^Z \end{bmatrix}$$

The exogenous states are denoted s:

$$s \equiv \begin{bmatrix} r^* \\ u \end{bmatrix}$$

and full state vector for relevant policy functions is, $\tilde{s}$:

$$\tilde{s} \equiv \begin{bmatrix} s \\ \hat{d} \end{bmatrix}$$

The exogenous state is defined as a set of fixed values for the cost push shock and natural rate. Specifically, $S_r \equiv \left\{ r_1^* \quad \ldots \quad r_{n_r}^* \right\}$ and $S_u \equiv \{u_1 \quad \ldots \quad u_{n_u}\}$. The transition matrices for the Markov processes are Ω_r and Ω_u.

The combined (exogenous) state-space is given by $S = S_u \times S_r$ with transition matrix $\Omega = \Omega_r \otimes \Omega_u$.[24] The endogenous state is $\hat{d}$, which is discretized on a grid $S_d \equiv \left\{ \hat{d}_1 \quad \ldots \quad \hat{d}_{n_d} \right\}$, with $\hat{d}_i > \hat{d}_{i-1}$, $i = 2, \ldots, n_d$. The endogenous state is assumed to be ordered last. So the full state space is given by $\tilde{S} = S \times S_d$.[25] Thus, $\tilde{S}$ is a $n_{\tilde{s}} \times 3$ matrix, where $n_{\tilde{s}} \equiv n_s \times n_d$. The index of the element $\left\{ u_i, r_j^*, \hat{d}_k \right\} \in \tilde{S}$ is $(k-1) \times n_s + (j-1) \times n_u + i$.

This implies that the combined state can be written as:

$$\tilde{S} = \begin{bmatrix} S & \hat{d}_1 \mathbf{1}_{n_s} \\ \vdots & \vdots \\ S & \hat{d}_{n_d} \mathbf{1}_{n_s} \end{bmatrix}$$

where $\mathbf{1}_{n_s}$ is a $n_s \times 1$ unit vector.

This representation of the state space is useful for subsequent computations since approximation of expectations requires interpolation between grid points for the endogenous state, while integrat-

[24] The first n_u elements of the state space are $\{u_1, r_1^*\}, \ldots, \{u_{n_u}, r_1^*\}$, followed by $\{u_1, r_2^*\}, \ldots, \{u_{n_u}, r_2^*\}$ and so on.

[25] Thus the first $n_s \equiv n_u \times n_r$ elements are given by the triples $\left\{ u_1, r_1^*, \hat{d}_1 \right\}, \ldots, \left\{ u_{n_u}, r_{n_r}^*, \hat{d}_1 \right\}$, the next n_s elements are $\left\{ u_1, r_1^*, \hat{d}_2 \right\}, \ldots, \left\{ u_{n_u}, r_{n_r}^*, \hat{d}_2 \right\}$ and so on.

ing across the exogenous state S. Similar methods are used for the estimation of derivatives of expectations.

The objects of interest are policy functions. These are $n_{\tilde{s}} \times n_z$ matrices. Let a generic policy function be denoted $\mathbf{f}$.

E.4 Expectations

It is useful to define an 'expectation' operator that integrates out exogenous state uncertainty but holds the endogenous state vector constant:

$$\mathbb{E}_S \mathbf{f} \equiv \overline{\mathbf{f}}^{\mathbf{S}} \equiv \begin{bmatrix} \Omega & 0 & \dots & 0 & 0 \\ 0 & \Omega & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega & 0 \\ 0 & 0 & \dots & 0 & \Omega \end{bmatrix} \mathbf{f}$$

so that the 'bar' is used as a summary notation for expectations and the S superscript indicates that the expectation is computed with respect to the exogenous state variables only.

To compute the actual expectation requires conditioning on the solution for $\hat{d}'$ at the particular point in the state space. This can be done as follows.

- Extract the relevant column of $\mathbf{f}$ that corresponds to debt. Let this column vector be denoted $\mathbf{d}$. The elements of this vector denote the solutions $\hat{d}' \in z$ for each state $1, \dots, n_{\tilde{s}}$.
- Let the elements of $\mathbf{d}$ be denoted $\mathbf{d}_k, k = 1, \dots, n_{\tilde{s}}$. Let the exogenous state corresponding to this solution be $S_{<k>}$. For each k, perform the following:
  - Compute the indices in S_d that bracket this element.[26] This gives two indices $i_1, i_2 \in S_d$ with $1 \leq i_1 < i_2 \, (= i_1 + 1) \leq n_d$.
  - Compute the weights that should apply to each of these gridpoints (using linear interpolation). This gives $\phi_1 = \frac{\mathbf{d}_k - S_d(i_1)}{S_d(i_2) - S_d(i_1)}$ and $\phi_2 = 1 - \phi_1$.
  - Compute the indices of the elements of $\tilde{S}$ corresponding to the elements in $\tilde{S}$ for which (a) $S = S_{<k>}$ and (b) $S_d = S_d\,(i_1)$ and $S_d = S_d\,(i_2)$. Denote these indices as $\tilde{i}_1$ and $\tilde{i}_2$.
  - Estimate the expectation using linear interpolation as:

    $$\overline{\mathbf{f}}_{\mathbf{k},\cdot} = \phi_1 \overline{\mathbf{f}}^{\mathbf{S}}_{\tilde{\mathbf{i}}_1,\cdot} + \phi_2 \overline{\mathbf{f}}^{\mathbf{S}}_{\tilde{\mathbf{i}}_2,\cdot}$$

    where the subscript '$j,\cdot$' denotes the j-th row of a matrix.

