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Collateral requirements in central bank lending

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Abstract

In periods of stress, acute liquidity squeeze can manifest in the riskier segments of the credit market, even amid a surplus of aggregate liquidity. In such scenarios, central bank interventions that directly lower the risky interest rate can be more effective than reductions in the risk-free interest rate. Specifically, the central bank lends to the market at more favourable interest rates while simultaneously reducing the haircuts imposed on eligible collateral. In doing so, the central bank takes on greater credit risk, but achieves an outcome that is more productively efficient than simply reducing the risk-free interest rate.

Key words: Collateral, leverage, credit conditions, monetary policy, general equilibrium.

JEL classification: D53, E44, E51, E52, E58.

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1 Introduction

Central bank lending during times of crisis traditionally followed Bagehot's rule: lend freely to solvent institutions, against collateral that is good in normal times, and at high interest rates. The rule is designed so that the central bank can improve credit conditions without taking on any credit risk while also limiting moral hazard. Lately, central banks have begun to deviate from this approach in their response to COVID-19, conducting more direct lending to firms, against a broader class of collateral, and reducing the haircuts imposed on the collateral posted. Which is the more appropriate response? Should central banks take on greater credit risk in order to provide a larger stimulus?

I develop a model of central bank intervention in collateralized credit markets that combines the Credit Cycles in Kiyotaki and Moore (1997) and the Leverage Cycle in Geanakoplos (1997). I find that when the downturn is severe, it is optimal for the central bank to take on greater credit risk. Specifically, the central bank should intervene by lending at more favorable interest rates compared to the private market, while simultaneously lowering collateral requirements. The potential losses for the central bank on the loans extended falls on taxpayers.

In the model, firms borrow in order to purchase capital for production. There are two sources of financial frictions. First, firms can only borrow using simple debt contracts that are non-state-contingent. Second, each debt contract must be backed by one unit of capital as collateral in order to enforce repayment. Crucially, firms can choose one or more contracts from an entire spectrum of such simple debt contracts that differ only in the size of the promised repayment. When the promise exceeds the value of the collateral at the point of delivery, firms default. Since all debt contracts are backed by one unit of capital as collateral, a debt contract with a higher promised repayment implies greater credit risk for the lender. The price of each debt contract, the firms' choice of contracts, and thus the credit risk faced by the lenders are fully endogenized in competitive equilibrium. This is in contrast to many papers in the literature where the amount the firm can borrow against each unit of collateral is exogenously given and the lenders face fixed value-at-risk when lending.

The borrowing constraints faced by firms amplify negative aggregate productivity shocks. During a downturn, firms experience an endogenous reduction in their liquid wealth and their ability to borrow. As a consequence, firms hold too little capital in the downturn relative to the socially efficient level.

A central bank can intervene in this case by lending to firms against collateral at more favorable terms relative to the market. Central bank loans are funded through the issuance of a public liability to households that carries the safe rate of interest, and crucially, without the need to post collateral. The central bank is able to borrow at the risk-free rate without posting collateral because it is backed by the ability of the government to tax. As such, any ex post losses incurred on central bank loans will need to be recouped through recourse to the Treasury.

Since the amount firms can borrow against each unit of collateral is fully endogenized in the model, we can assess two different types of central bank interventions. The central bank can either reduce the risk-free interest rate on low loan-to-value debt contracts, or subsidize riskier loans with high loan-to-value. If the central bank is unwilling to bear any credit risk, then the size of the stimulus may be limited. Instead, optimal intervention during severe downturns requires the central bank to take on greater risk in order to provide a larger stimulus. By offering contracts that private lenders are able, but unwilling, to make during a downturn, the central bank can achieve significant gains in productive efficiency through its intervention.

1.1 Background: Central bank response to COVID-19

Given the enormous effect of the COVID-19 pandemic on the real economy, central banks across the world are intervening aggressively in credit markets to cushion the impact. In addition to pushing the benchmark policy rate to historic lows and conducting very large asset purchases, the Federal Reserve launched a number of new credit facilities in March 2020. For instance, the Primary and Secondary Market Corporate Credit Facilities are joint programs set up by the Federal Reserve and the US Treasury, whereby a Special Purpose Vehicle (SPV) is established to purchase qualifying bonds from eligible issuers either directly, or through the secondary market. The Treasury made the initial $10bn equity investment and the Federal Reserve committed to lend to the SPV on a recourse basis.[1] The stated goal of these facilities is to "support credit to employers" through either bond issuance, or by providing liquidity to the market for outstanding corporate bonds.[2] Similarly, the Term Asset-Backed Securities Loan Facility extends loans secured against eligible asset-backed securities; and the Main Street

[1] Source: Primary Market Corporate Credit Facility Term Sheet (original version published on March 23, 2020), available at: https://www.federalreserve.gov/newsevents/pressreleases/files/monetary20200323b1.pdf

[2] More details can be found at: www.federalreserve.gov

Lending Program purchases participation in loans originated by eligible lenders. In the UK, the Bank of England and HM Treasury announced a similar suite of measures, including the COVID-19 Corporate Financing Facility and the Term Funding Scheme with additional incentives for SMEs.

This paper joins a growing literature that examines the design and efficacy of these dramatic interventions.

1.2 Key Themes and Related Literature

The analytical framework in this paper draws from two seminal models that study how financial frictions amplify shocks to the real economy: Credit Cycles by Kiyotaki and Moore (1997); and The Leverage Cycle by Geanakoplos (1997, 2003, 2010). In Credit Cycles, borrowers secure loans subject to a borrowing constraint that is tied to their net worth. During a downturn, the fall in borrower's net worth reduces their ability to borrow, leading to a fall in their capital holdings which reduces future revenue and net worth, thus completing a dynamic feedback loop to even lower asset prices and net worth today. Crucially however, in the Credit Cycles model the collateral constraint faced by borrowers is exogenously given: the loan to value (LTV) against each unit of collateral is fixed, and the interest rate always equals the lender's rate of time preference (i.e. the risk-free interest rate). Geanakoplos (1997) provides a natural framework for endogenizing the collateral constraint. In the Leverage Cycle, loan to value - and equivalently leverage - is too high in normal times, and crashes when bad news arrive. The bad news about the value of borrower's collateral also reduce their liquid wealth and further restricts their purchasing power today.

In this paper I combine the key features of both the Credit Cycle and the Leverage Cycle framework. Firms and households are differentially productive with a durable capital good, and firms face an *endogenous* borrowing constraint when they try to secure loans against capital as collateral. A key theme here is that firms have the option to choose from an entire spectrum of collateralized debt contracts that differ in LTV and the contractual/promised interest rate. For given collateral, a high LTV debt contract trades off a larger loan today at the cost of a higher promised rate of interest. Plotting the promised interest rate against the LTV on each debt contract that arises in the competitive collateral equilibrium generates the *credit surface* (Figure 1.1). The credit surface summarizes the prevailing credit condition at each point in time; and shifts in the credit surface reflect changing circumstances. During a downturn, credit conditions deteriorate and the corresponding credit surface shifts inwards and upwards. So for a firm that was originally highly leveraged, it must now either accept a much larger haircut on the same collateral in order to maintain the same interest rate; or retain a similar (albeit slightly reduced) LTV at the cost of a much higher interest rate. Figure 1.2 plots the option-adjusted spread on US corporate debt against credit rating, and highlights how credit conditions - especially for riskier loans - could deteriorate during recessions in this fashion, in spite of central bank interventions.

The analytical framework in this paper accounts for the richer picture of credit market conditions that we observe in practice. Firms optimize simultaneously over their desired capital holding and the type of debt contract they wish to issue. Choosing a high LTV contract means a lower haircut and a larger sized loan today, in return for a higher promised interest rate to compensate lenders for the increased credit risk. Importantly, firms' choices over leveraged debt contracts determine what the optimal central bank intervention should look like during a downturn (Figure 1.3). Interventions targeted at the low-LTV end of the credit surface can reduce the risk-free interest rate faced by firms without exposing the central bank to significant credit risk. In contrast, subsidizing riskier high-LTV loans can provide a larger stimulus, but at the cost of potential losses for the central bank that will ultimately fall on taxpayers. I show that when the downturn is severe firms demand larger and riskier loans against their dwindling pool of collateral. In such cases, it is optimal for the central bank to take on more credit risk.

Figure 1.4 shows that during the current COVID-19 recession high-yield corporate debt issuance in the US, as a proportion of investment grade issuance, first collapsed at the onset of the crisis before recovering quickly after the Federal Reserve intervened aggressively in the credit markets. Gilchrist et al. (2020) find that the Federal Reserve's Secondary Market Corporate Credit Facility (SMCCF) has been effective in reducing credit spreads on corporate bonds.

Hanson et al. (2020) is a recent and closely related paper which models the business credit programs conducted by the Federal Reserve during the current recession. The authors also conclude that "in contrast to the classic lender-of-last-resort thinking that underpinned much of the response to the 2007–2009 global financial crisis, an effective policy response to the pandemic will require the government to accept the prospect of significant losses on credit extended to private sector firms". Similarly, Koulischer and Struyven (2014) argued for looser central bank collateral requirements during credit crunches to reduce risk spreads and increase output. In both of these papers, the

Figure 1.1: Model generated Credit Surface

The Credit Surface plots the promised interest rate on a collateralized debt contract against its loan to value (LTV) - both of which are determined endogenously in equilibrium. The *haircut* imposed on the collateral posted is defined as one minus the loan to value. The credit surface is composed of three parts. The horizontal segment to the left represents the risk-free spectrum of the market where the haircut is so high on the collateral posted that the loan is effectively risk-free. The increasing function in the middle of the credit surface captures the fact that as the haircut falls (and the LTV rises), the lender starts to take on more credit risk and must be compensated through a higher promised interest rate. Lastly, the credit surface becomes vertical once the maximum loan-to-value is reached. During normal times, interest rates are low across the credit surface. In a downturn, credit conditions deteriorate and the corresponding credit surface shifts inwards and upwards. So for a firm that was originally highly leveraged, it must now either accept a much larger haircut on the same collateral in order to maintain the same interest rate; or retain a similar (albeit slightly reduced) LTV at the cost of a much higher interest rate.

Figure 1.2: US Corporate Index Option-Adjusted Spread

Source: Federal Reserve Economic Data.
Figure 1.2 plots the option-adjusted spread on US corporate debt against credit rating, for the three most recent recessions in the US. It compares the average spread during each recession against the average in the preceding years. The figure shows that, in spite of central bank interventions, credit conditions deteriorate during recessions. The interest rate spreads on riskier loans are especially elevated.

collateral constraints faced by borrowers are exogenously fixed. By endogenizing the size of the loan that can be secured against each unit of collateral, I provide a novel and richer framework to assess the design of central bank credit facilities during a crisis.

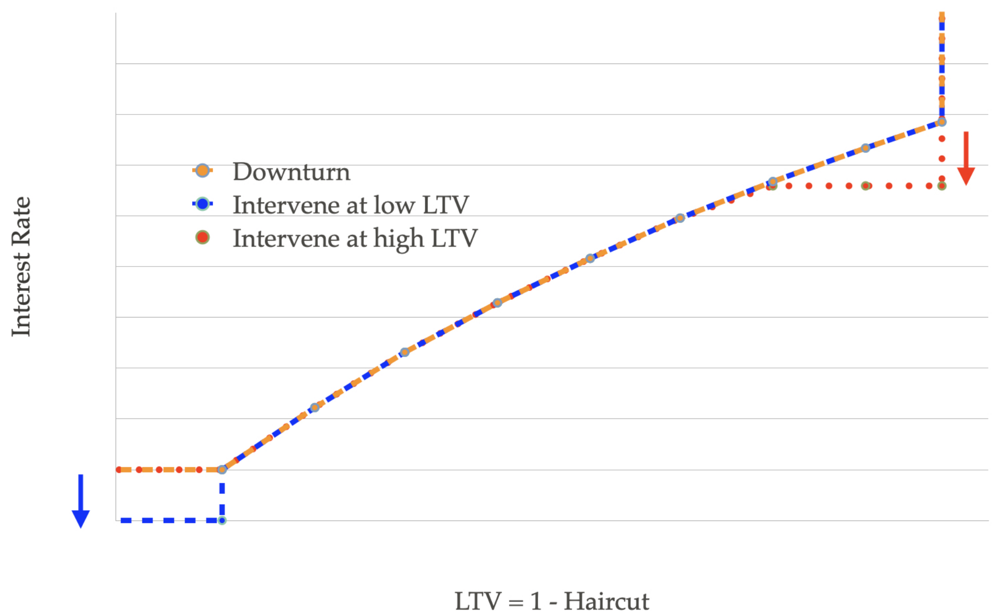
2 The Model

2.1 Agents, goods and uncertainty

Consider a discrete time general equilibrium model with two types of representative agents: households and firms, both risk-neutral and price taking. There are two goods: a numeraire consumption good which depreciates fully between periods (a "fruit") and a durable capital good (a "tree"). Let x_t and K_t denote the firms' holding of the consumption good and the capital good in period t respectively. The corresponding terms for the households are denoted by $\tilde{x}_t$ and $\tilde{K}_t$.[3] The total supply of capital in the economy is exogenously fixed at $\bar{K} = 1$.

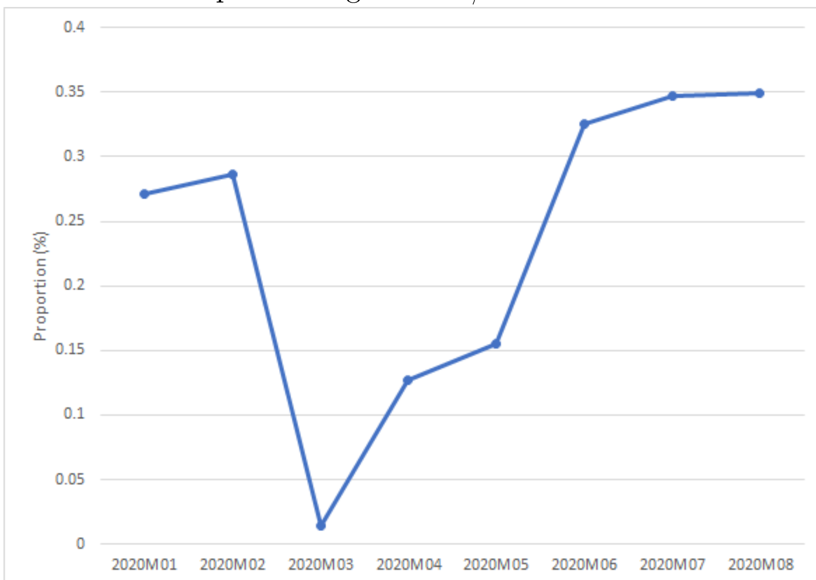
[3]Throughout the paper, I will use $\sim$ to differentiate variables associated with households and those associated with firms.

Figure 1.3: Central Bank Intervention during a downturn



In the model, a central bank can intervene in the collateralized debt market during a downturn by either reducing the interest rate on low loan-to-value (LTV) debt contracts (blue dashed line); or by reducing the interest rate on high LTV loans (red dotted line). Interventions aimed at the high LTV end of the market entail greater credit risk for the central bank.

