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Efficiency of central clearing under liquidity stress

Marco Bardoscia,[(1)] Fabio Caccioli [(2)] and Haotian Gao[(3)]

Abstract

We explore the impact of central clearing on the demand for collateral arising from variation margin calls in the derivatives market. We find that the aggregate demand for collateral is not necessarily minimal when all contracts are centrally cleared. Hence, at least in this respect, increasing the scope of central clearing is not always beneficial. Previous work finds instead that the demand for collateral is minimal when all contracts are centrally cleared, but rely on the crucial assumption that all institutions have centrally cleared and bilateral contracts with exactly the same underlying counterparties. In this special case we prove a stronger result: that the aggregate demand for collateral is (weakly) decreasing with the fraction of centrally cleared notional. We also prove that, in this case, the aggregate demand for collateral starts decreasing only when the fraction of centrally cleared notional is larger than a critical value, suggesting that the benefits of central clearing kick-in only when a sufficiently large portion of the market has moved to central clearing.

Key words: Financial network, systemic risk, derivatives, central counterparties, contagion.

JEL classification: G20, G28.

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1 Introduction

After the Global Financial Crisis in 2007 – 2008 derivatives contracts have become increasingly collateralised (Duffie, 2018). Typically, counterparties in a derivative contract exchange collateral both when the contract is set-up (initial margin) and daily to offset the variations in the value of the contract (variation margin). In this way, the two counterparties trade counterparty risk, i.e. the risk that either party defaults on its contractual obligations, with liquidity risk, i.e. the risk that either party is not able to source the collateral required to fulfil payment obligations arising from margins.[1] A shock that affects many market participants might result in a substantially heightened system-wide demand for collateral. Depending on how market participants react, this might lead, for example, to fire sales of their illiquid assets or to an increase of interest rates on funding markets. Initial accounts (see e.g. Bank of England (2021); Alstadheim et al. (2021); Czech et al. (2021)) seem to suggest that this was one of the channels at play during the dash-for-cash episode in March 2020, in which demand for liquid assets rose sharply in the major global markets. In particular, the increased volatility of asset prices due to the onset of the COVID-19 pandemic led to a significant increase in initial and variation margin calls, which generated a large demand for liquid assets.

The Dodd–Frank Wall Street Reform and Consumer Protection Act in the United Stated and the European Market Infrastructure Regulation (EMIR) in European Union have mandated that a large class of derivative contracts are centrally cleared. According to (BIS, 2024), in notional amounts and as of June 2024, almost 77% of interest rate derivatives, almost 68% of credit derivatives, and almost 5% of foreign exchange derivatives are centrally cleared. When a contract between two counterparties is cen-

[1]These risks are substantial. According to Jansen et al. (2023) variation margins on interest rate swaps faced by Dutch pension funds can easily exceed their cash buffers. Similarly, Alfaro et al. (2024) find that an 1% increase in interest rates can generate variation margins for UK pension funds and insurers nearly equal to their total cash holdings. Jukonis et al. (2024) estimate that up to one third of investment funds in the Euro area may not have enough liquidity to meet variation margins under adverse market shocks.

trally cleared, a third institution, the central counterparty or CCP, is interposed between them so that those two counterparties (also known as clearing members) have obligations to the CCP and not between them. In particular, payments arising from margins are exchanged between the CCP and clearing members rather than between the two clearing members. Crucially, payment obligations on centrally cleared contracts are netted multilaterally.[2] This means that the CCP nets the payment obligations of each clearing member against those of all other clearing members, resulting in much smaller *net* payments obligations. As a consequence, increasing the scope of central clearing can potentially reduce the demand for liquid assets.

Here we investigate how the demand for liquid assets from variation margin (VM) payments[3] changes with the fraction α of notional that is centrally cleared. We identify the demand for liquid assets of one institution with its liquidity shortfall, i.e. the amount of cash that it has to source to make payments both on centrally cleared and bilateral contracts and that it is not able to cover either with its cash buffers or with incoming VM payments. In order to quantify liquidity shortfalls we build a network model for clearing payments on both centrally cleared and bilateral obligations, similarly to Amini et al. (2016) and Cui et al. (2018). These models are based on Eisenberg and Noe (2001), in which realised payments are determined as the equilibrium outcome under the assumptions of limited liabilities, priority of debt over equity, and pro rata payments when obligations cannot be paid in full.

Our main finding is that, in many cases, the dependence of the aggregate shortfall on α is U-shaped. In those cases, it is neither optimal to centrally clear all contracts nor to centrally clear no contract, and there exists an optimal value of α, with $0 < \alpha < 1$, for which the aggregate shortfall is at its minimum. At the optimal value of α, improvements in aggregate liquidity shortfalls are economically significant in a large region of our

[2]The CCP nets only payments obligations within the same asset class. The tradeoff between multilateral netting within one asset class and bilateral netting across multi asset classes has been investigated for example in Duffie and Zhu (2011) and will be discussed later.

[3]Variation margins are typically settled in cash, see e.g. (ISDA, 2015; ISDA, 2017).

parameter space, both relative to full bilateral clearing ($\alpha = 0$) and relative to full central clearing ($\alpha = 1$). In particular, the U-shaped relationship emerges when institutions do not have bilateral contracts with the same underlying counterparties with which they have centrally cleared contracts.[4] Previous studies (Amini et al., 2016; Cui et al., 2018; Ahn, 2020; Amini et al., 2020) assume instead that all institutions have exactly the same underlying counterparties on centrally cleared and bilateral contracts and find that it is always optimal to centrally clear all contracts. We treat this as a special case and we prove a stronger result: that the aggregate shortfall is weakly decreasing with the fraction of notional that is centrally cleared. This means that, in this case, increasing the fraction of centrally cleared notional is always (weakly) beneficial.

The emergence of a U-shaped relationship can be understood in terms of the relationship between two competing effects. On the one hand, as α increases, VM payment obligations decrease due to the multilateral netting performed by the CCP. On the other hand, driven by the reduction in realised payments on bilateral contracts, also total realised payments decline. The minimum in the aggregate shortfall emerges when, as the fraction of centrally cleared notional becomes larger and larger, the gains from multilateral netting are not sufficient to offset the losses due to the reduced delivered payments. In fact, we observe that in almost all cases in which the aggregate shortfall displays a minimum, payments are delivered less efficiently in a fully centrally cleared market than in a fully bilateral market. This means that, when the aggregate shortfall has a minimum, the payments delivered per unit of net payment obligation are smaller in a fully centrally cleared market market than in a fully bilateral market.[5] When the network of counterparties is too densely interconnected the U-shaped relationship dis-

[4] For each centrally cleared contract the CCP interposes itself between two institutions. We call those institutions the *underlying* counterparties of the contract.

[5] We point out that bilateral payments received by one institution can be used by that institution to fulfil its own payment obligations. As a consequence, bilateral payments are "recycled" potentially many times as they flow through payment chains in the network of bilateral VM obligations. Centrally cleared payments are not recycled because they are only paid, by clearing members to other clearing members once, through the CCP.

appears because there are more netting opportunities and the gains from multilateral netting always dominate the reduction in realised (bilateral) payments.

We also investigate whether the existence of the U-shaped relationship can be linked to inefficiencies due to temporal constraints in the delivery of payments. In fact, payments from clearing members to the CCP are due early in the morning and shortly thereafter the CCP pays the clearing members. Payments on bilateral contracts are typically made later, possibly until close of business. As the fraction of notional that is centrally cleared increases, more and more payments are subject to those temporal constraints, which could be responsible for the reduction in realised bilateral payments. However, we find that in most cases payment sequencing is irrelevant for the existence of the U-shaped relationship. Moreover, while payment sequencing can exacerbate aggregate shortfalls at the optimal value of α, it typically does so only up to a small extent.

Finally, we prove that, when all institutions have centrally cleared and bilateral contracts with exactly the same underlying counterparties, the introduction of central clearing becomes beneficial only when the fraction of centrally cleared notional is sufficiently large. Indeed, for α smaller than or equal to a critical value α^*, liquidity shortfalls of all institutions are independent of α. This means that, as long as $\alpha \leq \alpha^*$, liquidity shortfalls are the same as they would be if no contract were centrally cleared. The critical value α^* is the smallest liquidity ratio across all institutions, defined as cash buffer divided by net obligations when all contracts are centrally cleared. As a consequence, central clearing becomes beneficial at smaller values of α when cash buffers are smaller, or when net obligations are larger, i.e. in the cases in which liquidity shortfalls are likely to be larger.

This study contributes to the literature on the design of derivatives markets, and more specifically to the strand investigating which portion of the market should be centrally cleared. Duffie and Zhu (2011) note that, while moving to central clearing increases the netting opportunities across counterparties, it also decreases netting opportunities across assess classes, as CCPs typically do not net across the different services they provide to clearing members. They conclude that introducing central clearing for one asset class reduces exposures only if the number of asset classes is sufficiently small when compared to the number of market participants. Cont and Kokholm (2014) extend Duffie and Zhu (2011) by introducing asset classes that are heterogeneous in riskiness and correlations of exposures across asset classes. They argue that, in empirically plausible scenarios, the benefits of multilateral netting dominates over the loss of netting efficiency across asset classes. Also in our case increasing the fraction of centrally cleared notional should increase multilateral netting opportunities. However, in our case increasing the fraction of centrally cleared notional does not reduce netting opportunities across asset classes, simply because we only consider one asset class. In other words, here we do not explore that tradeoff, but nevertheless we find that increasing central clearing is not always beneficial, even within a single asset class and indeed even when the number of market participants is constant.

A few studies directly investigate liquidity shortfalls in the context of central clearing. The model of Amini et al. (2016) is conceptually similar to ours, with some important differences. They assume that institutions that face a liquidity shortfall sell some of their illiquid assets, whereas we do not make specific assumptions on remedial actions that institutions might take and only record shortfalls; they account for default fund contributions; they allow for heterogeneous values of α that depend on the pair of counterparties, whereas we only consider the case of α equal for all pairs of counterparties; they do not consider the sequencing of payments. They conclude that full central clearing (the case in which $\alpha = 1$) always leads to weakly smaller shortfalls than the situation in which contracts are only partially centrally cleared. Amini et al. (2020) extend the model of Amini et al. (2016) to the case in which institutions also have liabilities to end users, whereas Ahn (2020) derives conditions that make central clearing beneficial for all institutions. Both studies consider only the case in which all contracts are centrally cleared and the one in which all contracts are bilateral, avoiding the case in which both kinds of contracts co-exist. To the best of our knowledge, Cui et al. (2018) is the only other theoretical study to consider the sequencing between centrally cleared and bilateral payments, and therefore the closest to our setting. The main difference is that in Cui et al. (2018) the CCP is allowed to default on its payments, whereas we assume that clearing members are always able to source the collateral needed to pay the CCP. They find that when central clearing and bilateral contracts co-exist ($\alpha > 0$) shortfalls of all institutions are weakly smaller than when no contracts are centrally cleared ($\alpha = 0$).

We point out that in all these studies every institution has exactly the same underlying counterparties both on centrally cleared and on bilateral contracts. Moreover, they consider either the effect of a transition from the state in which all contracts are bilateral ($\alpha = 0$) to the state in which all contracts are centrally cleared ($\alpha = 1$) (Amini et al., 2020; Ahn, 2020) or partially cleared ($0 < \alpha < 1$) (Cui et al., 2018; Amini et al., 2016), or the effect of a transition from the state in which contracts are partially centrally cleared ($0 < \alpha < 1$) to the state in which all contracts are centrally cleared ($\alpha = 1$) (Amini et al., 2016). In contrast, here we also look at the effect of increasing (or decreasing) the fraction of notional that is centrally cleared, i.e. we consider transitions between states in which centrally cleared and bilateral contracts co-exist, but in which the fractions of centrally cleared notional are different (say $0 \le \alpha_1 \le \alpha_2 \le 1$).