The penultimate step (finding $\tilde{i}_1$ and $\tilde{i}_2$) can be aided by a pre-computation operation. To see this, recall that for each $k \in \{1, \dots, n_{\tilde{s}}\}$, there is an exogenous state, $S_{<k>}$. The indices

[26] Extrapolation is conceptually identical, but for the purposes of exposition, I assume that interpolation is required.

corresponding to different values of $\hat{d}$ for the same value of $S_{<k>}$ are multiples of n_s away from from k. This allows us to form a $n_{\tilde{s}} \times n_d$ matrix of indices – a 'lookup matrix', denoted Λ – as follows.

For each $k \in \{1, \ldots, n_{\tilde{s}}\}$

- Compute j, defined as the index of the grid point $\hat{d}'_k$ within S_p. Recall that $\hat{d}'$ is the final (third) state.
- For $m = 1, \ldots, n_s$, form the k-th row of Λ as:

$$\Lambda_{k,m} = k - (j - m)\, n_s$$

Then, in the computation of expectations, for each k the indices are found by setting $\tilde{i}_1 = \Lambda_{k,i_1}$ and $\tilde{i}_2 = \Lambda_{k,i_2}$.

E.5 Derivatives

The first order conditions depend on derivatives of expectations of the policy functions. To approximate these derivatives, a two-sided finite difference approach is used. The derivatives are computed in two steps. In the first step, two-sided finite difference derivatives of the static expectations are computed, using adjacent gridpoints for $\hat{d}$. In the second step, linear interpolation is used to approximate the derivatives at the relevant values of $\mathbf{d}$.

The first step is to approximate the derivative of the static expectation function $\bar{\mathbf{f}}^{\mathbf{S}}$. We seek the finite difference approximation to the derivative of $\bar{\mathbf{f}}^{\mathbf{S}}$ for each row $m = 1, \ldots, n_{\tilde{s}}$. First note that S_d is assumed to be formed of an evenly-spaced grid of values: $S_d = \left\{\hat{d}_1 \quad \ldots \quad \hat{d}_{n_d}\right\}$, with $\hat{d}_{i+1} = \hat{d}_i + h_d$. So the difference between each grid point is h_d.

Consider an m for which the corresponding element of S_d is $\hat{d}_i$ with $1 < i < n_d$, that is, an interior point. Then the 'static derivative' at point m is given by:

$$\mathbf{D}^{\mathbf{S}}_{\mathbf{m},\cdot} = \frac{1}{2h_d}\left(\bar{\mathbf{f}}^{\mathbf{S}}_{\mathbf{m}+\mathbf{n_s},\cdot} - \bar{\mathbf{f}}^{\mathbf{S}}_{\mathbf{m}-\mathbf{n_s},\cdot}\right)$$

Now consider the endpoints. For $1 \leq m \leq n_s$, $i = 1$ and a one-sided difference is used:

$$\mathbf{D}^{\mathbf{S}}_{\mathbf{m},\cdot} = \frac{1}{h_d}\left(\bar{\mathbf{f}}^{\mathbf{S}}_{\mathbf{m}+\mathbf{n_s},\cdot} - \bar{\mathbf{f}}^{\mathbf{S}}_{\mathbf{m},\cdot}\right)$$

Similarly, for $n_{\tilde{s}} - n_s + 1 \leq m \leq n_{\tilde{s}}$, $i = n_d$ and the one-sided approximation is given by:

$$\mathbf{D}^{\mathbf{S}}_{\mathbf{m},\cdot} = \frac{1}{h_d}\left(\bar{\mathbf{f}}^{\mathbf{S}}_{\mathbf{m},\cdot} - \bar{\mathbf{f}}^{\mathbf{S}}_{\mathbf{m}-\mathbf{n_s},\cdot}\right)$$

The second step is to form an estimate of the derivative at $\mathbf{d}$ using linear interpolation. This step is set out in the description of the algorithm below.

E.6 Algorithm

The objective of the algorithm is to solve for the policy function $\mathbf{f}$ by iterating directly on it.