Figure 1.4: US Corporate High Yield / Investment Grade Issuance



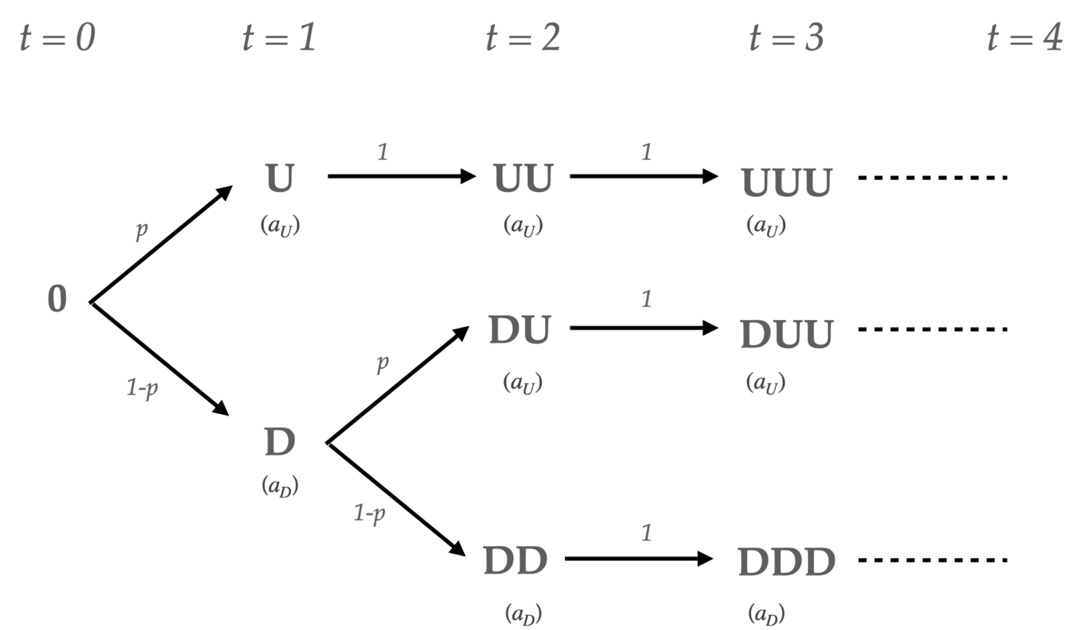
Source: SIFMA (Securities Industry and Financial Markets Association) and own computations. Figure 1.4 shows that during the current COVID-19 recession high-yield corporate debt issuance in the US, as a proportion of investment grade issuance, first collapsed at the onset of the crisis before recovering quickly after the Federal Reserve intervened aggressively in the credit markets.

Both households and firms can use the capital good as an input to produce the consumption good. The production technology is different between households and firms, but in both cases production occurs with one period delay. Specifically, households have a concave production technology $\tilde{y}_{t+1} = G\left(\tilde{K}_t\right)$, with $G \in \mathcal{C}^2$, $G'(\cdot) > 0$ and $G''(\cdot) < 0$, where $\tilde{y}_{t+1}$ is the units of the consumption good produced. Firms have a linear production technology, but one that is subject to uncertainty: $y_{t+1}\left(s^{t+1}\right) = a_{t+1}\left(s_{t+1}\right) K_t\left(s^t\right)$, where the productivity coefficient $a \in \{a_U, a_D\}$ can take either a high value a_U or a low value a_D depending on the state of nature in period $t+1$. The productivity of firms is the main source of uncertainty in the model. Figure 2.1 summarizes the timing and the structure of this uncertainty.

The model starts at period $t = 0$ with a certain state $S_0 = \{0\}$, before any production has occurred. From period $t = 1$ onward, there are at most two possible *states* $s_t \in S_t = \{U, D\}$: an Up-state with $a_t(U) = a_U$ and a Down-state with $a_t(D) = a_D$.[4] The path the economy can take is summarized by its *history* $s^t \in S^t$. In period $t = 1$,

[4]Technical side note: while the firm's productivity coefficient is state dependent and path/history independent: $a_t\left(s^t\right) = a_t\left(s_t\right) \quad \forall t, s^t$ (i.e. realized productivity today is independent of whether there was a Down-state previously); the price of capital *is* history dependent. Temporary shocks will have a persistent effect on real outcomes.

Figure 2.1: Timing and Uncertainty



there are two possible histories $S^1 = \{U, D\}$, that are reached with probability p and $(1-p)$ respectively. If the Up-state is reached at $t = 1$ (i.e. $s^1 = U$), then the economy will stay in state $s_t = U$, $\forall t > 1$. If instead the Down-state is reached at $t = 1$ (i.e. $s^1 = D$), then the state of the economy can either switch back to U with probability p in period $t = 2$ and stay there forever, or remain at D for all $t \geq 2$ with probability $(1-p)$. Therefore by $t = 2$ all uncertainty has resolved and there are three possible histories: $S^2 = \{UU, DU, DD\}$. These three histories captures the three possible paths for this economy. On the first path $0 \to U \to UU \to \dots$, the firm is found to be highly productive. On the second path $0 \to D \to DU \to \dots$, the firm suffers a temporary negative productivity shock in period 1, but recovers from period 2 onward. On the third path $0 \to D \to DD \to \dots$, the negative productivity shock is permanent.

2.2 Markets

Firms and households trade in three markets in each period t and history s^t. The first is a spot market for the numeraire consumption good, with price normalized to 1. The second is a spot market for the capital good, with price $q_t(s^t)$. Third, and most importantly, there is a credit market for one-period loans with the capital good

K serving as collateral. The key assumption here is that in this economy the only way to enforce repayment of loans is by requiring collateral.

Specifically, a loan contract at time t is composed of a *promise* to repay j units of the consumption good in all states of the world in period $t+1$, backed by 1 unit of the capital good as *collateral*. Given the limited enforcement of repayment, whenever the promised amount exceeds the price of the collateral posted, the borrower will simply default on the loan and hand over the collateral posted. The actual *delivery* on contract j at time $t+1$ is therefore given by:

$$\delta_{j,t+1}\left(s^{t+1}\right) = \min\left\{j, q_{t+1}\left(s^{t+1}\right)\right\} \tag{2.1}$$

At each period t and each history s^t an entire spectrum of such debt contracts are available, indexed by j the size of the promise (and all backed by 1 unit of the capital good as collateral). The price of each loan contract j, denoted by $\pi_{j,t}\left(s^t\right)$, is determined endogenously in equilibrium. Equivalently, one can interpret $\pi_{j,t}\left(s^t\right)$ as the size of the collateralized loan that promises to repay j next period.

A key feature of this set-up for the credit market is that for each loan j, we can also compute the *promised interest rate*:

$$1 + r_{j,t}\left(s^t\right) := \frac{j}{\pi_{j,t}\left(s^t\right)} \tag{2.2}$$

and the *loan to value* (or equivalently 1 minus the *haircut* imposed on the collateral):

$$\begin{aligned} LTV_{j,t}\left(s^t\right) &:= \frac{\pi_{j,t}\left(s^t\right)}{q_t\left(s^t\right)} \\ &=: 1 - Haircut_{j,t}\left(s^t\right) \end{aligned} \tag{2.3}$$

Plotting the loan to value on the x-axis against the promised interest rate on the y axis for each contract j generates the *credit surface* in Figure 2.2. The credit surface is composed of three parts.[5] The horizontal segment to the left represents the risk-free spectrum of the market where the haircut is so high on the collateral for given promise $j \leq \underline{j} := \min_{s^{t+1}}\left(q_{t+1}\left(s^{t+1}\right)\right)$ that the loan is effectively risk-free. I refer to $\underline{j}$ as the *max-min leverage contract*, because it is the maximum promise against one unit of collateral

[5]This concept of the "Credit Surface" was introduced in Geanakoplos and Zame (2014) and Fostel and Geanakoplos (2015). Geanakoplos (2016) discusses the implications of the credit surface for monetary policy.

that still minimizes credit risk to the lender. The increasing function in the middle of the credit surface captures the fact that as the haircut falls (and the LTV rises), the lender starts to take on more credit risk and must be compensated through a higher promised interest rate. Lastly, the credit surface becomes vertical once the maximum loan-to-value is reached. This cap on loan-to-value arises endogenously because the highest credible promise $\bar{j} := \max_{s^{t+1}} \left(q_{t+1}\left(s^{t+1}\right)\right)$ is given by the maximum possible valuation of the collateral next period. I refer to $\bar{j}$ as the *maximum leverage* contract. From the definition of $\underline{j}$ and $\bar{j}$ it is evident that how much a borrower can promise to repay on their collateralized debt contract depends on the future price of capital. So any changes in the expected price of capital tomorrow can influence credit conditions today. Firms' choice of contracts along the credit surface also determines the credit risk faced by lenders in equilibrium. The possibility, and the occurrence, of defaults play an essential role in the model.

Without loss of generality, I show later that the optimal contract for firms lies on the increasing segment of the credit surface: $j^* \in \left[\underline{j}, \bar{j}\right]$. Intuitively, when collateral is scarce, the max-min leverage contract $\underline{j}$ is preferable to contracts $j < \underline{j}$ because it allows the firm to borrow more at the same interest rate against the same unit of collateral. On the other side of the credit surface, any contract $j > \bar{j}$ will have the exact same expected delivery as contract $\bar{j}$ and will thus be priced the same: $\pi_{j>\bar{j}}\left(s^t\right) = \pi_{\bar{j}}\left(s^t\right)$. So using the maximum leverage contract $\bar{j}$ allows the firm to borrow the same amount as contracts $j > \bar{j}$ but with a lower promised interest rate.

Finally, let $\varphi_{j,t}\left(s^t\right) > 0$ denote that the firm is selling the contract j (i.e. borrowing an amount equal to $|\varphi_j|\,\pi_j$), and $\varphi_{j,t}\left(s^t\right) < 0$ indicate the firm is buying the contract j (i.e. lending $|\varphi_j|\,\pi_j$). The corresponding notation for households is $\tilde{\varphi}_{j,t}\left(s^t\right)$.

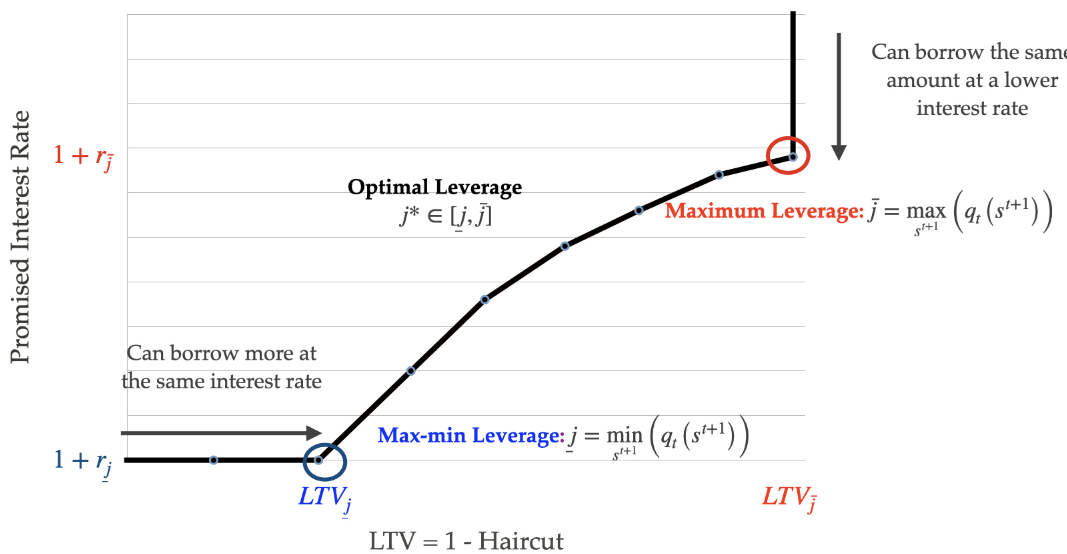
2.3 Agent Optimization

By assumption both the representative firm and household are price taking and risk neutral ($u\left(x\right) = x$). The firm's optimization problem is given by:

$$\max_{\{x_t(s^t), K_t(s^t), \{\varphi_{j,t}(s^t)\}\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t \in S^t} \left[u\left(x_t\left(s^t\right)\right) \Pr\left(s^t | s^{t-1}\right)\right] \tag{2.4}$$

subject to:

Figure 2.2: Credit Surface - Choice of Leverage



The credit surface plots the loan to value on the x-axis against the promised interest rate on the y axis for each collateralized contract j. Contract $\underline{j} := \min_{s^{t+1}} \left(q_{t+1}\left(s^{t+1}\right)\right)$ promises to repay the lender an amount equal to the minimum possible value of the collateral next period. Therefore $\underline{j}$ is the maximum promise that minimizes credit risk to the lender - a max-min leverage contract. Contract $\bar{j} := \max_{s^{t+1}} \left(q_{t+1}\left(s^{t+1}\right)\right)$ promises to repay the lender the maximum possible value of the collateral next period. $\bar{j}$ is the maximum credible promise the borrower can make, and is therefore the maximum leverage contract. Due to the scarcity of collateral, firms will optimal choose a contract in the interval $\left[\underline{j}, \bar{j}\right]$ in equilibrium.

1. Flow of funds constraint:

$$\begin{aligned}
&\overbrace{x_t\left(s^t\right)}^{\text{consumption}} + \overbrace{q_t\left(s^t\right)K_t\left(s^t\right)}^{\text{K expenditure}} - \overbrace{\sum_{j\in J}\varphi_{j,t}\left(s^t\right)\pi_{j,t}\left(s^t\right)}^{\text{loans}} \\
= &\underbrace{e_t\left(s^t\right)}_{\text{endowment}} + \underbrace{\left(a\left(s_t\right)+q_t\left(s^t\right)\right)K_{t-1}\left(s^{t-1}\right)}_{\text{K income}} - \underbrace{\sum_{j\in J}\varphi_{j,t-1}\left(s^{t-1}\right)\delta_{j,t}\left(s^t\right)}_{\text{delivery on loans}} \\
=: &w_t \quad \text{[liquid wealth at start of period]}
\end{aligned} \tag{2.5}$$

2. Collateral constraint:

$$\sum_j \max\left(\varphi_{j,t}\left(s^t\right),0\right) \leq K_t\left(s^t\right) \tag{2.6}$$

Relative to the standard general equilibrium model, the *collateral constraint* is the key addition here. The collateral constraint states that the total number of loans the firm takes out across all contracts j, $\sum_j \max\left(\varphi_{j,t}\left(s^t\right),0\right)$, must be weakly less than the units of capital it holds. This is due to the requirement that each loan must be backed by one unit of capital. The collateral constraint is asymmetric: only borrowers need to post collateral, and lenders do not; hence the max operator in the expression.