Another set of studies investigates the impact of introducing central clearing using real data. Duffie et al. (2015) use data on credit default swaps (CDS) exposures at the end of 2011. They show that, when more CDSs are centrally cleared, demand for collateral increases mostly due to increased initial margin requirements on dealers, but that it otherwise decreases. Heath et al. (2016) use data on total derivative assets and liabilities of 41 institutions at the end of 2012 and reconstruct individual exposures. They find that the number of institutions experiencing liquidity stress is smaller in the scenarios in which all derivatives are centrally cleared, when compared to the scenarios in which they are centrally cleared only partially. Cont and Minca (2016) calibrate their model using CDS data from 2010 and find that the liquidity shock needed to trigger a systemic illiquidity cascade is significantly larger when CDSs are centrally cleared, but only if interest rate derivatives are already centrally cleared.

Our model is also closely related to models of clearing and contagion on financial networks (Allen and Gale, 2000; Freixas et al., 2000; Furfine, 2003; Glasserman and Young, 2016; Bardoscia et al., 2021a). In particular, bilateral payments are cleared using the model in Eisenberg and Noe (2001).[6] Paddrik et al. (2020) and Paddrik and Young (2021) adapt this model to the derivatives market by introducing initial margins and central clearing and use it to carry out a stress test the US CDS market, but without accounting for the sequencing of payments. Bardoscia et al. (2021b) use a similar model that also accounts for the sequencing of payments (but without initial margins) to carry out a stress test of the global market of interest rate and foreign exchange derivatives. The model therein is essentially the model also used here to clear payments.

This paper is organized as follows: in Section 2 we introduce our model, in Section 4 we present our the results in the general case, in Section 6 we focus on the case in which the underlying counterparties with which institutions have centrally cleared contracts are independent of those with which they have bilateral contracts, in Section 5 we focus on the case in which institutions have centrally cleared and bilateral contracts with exactly the same underlying counterparties, and finally we draw our conclusions in Section 7.

[6] This model has been extended along several dimensions: e.g. in Rogers and Veraart (2013) to account for for bankruptcy costs, in Cifuentes et al. (2005) to consider the effect of illiquid assets and capital constraint, in to account for Kusnetsov and Veraart (2019) introduces multiple maturities.

2 Model

We consider a system of n financial institutions, which henceforth we simply call banks for brevity. Between each pair of banks, say between bank i and j, there are several derivative contracts, some of which are centrally cleared, and some of which are bilateral. We imagine that, following a shock to some risk factors, the value of some contracts between i and j changes and that, as a consequence, VM must be posted. VM obligations arising from bilateral contracts are settled between i and j. VM obligations arising from centrally cleared contracts result in payment obligations to and from the CCP.

We stress that all VM obligations depend on the initial shock. However, here we take VM obligations as given and do not investigate the specific nature of the shock that generates them. That said, we assume that gross (i.e. un-netted) VM obligations are proportional to the notional of the underlying contracts. In other words, if the notional of bilateral or centrally cleared contracts between i and j doubles, also the corresponding VM obligations between i and j doubles. Here we do not consider initial margins.

VM obligations are settled following a specific sequencing of payments that follows market protocols (Bardoscia et al., 2021b). First, banks pay their VM obligations to the CCP. Next, the CCP pays its VM obligations to banks. Finally, banks settle the bilateral VM obligations. At the end of each of these three payment rounds, the institutions that are not able to pay their VM obligations in full record a shortfall. The sum of shortfalls across all payment rounds — both of individual institutions and in aggregate — is the central quantity of our analysis. In fact, such liquidity shortfalls represent the amounts of cash that institutions would have to source immediately to avoid default.

In practice institutions might take a mix of remedial actions to source the cash needed to cover their shortfalls, such as borrowing on the repo market or selling illiquid assets. Here we do not make specific assumptions on which remedial actions are taken by institutions, and therefore we do not quantify the downstream impact on interest rates in funding markets or on asset prices. To that effect one could use the demand for cash, which is the output of our model, as the input of other models of those markets. However, we do assume that institutions take *some* remedial action if they do not have enough cash to cover their payment obligations to the CCP. As a consequence, clearing members do not default on their payment obligations to the CCP, and we can abstract from the details of the CCP's default waterfall. This also implies, see Section 2.4, that the CCP is always able to pay its obligations to clearing members in full.[7]

2.1 Fully centrally cleared market

We start from the case in which all derivative contracts are centrally cleared. For each centrally cleared contract, the CCP interposes itself between two banks, say i and j. We call i and j the *underlying* counterparties of the contract. We denote with $\mathbf{L}^c$ the matrix of gross centrally cleared VM obligations between underlying counterparties, i.e. L^c_{ij} is the VM obligation that i owes to the CCP arising from centrally cleared contracts for which the CCP has interposed between i and j, prior to any netting. Even though all VM payment obligations on centrally cleared contracts are to be paid to the CCP or to be received from the CCP, for brevity we often refer to L^c_{ij} as to the gross VM payment obligation on centrally cleared contracts that i owes to j. In general, since some contracts are "in-the-money" for i and some for j, we have that both L^c_{ij} and L^c_{ji} can be strictly larger than zero. We assume that banks do not have VM obligations to themselves, meaning that $L^c_{ii} = 0$, for all i.

The CCP performs multilateral netting. This means that, for each bank, the CCP offsets VM obligations due to be paid to and received from all other banks. If bank i's net VM obligations $\sum_j (L^c_{ij} - L^c_{ji})$ are positive, i has a VM obligation to the CCP.

[7] McLaughlin and Berndsen (2021) argue that CCPs are unlikely to fail. Using data from Woodhall (2020), Menkveld and Vuillemey (2021), report several large margin breaches to the world's largest CCPs in the first quarter of 2020. However, those did not lead to any CCP failure. That said, CCPs have failed in the past, see e.g. Bignon and Vuillemey (2020) for an account of the failure of a CCP clearing derivatives in 1974.

Otherwise, the CCP has a VM obligation to bank i. By denoting with $\bar{p}_{i\to CCP}$ the net VM obligation that i owes to the CCP and with $\bar{p}_{CCP\to i}$ the net VM obligation that the CCP owes to i we have:

$$\bar{p}_{i\to CCP} = \left(\sum_j L^c_{ij} - L^c_{ji}\right)^+ \tag{1a}$$

$$\begin{aligned}\bar{p}_{CCP\to i} &= \left(\sum_j L^c_{ji} - L^c_{ij}\right)^+ \\ &= \left(\sum_j L^c_{ij} - L^c_{ji}\right)^- \quad ,\end{aligned} \tag{1b}$$

where $(\dots)^+$ and $(\dots)^-$ are the positive and negative parts. If $\bar{p}_{i\to CCP} > 0$, then $\bar{p}_{CCP\to i} = 0$, and vice versa, i.e. when i owes to the CCP, the CCP does not owe to i, and vice versa.

2.2 Fully bilateral market

We denote with $\mathbf{L}^b$ the matrix of gross bilateral VM obligations, i.e. L^b_{ij} the VM obligation that i owes to j (arising from bilateral derivative contracts between them), prior to any netting. Also in this case both L^b_{ij} and L^b_{ji} can be strictly larger than zero. Banks do not have VM obligations to themselves, i.e. $L^b_{ii} = 0$, for all i.

In this case VM obligations are netted independently for each pair of institutions. By denoting with $\bar{p}_{ij}$ the net VM obligation that i owes to j, we have:

$$\bar{p}_{ij} = (L^b_{ij} - L^b_{ji})^+ \,, \tag{2}$$

where $(\dots)^+$ is the positive part. Clearly, if $\bar{p}_{ij} > 0$, then $\bar{p}_{ji} = 0$, and vice versa, i.e. when i owes to j, j does not owe to i, and vice versa.

2.3 Mixed market

Our main objective is to compare financial systems in which the amount of contracts that are centrally cleared varies. To this aim we assume that the gross VM obligations L_{ij}^{tot} that i owes to j are equal to the convex combination:

$$L_{ij}^{tot} = \alpha L_{ij}^{c} + (1-\alpha) L_{ij}^{b}\,, \tag{3}$$

with $0 \leq \alpha \leq 1$. $\alpha = 0$ corresponds to the case of a fully bilateral market, whereas $\alpha = 1$ to the case of a fully centrally cleared market. Eq. (3) allows to interpolate between these two cases with a single and easily interpretable parameter. When $0 < \alpha < 1$, the gross centrally cleared VM obligations of i to j are equal to αL_{ij}^{c}, while the gross bilateral VM obligations of i to j are equal to $(1-\alpha)L_{ij}^{b}$.

Net VM obligations are obtained from (1) and (2) by performing the substitutions $L_{ij}^{c} \to \alpha L_{ij}^{c}$ and $L_{ij}^{b} \to (1-\alpha)L_{ij}^{b}$:

$$\bar{p}_{i\to CCP} = \alpha \left(\sum_j L_{ij}^{c} - L_{ji}^{c} \right)^{+} \tag{4a}$$

$$\bar{p}_{CCP\to i} = \alpha \left(\sum_j L_{ij}^{c} - L_{ji}^{c} \right)^{-} \tag{4b}$$

$$\bar{p}_{ij} = (1-\alpha)(L_{ij}^{b} - L_{ji}^{b})^{+}\,. \tag{4c}$$

Additionally, we restrict ourselves to financial systems in which, for each bank i, total gross VM obligations L_{i}^{tot} are independent of α. This allows us to avoid any bias due to the difference in levels of bilateral and centrally cleared VM obligations and to focus on their relative importance. From (3) we have:

$$\begin{aligned} L_i^{tot} = \sum_j L_{ij}^{tot} &= \alpha \sum_j L_{ij}^c + (1-\alpha) \sum_j L_{ij}^b \\ &= \sum_j L_{ij}^b + \alpha \left(\sum_j L_{ij}^c - \sum_j L_{ij}^b \right) , \end{aligned} \tag{5}$$

which shows that this is possible if and only if $\sum_j L_{ij}^c = \sum_j L_{ij}^b$, for all i. In this case, the total gross VM obligations of bank i that are centrally cleared are equal to αL_i^{tot}, whereas the bilateral ones are equal to $(1-\alpha)L_i^{tot}$. Hence, for all banks, α is the fraction of total gross VM obligations that are centrally cleared. Since we assume that gross VM obligations are proportional to the notional of the underlying contracts, α is also equal to the fraction of notional that is centrally cleared. Even if the total gross VM obligations L_i^{tot} are independent of α, we explicitly note that L_{ij}^{tot}, the total gross VM obligation that i owes to *any individual* counterparty j in general does depend on α.

We explicitly note that, in the general case, $\mathbf{L}^c \neq \mathbf{L}^b$. We remark that, to the best of our knowledge, all the previous studies have investigated only the case $\mathbf{L}^c = \mathbf{L}^b = \mathbf{L}$. In this case, *all* banks have centrally cleared VM obligations *exactly* towards the same underlying counterparties to which they have bilateral VM obligations. Moreover, *each* individual gross centrally cleared VM obligation αL_{ij} is proportional to the corresponding gross bilateral cleared VM obligation $(1-\alpha)L_{ij}$. Therefore, hereafter we refer to this case as to the case of *perfectly correlated exposures.*

One way to justify perfectly correlated exposures is the following. Let us start from an initial state in which all contracts are bilateral ($\alpha = 0$) and the matrix of gross bilateral VM obligations is $\mathbf{L}^b = \mathbf{L}$. Let us further assume that all banks novate contracts with all their counterparties corresponding to a fraction α of their notional *at the same time.* At the end of this process the matrix of gross centrally cleared VM obligations would be $\alpha\mathbf{L}$ and the matrix of gross bilateral VM obligations would be $(1-\alpha)\mathbf{L}$, resulting in perfectly correlated exposures. Moreover, if one wants to compare results at α with results at $\alpha' = \alpha + \Delta\alpha$, one has to assume again that all banks novate a fraction $\Delta\alpha$ of notional of contracts with all their counterparties at the same time. In practice, it appears unrealistic to assume that all novations happen synchronously. Furthermore, over a sufficiently long time horizon some bilateral contracts will expire and may be replaced with centrally cleared contracts, potentially with different counterparties. As a consequence, $\mathbf{L}^c$ and $\mathbf{L}^b$ will unavoidably differ.