1. Initialize a guess, $\mathbf{f}^{<0>}$, for the policy function $\mathbf{f}$.

2. Build the 'lookup matrix', Λ, as described above.

3. For each iteration $j = 1, \ldots$

   Update expectations

   (a) Update the guess for 'static' expectations. As described above, this integrates out exogenous state uncertainty but holds the endogenous state vector constant:

$$\overline{\mathbf{f}}^{\mathbf{S}<\mathbf{j}>} = \begin{bmatrix} \Omega & 0 & \ldots & 0 & 0 \\ 0 & \Omega & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & \Omega & 0 \\ 0 & 0 & \ldots & 0 & \Omega \end{bmatrix} \mathbf{f}^{<\mathbf{j-1}>}$$

   (b) Extract the vector of $\tilde{d}'$ values, $\mathbf{d}^{<\mathbf{j-1}>}$ as the relevant column of $\mathbf{f}^{<\mathbf{j-1}>}$.

   (c) Compute the indices and weights of the elements of S_p that bracket the values in $\mathbf{d}^{<\mathbf{j-1}>}$. Use the lookup matrix Λ to convert these into $n_{\tilde{s}} \times 2$ matrices of indicators and interpolation/extrapolation weights, denoted Υ and Φ respectively.

   (d) For each $m = 1, \ldots, n_{\tilde{s}}$: Compute expectations by extracting interpolation indices $[\iota_1 \;\; \iota_2] = \Upsilon_m$ and weights $[\phi_1 \;\; \phi_2] = \Phi_m$. Translate the interpolation weights into $\tilde{S}$ space by setting $\tilde{\iota}_1 = \Lambda_{m,\iota_1}$ and $\tilde{\iota}_2 = \Lambda_{m,\iota_2}$. Now set

$$\overline{\mathbf{f}}_{\mathbf{m},\cdot} = \phi_1 \overline{\mathbf{f}}^{\mathbf{S}<\mathbf{j}>}_{\tilde{\iota}_1,\cdot} + \phi_2 \overline{\mathbf{f}}^{\mathbf{S}<\mathbf{j}>}_{\tilde{\iota}_2,\cdot}$$

   Update the estimate of the derivative of expectations

   (e) Update the estimate of the 'static' derivatives, $\mathbf{D}^{\mathbf{S}}$, as described in E.5.

   (f) Compute the derivatives prevailing at $\mathbf{d}$ by linear interpolation. For each $m = 1, \ldots, n_{\tilde{s}}$, set:

$$\mathbf{D}_{\mathbf{m},\cdot} = \phi_1 \mathbf{D}^{\mathbf{S}}_{\tilde{\iota}_1,\cdot} + \phi_2 \mathbf{D}^{\mathbf{S}}_{\tilde{\iota}_2,\cdot}$$

   where the indices $\tilde{\iota}_1, \tilde{\iota}_2$ and weights ϕ_1, ϕ_2 are the same as in step 3d.

   Update the guess for the policy function

   (g) For each $m = 1, \ldots, n_{\tilde{s}}$:

      i. Extract latest guesses for expectations and their derivatives:

$$\mathbb{E}z' = \overline{\mathbf{f}}_{\mathbf{m},\cdot} \qquad \mathcal{D} = \mathbf{D}_{\mathbf{m},\cdot}$$

ii. Assume that the zero bound does not bind. Form M and C using $\mathbb{E}z'$ and $\mathcal{D}$ and solve the system (75) as $z = M^{-1}C$.

iii. Check whether this solution is indeed consistent with a positive interest rate by checking whether the relevant element of z exceeds the zero bound. If it does, proceed to step 3(g)v, otherwise proceed to step 3(g)iv.

iv. Compute the solution imposing the lower bound. Form $\widetilde{M}$ and $\widetilde{C}$ using $\mathbb{E}z'$ and $\mathcal{D}$ and solving (76) as $z = \widetilde{M}^{-1}\widetilde{C}$.

v. Load the solutions into the latest guess for the policy function:

$$\mathbf{f}^{<\mathbf{j}>}_{\mathbf{m},\cdot} = z$$

4. Check for convergence. If $\left|\mathbf{f}^{<\mathbf{j}>} - \mathbf{f}^{<\mathbf{j-1}>}\right| < \varepsilon$, set $\mathbf{f} = \mathbf{f}^{<\mathbf{j}>}$ and stop, otherwise set $j = j + 1$ and return to step 3.

E.7 Practical implementation

Solutions for the policy functions were found using a heuristic iterative procedure. Before finding the solution, the equilibrium distribution of the endogenous state variable is unknown. So some iterative experimentation with the end points of the grid (ie $\hat{d}_1$ and $\hat{d}_{n_d}$) was required to ensure that the policy functions did not require extrapolation beyond these points.

Similarly, some experimentation was required to choose the increment h_d between grid points for $\hat{d}$ in way that provided a balance between computational efficiency and accurate computation of the derivatives of expected policy functions. In practice, the model was solved on a coarse grid for $\hat{d}$ and the resulting solution used to produce a guess (using linear interpolation) for the policy function on a finer grid.