The representative household solves a similar optimization problem, replacing only the uncertain CRTS production function for firms with households' certain but concave production function $G(\cdot)$. For brevity, I omit the (s^t) notation where appropriate from this point onward:

$$\max_{\left\{\tilde{x}_t,\tilde{K}_t,\{\tilde{\varphi}_{j,t}\}_j\right\}_t} E_0\left[\sum_{t=0}^{\infty}\beta^t u\left(\tilde{x}_t\right)\right] \tag{2.7}$$

subject to:

$$\tilde{x}_t + q_t\tilde{K}_t - \sum_{j\in J}\tilde{\varphi}_{j,t}\pi_{j,t} = \tilde{e}_t + G\left(\tilde{K}_{t-1}\right) + q_t\tilde{K}_{t-1} - \sum_{j\in J}\tilde{\varphi}_{j,t-1}\min\{j,q_t\} =: \tilde{w}_t \tag{2.8}$$

$$\sum_j \max\left(\tilde{\varphi}_{j,t},0\right) \leq \tilde{K}_t \tag{2.9}$$

2.4 Collateral Equilibrium

The solution concept of interest is a collateral equilibrium. Formally, a *Collateral Equilibrium* is a vector consisting of the price of capital, contract prices, consumption, capital holdings and contract trades $\left(q\left(s^{t}\right),\pi\left(s^{t}\right)\right),\left(x\left(s^{t}\right),K\left(s^{t}\right),\varphi\left(s^{t}\right)\right),\left(\tilde{x}\left(s^{t}\right),\tilde{K}\left(s^{t}\right),\tilde{\varphi}\left(s^{t}\right)\right)$ $\in\left(R_{+}\times R_{+}^{J}\right)\times\left(R_{+}\times R_{+}\times R^{J}\right)\times\left(R_{+}\times R_{+}\times R^{J}\right)$ s.t. at all period t and all history s^{t}:

1. All agents optimize (equations 2.4 to 2.9); and
2. All markets clear:
   (a) Consumption good market:

$$\left(x_{t}\left(s^{t}\right)-e_{t}\left(s^{t}\right)\right)+\left(\tilde{x}_{t}\left(s^{t}\right)-\tilde{e}_{t}\left(s^{t}\right)\right)=G\left(\tilde{K}_{t-1}\left(s^{t-1}\right)\right)+a\left(s_{t}\right)K_{t-1}\left(s^{t-1}\right) \tag{2.10}$$

   (b) Capital good market:

$$\tilde{K}_{t}\left(s^{t}\right)+K_{t}\left(s^{t}\right)=\bar{K}\equiv 1 \tag{2.11}$$

   (c) Collateralized debt market:

$$\tilde{\varphi}_{j,t}\left(s^{t}\right)+\varphi_{j,t}\left(s^{t}\right)=0 \quad \forall j \tag{2.12}$$

2.5 First-best Benchmark

Before characterizing the collateral equilibrium of the model, it is useful to establish the first-best outcome as a frame of comparison. In the first-best, I assume there is a central planner who assigns capital between firms and households in each period and each history in order to maximize the sum of their discounted utility (suppressing the $\left(s^{t}\right)$ notation again for brevity):

$$\begin{aligned} \max_{\left\{K_{t},\tilde{K}_{t}\right\}_{t\geq -1,s^{t}\in S^{t}}} \quad & E_{0}\left[\sum_{t=0}\beta^{t}\left(\tilde{x}_{t}+x_{t}\right)\right] \\ \text{s.t.} \quad & \tilde{x}_{t}+x_{t}=\tilde{e}_{t}+e_{t}+G\left(\tilde{K}_{t-1}\right)+a_{t}K_{t-1} \quad \forall t\geq 0 \\ & K_{t}+\tilde{K}_{t}=\bar{K}=1 \quad \forall t\geq -1 \end{aligned}$$

The first-best level of capital holdings therefore equalizes the marginal productivity of capital between households and firms:

$$G^{'}\left(\tilde{K}_t^{fb}\right) = E_t\left[a_{t+1}\right] \tag{2.13}$$

$$K_t^{fb} = \bar{K} - \tilde{K}_t^{fb} \tag{2.14}$$

The first-best benchmark also coincides with the decentralized solution when we remove the two main sources of productive inefficiencies in the collateral equilibrium: (1) the endogenous collateral constraint which restricts agents' ability to borrow; and (2) limitations in the firm's liquid wealth (which arise both exogenously at the start, and then endogenously in certain histories, e.g. during the downturn at $s^1 = D$). Specifically, if we assume (1) firms and households are sufficiently well-endowed in every history to ensure they can always consume in addition to any desired capital purchases and lending; and (2) there exists a perfect enforcement mechanism for debt repayments, so promises are always honored and there is no need for posting any collateral; then the equilibrium price of capital q_t will adjust to equate the marginal return of capital between firms and households: $\beta E_t\left[G^{'}\left(\tilde{K}_t\right) + q_{t+1}\right] = q_t\left(s^t\right) = \beta E_t\left[a_{t+1} + q_{t+1}\right]$, and it is easy to verify that $\tilde{K}_t = \tilde{K}_t^{fb} \quad \forall t \geq 0$.

3 Characterizing the Collateral Equilibrium

The first-order conditions for both the firm and household's optimization problems are reported in Appendix 6.1. I impose assumptions on preferences, production functions and endowments as follows. These assumptions are fairly weak (with risk neutrality arguably being the strongest). The assumptions provide analytical tractability and help restrict attention to equilibria of interest.

3.1 Assumptions and household's behavior in equilibrium

Assumption $A1$ [Risk neutrality]: $u^{'}\left(x\right) = u^{'}\left(\tilde{x}\right) = 1$.

Assumption $A2$ [Common discounting]: $1 > \tilde{\beta} = \beta > 0$.

Assumption $A3$ [Production functions]:

1. $1 > p > \frac{a_D}{a_U}$

2. $G \in \mathcal{C}^2$, with $G'(0) = a_U > a_D = G'\left(\bar{K}\right)$ and $G''(\cdot) < 0$.

Assumption A3.1 imposes an upper bound on the probability of the downstate $(1-p)$. Assumption A3.2 states that the households are as productive as the firm in the Up-state if they hold no capital; but if they hold the entire stock then their marginal productivity becomes as low as the firm in the Down-state.

Assumption *A4* [Endowments]:

1. $\tilde{K}_{-1} = \bar{K}$ and $K_{-1} = 0$.
2. $e_0 = w_0 \in (0, \beta^2 a_D K_0^{fb}]$ and $e_t(s^t) = 0 \; \forall s^t \in S^t$ and $\forall t \geq 1$.
3. $\tilde{e}_t(s^t) > \frac{\beta}{1-\beta} a_U$, $\forall t \geq 0$ and $\forall s^t \in S^t$.

The first part of assumption A4 states that households start period 0 with the entire stock of capital. The second part restricts the firms' endowment of consumption good in period 0 (their starting liquid wealth) and stipulates further that firms receive no more exogenous endowments from period 1 onward. These two parts combined imply that the firms will borrow from households in order to invest in the capital good. The third part of assumption A4 ensures that households are sufficiently well endowed to lend and consume in all periods and histories.

Lastly, I impose a *no-bubble condition* to rule out an ever increasing price for capital in equilibrium, whereby any arbitrary price for capital today can be justified by an sufficiently high expected price tomorrow:

$$\lim_{t \to \infty} q_t\left(s^t\right) < \infty \quad \forall s^t \in S^t \tag{3.1}$$

Under assumptions A1-4 and the no-bubble condition, we can characterize the behavior of the representative household as follows:

Lemma 1. ***[Household behavior in equilibrium]***

1. *The household always consumes:* $\tilde{x}_t(s^t) > 0 \quad \forall t \geq 0$ *and* $\forall s^t \in S^t$.
2. *The household never borrows:* $\tilde{\varphi}_{j,t}(s^t) < 0 \quad \forall t \geq 0$ *and* $\forall s^t \in S^t$.
3. *The household is always indifferent between consuming and purchasing another*

marginal unit of capital, so the equilibrium price of capital is given by:

$$
\begin{aligned}
q_t &= E_t\left[\frac{\tilde{\beta}u'\left(\tilde{x}_{t+1}\right)}{u'\left(\tilde{x}_t\right)}\left(G'\left(\tilde{K}_t\right)+q_{t+1}\right)\right] \\
&= E_t\left[\beta\left(G'\left(\tilde{K}_t\right)+q_{t+1}\right)\right]
\end{aligned}
\tag{3.2}
$$

4. *The household is always indifferent between consuming and lending, so the equilibrium price of contract j (i.e. the size of the loan granted for a promised repayment of j next period, collateralized by 1 unit of capital) is given by:*

$$
\begin{aligned}
\pi_{j,t} &= E_t\left[\frac{\tilde{\beta}u'\left(\tilde{x}_{t+1}\right)}{u'\left(\tilde{x}_t\right)}\delta_{j,t+1}\right] \\
&= E_t\left[\beta\min\left\{j,q_{t+1}\right\}\right] \quad \forall j
\end{aligned}
\tag{3.3}
$$

Proof. Very briefly, the household always consumes and never borrows because by assumption they are given very large exogenous endowments in every period and every history. The price of capital and the price of contract j are both expressed in the form of standard asset pricing equations (stochastically discounted cash flows, from the household's perspective), which can be derived directly from household's first order conditions (Appendix 6.1). More details can be found in Appendix 6.2. □

3.2 Collateral Value and the Optimal Choice of Leverage for firms

For households, who never need to borrow, the capital good is simply a means to transfer wealth into the next period through production or resale. For firms however, the capital good serves as both an investment opportunity and the only means to secure loans (courtesy of the collateral constraint - equation 2.6). Consequently, from the firm's perspective, the price of capital reflects both its stochastically discounted cash flow and its value as collateral.

When the firm purchases capital on leverage using contract j, the standard asset pricing equation yields:

$$
q_t-\pi_{j,t}=E_t\left[\frac{\beta\gamma_{t+1}}{\gamma_t}\left(a_{t+1}+q_{t+1}-\delta_{j,t+1}\right)\right]
$$

where the left-hand-side of the expression is the down-payment required, and the right-hand-side gives the stochastically discounted cash flow from the transaction, composed of dividends plus the price next period and minus actual delivery on debt next period. γ_t denotes the marginal utility of income of firms in period t and history s^t. Re-arranging this equation illustrates how the price of capital can be decomposed into its fundamental value and its collateral value to firms.

Lemma 2. *[**Collateral Value**] When firms' capital holding is strictly positive $K_t > 0$ and the collateral constraint is binding, the price of capital can be expressed as:*

$$
\begin{aligned}
q_t &= E_t\left[\frac{\beta\gamma_{t+1}}{\gamma_t}\left(a_{t+1}+q_{t+1}\right)\right] + \left\{\pi_{j,t} - E_t\left[\frac{\beta\gamma_{t+1}}{\gamma_t}\delta_{j,t+1}\right]\right\} \\
&= \underbrace{E_t\left[\frac{\beta\gamma_{t+1}}{\gamma_t}\left(a_{t+1}+q_{t+1}\right)\right]}_{\text{fundamental value}} + \underbrace{\left\{E_t\left[\frac{\beta\tilde{\gamma}_{t+1}}{\tilde{\gamma}_t}\delta_{j,t+1}\right] - E_t\left[\frac{\beta\gamma_{t+1}}{\gamma_t}\delta_{j,t+1}\right]\right\}}_{\text{collateral value when using contract } j}
\end{aligned} \tag{3.4}
$$

where γ_t denotes the marginal utility of income of firms in period t and history s^t; and $\tilde{\gamma}_t = \tilde{\gamma}_{t+1} = 1$ for households. Note that since the firm does not necessarily consume in every history, in general $\gamma_t \neq u'(x_t) = 1$.

Let λ_t denote the Lagrangian multiplier for the firm's collateral constraint at period t and history s^t, and γ_t the multiplier for its budget constraint, then the equilibrium collateral value can be expressed as:

$$
\textit{Collateral Value} := \max_j\left\{E_t\left[\left(\frac{\beta\tilde{\gamma}_{t+1}}{\tilde{\gamma}_t} - \frac{\beta\gamma_{t+1}}{\gamma_t}\right)\delta_{j,t+1}\right], 0\right\} = \frac{\lambda_t}{\gamma_t} \tag{3.5}
$$

In other words, collateral value is the dollar value the firm attaches to a marginal relaxation of its collateral constraint.

Proof. Both equations can be derived directly from firm's first order conditions (Appendix 6.1). □

Equation 3.4 highlights the dual role capital plays for firms, and equation 3.5 shows that the collateral value is positive whenever the firm's collateral constraint is binding ($\lambda_t > 0$). Furthermore, when the collateral constraint is binding, the firm will choose the debt contract j that maximizes collateral value. Thus even though an entire spectrum of contracts $j \in R_+$ is priced, potentially only a single contract (if any) will be actively traded in equilibrium. Lemma 3 below examines how the firm chooses the optimal contract j^*.

Lemma 3. ***[Optimal leverage]*** *Suppose in equilibrium the collateral constraint is binding* ($\lambda_t > 0$)*, then:*

1. *Without loss of generality, the firm will only consider contracts:*

$$j \in \left[\underline{j}, \bar{j}\right]$$

where $\underline{j} := \min_{s^{t+1}} \left(q_t \left(s^{t+1}\right)\right)$ *is the max-min leverage contract, and* $\bar{j} := \max_{s^{t+1}} \left(q_t \left(s^{t+1}\right)\right)$ *is the maximum leverage contract.*

2. *The firm's optimal choice of debt contract is given by:*

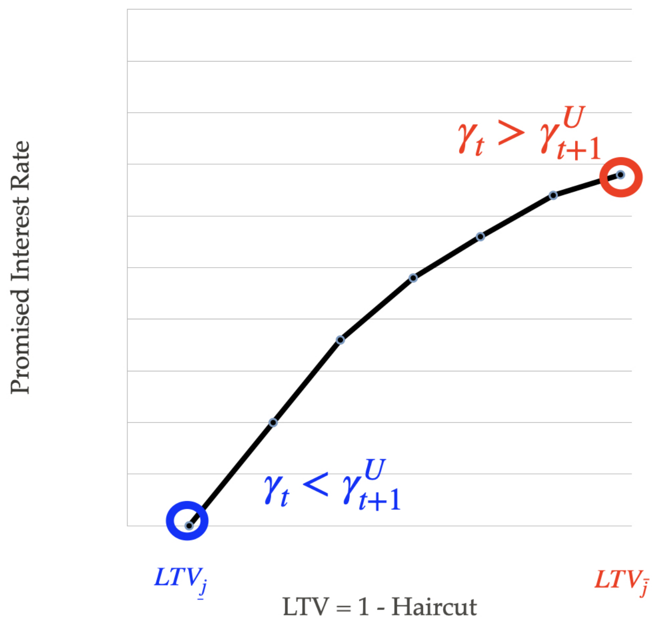
$$j^* = \begin{cases} \bar{j} & \text{if } \gamma_t > \gamma_{t+1}^U \\ \underline{j} & \text{if } \gamma_t < \gamma_{t+1}^U \\ \left[\underline{j}, \bar{j}\right] & \text{if } \gamma_t = \gamma_{t+1}^U \end{cases}$$

where γ_{t+1}^U *is the firm's marginal utility of income in period* $t+1$ *if the Up-state is realized (i.e.* $s_{t+1} = U$*).*

Lemma 3 states that in general the firm will choose between either the left-hand or the right-hand side kink of the credit surface (see Figure 3.1). In the knife edge case where $\gamma_t = \gamma_{t+1}^U$, the firm is indifferent between all debt contracts $j \in \left[\underline{j}, \bar{j}\right]$. A formal proof can be found in Appendix 6.3.

To see why the choice between the max-min leverage contract $\underline{j}$ and the maximum leverage contract $\bar{j}$ depends only on the marginal utility of income in the Up-state tomorrow γ_{t+1}^U, and not the Down-state γ_{t+1}^D, note that with buying K using contract $\underline{j} = q_{t+1}^D$ the firm's cash flow in period $t+1$ is given by: $\begin{pmatrix} a_U + q_{t+1}^U - q_{t+1}^D \\ a_D \end{pmatrix}$. In comparison, if the firm bought K using contract $\bar{j} = q_{t+1}^U$ instead, its cash flow would be given by $\begin{pmatrix} a_U \\ a_D \end{pmatrix}$. The net difference between the two is $\begin{pmatrix} q_{t+1}^U - q_{t+1}^D \\ 0 \end{pmatrix}$, an Up-Arrow security that delivers only in the Up-state. Therefore the firm would choose the max-min contract $\underline{j}$ when the marginal utility of income in the Up-state tomorrow is sufficiently high.