2.4 Payments and shortfalls

We assume that each bank i is initially endowed with $e_i^{(1)}$ units of cash. In the first payment round banks pay the CCP. Since they have not received any other payment, banks can only rely on their initial cash endowment. If this is larger than their VM obligation to the CCP, banks immediately pay the CCP in full. Otherwise, without taking any further action, they can pay only up to their initial cash endowment:

$$p_{i\to CCP} = \min(e_i^{(1)}, \bar{p}_{i\to CCP})\,. \tag{6}$$

The shortfall of bank i at the end of the first round, i.e. the shortfall recorded by i on its centrally cleared VM obligations, is denoted with s_i^c and is defined as the difference between the VM obligation and payment that i can make without taking further actions:

$$\begin{aligned} s_i^c &= \bar{p}_{i\to CCP} - p_{i\to CCP} \\ &= (\bar{p}_{i\to CCP} - e_i^{(1)})^+\,. \end{aligned} \tag{7}$$

Importantly, we assume that banks that record a shortfall in the first round do take some action to source the corresponding amount of cash. This means that, after taking action, they are able to pay the CCP in full. By doing so they use both their initial endowment and the additional amount of cash sourced. Hence, their cash at the beginning of the second round is equal to zero. Banks that do not record a shortfall in the first round do not take further actions. As a consequence, their cash at the beginning of the second round is their initial cash endowment minus the payment made in the first round, i.e. their VM obligation to the CCP. Putting both cases together we have:

$$\begin{aligned} e_i^{(2)} &= (e_i^{(1)} - \bar{p}_{i\to CCP})^+ \\ &= e_i^{(1)} - p_{i\to CCP}\,. \end{aligned} \tag{8}$$

In the second payment round the CCP pays the banks. We note that the CCP has a perfectly matched trading book, meaning that for each incoming VM obligation from a bank there is a matching outgoing VM obligation to another bank. Therefore, the CCP's total outgoing VM obligation is equal to its total incoming VM obligation:

$$\sum_i \bar{p}_{i\to CCP} = \sum_i \bar{p}_{CCP\to i}\,. \tag{9}$$

Since all banks have paid the CCP in full (after taking actions), at the end of the first round the CCP has received a total amount of cash equal to $\sum_i \bar{p}_{i\to CCP}$. This means that the CCP always has enough cash to pay all banks in full, regardless of its initial cash endowment and without taking further actions. Therefore, no shortfalls are recorded in the second round. Banks' cash at the beginning of the third round is equal to their cash at the beginning of the second round plus the payments received from the CCP:

$$\begin{aligned} e_i^{(3)} &= e_i^{(2)} + \bar{p}_{CCP\to i} \\ &= e_i^{(1)} - p_{i\to CCP} + \bar{p}_{CCP\to i}\,. \end{aligned} \tag{10}$$

In the third payment round banks settle their bilateral VM obligations by using the Eisenberg and Noe model, which allows to compute clearing payments under the assumptions of i) limited liabilities, ii) proportionality of payments and iii) priority of debt over equity. We denote the total bilateral VM obligations of bank i with $\bar{p}_i$:

$$\bar{p}_i = \sum_j \bar{p}_{ij} = (1-\alpha) \sum_j (L^b_{ij} - L^b_{ji})^+ \tag{11}$$

and we introduce the relative liability matrix:

$$\Pi_{ij} = \begin{cases} \frac{\bar{p}_{ij}}{\bar{p}_i} & \text{if } \bar{p}_i > 0 \\ 0 & \text{otherwise} \end{cases}, \tag{12}$$

whose element ij represents the fraction of bilateral VM obligations that i owes to j. Eq. (12) can be re-written as:

$$\Pi_{ij} = \begin{cases} \frac{(1-\alpha)(L^b_{ij}-L^b_{ji})^+}{(1-\alpha)\sum_k (L^b_{ik}-L^b_{ki})^+} = \frac{(L^b_{ij}-L^b_{ji})^+}{\sum_k (L^b_{ik}-L^b_{ki})^+} & \text{if } \sum_k (L^b_{ik} - L^b_{ki})^+ > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{13}$$

According to the Eisenberg and Noe model, the clearing (or realised) payment of bank i is determined by the equilibrium condition:

$$p_i = \min\left(\bar{p}_i, e^{(3)}_i + \sum_j \Pi_{ji} p_j\right), \tag{14}$$

while individual payments are $p_{ij} = \Pi_{ij} p_i$. Also in this round p_i is the payment that bank i can make without taking any further actions. Eq. (14) is known to admit a greatest and a least solution.[8] Here we focus on the least solution. In fact, in Bardoscia et al. (2019) it is shown that, when using the Eisenberg and Noe in a liquidity setting, computing the least solution corresponds to tracking subsequent payment rounds.

[8]In many cases (14) admits a unique solution, for example when $e^{(3)}_i > 0$, for all i. In our case, this condition is not necessarily satisfied. In particular, all banks that record a shortfall at the end of the first round will start the third round with zero cash. Since they had strictly positive VM obligations to the CCP in the first round, the CCP has zero VM obligations to them in the second round. Hence, their cash at the beginning of the third round cannot be increased by any incoming payment from the CCP.

The shortfall of bank i at the end of the third round, i.e. the shortfall recorded by i on its bilateral VM obligations, is denoted with s_i^b and is defined as the difference between its bilateral VM obligations and its realised payment when no further actions are taken:

$$s_i^b = \bar{p}_i - p_i \, . \tag{15}$$

The shortfall of bank i is simply defined as the sum of shortfalls recorded in the first and third payment round, i.e. on both centrally cleared and bilateral VM obligations:

$$s_i = s_i^c + s_i^b \, . \tag{16}$$

As shorthands, we define the following aggregate quantities: $\bar{p}^c = \sum_i \bar{p}_{i \to CCP}$ and $\bar{p}^b = \sum_i \bar{p}_i$ denote respectively aggregate centrally cleared and bilateral obligations. Similarly, $p^c = \sum_i p_{i \to CCP}$ and $p^b = \sum_i p_i$ denote aggregate centrally cleared and bilateral payments and $s^c = \sum_i s_i^c$ and $s^b = \sum_i s_i^b$ denote aggregate centrally cleared and bilateral shortfalls. We also define the efficiencies with which bilateral and centrally cleared are delivered as:

$$\epsilon^c = \frac{p^c}{\bar{p}^c} \qquad \epsilon^b = \frac{p^b}{\bar{p}^b} \, . \tag{17}$$

Both ratios capture the amount of payments delivered per unit of (centrally cleared or bilateral) net payment obligation and range between zero and one.

When we compare quantities for different values of α we explicitly indicate their dependence on α, e.g. $s_i(\alpha)$ denotes the shortfall of bank i when the fraction of centrally cleared notional is equal to α.

2.5 Clearing systems

We now summarise the considerations made in the previous paragraphs in the following definitions.

Definition 1 (Mixed clearing system)**.** *Let n (the number of banks) be a strictly positive integer and let:*

- $\mathbf{L}^b$ *(the matrix of gross VM bilateral obligations) and* $\mathbf{L}^c$ *(the matrix of gross VM centrally cleared obligations) be two $n \times n$ matrices such that $L^b_{ij} \geq 0$, $L^c_{ii} \geq 0$, and $L^b_{ii} = L^c_{ii} = 0$, for all i and j.*
- $\mathbf{e}^{(1)}$ *(the cash endowments at the beginning of the first stage) be a vector of length n, such that $e_i \geq 0$, for all i.*
- $\alpha \in [0, 1]$ *(the parameter interpolating between centrally cleared and bilateral gross VM obligations).*

Then the tuple $(\mathbf{L}^b, \mathbf{L}^c, \mathbf{e}^{(1)}, \alpha)$ is a mixed clearing system.

For mixed clearing systems we can introduce total gross VM obligations L^{tot}_i as in (3) and net VM obligations $\bar{p}_{i\to CCP}$, $\bar{p}_{CCP\to i}$, $\bar{p}_{ij}$ as in (4), for all i and j. We can also introduce all quantities introduced in Section 2.4 and in particular: (sequenced) payments to the CCP $p_{i\to CCP}$ as in (6), (sequenced) bilateral payments p_{ij} as in (14), shortfalls s^c_i on centrally cleared obligations as in (7), shortfalls s^b_i on bilateral obligations as in (15), and shortfalls s_i, for all i and j.

As anticipated in Section 2.3 we will focus on the case in which total gross VM bilateral obligations and total gross VM centrally cleared obligations are the same for all banks, which is formalised by the following definition.

Definition 2 (Balanced clearing system)**.** *Let $(\mathbf{L}^b, \mathbf{L}^c, \mathbf{e}^{(1)}, \alpha)$ be a mixed clearing system such that:*

$$\sum_j L^b_{ij} = \sum_j L^c_{ij}\,,$$

for all i. Then the tuple $(\mathbf{L}^b, \mathbf{L}^c, \mathbf{e}^{(1)}, \alpha)$ is a balanced clearing system.

For balanced clearing systems, α corresponds to the fraction of gross VM obligations that is centrally cleared. Assuming that gross VM obligations are proportional to the notional of the underlying contracts, α is also the fraction of notional that is centrally cleared.

Within balanced clearing systems a special case is the one in which the matrices of gross bilateral and centrally cleared VM obligations are equal, i.e. the case of perfectly correlated exposures. It is convenient to introduce the following shorthand notation.

Definition 3 (Clearing system with perfectly correlated exposures)**.** *Let* $(\mathbf{L}^b, \mathbf{L}^c, \mathbf{e}^{(1)}, \alpha)$ *be a mixed clearing system such that:*

$$\mathbf{L}^b = \mathbf{L}^c = \mathbf{L}\,.$$

Then the tuple $(\mathbf{L}, \mathbf{e}^{(1)}, \alpha)$ *is a clearing system with perfectly correlated exposures.*

Since we are mainly concerned with how payments and shortfalls vary as α increases, it is useful to introduce the following definition.

Definition 4 (Family of mixed clearing systems)**.** *Let* $\mathcal{S}(\mathbf{L}^b, \mathbf{L}^c, \mathbf{e}^{(1)})$ *be the set of all mixed clearing systems* $(\mathbf{L}^b, \mathbf{L}^c, \mathbf{e}^{(1)}, \alpha)$ *as* α *varies in the interval* $[0, 1]$*:*

$$\mathcal{S}(\mathbf{L}^b, \mathbf{L}^c, \mathbf{e}^{(1)}) = \{(\mathbf{L}^b, \mathbf{L}^c, \mathbf{e}^{(1)}, \alpha) : \alpha \in [0, 1]\}\,.$$

Then $\mathcal{S}(\mathbf{L}^b, \mathbf{L}^c, \mathbf{e}^{(1)})$ *is a family of mixed clearing systems.*

By combining Definition 4 with Definitions 2 and 3 respectively one can introduce families of balanced clearing systems and families of clearing systems with perfectly correlated exposures. The latter will be denoted with $\mathcal{S}(\mathbf{L}, \mathbf{e}^{(1)})$.

3 Examples

In order to familiarise with the model, we now compute payments and shortfalls for three small clearing systems of $N = 4$ banks and provide some intuition as to why the aggregate shortfall can have a minimum with respect to the fraction of centrally cleared notional α. All three clearing systems are balanced, meaning that gross bilateral (in a fully bilateral market) and centrally cleared obligations (in a fully centrally cleared market) are equal, i.e. $\sum_j L^b_{ij} = \sum_j L^c_{ij} = L$, for all i. Therefore, they are also equal in aggregate: $\sum_{i,j} L^b_{ij} = \sum_{i,j} L^c_{ij} = N \cdot L$. Moreover, all initial cash endowments are equal to one unit, i.e. $\mathbf{e}^{(1)} = 1$.