Figure 3.1: Choosing a point on the credit surface



Due to linear preference, when firms borrow against capital as collateral, they do so with either the max-min leverage contract $\underline{j}$ or the maximum leverage contract $\bar{j}$. Firms prefer the max-min leverage contract when their marginal utility of income today γ_t is lower than their marginal utility of income tomorrow in the Up-state γ_{t+1}^U. Intuitively, this is because the max-min leverage contract entails a larger downpayment today, but allows the firm to transfer more resources into the Up-state tomorrow. In contrast, in any ensuing Down-state, firms default - regardless of whether they borrowed using the max-min leverage contract or the maximum leverage contract today.

3.3 Illustrative Example

Having described households' behavior and the choice of firms with regard to collateralized debt contracts, we can now proceed to characterize the collateral equilibrium along the different possible paths of the economy. The main complication that arises when solving the model is that prices and choices today depend on the expectation of prices tomorrow. The model is thus solved through backward induction, from deterministic steady states that can be eventually reached after the uncertainty has been fully resolved[6], back to the uncertain histories in periods $t = 0, 1$. The path $(0 \rightarrow D \rightarrow DU \rightarrow DUU \ldots)$, whereby the firm's productivity suffers a temporary negative shock in history D before recovering permanently to a_U in period $t \geq 2$, is the most interesting trajectory for our purpose. I illustrate the key features of the collateral equilibrium with a simple numerical example below, before extending the key results in the form of general propositions.

Consider an economy characterized by the set of parameters shown in Table 1. Restricting attention to the path where firms suffer a negative, but temporary, productivity shock $(0 \rightarrow D \rightarrow DU \rightarrow DUU \ldots)$, Figure 3.2 plots the dynamics of the firms' capital holding in the collateral equilibrium against the first-best benchmark. In the initial period 0, firms start with no capital and very limited endowment of the numeraire consumption good, so they borrow using the maximum leverage contract to purchase as much capital as they can. Endogenously, the model generates two further instances of productive inefficiency along this temporary downturn path. First, firms default in history D, their ability to borrow and leverage collapse and firms' holding of capital falls even further. Second, even when the uncertainty has been fully resolved at history DU and beyond, the recovery to the first-best level is gradual. I address the causes of each of these two distortions in turn.

3.4 The Downturn $\left(s^1 = D\right)$

During the downturn (history D), firms experience low productivity. The price of capital falls because a larger proportion of capital is transferred to households, whose production function exhibits diminishing returns. Firms would like to hold more capital in order to produce next period during the potential recovery phase, but cannot due to a tightening borrowing constraint arising from the reduction in their liquid wealth and

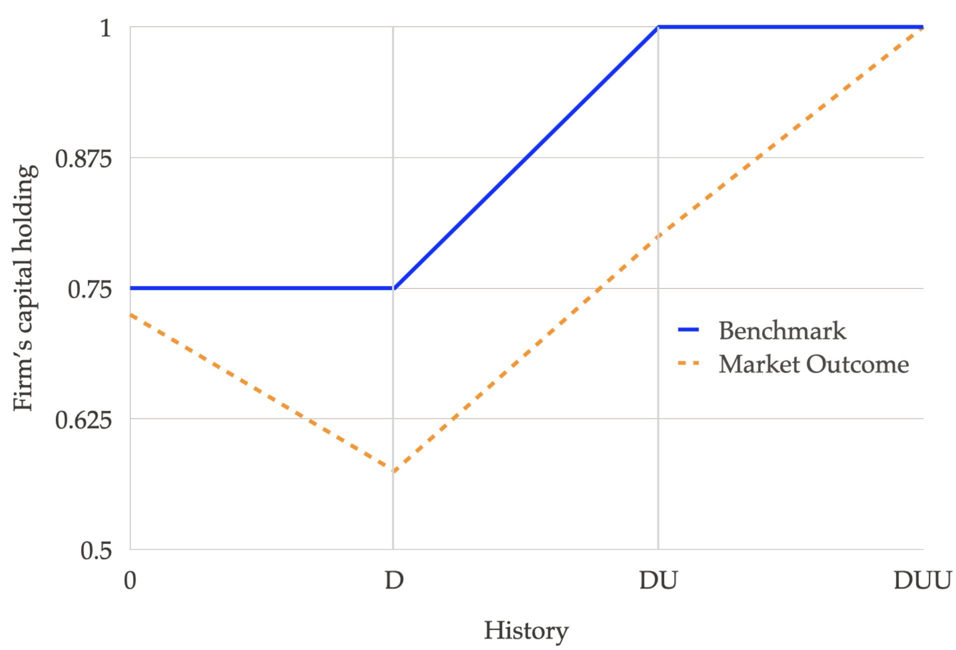
[6] A full characterization of the deterministic steady states can be found in Appendix 6.4.

Table 1: Parameters for Illustrative Example

Parameter	Value	Parameter	Value
p	0.75	$\tilde{\beta}$	0.8
a_U	1.5	β	0.8
a_D	0.75	e_0	0.75
$G\left(\tilde{K}\right)$	$a_U\tilde{K} - \left(\frac{a_U - a_D}{2}\right)\tilde{K}^2$	$\tilde{e}_{t\geq 0}$	6

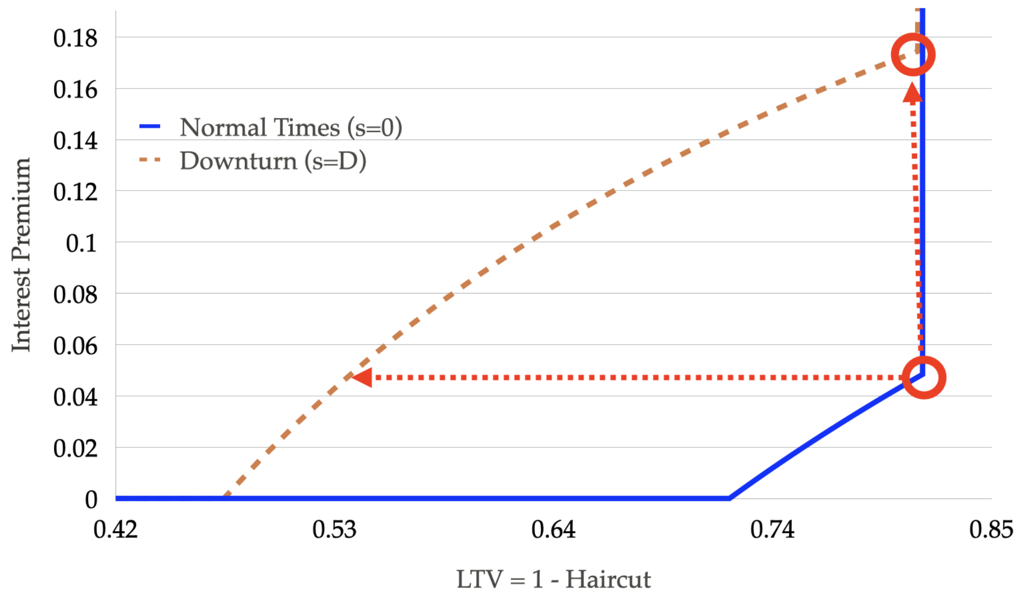
Note: While assumption A4.2: $e_0 < \beta^2 a_D K_0^{fb}$, which restricts firms' starting endowment in period 0, is useful in the analytical proofs of my Propositions, the assumption is much stricter than necessary. In this illustrative example, I relax this assumption significantly and demonstrate that the key implications of the model are robust.

Figure 3.2: Proportion of capital held by firms



The representative firm's capital holding collapses during the downturn (history D) before gradually recovering in subsequent periods towards the first-best benchmark. Even though by history DU all aggregate productivity uncertainty has been resolved, a full recovery is not achieved immediately.

Figure 3.3: Credit Surface during the downturn



Credit conditions deteriorate during the downturn (orange dashed line) relative to normal times (blue solid line). Firms that wish to borrow using the maximum leverage contract face both a higher interest rate and a higher haircut for each unit of capital posted as collateral.

a fall in the borrowing capacity of capital as collateral. Crucially, even though at both history 0 and D there is a common probability p that aggregate productivity will be high (a_U) next period and $(1-p)$ that it will be low, the situation for firms at history D is much worse. This is because in history DD the negative aggregate productivity shock would be permanent. The price of capital at history DD, q_{DD}, is significantly lower than that in history D. Consequently, each unit of capital is much more valuable as collateral at history 0 than at history D. Thus credit conditions deteriorate during the downturn relative to normal times (Figure 3.3). Firms that wish to borrow using the maximum leverage contract face both a higher interest rate and a higher haircut for each unit of capital posted as collateral. The collateral constraint faced by firms and this endogenous fall in the borrowing capacity of their collateral amplify the adverse aggregate productivity shock to the real economy, and feed back into even lower capital holding by firms.

I generalize these findings in the two propositions below. In the statement of these propositions (and corresponding proofs) I adopt a simplified set of notation, combining the time subscript t and the dependence of variables on histories (s^t) into a single subscript whenever the context is clear. For instance, let $K_D \coloneqq K_{t=1}(s^1 = D)$.

Proposition 1. ***[Productive inefficiency and leverage during the downturn]*** *Under assumptions $A1-4$ and given the no-bubble condition (equation 3.1), at history D:*

1. $K_D < K_D^{fb}$ *: firms hold too little capital relative to the first best; and*
2. $\sum_{j\in J} \max\{\varphi_{D,j}, 0\} = K_D$*: the collateral constraint is binding (i.e. firms purchase capital using leveraged debt contracts).*

Proof. Intuitively, at history D credit conditions are so tight and firms' liquid wealth are so low that firms are unable to purchase the first-best level of capital even if they borrow using the maximum leverage contract. Credit conditions tighten at history D because the negative productivity shock significantly reduces the expected price of capital in the next period. Specifically, if the shock proves to be permanent, the price of capital in history DD would be significantly lower. The arrival of the bad news at D therefore endogenously reduces the borrowing capacity of firms against each unit of collateral. Moreover, firms have limited means to transfer their limited liquid wealth from the initial history 0 to history D. If firms used leverage during history 0 to purchase capital, they would default at history D, hand over the collateral to lenders and retain only the consumption goods produced. The resulting amount of liquid wealth is insufficient to purchase the first best level of capital at D given the prevailing credit conditions. On the other hand, firms may try to transfer more resources into history D by purchasing capital without leverage at history 0 (and thus avoiding default at D).[7] But for firms to have enough liquid wealth at history D to purchase K_D^{fb} with this strategy, the price of capital in the downturn q_D must be sufficiently high relative to its initial price q_0. In such situations, the household would also like to purchase more capital in period 0, which would push q_0 beyond the level required for the firms' strategy to succeed. For the second part of the proposition, because firms hold too little capital relative to the first-best benchmark at history D, they are in expectations more productive than households and would like to utilize leveraged debt contracts to increase their holding of capital. A formal proof can be found in Appendix 6.5. □

[7] Another way for firms to transfer resources into history D is by lending to households at history 0. However, since households start with the entire stock of capital and their productivity is concave, it is easy to show that firms strictly prefer purchasing capital without leverage to lending at history 0.

Proposition 1 states that firms would use leveraged debt contracts during the downturn, but is silent on the optimal choice of contracts j_D^*. From corollary 3 we know that the choice is essentially between the maximum leverage contract $\bar{j}_D = q_{DU}$ and the max-min (risk-free) leverage contract $\underline{j}_D = q_{DD}$. The following proposition shows that firms will use maximum leverage when the downturn is "severe", and max-min leverage when it is more "moderate".

Proposition 2. ***[Optimal leverage during the downturn]*** *Under assumptions A1–4 and given the no-bubble condition (equation 3.1), at history D* $\exists \hat{K}_D \in (0, K_D^{fb}]$ *s.t.*

$$j_D^* = \begin{cases} \bar{j}_D := q_{DU} & \text{if } K_D < \hat{K}_D \\ \underline{j}_D := q_{DD} & \text{if } K_D > \hat{K}_D \\ \left\{\bar{j}_D, \underline{j}_D\right\} & \text{if } K_D = \hat{K}_D \end{cases} \tag{3.6}$$

Proof. From Lemma 3, we have previously established that firms prefer the maximum leverage contract over the max-min leverage contract when their marginal utility of income today is sufficiently high relative to their marginal utility of income tomorrow in the Up-State. When firms start history D with very limited liquid wealth, households will hold the majority of capital in equilibrium and firms will hold very little (high $\tilde{K}_D$ and low K_D). This implies that the equilibrium price of capital will be very low from the perspective of the firm, so firms' marginal utility of income at history D is very high and they would like to borrow the maximum amount possible to take advantage of these fire sale prices. In contrast, if firms have access to more resources at history D, the price of capital would be higher and firms may find it optimal to use the max-min leverage contract to transfer more resources into the Up-state at history DU where the uncertainty surrounding their productivity has been resolved favorably. Given the linearity in preferences and the continuity of households' production function $G(\cdot)$, there exists a threshold value of K_D whereby the firms are indifferent between the max-min leverage contract and the maximum leverage contract. A formal proof can be found in Appendix 6.6. □

In summary, Proposition 2 shows that when the production distortions in the downturn is severe (i.e. very low K_D) firms would like to use maximum leverage contracts to maximize their purchasing power. But when the production distortion is more moderate (K_D closer to K_D^{fb}), then the max-min leverage contract is optimal instead.

3.5 The Recovery Phase ($s^2 = DU$)

In the illustrative example (Figure 3.2) above, in the recovery phase firms honor the promise made in history D and repay the loan ($\bar{j}_D := q_{DU}$) by either selling their capital holding or equivalently handing over the collateral posted. This leaves firms with just the output from production $w_{DU} = a_U K_D$ with which to rebuild their stock of capital. Unfortunately, this amount of liquid wealth is insufficient to purchase the first-best level of capital, even if they use the maximum leverage contract to minimize the size of the down-payment required. Thus full recovery is not achieved at history DU even though the uncertainty surrounding aggregate productivity has been fully resolved. Lemma 4 generalizes this result.

Lemma 4. ***[Gradual Recovery]:***

1. *If firms used maximum leverage at $s^1 = D$, then $K_{DU} < K_{DU}^{fb}$ if:*

$$K_D < \beta K_{DU}^{fb} \equiv \beta \left[1 - G'^{-1}(a_U)\right] \tag{3.7}$$

2. *If firms used max-min leverage at $s^1 = D$, then $K_{DU} < K_{DU}^{fb}$ if:*

$$K_D < \beta K_{DU}^{fb} \left[\frac{1-\beta}{1-\beta\frac{a_D}{a_U}}\right] \tag{3.8}$$

Proof. If firms used maximum leverage contracts at $s^1 = D$, then their liquid wealth at history DU is given by $w_{DU,\left(j_D^* = \bar{j}_D\right)} = a_U K_D$. In order to purchase the first-best level of capital K_{DU}^{fb}, the minimum down-payment required in equilibrium is given by: $q_{DU} - \pi_{\bar{j}_{DU}} = \beta G'\left(\tilde{K}_{DU}^{fb}\right) = \beta a_U$ per unit of capital. So K_{DU} is less than K_{DU}^{fb} whenever $K_D < \beta K_{DU}^{fb}$. If instead firms used maximum leverage contracts at $s^1 = D$, then their liquid wealth at history DU is given by $w_{DU,,\left(j_D^* = \underline{j}_D\right)} = (a_U + q_{DU} - q_{DD}) K_D$, where $(q_{DU} - q_{DD})$ is bounded above by $\left(\frac{\beta}{1-\beta} a_U - \frac{\beta}{1-\beta} a_D\right)$. Comparing w_{DU} with the minimum down-payment required again yields inequality 3.8. □

Recovery from a downturn will typically be gradual because firms held too little capital in history D and will need more time to build up sufficient liquid wealth to purchase the efficient level of capital at equilibrium prices. Recovery from a moderate downturn will be faster than that from a severe downturns for two reasons. First, trivially, firms retain a higher stock of capital in a moderate downturn (by definition) and this increases their production in history DU. Second, more subtly, firms optimally choose a lower level of leverage during moderate downturns (the max-min leverage contract $\underline{j}_D$ instead of the maximum leverage contract $\bar{j}_D$ - see Proposition 2). This more conservative choice of leverage helps firms transfer a higher level of liquid wealth into the recovery phase, reducing the time required to achieve the efficient level of production.

4 Central Bank Intervention

In the previous section, we saw how the negative aggregate productivity shock to the real economy is amplified by the borrowing constraints faced by firms. A combination of reduced liquid wealth and decline in the borrowing capacity of collateral leads to firms holding too little capital during the downturn relative to the first-best. A central bank can intervene in this case by lending to firms against collateral at more favorable terms relative to the market ($\pi^{CB}_{j,s^t} := (1 + \chi_{j,s^t})\, \pi_{j,s^t}$). Central bank lending is funded through the issuance of a public liability m to households that carries the risk-free rate of interest, and crucially, without the need to post collateral. The central bank is able to borrow at the risk-free rate without posting collateral because it is backed by the ability of the government to tax. As such, any ex post losses incurred on central bank loans will need to be recouped through recourse to the Treasury. By circumventing the collateral constraints faced by firms, a central bank can fully alleviate the aggregate productive inefficiencies in this model if it is willing to take on the necessary level of credit risks.

Formally, the central bank aims to achieve the first-best efficient level of aggregate production, by choosing subsidy $\{\chi_{j,t}\}$ on loan j at history s^t and the amount of public liability issued m_t at history s^t (omitting the (s^t) notation henceforth for brevity):

$$\max_{\{\chi_{j,t}\},\{m_t\}} \sum_{\tau=0}^{\infty} \beta^{\tau+1} E_t \left[G\left(\tilde{K}_{t+\tau}\right) + a_{t+\tau+1} K_{t+\tau} \right] \tag{4.1}$$

subject to households' optimization (equations 2.4 - 2.6), firms' optimization (equations

2.7 - 2.9) and the public sector budget constraint $\forall \tau \geq 0$:

$$\begin{aligned} &\underbrace{\frac{1}{\tilde{\beta}} m_{t+\tau-1}}_{\text{repayment on outstanding public liability}} \underbrace{- \sum_j \varphi^{CB}_{j,t+\tau} \pi^{CB}_{j,t+\tau}}_{\text{new lending (to firms)}} \\ \leq &\underbrace{m_{t+\tau}}_{\text{issuance of new public liability}} \underbrace{- \sum_j \varphi^{CB}_{j,t+\tau-1} \delta_{j,t+\tau}}_{\text{delivery on outstanding loans}} + \underbrace{T_{t+\tau}}_{\text{transfer from/to Treasury}} \end{aligned} \tag{4.2}$$

where φ^{CB}_j denotes the number of collateralized debt contract j held by the central bank, with $\varphi^{CB}_j < 0$ indicating the central bank is selling contract j (i.e. lending). The size of the loan offered on contract j: $\pi^{CB}_j := (1 + \chi_j)\,\pi_j$ depends on the size of the subsidy offered (χ_j) relative to prevailing market rates. Lastly, T is the lump sum tax required to balance the budget in every history (representing recourse to the Treasury when positive and transfer of surplus when negative). The new market clearing condition for collateralized debt contracts is given by:

$$\varphi^{CB}_{j,t+\tau} + \varphi_{j,t+\tau} + \tilde{\varphi}_{j,t+\tau} = 0 \quad \forall \tau \geq 0, \forall j \in J \tag{4.3}$$

A key point here is that the central bank can intervene both across time t and across the credit surface. By choosing which segment of the credit surface, or which specific contract j, it is willing to subsidize $\{\chi_{j,t}\}$, the central bank is simultaneously setting both the interest rate and the haircut it imposes on the collateralized loan to firms:

$$(1 + r_{j,t}) := \frac{j}{\pi^{CB}_{j,t}} \tag{4.4}$$

$$haircut_{j,t} := 1 - LTV_{j,t} = 1 - \frac{\pi^{CB}_{j,t}}{q_t} \tag{4.5}$$

The central bank can also choose the timing of the intervention. Since period 0 production is heavily dependent on starting endowments (which are exogenous parameters in the model), we will instead focus attention on the optimal policy intervention during the downturn, when credit conditions tighten and firms' holding of capital falls endogenously. In the following sections, we will first examine the optimal policy intervention during the downturn (history D) when the central bank's actions are unanticipated, before turning to the period 0 impact of such policies if they were anticipated instead.

4.1 Unanticipated intervention at history D

From Proposition 2 we see that firms will choose either the max-min leverage contract $\underline{j}_D$ or the maximum leverage contract $\bar{j}_D$ during the downturn depending on the severity of the recession. Correspondingly, central bank responses can be grouped into two broad categories: (1) intervene at the risk-free, low LTV end of the credit surface $\underline{j}_D$; or (2) intervene at the high LTV end $\bar{j}_D$.

Proposition 3 states that when the downturn is severe and firms wish to borrow using the maximally leveraged loans, then the central bank can bring about the first-best level of capital holding by subsidizing contract $\bar{j}_D$. In contrast, intervening at the risk-free end of the credit surface may not be sufficient to achieve the first-best outcome, when the central bank abides by an effective-lower-bound on interest rates[8] and never lends more than the promised repayment amount ($\pi_j^{CB} \leq j$, $\forall j$).

Proposition 3. ***[Unanticipated Intervention]*** *Under assumptions $A1-4$ and given the no-bubble condition (equation 3.1), at history D if $j_D^* = \bar{j}_D$, then:*

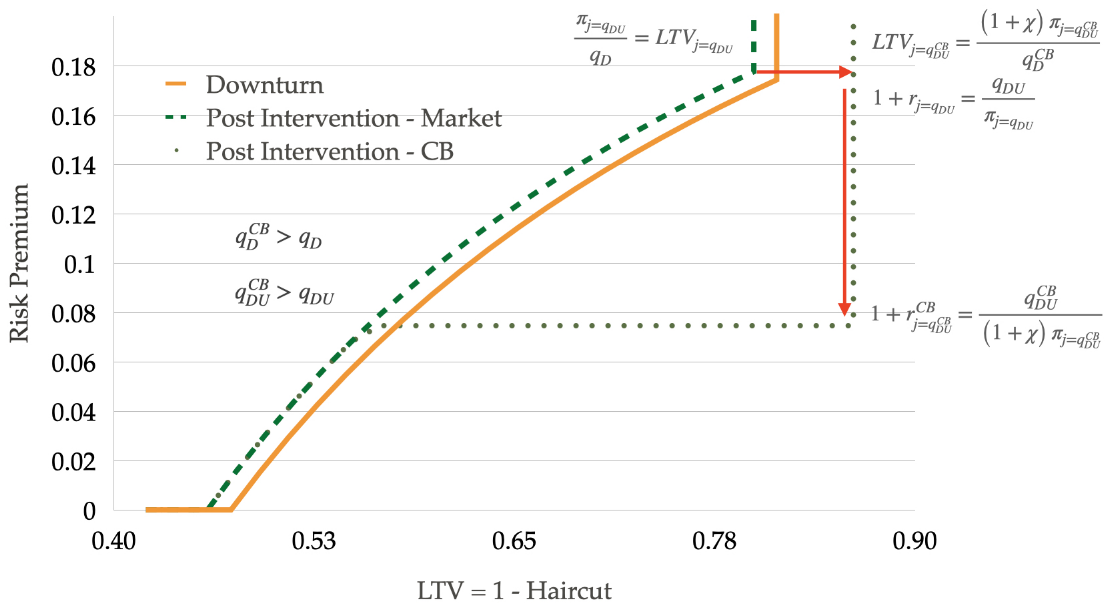
1. *There exists $\chi_{\bar{j}_D}^* > 0$ s.t. $\pi_{\bar{j}_D}^{CB} \leq \bar{j}_D$ and $K_D^{CB} = K_D^{fb}$; and*

2. *There may not exist $\chi_{\underline{j}_D}^* > 0$ s.t. $\pi_{\underline{j}_D}^{CB} \leq \underline{j}_D$ and $K_D^{CB} = K_D^{fb}$.*

Proof. Intuitively, when firms find it optimal to use the maximum leverage contract $j_D^* = \bar{j}_D$ during the downturn, they will purchase as much capital as they can with their available liquid wealth w_D. The central bank can increase the amount of capital the firms can buy for given w_D by subsidizing loans to the firms and thus effectively reducing the down-payment required on each unit of capital. The intervention will raise the equilibrium price of capital during the downturn ($q_D^{CB} > q_D$) as well as during any subsequent recovery ($q_{DU}^{CB} > q_{DU}$), which will affect credit conditions in the private market during the downturn (see Figure 4.1, dashed line). The central bank can offset the impact of these price increases on the down-payment required for firms by simultaneously reducing the haircuts and the interest rates on high LTV loans, pushing the credit surface downwards and outwards (Figure 4.1, dotted line). In fact, when the central bank is willing to intervene at the high LTV segment of the market, it is always possible to reduce the down-payment faced by firms such that they can purchase the first-best level of capital for given liquid wealth w_D. In contrast, when the central bank

[8] The effective-lower-bound here is not a constraint in the strict sense, because the central bank can always choose to lend at an negative interest rate that is below its cost of funding. But doing so *guarantees* a loss of public funds on every loan made.

Figure 4.1: Unanticipated Intervention: Credit Surface



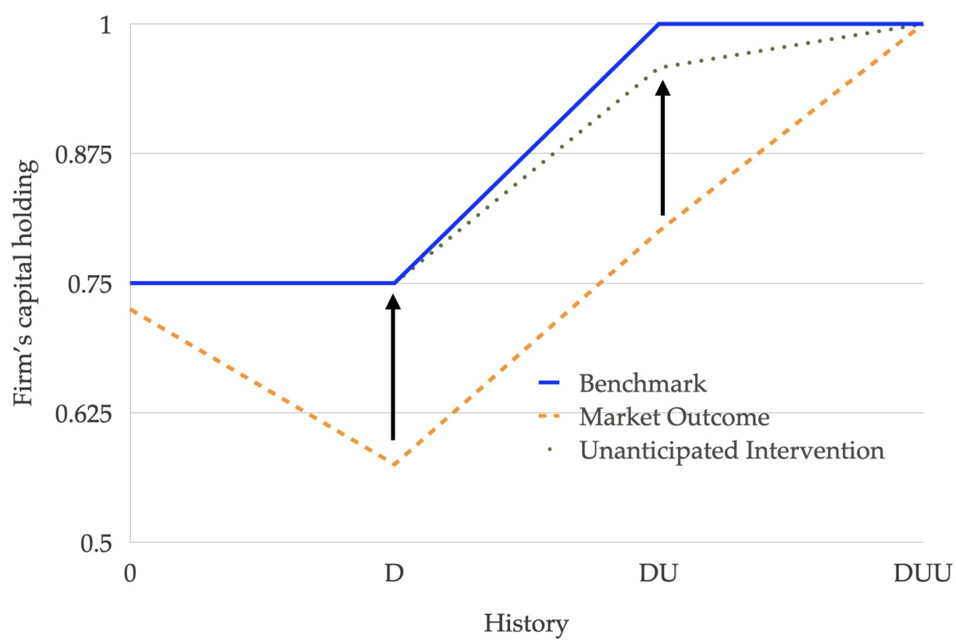
The solid orange line plots the credit surface during the downturn in the absence of any central bank intervention. Given this deterioration in credit conditions, the central bank intervenes at the riskier segment of the credit market. This intervention raises the price of capital immediately ($q_D^{CB} > q_D$) as well as during any subsequent recovery ($q_{DU}^{CB} > q_{DU}$). The net effect of the price increase is that for given contract $j < \bar{j}_D$ on the private market, the interest rate remains unchanged but the loan to value (LTV) falls (green dashed line). But the central bank intervention simultaneously lowers the interest rate and the haircut on high LTV loans, so the actual credit surface faced by firms is given by the green dotted line. As a consequence of the intervention, firms can borrow at a higher LTV and at a lower interest rate.

intervenes only at the risk-free segment of the market, the maximum loan amount it can offer while ensuring full repayment is given by the price of capital in history DD, where the firms are permanently unproductive. In general, the price of capital at DD, q_{DD}, is so low that the loan amount - even with the central bank subsidy - is insufficient to allow firms to purchase the first-best level of capital. The full proof can be found in Appendix 6.7.

$\square$

Figure 4.2 plots firms' capital holding given optimal central bank intervention, along the path where the productivity shock is temporary, using the same illustrative parameters as previously. It shows that when the central bank is willing to take on credit risks, it becomes possible for firms to hold the socially efficient level of capital. Moreover, the

Figure 4.2: Unanticipated Intervention: Firms' Capital Holding



The unanticipated central bank intervention during the downturn ensures that firms can purchase the first-best benchmark level of capital at history D. This increase in capital holding improves firms' financial position at history DU as well. Consequently, the recovery at history DU (green dotted line) is more robust than that under the no-intervention case (orange dashed line).

recovery is more robust in the following period because firms hold more capital during the downturn.

4.2 Sustained support at history DU and its announcement effect at history D

From Figure 4.2 we see that even though the recovery at history DU is stronger with central bank intervention at D than without, firms' capital holding does not immediately reach the first-best benchmark once the uncertainty surrounding aggregate productivity has been resolved.[9] This is because even if the firms held the first-best level of capital during the downturn ($K_D^{CB} = K_D^{fb}$), their available liquid wealth during the recovery may not be sufficient, in general, to ramp up production fully to the new,

[9] In the illustrative example, full recovery occurs at period $t = 3$ and history DUU, one period after the resolution of uncertainty at period $t = 2$. A more protracted recovery period is possible with alternative parameterizations, with full recovery not reached until period $t = 4$ or beyond.

higher, efficient level ($K_{DU}^{fb} > K_{D}^{fb}$).

An immediate implication of the above observation is that it is optimal for the central bank to continue its credit support during the recovery phase: $\chi^*_{j_{DU}=q_{DUU}} > 0$.

Lemma 5. ***[Sustained credit support]*** *If $K_{DU} < K_{DU}^{fb}$, then $\exists \chi^*_{j_{DU}=q_{DUU}} > 0$ s.t. $\pi^{CB}_{j_{DU}} \leq q_{DUU}$ and $K_{DU}^{CB} = K_{DU}^{fb}$.*

Proof. The proof of Lemma 5 is analogous to that for the first part of Proposition 3. □

A second, more subtle, implication is that whether or not the private sector anticipates continued credit support at history DU can have important fiscal implications for the public sector. Specifically, suppose the central bank credibly announces its commitment to sustained credit support if and only if the resolution of uncertainty next period is favorable (i.e. at history DU but not at DD). The anticipation of credit support during the recovery raises the expected price of capital at DU, which in turn increases the borrowing capacity of each unit of capital at D. Credit conditions in the private market ease and capital prices rise immediately at D. The net result is a reduction in the rate at which central bank loans must be subsidized (χ falls), but an overall increase in the total loan amount ($(1+\chi)\,\pi$ increases). With or without the announcement, the central bank still ensures firms hold the first-best level of capital at D; but with the announcement the larger total loan size means that the return on public funds in the following period becomes a mean-preserving spread.