3.1 Example 1

In the left panels of Figure 1 we show gross and net bilateral obligations in a fully bilateral market ($\alpha = 0$). In this case $L = 2$ and, since bank A can net its bilateral obligations with banks B and C, net aggregate bilateral obligations are equal to: $\bar{p}^b = 4/3 + 4/3 + 2/3 + 2 = 16/3$ (see bottom left panel). Gross and net centrally-cleared obligations in a fully centrally cleared market ($\alpha = 1$) are displayed in the right panels. Even though all centrally cleared obligations are between clearing members and the CCP, in order to facilitate the comparison with bilateral obligations, we show the underlying counterparties for gross centrally cleared obligations (top right panel). After multilateral netting all obligations are explicitly shown to and from the CCP (bottom right panel). Banks B, C, and D can fully offset their centrally cleared obligations to and from the CCP. Only bank A is left with obligations to the CCP, meaning that aggregate centrally cleared obligations are equal to: $\bar{p}^c = \bar{p}_{A \to CCP} = 2$.[9] We note that here multilateral netting is working as intended. In fact, no bilateral netting of centrally cleared obligations would have been possible in this case. Moreover, in aggregate, net centrally cleared obligations are far smaller than net bilateral obligations: $\bar{p}^c < \bar{p}^b$.

Let us now work out shortfalls in the fully bilateral market, using the Eisenberg and Noe model. In order compute the least solution, we start from the initial condition in

[9] Note that we are only counting obligations from banks to the CCP. This is to avoid double-counting: in fact, when a bilateral obligation is novated through the CCP, it is replaced by two obligations, one from a bank to the CCP and one from the CCP to another bank.

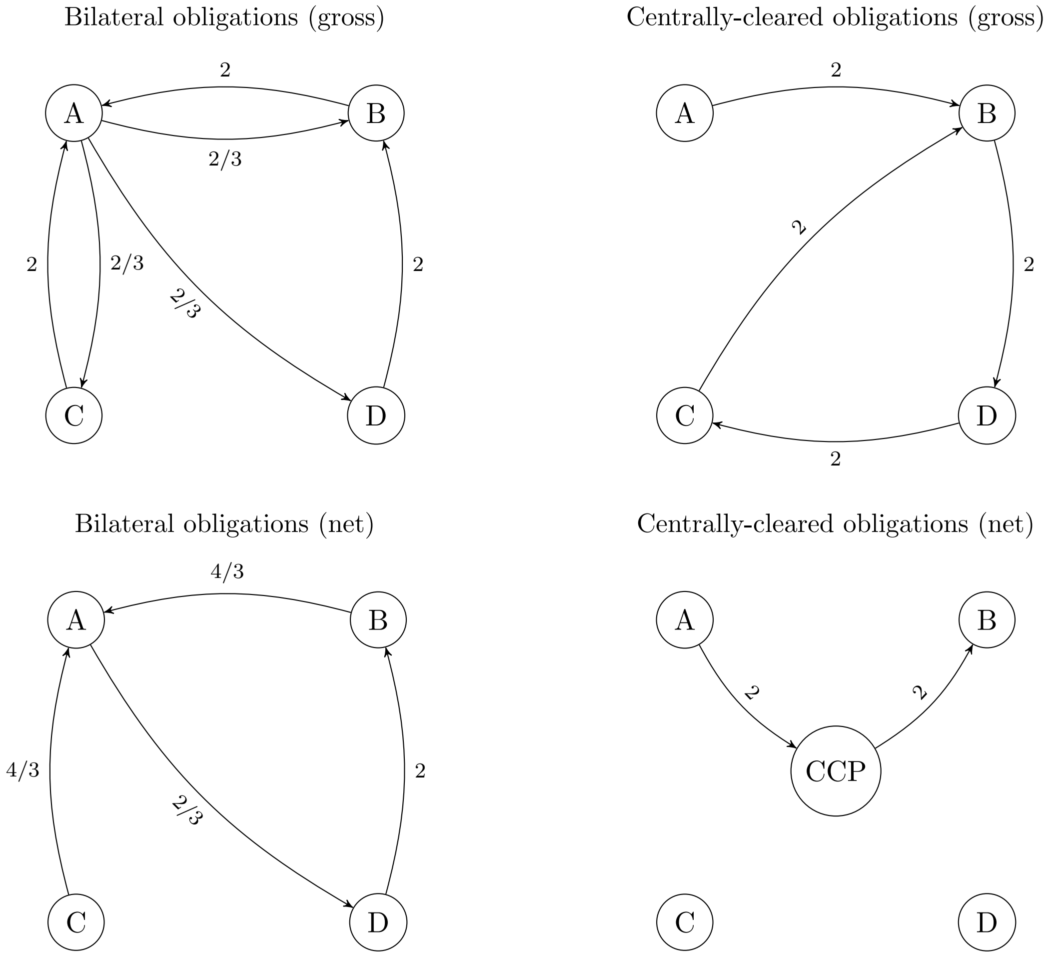

Figure 1: Gross and net bilateral and centrally cleared obligations for example 1.

which all payments are zero $\mathbf{p}^{(t=0)} = 0$. We also note that, since there are no centrally cleared obligations, $\mathbf{e}^{(3)} = \mathbf{e}^{(1)}$. From (14) we have:

$$\begin{aligned}
p_{\mathrm{A}}^{(t=1)} &= \min(\bar{p}_A,\, e_A^{(1)} + p_B^{(t=0)} + p_C^{(t=0)}) = \min(2/3, 1+0) &&= 2/3 \\
p_{\mathrm{B}}^{(t=1)} &= \min(\bar{p}_B,\, e_B^{(1)} + \qquad\quad p_D^{(t=0)}) = \min(4/3, 1+0) &&= 1 \\
p_{\mathrm{C}}^{(t=1)} &= \min(\bar{p}_C,\, e_C^{(1)} + \qquad\qquad\quad 0) = \min(4/3, 1+0) &&= 1 \\
p_{\mathrm{D}}^{(t=1)} &= \min(\bar{p}_D,\, e_D^{(1)} + \qquad\quad p_A^{(t=0)}) = \min(2, \quad 1+0) &&= 1
\end{aligned}$$

$$
\begin{aligned}
p_{\mathrm{A}}^{(t=2)} &= \min(\bar{p}_A,\, e_A^{(1)} + p_B^{(t=1)} + p_C^{(t=1)}) = \min(2/3, 1+1+1) &= 2/3\\
p_{\mathrm{B}}^{(t=2)} &= \min(\bar{p}_B,\, e_B^{(1)} + \qquad\quad p_D^{(t=1)}) = \min(4/3, 1+1) &= 4/3\\
p_{\mathrm{C}}^{(t=2)} &= \min(\bar{p}_C,\, e_C^{(1)} + \qquad\qquad\quad 0) = \min(4/3, 1+0) &= 1\\
p_{\mathrm{D}}^{(t=2)} &= \min(\bar{p}_D,\, e_D^{(1)} + \qquad\quad p_A^{(t=1)}) = \min(2, \quad 1+2/3) &= 5/3
\end{aligned}
$$

$$
\begin{aligned}
p_{\mathrm{A}}^{(t=3)} &= \min(\bar{p}_A,\, e_A^{(1)} + p_B^{(t=2)} + p_C^{(t=2)}) = \min(2/3, 1+4/3+1) &= 2/3\\
p_{\mathrm{B}}^{(t=3)} &= \min(\bar{p}_B,\, e_B^{(1)} + \qquad\quad p_D^{(t=2)}) = \min(4/3, 1+5/3) &= 4/3\\
p_{\mathrm{C}}^{(t=3)} &= \min(\bar{p}_C,\, e_C^{(1)} + \qquad\qquad\quad 0) = \min(4/3, 1+0) &= 1\\
p_{\mathrm{D}}^{(t=3)} &= \min(\bar{p}_D,\, e_D^{(1)} + \qquad\quad p_A^{(t=2)}) = \min(2, \quad 1+2/3) &= 5/3\,.
\end{aligned}
$$

Since $\mathbf{p}^{(t=2)} = \mathbf{p}^{(t=3)}$, at $t = 2$ payments have reached the stationary state, i.e. $\mathbf{p}^{(t=2)} = \mathbf{p}$. Therefore, the aggregate bilateral payment at $\alpha = 0$ is equal to: $p^b = 2/3 + 4/3 + 1 + 5/3 = 14/3$. We can then compute shortfalls:

$$
\begin{aligned}
s_{\mathrm{A}}^b &= \bar{p}_{\mathrm{A}} - p_{\mathrm{A}} = 2/3 - 2/3 = 0\\
s_{\mathrm{B}}^b &= \bar{p}_{\mathrm{B}} - p_{\mathrm{B}} = 4/3 - 4/3 = 0\\
s_{\mathrm{C}}^b &= \bar{p}_{\mathrm{C}} - p_{\mathrm{C}} = 4/3 - 1 \quad = 1/3\\
s_{\mathrm{D}}^b &= \bar{p}_{\mathrm{D}} - p_{\mathrm{D}} = 2 - 5/3 \quad = 1/3
\end{aligned}
$$

which leads to an aggregate bilateral shortfall equal to $s^b = 2/3$. Payments and shortfalls in the fully centrally cleared market are much easier to compute. After netting, bank A is the only bank with centrally cleared obligations. From (6) we have that $p_{A\to CCP} = e_A^{(1)} = 1$ and:

$$
s_A^c = (\bar{p}_{A\to CCP} - p_{A\to CCP})^+ = (2-1)^+ = 1\,,
$$

which leads to an aggregate centrally cleared payment and shortfall equal to: $p^c = p_{A\to CCP} = 1$ and $s^c = 1$. In this example, the aggregate shortfall at $\alpha = 0$ is smaller

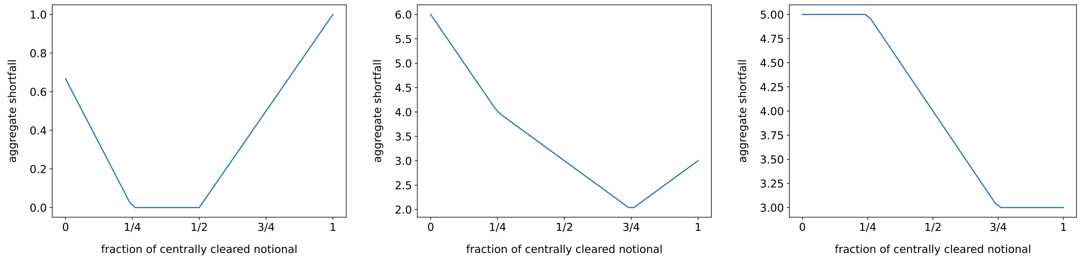

Figure 2: Aggregate shortfall as a fraction of the gross aggregate VM obligation for example 1 (left panel), example 2 (centre panel), and example 3 (right panel).

than at $\alpha = 1$. As a consequence, the aggregate shortfall must eventually become increasing with α. Indeed, we can repeat the exercise above for several values of α. Results are shown in the left panel of Figure 2. We can see that, while the aggregate shortfall is initially decreasing with α, it eventually starts to increase.

Since the aggregate shortfall at $\alpha = 0$ is smaller than at $\alpha = 1$, from (7) and (15) we have that:

$$
\begin{aligned}
s^b(0) < s^c(1) \Leftrightarrow \\
\bar{p}^b(0) - p^b(0) < \bar{p}^c(1) - p^c(1) \Leftrightarrow \\
\bar{p}^b(0) - \bar{p}^c(1) < p^b(0) - p^c(1)\,.
\end{aligned}
\tag{18}
$$

We have already mentioned that, when going from $\alpha = 0$ to $\alpha = 1$, shortfalls benefit from gains due to multilateral netting ($\bar{p}^b(0) > \bar{p}^c(1)$). Using (18), this implies that payments in the fully centrally cleared market are smaller than payments in the fully bilateral market ($p^b(0) > p^c(1)$). In fact, the meaning of (18) is precisely that the loss in the amount of payments delivered outweighs the gain from multilateral netting.