Formally, when the central bank makes the announcement, let q_D^{An} denote the price of capital at history D; q_{DU}^{An} the price at history DU; $\pi_{\bar{j}_D=q_{DU}^{An}}$ the price of the maximum leverage contract on the private credit market at D; and $\chi^{An}_{\bar{j}_D=q_{DU}^{An}}$ the level of central bank subsidy required to achieve the first-best benchmark at D. Furthermore, let central bank lending at D be financed through the issuance of the risk-free public liability: $m_D^{An} = \left(1+\chi^{An}_{\bar{j}_D=q_{DU}^{An}}\right)\pi_{\bar{j}_D=q_{DU}^{An}}$. And let any shortfalls/windfalls in the following period ($t=2$) be met through taxation/transfers: $E_D\left[T_2|\chi^{An}_{\bar{j}_D}\right] = \left[\frac{1}{\beta}\left(1+\chi^{An}_{\bar{j}_{D=q_{DU}^{An}}}\right)\pi_{\bar{j}_D=q_{DU}^{An}}\right] - E\left[q_2^{An}\right]$, where $E\left[q_2^{An}\right]$ is the expected price of capital next period and is equal to the expected delivery on the maximum leverage contract. I prove the following proposition:

Proposition 4. ***[The Announcement Effect]*** *If firms used leverage at history 0: $j_0^* \in \left[\underline{j}_0, \bar{j}_0\right]$, and prefer the maximum leverage contract during the downturn $j_D^* = \bar{j}_D$,* *then, when the central bank announces conditional credit support at history D, such that $\chi^*_{\bar{j}_{DU}=q_{DUU}} > 0$ and $\chi^*_{\bar{j}_{DD}} = 0$:*

1. *The optimal rate of subsidy falls, $\frac{\chi^{An}_{\bar{j}_D}}{\chi^*_{\bar{j}_D}} = \frac{\pi_{\bar{j}_D=q^{CB}_{DU}}}{\pi_{\bar{j}_D=q^{An}_{DU}}} < 1$, but the optimal total loan size increases, $\left(1+\chi^{An}_{\bar{j}_D}\right)\pi_{\bar{j}_D=q^{An}_{DU}} > \left(1+\chi^*_{\bar{j}_D}\right)\pi_{\bar{j}_D=q^{CB}_{DU}}$.*
2. *The return on public funds becomes a mean-preserving spread relative to the no announcement case: $E\left[T_2|\chi^{An}_{\bar{j}_D}\right] = E\left[T_2|\chi^*_{\bar{j}_D}\right]$, and $T_{DU}|\chi^{An}_{\bar{j}_D} < T_{DU}|\chi^*_{\bar{j}_D}$ (larger windfall in the recovery phase) and $T_{DD}|\chi^{An}_{\bar{j}_D} > T_{DD}|\chi^*_{\bar{j}_D}$ (larger shortfall in the permanent downturn).*

Proof. The intuition of the proof is as follows. A credible announcement of continued credit support during any subsequent recovery raises the price of capital during the down-state. Conventionally, one would expect this price increase to improve the balance sheet position of firms and reduce the amount of central bank lending required. However, if firms used leverage at history 0: $j^*_0 \in \left[\underline{j}_0, \bar{j}_0\right]$, they would default in history D and hand over their entire stock of capital to lenders. As a result, firms must rebuild their stock of capital from scratch at history D. The increase in capital prices therefore increases the total amount the central bank must lend to firms, even as private market credit conditions improve and the required *rate* of subsidy falls. A larger initial loan implies a larger windfall in the recovery phase but also a larger shortfall in any permanent downturns. Lastly, since with or without the announcement the central bank will ensure the firms hold the first-best level of capital at history D, the expected cost to taxpayers is unchanged. A formal proof can be found in Appendix 6.8. □

Proposition 4 shows that even though a credible announcement of sustained credit support in the future can have significant immediate effects on asset prices and credit conditions, the expected return/loss on public funds may stay the same with or without the announcement. This is a somewhat surprising result. In traditional Diamond-Dybvig style models with multiple equilibria, a commitment to "do whatever it takes" can shift the economy to a more virtuous equilibrium and reduce the initial stimulus required. In my model, having abstracted away from the asymmetry of information that is central to models of financial crises and panics, we see that a credible announcement of future support provides no additional gains when responding to big shocks to the real economy. In fact, if the public sector is instead risk-averse with taxpayer funds, it might prefer to "surprise" the market with additional stimulus instead of announcing them in advance. The model therefore suggests that the optimal central bank response might look very different when the shocks are predominantly hitting the real economy (such as during the COVID-19 crisis) as opposed to the financial sector.

4.3 Anticipated Intervention

What if the intervention at history D and beyond are anticipated by market participants in advance? Conventional wisdom suggests that if agents expect the central bank to provide credit support during a downturn, they may take on excessive leverage during normal times, potentially leading to greater productive distortions and exacerbating the severity of the downturn. There exists however an alternative narrative. Anticipating low asset prices and central bank credit support during the downturn, firms may be incentivized to reduce leverage in normal times in order to exploit more favorable investment opportunities during the downturn.

In the following Proposition, I show that if firms are using maximum leverage contracts $j_0^* = \bar{j}_0 := q_U$ at history 0, then the anticipation of central bank intervention at history D will lead to either: (1) firms continuing to use maximum leverage contracts at period 0 and hold the same level of capital at history 0; or (2) a jump to an equilibrium whereby firms are purchasing capital without leverage at history 0 .

Proposition 5. ***[Waiting for the downturn]*** *If $j_0^* = \bar{j}_0 := q_U$ in the absence of central bank intervention, and it becomes common knowledge that $\chi^*_{\bar{j}_D} > 0$, then either:*

1. $j_0^* = \bar{j}_0^{CB} := q_U^{CB}$ *and* $K_0 = K_0^{CB}$*; or*
2. $\varphi_{0,j}^{CB} = 0$*,* $\forall j$.

where q_U^{CB}, K_0^{CB} and $\varphi_{0,j}^{CB}$ denote respectively the equilibrium price for capital in history U, the firms' optimal capital holding and contract sales at history 0, when central bank intervention is fully anticipated.

Proof. For the first part of the proposition, I show that if firms continue to find it optimal to use the maximum leverage contract at $t = 0$, then their capital holding in equilibrium remains unchanged. This is because when firms use maximum leverage contracts, the required down-payment is given by the opportunity cost of production faced by households (see Corollary 1 and equation 6.14). Therefore $K_0 = \frac{e_0}{q_0 - \pi_{\bar{j}_0}} = \frac{e_0}{\beta G'\left(\bar{K} - K_0\right)}$ and $K_0^{CB} = \frac{e_0}{q_0^{CB} - \pi_{\bar{j}_0^{CB}}} = \frac{e_0}{\beta G'\left(\bar{K} - K_0^{CB}\right)}$. And given $G'(\cdot) < 0$, we have $K_0 =$

K_0^{CB}. The capital holding of firms remains unchanged despite the change in equilibrium prices and credit conditions.

For the second part of the proposition, recall that the payoff to the firm from purchasing capital at history 0 without leverage is given by $\begin{pmatrix} a_U + q_U \\ a_D + q_D \end{pmatrix}$ in the two possible states. Correspondingly the payoff to the firm from purchasing capital on leverage is $\begin{pmatrix} a_U + q_U - q_D \\ a_D \end{pmatrix}$ with the max-min leverage contract, and $\begin{pmatrix} a_U \\ a_D \end{pmatrix}$ with the maximum leverage contract. We see that the difference in payoffs between the max-min leverage contract and the maximum leverage contract is simply an Up-Arrow security $\begin{pmatrix} q_U - q_D \\ 0 \end{pmatrix}$, which is reflected in a higher down-payment required with the max-min leverage contract. Consequently, firms will only switch from the maximum leverage contract to the max-min leverage contract when their marginal utility of income in the up-state is sufficiently higher than their marginal utility of income in the current period[10]. The central bank intervention at history D is aimed at reducing the down-payment required when purchasing capital on leverage during the downturn. This drives up the firms' marginal utility of income at D: $\gamma_D^{CB} > \gamma_D$, but leaves their marginal utility of income at U unchanged. Thus the firm will either continue with the maximum leverage contract at history 0, or switch to purchasing capital without leverage. Purchasing capital without leverage generates the highest payoff in the down-state for each unit of capital, and may become the optimal choice for firms when their marginal utility of income at D becomes sufficiently high. When firms purchase capital without leverage: $\varphi_{0,j}^{CB} = 0$, $\forall j$. □

A subtle implication of Proposition 5 is that even when firms continue to use the maximum leverage contract at period 0 and their holding of capital remain unchanged, the loan to value on their collateralized debt contract may actually increase due to the change in equilibrium asset prices. The loan to value on the maximum leverage contract at history 0 is the ratio between the size of the loan $\pi_{\bar{j}_0} = \beta\left[pq_U + (1-p)\,q_D\right]$ and the price of capital q_0. The anticipation of central bank interventions increases both q_0 and q_D, the former reduces LTV on debt contracts and the latter increases it. For broad parameterizations, the proportional increase in q_D typically outweigh the increase in q_0, so the net effect is often an increase in the period 0 leverage of the firm. The proposition shows that when the economy is already highly leveraged in normal

[10]For more details, see Lemma 3

times, the anticipation of central bank interventions during any potential downturns may indeed increase leverage further. However firms may be borrowing more just to purchase the same amount of capital, because the price of capital is higher when central bank interventions are anticipated. Therefore the increase in leverage during normal times does not always translate directly into greater productive inefficiencies.

5 Concluding Remarks

A salient feature of the "central bank intervention" examined in this paper is the potential need for recourse to taxpayer funds, when losses are incurred on the loans extended. In this sense, the intervention is a joint public sector effort - similar to the style of the programs we are seeing during the current pandemic. For instance, in both the Primary and Secondary Market Corporate Credit Facility, the US Treasury provides the equity investment in the facility, and the Federal Reserve lends against the assets held in the facility.

A second point is that although every debt contract is collateralized by a durable asset in the model, in practice a substantial portion of the underlying bonds/loans in the credit facilities may not be explicitly secured. Nevertheless, such loans are still implicitly backed by the value of the tangible assets of the firm. So a high-LTV collateralized debt contract in the model simply corresponds to a loan with a higher degree of credit risk to the lender. The central message of the paper - that central banks should take on greater credit risk when the downturn is severe - remains unchanged regardless of whether the loan is contractually collateralized. The analytical framework presented here highlights the role of defaults and limited enforcement of repayment in equilibrium outcomes. It can be extended into more general settings where, instead of posting collateral explicitly, promises on debt contracts are backed by the assets of the firm.[11]

Third, the central bank in the model lends directly to firms that engage in production, whereas traditionally central banks preferred to lend to financial intermediaries.

[11] In a separate working paper, Du (2020), I analyze the collateral equilibrium in economies with more than two states of nature when debt contracts can be ordered by seniority and backed by financial assets. For instance, with three states of nature, borrowers can issue both senior secured debt and junior subordinated debt, where the junior subordinated debt is backed by the residual value of the firm after senior creditors are repaid. I show that any equilibrium in this economy is equivalent to another equilibrium where the senior tranche never defaults, and the junior tranche only defaults in the worst state of the world.

Extending the model to encompass a third sector functioning as the financial intermediary is an interesting direction for future research. Nevertheless, current credit facilities like the Main Street Lending Program in the US demonstrate a policy shift towards more direct lending to the real economy.

Most importantly, the paper emphasizes that central banks have much more than just interest rates in their policy toolkit. Collateral requirements (and risk appetites) are also an important part of the transmission mechanism. Under the Bagehot rule and as standard practice during normal times, collateral requirements at central banks are set mechanically to ensure the central bank never takes on significant credit risk. This leaves the risk-free interest rate as the main policy instrument. In my model - as in real life - variations in collateral requirements allow central banks to intervene across the credit surface, and to provide support to riskier borrowers who may be shut out of the credit markets during a downturn. As such, collateral requirements in central bank lending should play an integral role in policy discussions.

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6 Appendix

6.1 First-order conditions to agents' optimization

The Lagrangian for the firm's optimization problem (eqns 2.4 to 2.6) is given by:

$$\begin{aligned}
\mathcal{L} \;=\; & E_t\{\sum_{t=0}^{\infty} \beta^t u\left(x_t\right) \\
& -\sum_{t=0}^{\infty} \beta^t \gamma_t \left[\left(x_t - a_t K_{t-1} - e_t\right) + q_t\left(K_t - K_{t-1}\right) + \sum_{j\in J} \varphi_{j,t-1} \min\{j, q_t\} - \sum_{j\in J} \varphi_{j,t}\pi_{j,t}\right] \\
& -\sum_{t=0}^{\infty} \beta^t \lambda_t \left[\sum_j \max\left(\varphi_{j,t}, 0\right) - K_t\right]\}
\end{aligned}$$

where γ_t denotes the firm's *marginal utility of income* at period t, and λ_t denotes the *marginal utility from relaxing the firm's collateral constraint.*

The first-order conditions for the firm can be derived as follows (omitting the notation (s^t) for brevity):

$$\frac{\partial \mathcal{L}_t}{\partial x_t} = \beta^t u'\left(x_t\right) - \beta^t \gamma_t \leq 0 \tag{6.1}$$

$$\frac{\partial \mathcal{L}_t}{\partial K_t} = -\beta^t \gamma_t q_t + E_t\left[\beta^{t+1}\gamma_{t+1}\left(a_{t+1} + q_{t+1}\right)\right] + \beta^t \lambda_t \leq 0 \tag{6.2}$$

$$\frac{\partial \mathcal{L}_t}{\partial \varphi_{j,t}} = \begin{cases} \beta^t \gamma_t \pi_{j,t} - E_t\left[\beta^{t+1}\gamma_{t+1}\delta_{j,t+1}\right] - \beta^t \lambda_t \leq 0 & \text{if } \varphi_{j,t} \geq 0 \\ \beta^t \gamma_t \pi_{j,t} - E_t\left[\beta^{t+1}\gamma_{t+1}\delta_{j,t+1}\right] \geq 0 & \text{if } \varphi_{j,t} \leq 0 \end{cases} \quad \forall j \tag{6.3}$$

$$x_t \frac{\partial \mathcal{L}_t}{\partial x_t} = 0 \tag{6.4}$$

$$K_t \frac{\partial \mathcal{L}_t}{\partial K_t} = 0 \tag{6.5}$$

$$\varphi_{j,t}\left(\frac{\partial \mathcal{L}_t}{\partial \varphi_{j,t}}\right) = 0 \quad \forall j \tag{6.6}$$

Correspondingly, the first-order conditions for the household are given by:

$$\frac{\partial \tilde{\mathcal{L}}_t}{\partial \tilde{x}_t} = \tilde{\beta}^t u'(\tilde{x}_t) - \tilde{\beta}^t \tilde{\gamma}_t \leq 0 \tag{6.7}$$

$$\frac{\partial \tilde{\mathcal{L}}_t}{\partial \tilde{K}_t} = -\tilde{\beta}^t \tilde{\gamma}_t q_t + E_t\left[\tilde{\beta}^{t+1}\tilde{\gamma}_{t+1}\left(G'\left(\tilde{K}_t\right) + q_{t+1}\right)\right] + \tilde{\beta}^t \tilde{\lambda}_t \leq 0 \tag{6.8}$$

$$\frac{\partial \tilde{\mathcal{L}}_t}{\partial \tilde{\varphi}_{j,t}} = \begin{cases} \tilde{\beta}^t \tilde{\gamma}_t \pi_{j,t} - E_t\left[\tilde{\beta}^{t+1}\tilde{\gamma}_{t+1}\delta_{j,t+1}\right] - \tilde{\beta}^t \tilde{\lambda}_t \leq 0 & \text{if } \tilde{\varphi}_{j,t} \geq 0 \\ \tilde{\beta}^t \tilde{\gamma}_t \pi_{j,t} - E_t\left[\tilde{\beta}^{t+1}\tilde{\gamma}_{t+1}\delta_{j,t+1}\right] \geq 0 & \text{if } \tilde{\varphi}_{j,t} \leq 0 \end{cases} \quad \forall j \tag{6.9}$$

$$\tilde{x}_t \frac{\partial \tilde{\mathcal{L}}_t}{\partial \tilde{x}_t} = 0 \tag{6.10}$$

$$\tilde{K}_t \frac{\partial \tilde{\mathcal{L}}_t}{\partial \tilde{K}_t} = 0 \tag{6.11}$$

$$\tilde{\varphi}_{j,t}\left(\frac{\partial \tilde{\mathcal{L}}_t}{\partial \tilde{\varphi}_{j,t}}\right) = 0 \quad \forall j \tag{6.12}$$

6.2 Household behavior

In this subsection I provide a sketch of the proof for Lemma 1, as well as further discussions on its implications.