3.2 Example 2

In our second example we have $L = 4$. From Figure 3 we can see that banks A and D can net their bilateral obligations, meaning that aggregate bilateral obligations are equal

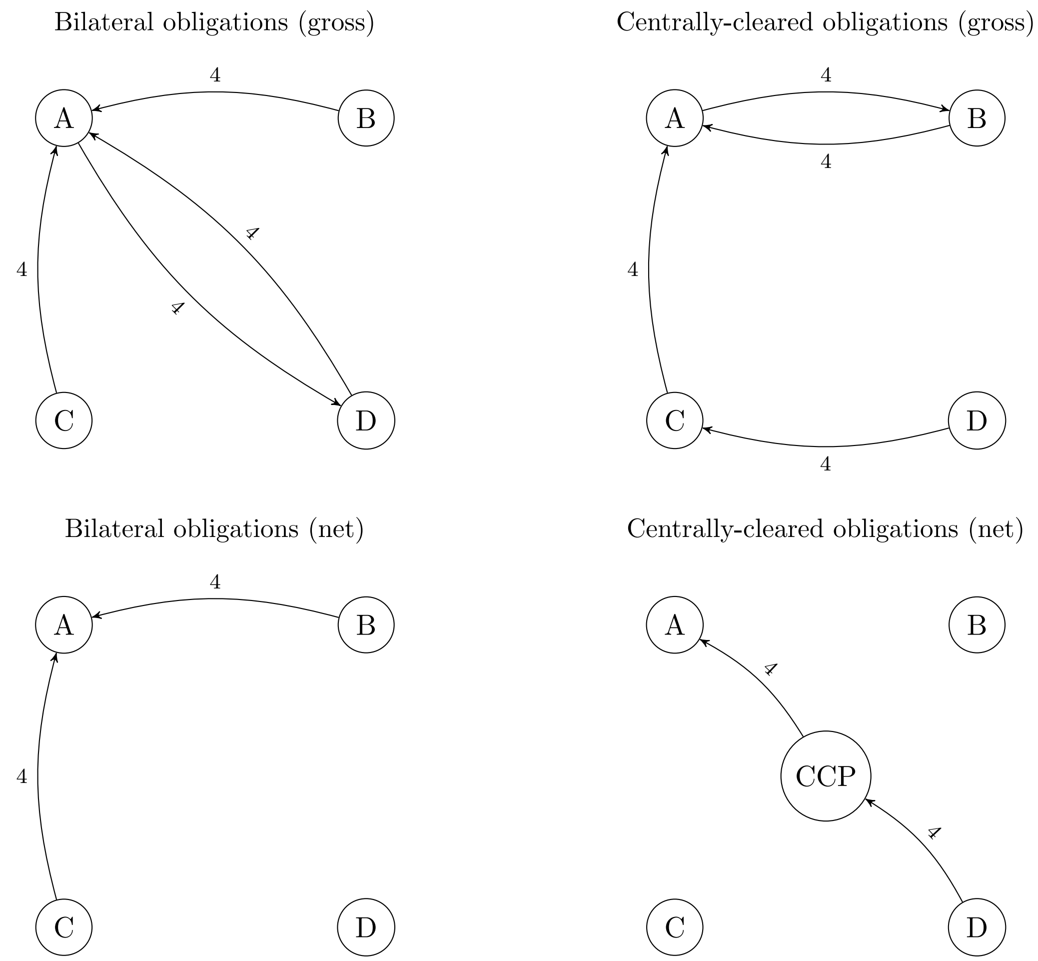

Figure 3: Gross and net bilateral and centrally cleared obligations for example 2.

to: $\bar{p}^b = 4 + 4 = 8$. Banks A and B fully offset their mutual centrally cleared obligations and bank C can fully net its centrally cleared obligations multilaterally. Only bank D is left with obligations to the CCP, meaning that aggregate centrally cleared obligations are equal to: $\bar{p}^c = \bar{p}_{D \to CCP} = 4$. As for example 1, in aggregate, *net* centrally cleared obligations are smaller than *net* bilateral obligations.

Let us now explicitly work out the shortfalls for some values of α. For reference, in Figure 4 we show net bilateral and centrally cleared obligations for several values of α. In a fully bilateral market ($\alpha = 0$), it is easy to check that both banks B and C are

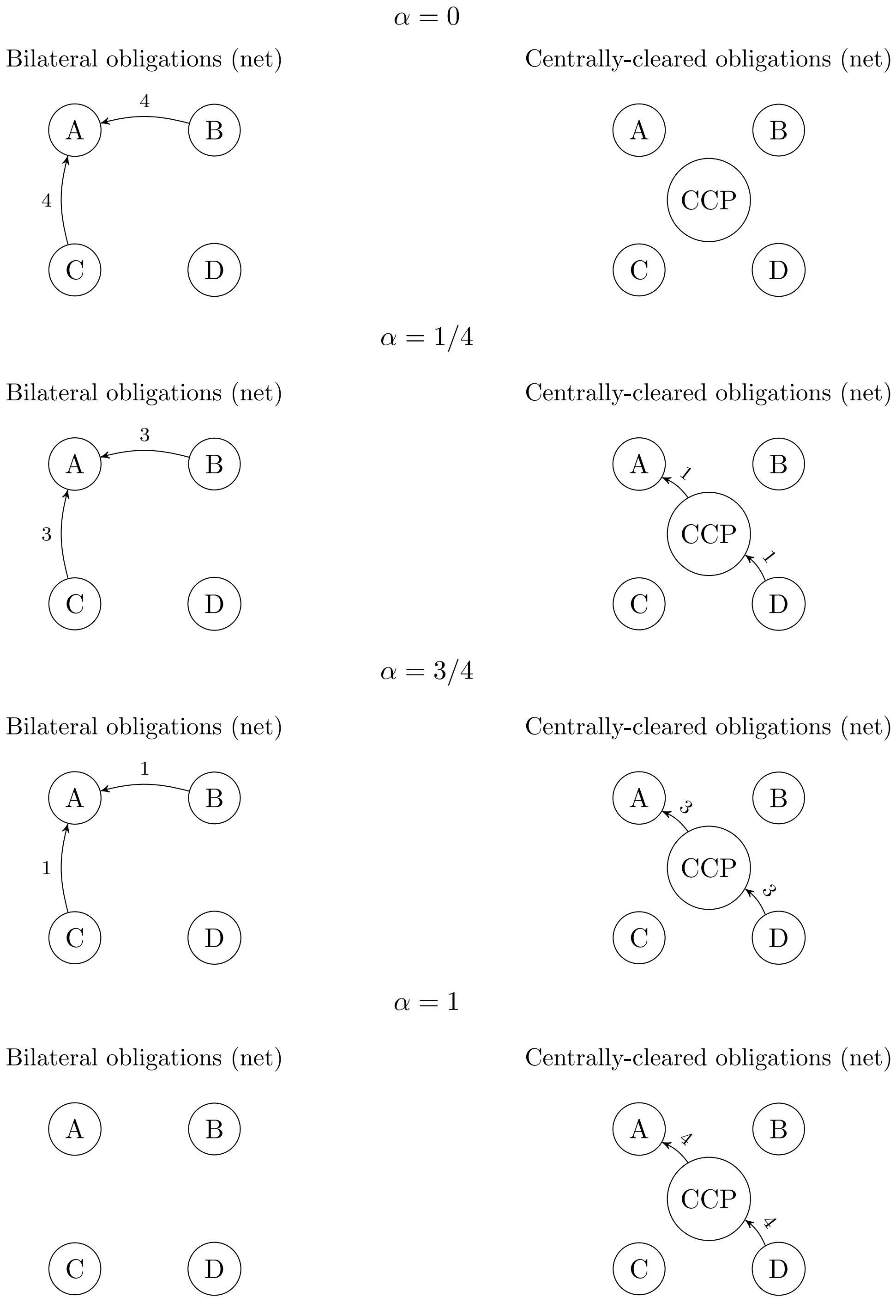

Figure 4: Net bilateral and centrally cleared obligations for example 2, for several values of α.

able to pay only 1 to bank A, thereby each facing a shortfall equal to 3. The aggregate payment is equal to $p^b_B + p^b_C = 2$ and the aggregate shortfall is equal to $s^b_B + s^b_C = 6$. For $\alpha = 1/4$, bank D is able to pay fully its obligations to the CCP, meaning that there are no shortfalls on centrally cleared obligations. On bilateral obligations, both banks B and C are able to pay only 1 to bank A, thereby each facing a shortfall equal to 2. As a consequence, the aggregate payment is equal to $p_{A\to CCP} + p^b_B + p^b_C = 3$ and the aggregate shortfall is equal to $s^b_B + s^b_C = 4$. For $\alpha = 3/4$, bank D is able to pay only 1 to the CCP, and faces a shortfall equal to 2 on its centrally cleared obligations. Both banks B and C are able to fully pay their bilateral obligations. Hence, the aggregate payment is equal to $p_{A\to CCP} + p^b_B + p^b_C = 3$ and the aggregate shortfall is equal to: $s^c_D = 2$. In a fully centrally cleared market ($\alpha = 1$), bank A is able to pay only 1 to the CCP ($p_{A\to CCP} = 1$), which is also the aggregate payment, and faces a shortfall equal to 3 ($s^c_A = 3$), which is also the aggregate shortfall. In the centre panel of Figure 2 we show the aggregate shortfall for many values of α. In fact, the aggregate shortfall has a minimum precisely at $\alpha = 3/4$.

The aggregate shortfall at $\alpha = 0$ is larger than at $\alpha = 1$, and therefore it is not obvious that at some point the shortfall would become increasing with α, as for the previous example. We note that, when going from $\alpha = 0$ to $\alpha = 1/4$, aggregate obligations decrease (from 8 to 7) and aggregate payments increase (from 2 to 3), both leading to aggregate shortfalls decreasing (from 6 to 4). When going from $\alpha = 1/4$ to $\alpha = 3/4$, aggregate net obligations decrease (from 7 to 5) and aggregate payments are constant (at 3). Hence, shortfalls decrease (from 5 to 2) . However, when going from $\alpha = 3/4$ to $\alpha = 1$, while aggregate net obligations keep decreasing (from 5 to 4), aggregate payments decrease *more* (from 3 to 1), leading to aggregate shortfalls actually increasing (from 2 to 3). Similarly to the previous example, this loss in the amount of payments delivered more than compensates for the gain from multilateral netting. Zooming in individual banks, when going from $\alpha = 3/4$ to $\alpha = 1$, on the one hand we are replacing two units of bilateral obligations (from bank B and C to bank A with one unit of centrally cleared obligations (from bank D to the CCP). On the other hand, both banks B and C had sufficient liquidity to pay those two units of bilateral obligations in full, whereas bank D had already exhausted its liquidity before receiving the additional unit of centrally cleared obligations, which therefore fully translate into a shortfall. The aggregate shortfall increases because obligations are reallocated from banks with sufficient liquidity (banks B and C) to banks without liquidity (bank D).

3.3 Example 3

In our last example gross bilateral and centrally cleared obligations are perfectly correlated, i.e. $\mathbf{L}^b = \mathbf{L}^c = \mathbf{L}$. In order to facilitate the comparison with the case in which $\mathbf{L}^b \neq \mathbf{L}^c$, we take the liability matrix $\mathbf{L}$ to be equal to the liability matrix of centrally cleared obligations in example 2 (see the top right panel of Figure 3). Therefore, also in this case $L = 4$.