Lemma 1 follows quite naturally from assumptions A1-A4. In particular, assumption A4.3 ensures that households always have sufficient endowments to consume after any desired capital purchases, current lending and repayment on existing debt obligations. Households would not borrow from firms in order purchase capital on leverage because (1) households are well endowed in every history (A4.3); and (2) household production exhibits diminishing return to scale (A3.2).

Households are always indifferent between consumption and holding capital. Suppose not, then households are consuming yet strictly prefer holding capital. By linearity of preferences, households must be holding the entire stock of capital. But by assumption $G'\left(\bar{K}\right) = a_D$, so the price of capital must be so low that firms also want to hold capital, thus violating the market clearing condition for equilibrium. Similarly, if households strictly prefers consumption over holding capital, their marginal productivity is $G'(0) = a_U$, and the price of capital would be too high for firms to want to hold capital either in equilibrium, leading to another contradiction.

Lastly, when households lend (i.e. purchase debt contract j), they do so at a price that equates their marginal utility from lending to their marginal utility of consumption. Consequently, both the price of capital q (equation 3.2) and the price of loans π_j (equation 3.3) can be derived from the household's first-order conditions.

A consequence of these two asset pricing equations is that a strictly positive down-payment is always required when purchasing capital on leverage.

Corollary 1. *Down-payments are strictly positive in every history and for every contract j:*

$$\begin{aligned} q_t - \pi_{j,t} &= E_t\left[\beta\left(G'\left(\tilde{K}_t\right) + q_{t+1} - \min\left\{j, q_{t+1}\right\}\right)\right] \\ &\geq \beta G'\left(\tilde{K}_t\right) > 0 \quad \forall j, t, s^t \end{aligned} \tag{6.13}$$

Specifically, the down-payment on the maximum leverage contract $\bar{j} := \max_{s^{t+1}}\left\{q_{t+1}\left(s^{t+1}\right)\right\}$ is given by:

$$q_t - \pi_{\bar{j},t} = \beta G'\left(\tilde{K}_t\right) > 0 \tag{6.14}$$

Corollary 1 shows that the loan will never exceed the price of the capital good posted as collateral. Intuitively, this is because the price of the capital good today reflects both its price next period as well as its expected production next period, but the output from production cannot be collateralized, so the price of the loan reflects only the price of the capital good next period. When the firm defaults, it hands over only the capital good, and not the output from its production.

For the maximally leveraged contract, $\bar{j} = \max_{s^{t+1}} q_{t+1}\left(s^{t+1}\right)$, we have: $q_t - \pi_{\bar{j}} = \beta G'\left(\tilde{K}_t\right)$. The down-payment on this maximum leverage contract is equal to the value of foregone production for households, and depends on households' preferences, production function, and capital holding.

6.3 Optimal Leverage for Firms (Proof for Lemma 3)

The first part of Lemma 3 states that we can restrict attention to $j \in J = \left[\underline{j}, \bar{j}\right]$. Recall $\underline{j} := \min_{s^{t+1}}\left(q_t\left(s^{t+1}\right)\right)$ is the max-min leverage contract. When the firm is borrowing using contract $\underline{j}$, it is promising to repay an amount equal to the lowest possible price of the collateral next period. Since delivery on contract $\underline{j}$ is given by $\delta_{\underline{j},t+1} = \min\left\{\underline{j}, q_{t+1}\right\} = \underline{j}$, the contract is risk-free and will be priced at $\pi_{\underline{j},t} = \beta\underline{j}$ with implied interest rate $1 + r_{\underline{j},t} := \frac{\underline{j}}{\pi_{\underline{j},t}} = \frac{1}{\beta}$. Selling any other debt contracts $j < \underline{j}$ will also incur the same risk-free interest rate but raise a lower amount of funds at time t: $\pi_{j,t} = \beta j < \pi_{\underline{j},t} \quad \forall j < \underline{j}$. Since every contract j must be backed by one unit of collateral, the firm will always choose $\underline{j}$ over any $j < \underline{j}$ when the collateral constraint is binding.

Suppose instead the firm is borrowing using contract $\bar{j} := \max_{s^{t+1}} \left(q_t\left(s^{t+1}\right)\right)$. This is the *maximum leverage contract*, because with any promise $j > \bar{j}$ the firm will always default and the promise is no longer credible. Consequently any contract $j > \bar{j}$ generates the same expected delivery as $\bar{j}$ and will be priced identically. It is therefore possible to restrict attention to $j \leq \bar{j}$ without loss of generality.

The second and third part of Lemma 3 state that if the firm's marginal utility of income is high today relative to tomorrow in the Up-state, then the firm will choose the maximum leverage contract; and conversely the max-min leverage contract if the inequality is reversed. To see this formally, note that for $j \in J = \left[\underline{j}, \bar{j}\right]$ and $\lambda_t > 0$:

$$\begin{aligned}\frac{\lambda_t}{\gamma_t} &= \max_{j \in J}\left\{E_t\left[\left(\frac{\tilde{\beta}\tilde{\gamma}_{t+1}}{\tilde{\gamma}_t} - \frac{\beta\gamma_{t+1}}{\gamma_t}\right)\delta_{j,t+1}\right]\right\} \\ &= \max_{j \in J}\left\{p\left(\frac{\tilde{\beta}\tilde{\gamma}^U_{t+1}}{\tilde{\gamma}_t} - \frac{\beta\gamma^U_{t+1}}{\gamma_t}\right)j + (1-p)\left(\frac{\tilde{\beta}\tilde{\gamma}^D_{t+1}}{\tilde{\gamma}_t} - \frac{\beta\gamma^D_{t+1}}{\gamma_t}\right)q^D_{t+1}\right\}\end{aligned} \tag{6.15}$$

so collateral value is maximized for $j_t = \bar{j}_t$ when $\frac{\tilde{\beta}\tilde{\gamma}^U_{t+1}}{\tilde{\gamma}_t} > \frac{\beta\gamma^U_{t+1}}{\gamma_t}$; and for $j_t = \underline{j}_t$ when the inequality is reversed. Setting $\tilde{\beta} = \beta$ and $\tilde{\gamma}_t = \tilde{\gamma}^U_{t+1} = 1$ simplifies the condition to $\gamma_t > \gamma^U_{t+1} \Rightarrow j^*_t = \bar{j}_t$ as required.

6.4 Deterministic Steady States

There are two possible deterministic steady states. One where the firm's productivity is known to be a_U permanently, and another where it is a_D permanently. Let's denote these two steady states by U^∞ and D^∞ respectively.

In the Up-steady-state U^∞, the firm will consume and hold the entire stock of capital (as per the first-best benchmark). In the absence of uncertainty, there is only one risk-free collateralized debt contract available to trade (with promised repayment $\underline{j} = \bar{j} = q_{U^\infty}$) and firms are indifferent between purchasing the capital with and without leverage. In the following lemma, I summarize the equilibrium outcomes in U^∞ when the firm purchases capital on leverage.

Lemma 6. ***[The Up-steady-state]*** *In the deterministic steady state U^∞ where $a = a_U$, prices and allocations in the collateral equilibrium are given by:*

1. *Capital and contract prices:* $q_{U^\infty} = \frac{\beta}{1-\beta} a_U$; $\pi_{j=q_{U^\infty}} = \beta q_{U^\infty}$

2. *Capital holdings:* $\tilde{K}_{U^\infty} = \tilde{K}_{fb} = G^{'-1}(a_U)$; $K_{U^\infty} = \bar{K} - \tilde{K}_{U^\infty}$

3. *Contract trades:* $\varphi_{j=q_{U^\infty}} = K_{U^\infty} = -\tilde{\varphi}_{j=q_{U^\infty}}$

4. *Consumption:* $\tilde{x}_{U^\infty} = \tilde{e}_{U^\infty} + G\left(\tilde{K}_{U^\infty}\right) + \beta a_U \tilde{K}_{U^\infty}$; $x_{U^\infty} = (1-\beta)\, a_U K_{U^\infty}$

In the Down-steady-state D^∞, the household will end up holding the entire stock of capital. The firms that are shown to be permanently unproductive exit the market.

Lemma 7. ***[The Down-steady-state]*** *In the deterministic steady state D^∞ where $a = a_D$, prices and allocations in the collateral equilibrium are given by:*

1. *Capital and contract prices:* $q_{D^\infty} = \frac{\beta}{1-\beta} a_D$; $\pi_{j=q_{D^\infty}} = \beta q_{D^\infty}$

2. *Capital holdings:* $\tilde{K}_{D^\infty} = \bar{K}$; $K_{D^\infty} = 0$

3. *Contract trades:* $\varphi_{j=q_{D^\infty}} = 0 = -\tilde{\varphi}_{j=q_{D^\infty}}$

4. *Consumption:* $\tilde{x}_{D^\infty} = \tilde{e}_{D^\infty} + a_D$; $x_{D^\infty} = 0$

I omit proofs for Lemma 6 and 7 as they can both be readily derived from the relevant first-order conditions.

It is also clear that neither steady states will be reached immediately upon the resolution of uncertainty in the model. Even when the firm is known to be permanently productive at time t, ($a_{t+\tau} = a_U \ \forall \tau \geq 0$), in general the firm might not have enough liquid wealth to purchase K_{U^∞} and consume x_{U^∞}. A period of transition is required whereby the firm gradually builds up its liquid wealth and capital stock towards the steady state. During this transition towards deterministic Up-steady-state U^∞, it is possible to show that the firm will use leverage in order to accumulate capital faster. The transition to the Down-steady-state D^∞ is straight-forward. Once the firm is known to be permanently unproductive at history $s^2 = DD$, the firm will exit the market by handing over any collateral, liquidating its remaining capital stock, and consuming the entirety of its net worth. The deterministic steady state D^∞ will then be reached immediately in the following period.

6.5 Proof of proposition 1 [Productive inefficiency and leverage during the downturn]

For the first part of Proposition 1 I show that firms will not begin history $s^1 = D$ with enough liquid wealth to purchase the first-best benchmark level of capital K_D^{fb} given equilibrium prices for capital and debt contracts.

1. Suppose to the contrary that $K_D \geq K_D^{fb}$, then firm's liquid wealth at history D must be high enough to at least purchase K_D^{fb} using the maximum leverage debt contract $\bar{j} = q_{DU}$ (since the maximum leverage contract demands the lowest down-payment):

$$\begin{aligned} w_D &\geq \left(q_D - \pi_{D,\bar{j}}\right) K_D^{fb} \\ &= \beta G'\left(\tilde{K}_D^{fb}\right) K_D^{fb} \quad \text{given corollary 1} \\ &= \beta E\left[a\right] K_D^{fb} \quad \text{by definition of } \tilde{K}_D^{fb} \text{ (eqn 2.13)} \\ &> \beta a_D K_D^{fb} \end{aligned}$$

2. If the firm used any leverage debt contracts in period 0 ($j \in J = [q_D, q_U]$), then during the downturn it will surrender the collateral posted and retain only the products of its capital holding, so $w_D = a_D K_0$. For the firm to be able to afford K_D^{fb} in history D, we must have:

$$\begin{aligned} K_0 &\geq \beta K_D^{fb} \\ &= \beta K_0^{fb} \quad \text{by definition of } K_t^{fb} \text{ (eqn 2.13)} \end{aligned}$$

However this level of capital holding in period 0 cannot be achieved given the firm's starting endowment even if it used maximum leverage in period 0:

$$\begin{aligned} K_0 &\leq \frac{e_0}{\left(q_0 - \pi_{0,\bar{j}}\right)} = \frac{e_0}{\beta G'\left(\tilde{K}_0\right)} \quad \text{given corollary 1} \\ &\leq \frac{e_0}{\beta a_D} \quad \text{by assumption A3.2} \\ &< \frac{\beta^2 a_D K_0^{fb}}{\beta a_D} = \beta K_0^{fb} \quad \text{by assumption } A4.2 \end{aligned}$$

So we reach a contradiction $K_0 < \beta K_0^{fb} \leq K_0$.

3. Suppose instead the firm purchased capital without leverage in period 0. Then $w_D = (a_D + q_D) K_0$, where K_0 is bounded above by $K_0 \leq \frac{e_0}{q_0} < \frac{\beta^2 a_D K_0^{fb}}{q_0}$. Consequently for firm to purchase at least K_D^{fb} in history D we need:

$$\begin{aligned}(a_D + q_D) \frac{\beta^2 a_D K_0^{fb}}{q_0} &\geq w_D > \beta a_D K_D^{fb} \\ \Leftrightarrow \frac{\beta (a_D + q_D)}{q_0} &> 1\end{aligned}$$

but if $\frac{\beta(a_D+q_D)}{q_0} > 1$ then the household would also want to purchase more capital instead of consuming, leading to a contradiction with respect to Lemma 1. Intuitively, for the firm to transfer enough resources into state D to purchase K_D^{fb} without using leverage in period 0, the price of capital in the downturn q_D must be sufficiently high relative to the initial price q_0. But in such situations, the household would also like to purchase more capital in period 0, which would push q_0 beyond the level required for the firm.

To prove the second part of Proposition 1, I show that when firms hold less capital than under the first-best benchmark, they will prefer to purchase capital using leverage than without leverage.

Lemma 8. *If $K_t(s^t) < K_t^{fb}(s^t)$, then $\sum_{j \in J(s^t)} \max \{\varphi_{t,j}(s^t), 0\} = K_t(s^t)$, $\forall t \geq 0$ and $\forall s^t \in S^t$.*

Proof. It is sufficient to show that the maximum leverage contract $\bar{j}_t$ is preferable to purchasing capital without leverage, when $K_t < K_t^{fb}$. The expected rate of return from buying capital with contract $\bar{j}$ at time t is given by:

$$\begin{aligned}\gamma_{t,\bar{j}} &:= \frac{E_t[\beta \gamma_{t+1}(a_{t+1})]}{q_t - \beta E_t[q_{t+1}]} \\ &= \frac{\beta E_t[\gamma_{t+1} a_{t+1}]}{\beta G'\left(\tilde{K}_t\right)} \quad \text{by corollary 1} \\ &= \frac{E_t[\gamma_{t+1} a_{t+1}]}{G'\left(\tilde{K}_t\right)}\end{aligned}$$

The expected rate of return from buying capital without leverage is given by:

$$\begin{aligned}\gamma_{t,no} &:= \frac{E_t\left[\beta\gamma_{t+1}\left(a_{t+1}+q_{t+1}\right)\right]}{\beta\left[G'\left(\tilde{K}_t\right)+E\left[q_{t+1}\right]\right]}\\ &= \frac{E_t\left[\gamma_{t+1}a_{t+1}\right]+E_t\left[\gamma_{t+1}q_{t+1}\right]}{G'\left(\tilde{K}_t\right)+E_t\left[q_{t+1}\right]}\end{aligned}$$

Since $\frac{a}{b} > \frac{a+c}{b+d} \Leftrightarrow \frac{a}{b} > \frac{c}{d}\ \forall a,b,c,d > 0$, we have $\gamma_{t,\bar{j}} > \gamma_{t,no}$ if and only if $\frac{E_t[\gamma_{t+1}a_{t+1}]}{G'\left(\tilde{K}\right)} > \frac{E_t[\gamma_{t+1}q_{t+1}]}{E_t[q_{t+1}]}$. Furthermore, since $K_t < K_t^{fb} \Leftrightarrow \tilde{K}_t > \tilde{K}_t^{fb}$, we know $G'\left(\tilde{K}\right) < E\left[a_{t+1}\right]$ and $\frac{E_t[\gamma_{t+1}a_{t+1}]}{G'\left(\tilde{K}\right)} > \frac{E_t[\gamma_{t+1}a_{t+1}]}{E[a_{t+1}]} \geq \frac{E_t[\gamma_{t+1}a_{t+1}]}{E_t[a_{t+1}]}$. Therefore a sufficient condition for $\gamma_{t,\bar{j}} > \gamma_{t,no}$ is:

$$\begin{aligned}\frac{E_t\left[\gamma_{t+1}a_{t+1}\right]}{E_t\left[a_{t+1}\right]} &\equiv \frac{p\gamma_{t+1}^U a_U+(1-p)\,\gamma_{t+1}^D a_D}{pa_U+(1-p)\,a_D}\\ \geq\frac{E_t\left[\gamma_{t+1}q_{t+1}\right]}{E_t\left[q_{t+1}\right]} &\equiv \frac{p\gamma_{t+1}^U q_{t+1}^U+(1-p)\,\gamma_{t+1}^D q_{t+1}^D}{pq_{t+1}^U+(1-p)\,q_{t+1}^D}\end{aligned}$$

With a little algebra we can show that the sufficient condition captured in the inequality above is equivalent to:

$$\frac{a_U}{a_D} \geq \frac{q_{t+1}^U}{q_{t+1}^D}$$

Since the price of capital is always given by households' asset pricing equation, it is bounded above and below by its value in the deterministic Up and Down steady states respectively $q_t \in \left[\frac{\beta}{1-\beta}a_D, \frac{\beta}{1-\beta}a_U\right], \forall t \geq 0, \forall s^t \in S^t$. Therefore we have

$$\frac{a_U}{a_D} \geq \frac{q_{t+1}^U}{q_{t+1}^D} \Rightarrow \gamma_{t,\bar{j}} > \gamma_{t,no}$$

as required. □

Lemma 8 concludes the proof for Proposition 1. In summary, during the downturn, firms hold too little capital relative to the first-best benchmark. Consequently they are in expectations more productive than households and would like to utilize leveraged debt contracts to increase their holding of capital. But a combination of fallen liquid wealth and worsened credit conditions constrain the firm's purchasing power. These two financial frictions work together to exacerbate the downturn.

6.6 Proof to Proposition 2 [Optimal leverage during the downturn]

From Lemma 3 we know that $\bar{j}_D$ is weakly preferred to $\underline{j}_D$ if and only if firms' marginal utility of income during the downturn is weakly greater than its marginal utility of income during the subsequent recovery: $\gamma_D \geq \gamma_{DU}$. So to determine the optimal debt contract for firms at history D, we need to sign the difference between γ_D and γ_{DU}, which is given by:

$$\begin{aligned}
\gamma_D - \gamma_{DU} &= \frac{E\left[\beta\gamma_{t+1}a_{t+1}\right]}{q_D - \pi_{D,\bar{j}_D}} - \gamma_{DU} \\
&= \frac{E\left[\gamma_{t+1}a_{t+1}\right]}{G'\left(\tilde{K}_D\right)} - \gamma_{DU} \\
&= \frac{\left[E\left[\gamma_{t+1}a_{t+1}\right] - \gamma_{DU}G'\left(\tilde{K}_D\right)\right]}{G'\left(\tilde{K}_D\right)} \\
&= \frac{\left[p\gamma_{DU}a_U + (1-p)\gamma_{DD}a_D - \gamma_{DU}G'\left(\tilde{K}_D\right)\right]}{G'\left(\tilde{K}_D\right)} \\
&= \frac{\left[(1-p)a_D - \gamma_{DU}\left(G'\left(\tilde{K}_D\right) - pa_U\right)\right]}{G'\left(\tilde{K}_D\right)} \quad \text{since } \gamma_{DD} = 1
\end{aligned}$$

First, when $K_D = K_D^{fb}$ (and $\tilde{K}_D = \tilde{K}_D^{fb}$), we have $\gamma_D - \gamma_{DU} \leq 0$:

$$\begin{aligned}
\gamma_D - \gamma_{DU} &= \frac{\left[(1-p)a_D - \gamma_{DU}\left(G'\left(\tilde{K}_D^{fb}\right) - pa_U\right)\right]}{G'\left(\tilde{K}_D\right)} \\
&\leq \frac{\left[(1-p)a_D - \left(G'\left(K_D^{fb}\right) - pa_U\right)\right]}{G'\left(\tilde{K}_D\right)} \quad \text{since } \gamma_{DU} \geq 1 \\
&= \frac{\left[(1-p)a_D - (1-p)a_D\right]}{G'\left(\tilde{K}_D\right)} \quad \text{since } G'\left(K_D^{fb}\right) = E\left[a\right] \\
&= 0
\end{aligned}$$

Second, when $K_D = 0$ (and $\tilde{K}_D = 1$), we have $\gamma_D - \gamma_{DU} > 0$

$$\begin{aligned}
\gamma_D - \gamma_{DU} &= \frac{\left[(1-p)\, a_D - \gamma_{DU}\left(G'(1) - pa_U\right)\right]}{G'\left(\tilde{K}_D\right)} \\
&= \frac{\left[(1-p)\, a_D - \gamma_{DU}\left(a_D - pa_U\right)\right]}{G'\left(\tilde{K}_D\right)} \\
&> 0 \quad \text{since } p \geq \frac{a_D}{a_U} \text{ by assumption } A3.1
\end{aligned}$$

Since $G \in \mathcal{C}^2$, by the intermediate value theorem $\exists \hat{K}_D \in (0, K_D^{fb}]$ s.t. $\gamma_D - \gamma_{DU} = 0$ and the firm is indifferent between the maximum leverage contract and the max-min leverage contract. This concludes the proof.

6.7 Proof of proposition 3 [Unanticipated Intervention at D]

When firms find it optimal to use the maximum leverage contract $j_D^* = \bar{j}_D$, they will purchase as much capital as they can with their available liquid wealth w_D. So firms' capital holding is given by $K_D = \frac{w_D}{q_D - \pi_{\bar{j}_D = q_{DU}}} < K_D^{fb}$ (inequality by Proposition 1).

Suppose the central bank subsidizes the maximum leverage contract $\bar{j}_D$. This will allow the firms to hold a larger stock of capital in equilibrium, and the equilibrium price for capital will rise both immediately and in any ensuing recovery period: $\bar{q}_D^{CB} > q_D$ and $\bar{q}_{DU}^{CB} > q_{DU}$. The required level of subsidy to achieve the first-best benchmark level of capital holding by firms is thus given by:

$$\begin{aligned}
K_D^{fb} &= \frac{w_D}{\bar{q}_D^{CB} - \left(1 + \chi^*_{\bar{j}_D = \bar{q}_{DU}^{CB}}\right) \pi_{\bar{j}_D = \bar{q}_{DU}^{CB}}} \\
&=: \frac{w_D}{\bar{q}_D^{CB} - \pi^{CB}_{\bar{j}_D = \bar{q}_{DU}^{CB}}}
\end{aligned} \tag{6.16}$$

It remains to show that this subsidized loan entails a positive interest rate $\pi^{CB}_{\bar{j}_D = \bar{q}_{DU}^{CB}} \leq \bar{q}_{DU}^{CB}$, which follows immediately given $K_D^{fb} > 0$, $w_D > 0$ and $\bar{q}_{DU}^{CB} \geq \bar{q}_D^{CB}$. The last condition $\bar{q}_{DU}^{CB} \geq \bar{q}_D^{CB}$ claims that the price of capital will be (weakly) higher during the recovery (history DU) than during the downturn (history D). This is equivalent to showing that firms will hold a higher proportion of capital in the recovery: $K_{DU} > K_D^{fb}$. We do this in two steps. First, observe that in history DU the uncertainty surrounding firms' productivity has been fully resolved and the economy is transitioning towards the

Up-steady-state. Second, firms use leveraged debt contracts along the transition path to build up capital faster[12] so $K_{DU} = \frac{w_{DU}}{q_{DU} - \pi_{j=q_{DUU}}} = \frac{a_U K_D^{fb}}{\beta G'\left(\tilde{K}_{DU}\right)} \geq \frac{a_U K_D^{fb}}{\beta a_U} = \frac{1}{\beta} K_D^{fb} > K_D^{fb}$ as required.

For the second part of the proposition, suppose the central bank subsidizes the max-min leverage contract $\underline{j}_D$. The equilibrium price for capital will be (weakly) higher during the downturn: $q_{\underline{D}}^{CB} \geq q_D$; and remains unchanged if the productivity shock turns out to be permanent and the households hold the entire stock of capital: $q_{DD}^{CB} = q_{DD}$. The required level of subsidy to achieve the first-best benchmark becomes:

$$\begin{aligned} K_D^{fb} &= \frac{w_D}{q_{\underline{D}}^{CB} - \left(1 + \chi^*_{\underline{j}_D = q_{DD}}\right) \pi_{\underline{j}_D = q_{DD}}} \\ &=: \frac{w_D}{q_{\underline{D}}^{CB} - \pi^{CB}_{\underline{j}_D = q_{DD}}} \end{aligned} \tag{6.17}$$

Consequently, for sufficiently low w_D and sufficiently large gap in the price of capital between periods $q_{\underline{D}}^{CB} - q_{DD}$, both of which depend in part on the specifications of the initial endowment for firms e_0 and on the household's production function $G(\cdot)$, the size of the central bank loan required to achieve the first-best outcome may exceed the promised value of repayment: $\pi^{CB}_{\underline{j}_D = q_{DD}} > q_{DD}$, thus violating the effective-lower-bound condition.

6.8 Proof to Proposition 4 [The Announcement Effect]

When the central bank optimally provides credit support in history DU, $\chi^*_{j_{DU} = q_{DUU}} > 0$, we know that the firms' capital holding at DU is raised to the first-best benchmark K_{DU}^{fb}. Consequently, the price of capital rises relative to the scenario where the central bank is only expected to intervene at D: $q_{DU}^{An} > q_{DU}^{CB}$. The price of capital at history DD remains unchanged: $q_{DD}^{An} = q_{DD}^{CB}$ because the firms will always default and the households hold the entire stock of capital at DD. Higher capital prices during the recovery eases credit conditions in the private market at history D. For the maximum leverage contract: $\pi_{\bar{j}_D = q_{DU}^{An}} > \pi_{\bar{j}_D = q_{DU}^{CB}}$, so the firms can secure a larger loan against the same unit of capital. The improved access to credit leads to an immediate increase in capital prices: $q_D^{An} > q_D^{CB}$. With and without the announcement, the central bank will set subsidy, χ_D^{An} and $\chi_{\bar{j}_D}^{CB}$ respectively, such that firms hold the first-best level of capital

[12] Note that all debt contracts along the transition path are riskless because all uncertainty has already been resolved.

at history D:

$$q_D^{CB} - \left(1+\chi_{\bar{j}_D}^*\right)\pi_{\bar{j}_D=q_{DU}^{CB}} = \frac{w_D}{K_D^{fb}} = q_D^{An} - \left(1+\chi_{\bar{j}_D}^{An}\right)\pi_{\bar{j}_D=q_{DU}^{An}}$$
$$\Leftrightarrow q_D^{An} - q_D^{CB} = \left(1+\chi_{\bar{j}_D}^{An}\right)\pi_{\bar{j}_D=q_{DU}^{An}} - \left(1+\chi_{\bar{j}_D}^*\right)\pi_{\bar{j}_D=q_{DU}^{CB}} > 0 \tag{6.18}$$

where $w_D = a_D K_0$ because the firm used leverage at history 0.

Since agents' capital holdings are unchanged, from households' asset pricing equations (see equation 6.14) we know that the net down-payment required on the private market for maximum leverage contracts also remains unchanged, despite the changes in the price of capital and the size of the loan:

$$q_D^{CB} - \pi_{\bar{j}_D=q_{DU}^{CB}} = \beta G'\left(\tilde{K}_D^{fb}\right) = q_D^{An} - \pi_{\bar{j}_D=q_{DU}^{An}} \tag{6.19}$$

With a little algebra, combining equations 6.18 and 6.19 above yields: $\frac{\chi_{\bar{j}_D}^{An}}{\chi_{\bar{j}_D}^*} = \frac{\pi_{\bar{j}_D=q_{DU}^{CB}}}{\pi_{\bar{j}_D=q_{DU}^{An}}} < 1$ as required.

To see why the balancing tax transfers become a mean-preserving spread, note that:

$$\begin{aligned}
&E\left[T_2|\chi_{\bar{j}_D}^*\right] - E\left[T_2|\chi_{\bar{j}_D}^{An}\right] \\
=&\left\{\frac{1}{\beta}\left(1+\chi_{\bar{j}_D}^*\right)\pi_{\bar{j}_D=q_{DU}^{CB}} - E\left[q_2^{CB}\right]\right\} - \left\{\frac{1}{\beta}\left(1+\chi_{\bar{j}_D}^{An}\right)\pi_{\bar{j}_D=q_{DU}^{FG}} - E\left[q_2^{An}\right]\right\} \\
=&\frac{1}{\beta}\left\{\left(1+\chi_{\bar{j}_D}^*\right)\pi_{\bar{j}_D=q_{DU}^{CB}} - \left(1+\chi_{\bar{j}_D}^{An}\right)\pi_{\bar{j}_D=q_{DU}^{An}}\right\} + \left\{E\left[q_2^{An} - q_2^{CB}\right]\right\} \\
=&-\frac{1}{\beta}\left\{q_D^{An} - q_D^{CB}\right\} + \left\{E\left[q_2^{An} - q_2^{CB}\right]\right\} \\
=&-\frac{1}{\beta}\left\{\beta E\left[G'\left(\tilde{K}_D^{fb}\right) + q_2^{An}\right] - \beta E\left[G'\left(\tilde{K}_D^{fb}\right) + q_2^{CB}\right]\right\} + \left\{E\left[q_2^{An} - q_2^{CB}\right]\right\} \\
=&0
\end{aligned}$$

Lastly, identical liquidation value $q_{DD}^{AD} = q_{DD}^{CB}$ at history DD but larger public liability given announcement $\frac{1}{\beta}\left(1+\chi_{\bar{j}_D}^{AD}\right)\pi_{\bar{j}_D=q_{DU}^{AN}} > \frac{1}{\beta}\left(1+\chi_{\bar{j}_D}^*\right)\pi_{\bar{j}_D=q_{DU}^{CB}}$ means a larger shortfall at DD: $T_{DD}|\chi^{AN} > T_{DD}|\chi_{\bar{j}_D}^*$; and correspondingly a larger windfall at DU.