In Figure 5 we show net bilateral and centrally cleared obligations for several values of α. Net centrally cleared obligations are obviously equal to those in example 2. For $\alpha = 0$, bank D can pay only 1 to bank C and faces a shortfall equal to 3. Bank C can pay only 2 to bank A (1 unit of initial cash endowment plus one unit received from bank D) and faces a shortfall equal to 2. The aggregate payment is equal to $p^b_C + p^b_D = 3$ and the aggregate shortfall is equal to $s^b_C + s^b_D = 5$. For $\alpha = 1/4$, bank D uses its initial endowment to fully pay its centrally cleared obligations, but it is left without cash to pay its bilateral obligations. As a consequence, it faces a bilateral shortfall equal to 3. Bank C can pay only 1 to bank A and faces a shortfall equal to 2. Therefore, the aggregate payment is equal to $p_{D \to CCP} + p^b_C = 2$ and the aggregate shortfall is equal to $s^b_C + s^b_D = 5$. For $\alpha = 3/4$, bank D can pay only 1 to the CCP and faces a centrally cleared shortfall equal to 2. Since bank D is left without cash to pay its bilateral obligations, it also faces a bilateral shortfall equal to 1. Instead, bank C can fully pay its bilateral obligations

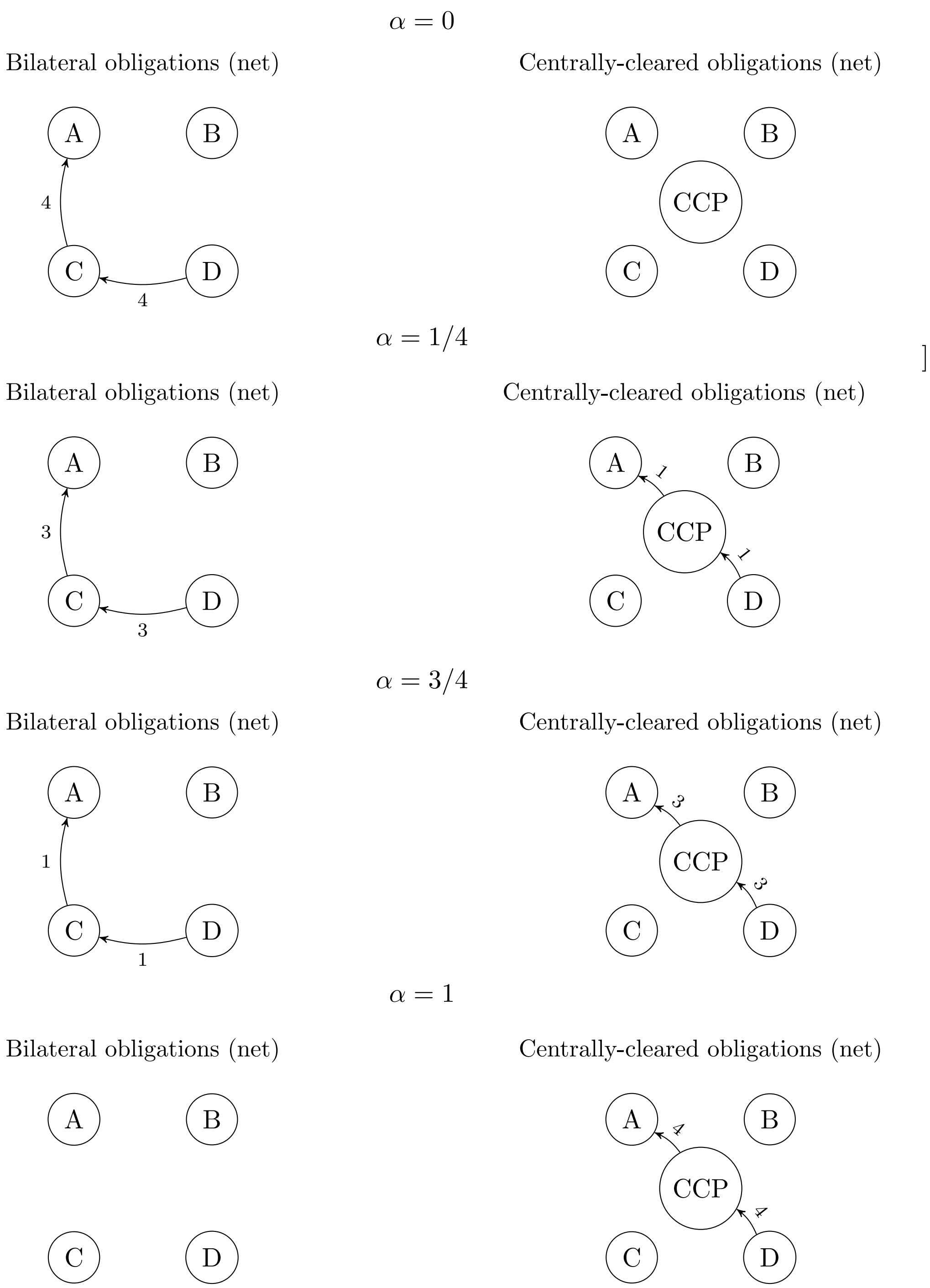

Figure 5: Net bilateral and centrally cleared obligations for example 3, for several values of α.

to bank A. Therefore, the aggregate payment is equal to $p_{D\to CCP} + p^b_C = 2$ and the aggregate shortfall is equal to $s^c_D + s^b_D = 3$. Finally, for $\alpha = 1$ bank D can pay only 1 to the CCP ($p_{D\to CCP} = 1$), which is also the aggregate payment, and faces a centrally cleared shortfall equal to 3 ($s^c_D = 3$), which is also the aggregate shortfall. In the right panel of Figure 2 we show the aggregate shortfall for many values of α. We can see that the aggregate shortfall is non-increasing with α.

When moving from $\alpha = 3/4$ to $\alpha = 1$, the aggregate shortfall is constant, and equal to the shortfall faced by bank D. In this case, we are replacing two units of bilateral obligations (from bank D to bank C and from bank C to bank A) with one unit of centrally cleared obligations (from bank D to the CCP). We note that all obligations from banks without liquidity, and thereby facing a shortfall, at $\alpha = 1$ (bank D) are allocated to the same banks at $\alpha = 3/4$, albeit split between centrally cleared and bilateral obligations. Hence, in contrast with example 2, when moving from $\alpha = 3/4$ to $\alpha = 1$, there is no reallocation of obligations from banks with sufficient liquidity to banks without liquidity.

We note that for $\alpha \leq 1/4$, i.e. in the region in which aggregate centrally cleared shortfalls are equal to zero, aggregate shortfalls (and therefore also bilateral shortfalls) are constant. In Section 5 we will see that this is not a coincidence, but indeed it is a general result for the case of perfectly correlated exposures.

4 General case

In this section, we discuss the results for the general case in which $\mathbf{L}^b \neq \mathbf{L}^c$, i.e. exposures are not necessarily perfectly correlated, and no further assumption is made on the correlation between $\mathbf{L}^c$ and $\mathbf{L}^b$.

For centrally cleared obligations, it is easy to compute payments and shortfalls explicitly from (4a) and (6).

Proposition 1. *Let $\mathcal{S}(\mathbf{L}^c, \mathbf{L}^b, \mathbf{e}^{(1)})$ be a family of mixed clearing systems. For all banks i we have:*

$$p_{i\to CCP} = \begin{cases} \alpha\left(\sum_j L^c_{ij} - L^c_{ji}\right)^+ & \textit{for } \alpha \le \frac{e_i^{(1)}}{\left(\sum_j L^c_{ij} - L^c_{ji}\right)^+} \\ e_i^{(1)} & \textit{for } \alpha > \frac{e_i^{(1)}}{\left(\sum_j L^c_{ij} - L^c_{ji}\right)^+} \end{cases} \tag{19}$$

and:

$$s_i^c = \begin{cases} 0 & \textit{for } \alpha \le \frac{e_i^{(1)}}{\left(\sum_j L^c_{ij} - L^c_{ji}\right)^+} \\ \alpha\left(\sum_j L^c_{ij} - L^c_{ji}\right)^+ - e_i^{(1)} & \textit{for } \alpha > \frac{e_i^{(1)}}{\left(\sum_j L^c_{ij} - L^c_{ji}\right)^+}\,. \end{cases} \tag{20}$$

This means that the payments of all banks to the CCP increase linearly in α up to $\alpha = e_i^{(1)}/\left(\sum_j L^c_{ij} - L^c_{ji}\right)^+$ and that they saturate to $e_i^{(1)}$ for larger values of α. Correspondingly, shortfalls in centrally cleared obligations are zero for $\alpha < e_i^{(1)}/\left(\sum_j L^c_{ij} - L^c_{ji}\right)^+$ and increase linearly for larger values of α. As a consequence, net centrally cleared VM obligations (from (4a)), payments to the CCP, and shortfalls in centrally cleared VM obligations are, for all banks and therefore also in aggregate, all non-decreasing functions of α.

From (11) we know that for all banks total bilateral VM obligations are decreasing with α. However, from (10), (4a), and (6) we have:

$$\begin{aligned} e_i^{(3)} &= e_i^{(1)} - \min\left(e_i^{(1)}, \alpha\Big(\sum_j L^c_{ij} - L^c_{ji}\Big)^+\right) + \alpha\left(\sum_j L^c_{ij} - L^c_{ji}\right)^- \\ &= \begin{cases} e_i^{(1)} - \min\left(e_i^{(1)}, \alpha\Big(\sum_j L^c_{ij} - L^c_{ji}\Big)^+\right) & \text{for } \sum_j L^c_{ij} - L^c_{ji} \ge 0 \\ e_i^{(1)} + \alpha\left(\sum_j L^c_{ij} - L^c_{ji}\right)^- & \text{for } \sum_j L^c_{ij} - L^c_{ji} < 0\,. \end{cases} \end{aligned} \tag{21}$$

If bank i has positive net centrally cleared VM obligations, i.e. $\sum_j L^c_{ij} - L^c_{ji} \ge 0$:

$$e_i^{(3)} = \begin{cases} e_i^{(1)} - \alpha\left(\sum_j L^c_{ij} - L^c_{ji}\right) & \text{for } \alpha \le \frac{e_i^{(1)}}{\left(\sum_j L^c_{ij} - L^c_{ji}\right)} \\ 0 & \text{for } \alpha > \frac{e_i^{(1)}}{\left(\sum_j L^c_{ij} - L^c_{ji}\right)}\,, \end{cases} \tag{22}$$

meaning that $e_i^{(3)}$ is non-increasing with α. Instead, if bank i has negative net centrally cleared VM obligations, i.e. $\sum_j L_{ij}^c - L_{ji}^c < 0$, we have:

$$\begin{aligned} e_i^{(3)} &= e_i^{(1)} - \alpha\Big(\sum_j L_{ij}^c - L_{ji}^c\Big) \\ &= e_i^{(1)} + \alpha\Big|\sum_j L_{ij}^c - L_{ji}^c\Big|, \end{aligned} \tag{23}$$

meaning that $e_i^{(3)}$ is increasing with α. The fact that $e_i^{(3)}$, the cash available to pay VM bilateral obligations, is non-increasing with α for some banks and increasing with α for others prevents from applying results on comparative statics of payments in the Eisenberg and Noe model. Therefore, it is not straightforward to derive the general behaviour of bilateral payments and shortfalls. While this is indeed possible for the special case of perfectly correlated exposures $\mathbf{L}^c = \mathbf{L}^b = \mathbf{L}$ (see Section 5), in the more general case one has to resort to numerical simulations (see Section 6).

Let us now denote with Δ the differentials of key quantities when transitioning from α_1 to $\alpha_2 > \alpha_1$. For example: $\Delta s_i^c = s_i^c(\alpha_2) - s_i^c(\alpha_1)$, and similarly for all other quantities. The change in shortfall for bank i is: $\Delta s_i^c + \Delta s_i^b = \Delta\bar{p}_{i\to CCP} + \Delta\bar{p}_i - \Delta p_{i\to CCP} - \Delta p_i$. In aggregate:

$$\Delta s^c + \Delta s^b = \Delta\bar{p}^c + \Delta\bar{p}^b - \Big(\Delta p^c + \Delta p^b\Big)\ . \tag{24}$$

Proposition 1 tells us that $\Delta\bar{p}^c \geq 0$, $\Delta p^c \geq 0$, and $\Delta s^c \geq 0$. From (4a) and (11) we can see that net payment obligations $\bar{p}^c + \bar{p}^b$ are a linear function of α, therefore they can only be increasing, decreasing, or constant. In Section 6 we will see that, in practice, multilateral netting is always beneficial and therefore that net payment obligations are always decreasing with α, i.e. $\Delta\bar{p}^c + \Delta\bar{p}^b < 0$. This has the following interesting implications.

Proposition 2. *Let $\mathcal{S}(\mathbf{L}^c, \mathbf{L}^b, \mathbf{e}^{(1)})$ be a family of mixed clearing systems. Let $0 < \alpha_1 <$*

$\alpha_2 < 1$, *and let us assume that* $\Delta\bar{p}^c + \Delta\bar{p}^b < 0$. *If:*

$$\Delta s^c + \Delta s^b < 0\,,$$

then either:

$$\Delta p^c + \Delta p^b \geq 0\,,$$

or:

$$\Delta p^c + \Delta p^b < 0 \qquad \textit{and} \qquad |\Delta p^c + \Delta p^b| < |\Delta\bar{p}^c + \Delta\bar{p}^b|\,.$$

If:

$$\Delta s^c + \Delta s^b > 0\,,$$

then:

$$\Delta p^c + \Delta p^b < 0 \qquad \textit{and} \qquad |\Delta p^c + \Delta p^b| > |\Delta\bar{p}^c + \Delta\bar{p}^b|\,.$$

This means that, if aggregate shortfalls are decreasing with α, as long as aggregate net payment obligations are also decreasing, there are two possibilities. First, aggregate payments are increasing. In this case, there are both gains due to multilateral netting and gains due to the amount of payments delivered. Second, aggregate payments are decreasing, but *less* than net payment obligations. In this case, fewer payments are delivered, but the gains due to multilateral netting are still larger then the loss from reduced payments. Instead, if aggregate shortfalls are increasing, then aggregate payments must be decreasing, and they must decrease *more* than aggregate net payment obligations. In this case, fewer payments are delivered, and this loss outweighs the gains due to multilateral netting.

5 Perfectly correlated exposures

In this section we present results for the special case of perfectly correlated exposures, i.e. $\mathbf{L}^c = \mathbf{L}^b = \mathbf{L}$. As discussed in Section 2.3, going from α to $\alpha' = \alpha + \Delta\alpha$ corresponds to all banks novating synchronously a fraction $\Delta\alpha$ of their notional with all their counterparties. In this case it is possible to prove several results on the behaviour of liquidity shortfalls without resorting to numerical simulations.

The first result we present compares shortfalls in a fully bilateral setting with shortfalls in a fully centrally cleared setting. In this section we often refer to quantities at a certain value of α. For example, with $s_i(\alpha)$ we denote the total shortfall of bank i at α. We explicitly note that with $s_i^b(\alpha)$ we denote the shortfall of bank i on bilateral VM obligations at α (but the fraction of VM obligations that are bilaterally cleared at α is $1 - \alpha$).

Theorem 1. *Let $\mathcal{S}(\mathbf{L}, \mathbf{e}^{(1)})$ be a family of clearing systems with perfectly correlated exposures. For all banks, the shortfall in the fully bilateral setting is larger than or equal to the shortfall in the fully centrally cleared setting:*

$$s_i(0) \geq s_i(1) \quad \forall i\,. \tag{25}$$

Theorem 1 formalises the intuition according to which a system in which all VM obligations are centrally cleared is more efficient (in the sense that liquidity shortfalls are smaller) than a system in which all VM obligations are bilateral. For families of clearing systems with perfectly correlated exposures this property holds not only in aggregate, but for individual banks. Nevertheless, this is not true for all families of balanced clearing systems, see e.g. the first example in Section 3, in which $\sum_i s_i(0) < \sum_i s_i(1)$.

The next result generalises Theorem 1 by showing that a system in which all VM obligations are centrally cleared is more efficient, in the sense that liquidity shortfalls of all banks are smaller than a system in which only a fraction of all VM obligations is centrally cleared. Amini et al. (2020) and Ahn (2020) prove similar results in richer models. Amini et al. (2020) introduce liquidation costs, end users, and default fund contributions. In Ahn (2020) there are bankruptcy costs and institutions can have external liabilities.

Theorem 2. *Let $\mathcal{S}(\mathbf{L}, \mathbf{e}^{(1)})$ be a family of clearing systems with perfectly correlated exposures and let $\alpha \in [0, 1)$. For all banks, the shortfall at $\alpha < 1$ is larger than or equal to the shortfall in the fully centrally cleared setting:*

$$s_i(\alpha) \geq s_i(1) \quad \forall i\,. \tag{26}$$

To the best of our knowledge, previous studies compare fully centrally cleared markets either with fully bilateral markets (Amini et al., 2020; Ahn, 2020) (as in Theorem 1) or with mixed markets (Amini et al., 2016) (as in Theorem 2). While those results allow to gauge what happens when we move to a fully centrally cleared market, they do not tell anything about the case in which the fraction of notional that is centrally cleared increases (or decreases). To this end we now compare two different mixed markets corresponding to two fractions of centrally cleared notional, e.g. α_1 and α_2.

Theorem 3. *Let $\mathcal{S}(\mathbf{L}, \mathbf{e}^{(1)})$ be a family of clearing systems with perfectly correlated exposures and let $\alpha_1, \alpha_2 \in [0, 1]$, with $\alpha_1 \leq \alpha_2$. The aggregate shortfall is a non-increasing function of α:*

$$\sum_i s_i(\alpha_1) \geq \sum_i s_i(\alpha_2)\,. \tag{27}$$

Theorem 3 shows that, for perfectly correlated exposures, increasing the fraction of centrally cleared notional from α to $\alpha' = \alpha + \Delta\alpha$, is always (weakly) beneficial in aggregate, independently of the starting fraction of notional α that is centrally cleared and of the additional fraction of notional $\Delta\alpha$ that becomes centrally cleared. In other words, for perfectly correlated exposures, in aggregate there are no unintended consequences of increasing the fraction of notional that is centrally cleared.

Theorem 3 tells us that increasing the fraction of centrally cleared notional is weakly beneficial in aggregate. But it does not exclude that, at least in some interval, increasing the fraction of centrally cleared notional is not strictly beneficial. That is, it does not exclude that, in some interval, the aggregate shortfall does not decrease as the fraction of centrally cleared notional increases. We find that this is indeed the case. More precisely, shortfalls of all banks do not decrease in the interval $[0, \alpha^*)$, where α^* is a critical value that depends on cash buffers and net obligations of individual banks.

Theorem 4. *Let $\mathcal{S}(\mathbf{L}, \mathbf{e}^{(1)})$ be a family of clearing systems with perfectly correlated exposures and let:*

$$\alpha^* = \min_i \frac{e_i^{(1)}}{\left(\sum_j L_{ij} - L_{ji}\right)^+} . \tag{28}$$

Then, for all $\alpha < \alpha^*$, $s_i(\alpha)$ is independent of α, i.e. $s_i(0) = s_i(\alpha)$, for all i.

From (20) we have that, if $\alpha \leq e_i^{(1)} / (\sum_j L_{ij} - L_{ji})^+$, bank i has zero shortfall on centrally cleared obligations, i.e. $s_i^c = 0$. When $\alpha < \alpha^*$, then no bank has a shortfall on centrally cleared obligations, and $s_i = s_i^b$, for all i. Therefore, Theorem 4 tells us that, when no bank has a shortfall on centrally cleared obligations, then shortfalls of individual banks on bilateral obligations (which are equal to total shortfalls) do not depend on the fraction of notional that is centrally cleared. Clearly, an analogous result holds for the aggregate shortfall:

Corollary 1. *Let $\mathcal{S}(\mathbf{L}, \mathbf{e}^{(1)})$ be a family of clearing systems with perfectly correlated exposures. Then, for all $\alpha < \alpha^*$, $\sum_i s_i(\alpha)$ is independent of α, i.e. $\sum_i s_i(0) = \sum_i s_i(\alpha) = \sum_i s_i(\alpha^*)$.*

From Theorem 3 we know that the aggregate shortfall is non-increasing, which means that $\sum_i s_i(0)$ is the largest aggregate shortfall with respect to α. Apart from the corner case in which all individual shortfalls are independent from α (i.e. unless $s_i(0) = s_i(1)$, for all i), the aggregate shortfall will eventually start decreasing. In addition, Corollary 1 tells us that the aggregate shortfall will not start decreasing until the fraction of centrally cleared notional is *at least* as large as the α^*. This implies that, in the case of perfectly correlated exposures, central clearing becomes beneficial only when a sufficiently large fraction of notional is centrally cleared. However, we explicitly note that Corollary 1 does not tell us that the aggregate shortfall starts decreasing *at* α^*. In principle it may exist $\delta \in [0, 1 - \alpha^*)$ such that $\sum_i s_i(0) = \sum_i s_i(\alpha^* + \delta)$, meaning that the aggregate shortfall could start decreasing for any value of α in the interval $[\alpha^*, 1)$.

We note that α^* is the smallest ratio between cash buffers and net VM payment obligations at $\alpha = 1$ (that is when all VM payment obligations are netted multilaterally). As a consequence, when cash endowments are smaller or net VM payment obligations are lager, α^* becomes smaller and central clearing becomes strictly beneficial at smaller values of centrally cleared notional.

To recap, in the case of perfectly correlated exposures, the aggregate shortfall is constant in the interval $[0, \alpha^*)$. For $\alpha \geq \alpha^*$, either the aggregate shortfall remains constant up to $\alpha = 1$ (i.e. $\sum_i s_i(0) = \sum_i s_i(\alpha^*) = \sum_i s_i(\alpha) = \sum_i s_i(1)$, for all α), or it eventually starts decreasing so that $\sum_i s_i(\alpha^*) > \sum_i s_i(1)$. This implies that, for perfectly correlated exposures, increasing the scope of central clearing never leads to larger liquidity shortfalls. The existence of an interval of values of α where the shortfall is constant implies however the existence of a critical fraction of centrally cleared obligations below which the presence of a CCP is not beneficial from a liquidity standpoint.

6 Independent exposures

Now we consider the case in which $\mathbf{L}^c \neq \mathbf{L}^b$. As anticipated in Section 4, in order to overcome the difficulty of characterising bilateral shortfalls analytically, in this section we resort to numerical experiments. In a nutshell: we generate networks of random VM obligations, we simulate the three stages of the payment algorithm in Section 2.4, and we compute liquidity shortfalls. For simplicity we focus on the case in which $\mathbf{L}^c$ and $\mathbf{L}^b$ are independent, which we refer to as the case of *independent exposures*. Our main result is that increasing the fraction of centrally cleared notional α is not always beneficial, in the sense that it does not necessarily lead to smaller aggregate liquidity shortfalls.

We fix the initial cash endowment of all banks to one unit (i.e. $\mathbf{e}^{(1)} = 1$). Conceptually the generation of random VM obligations consists of two steps. First, generating the network of counterparties, i.e. for each bank i generating the set of banks i owes to. Second, generating the amounts of the individual VM obligations. In the language of network theory, in the first step we generate the topology of the network of VM obligations, while in the second step we generate its weights. Formally, we write centrally cleared and bilateral VM obligations as follows:

$$L_{ij}^c = A_{ij}^c w_{ij}^c \tag{29a}$$

$$L_{ij}^b = A_{ij}^b w_{ij}^b \,, \tag{29b}$$

where $A_{ij}^c \in \{0, 1\}$ and $w_{ij}^c > 0$, for all i and j. If $A_{ij}^c = 1$, i has a centrally cleared VM obligation to j of amount w_{ij}^c. The same applies to bilateral VM obligations. $\mathbf{A}^c$ and $\mathbf{A}^b$ are called adjacency matrices for centrally cleared and bilateral VM obligations.

Regarding the topology we assume that underlying counterparty relationships for centrally cleared and bilateral contracts are independent of each other, i.e. that $\mathbf{A}^c$ and $\mathbf{A}^b$ are independent. Moreover, within both centrally cleared and bilateral contracts, we further assume that all counterparty relationships exist independently of each other with probability c. Networks with such topology are known as Erdős-Rényi networks. This means that, on average, each bank i has centrally cleared VM obligations to $c \cdot (n - 1)$ banks, but also that $c \cdot (n - 1)$ banks have VM obligations to i. Analogously for bilateral VM obligations. The case $c = 0$ corresponds to the trivial empty network in which there are no relationships, and the case $c = 1$ to the fully connected network in which all banks have relationships with all other banks. The larger c, the denser the network of VM obligations, which is why we refer to c as the *density* of the network. We also discard any topology in which there is at least one bank without VM obligations.[10]

As regards the weights, for all banks i we generate total obligations $L_i^{tot} = \sum_j L_{ij}^c = \sum_j L_{ij}^b$ and we partition those uniformly across its counterparties:

$$w_{ij}^c = \frac{L_i^{tot}}{\sum_j A_{ij}^c} \tag{30a}$$

$$w_{ij}^b = \frac{L_i^{tot}}{\sum_j A_{ij}^b}\,, \tag{30b}$$

so that

$$L_{ij}^c = \begin{cases} \frac{L_i^{tot}}{\sum_j A_{ij}^c} & \text{for } A_{ij}^c = 1 \\ 0 & \text{for } A_{ij}^c = 0 \end{cases} \tag{31a}$$

$$L_{ij}^b = \begin{cases} \frac{L_i^{tot}}{\sum_j A_{ij}^b} & \text{for } A_{ij}^b = 1 \\ 0 & \text{for } A_{ij}^b = 0\,. \end{cases} \tag{31b}$$

We note that, since the two adjacency matrices $\mathbf{A}^c$ and $\mathbf{A}^b$ are different, also the two VM obligation matrices $\mathbf{L}^c$ and $\mathbf{L}^b$ are different. This set-up allows us to express the variability of VM obligations in terms of the density c and the parameters of the distribution of total obligations.

For simplicity we focus on the case in which total gross VM obligations are the same for all banks, i.e. $L_i^{tot} = L$, for all i. The case of heterogenous VM obligations is discussed in detail in Appendix B. In Figure 6 we show the aggregate shortfall $\sum_i s_i$ normalised to the aggregate gross VM obligation nL, for three different realizations of $\mathbf{L}^c$

[10]This means that, as long as there exists at least one i such that $\sum_j A_{ij}^c = 0$, we discard $\mathbf{A}^c$ and we generate another one, and similarly for $\mathbf{A}^b$. This becomes increasingly less likely for larger and larger values of c.

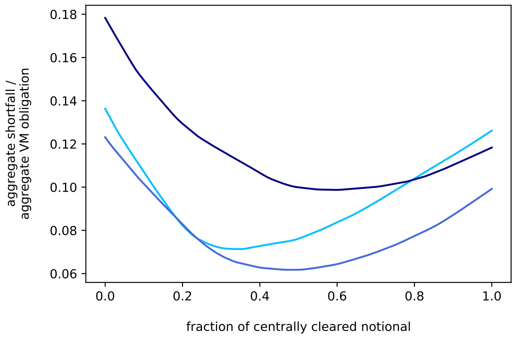

Figure 6: Aggregate shortfall as a fraction of the gross aggregate VM obligation for three realisations of VM obligations. $n = 100$, $c = 0.04$, and $L_i^{tot} = L$, for all i with $L = 4$.

and $\mathbf{L}^b$.[11] In this specific example, $n = 100$ and $c = 0.04$, meaning that each bank has, on average, centrally cleared VM obligations to four banks and bilateral obligations also to four banks. Moreover, $L = 4$, meaning that the total VM obligations of each bank are equal to four times its initial cash endowment. For all three realisations the aggregate shortfall starts to decrease as α increases, but after having reached a minimum at α_{min}, it starts to increase again. This means that there is an optimal value of α at which the aggregate liquidity shortfall is minimal. Both the position of the minimum and its value are variable and depend on the individual realisation of VM obligations. For the realisations in Figure 6, the optimal fraction of centrally cleared notional α_{min} ranges from about 0.35 to about 0.6, while the aggregate shortfall at the minimum ranges from about 6% to about 10% of aggregate gross VM obligations.

In Figure 7 we focus on one individual realisation of VM obligations. In the top panel we show the aggregate normalised shortfall $\sum_i s_i/nL$ alongside the aggregate normalised centrally cleared and bilateral shortfalls, i.e. $\sum_i s_i^c/nL$ and $\sum_i s_i^b/nL$. The minimum in the aggregate normalised shortfall emerges as the result of two competing trends: the aggregate normalised centrally cleared shortfall increases with α (as expected from

[11] We normalise to the (aggregate) *gross* VM obligation, as this quantity does not depend on α. As a consequence, the behaviour of normalised and un-normalised shortfalls is exactly the same, but the normalised shortfall is between zero and one.

Proposition 1), while the aggregate normalised bilateral shortfall decreases with it.

From the bottom left panel of Figure 7, we can see that net centrally cleared VM obligations increase with α (as expected from Proposition 1) and net bilateral VM obligations decrease with α (from (11)). Overall, total net VM obligations decrease with α. Since total gross VM obligations do not depend on α, this means that netting is more and more efficient as α increases, indicating that multilateral netting (on centrally cleared VM obligations) dominates bilateral netting more and more. From the bottom right panel of Figure 7, we can see that the aggregate normalised payment on centrally cleared contracts, i.e. $\sum_i p_{i\to CCP}/nL$, increases with α (as expected from Proposition 1). But the aggregate normalised payment on bilateral contracts, i.e. $\sum_i p_i/nL$, decreases with α *more*, resulting in an overall trend of the aggregate normalised payment on all contracts that is decreasing with α.

We now look at the statistical properties of a large sample of 1 000 realisations. First, we ask how frequently the aggregate shortfall has one minimum with respect to α. We find that, when the density c is sufficiently small, the aggregate shortfall always has a minimum (see Figure 8). By increasing the density, the fraction of realisations with a minimum starts to decrease. More precisely, the larger the total gross VM obligations L, the larger the density c at which the fraction of realisations with a minimum drops. By increasing the density further, one reaches a point at which the aggregate shortfall never has a minimum.

We also find that the trends of both VM obligations and payments displayed in Figure 7 for an individual realisation are confirmed for all realisations in our sample. In particular, total net VM obligations decrease strictly with α. Again, since net centrally cleared VM obligations $\sum_i \bar{p}_{i\to CCP}$ increase with α (see Proposition 1), this means that net bilateral VM obligations $\sum_i \bar{p}_i$ decrease *more*. In other words, as α increases, net bilateral VM obligations are replaced by smaller centrally cleared VM obligations due to the multilateral netting performed by the CCP. Similarly, total payments $\sum_i p_{i\to CCP} + \sum_i p_i$

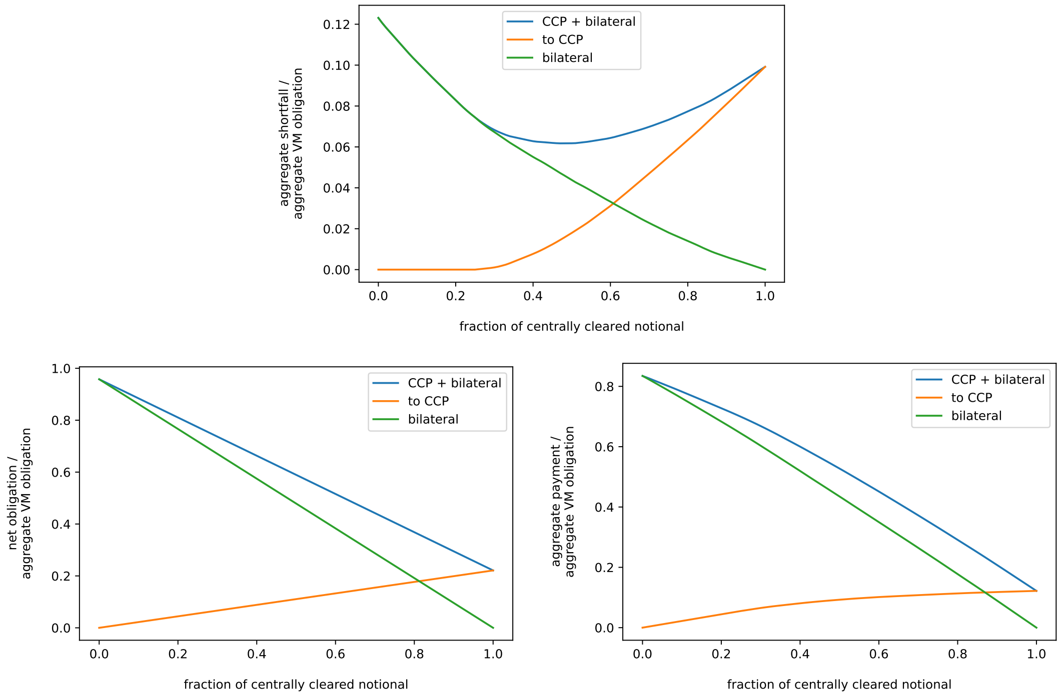

Figure 7: Decomposition of normalised aggregate shortfall (top panel), net VM obligation (bottom left panel), and aggregate normalised payments (bottom right panel) for one realisation of VM obligations. $n = 100$, $c = 0.04$, and $L_i^{tot} = L$, for all i with $L = 4$.

decrease strictly with α and, since payments on centrally cleared obligations $\sum_i p_{i \to CCP}$ increase with α (see Proposition 1), this means that payments on bilateral VM obligations $\sum_i p_i$ decrease *more*. As α increases, bilateral payments are replaced by smaller payments on centrally cleared obligations and, as a result, fewer payments as a whole are delivered.

We can interpret these results through the lens of Proposition 2. For $\alpha < \alpha_{\text{mins}}$ the aggregate shortfall is decreasing because the gains from multilateral netting (reflected in smaller net VM obligations) are larger than the losses due to reduced delivered payments. Conversely, for $\alpha > \alpha_{\text{mins}}$, the aggregate shortfall is increasing because the gains from multilateral netting are not sufficient to offset the losses due to reduced delivered payments. We also find that in almost all (99.78%) realisations with a minimum, not only

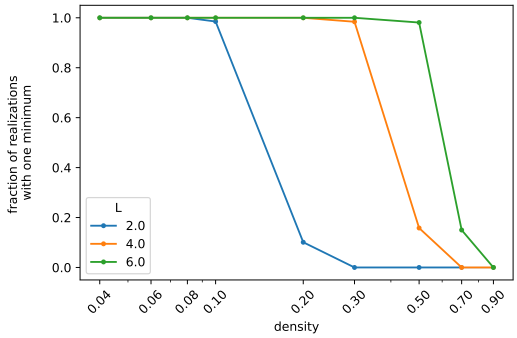

Figure 8: Fraction of realisations in which the aggregate shortfall has one minimum with respect to α. $n = 100$, 1 000 realisations, and $L_i^{tot} = L$, for all i.

fewer payments are delivered in a fully bilateral market than in a fully centrally cleared market, but actually fewer payments are delivered per unit of net payment obligation. In other words, when there is a minimum in the aggregate shortfall, payments are almost always delivered more efficiently in a fully bilateral market than in a fully centrally cleared market ($\epsilon^b(0) > \epsilon^c(1)$). While central clearing reduces payment obligations due to multilateral netting, whenever there is a minimum in the aggregate shortfall, it does also reduce the efficiency with which payments are delivered.

What about the realisations in which the aggregate shortfall has no minimum with α? By increasing the density of the network, realisations with one minimum decrease and monotonic (non-increasing) realisations start to increase (see Figures 8 and 9). In these realisations, VM obligations decrease more than payments for all values of α, suggesting that gains from multilateral netting dominate the reduction in (bilateral) payments. By further increasing the density of the network, also monotonic (non-increasing) realisations decrease and realisations in which the aggregate shortfall is zero for all values of α start to appear. Eventually, for even larger densities, in most realisations the aggregate shortfall is zero for all values of α.

In the left panel of Figure 10 we show the mean value of $\alpha_{\min}$ and its standard deviation across network realisations, for several values of the density c and total gross

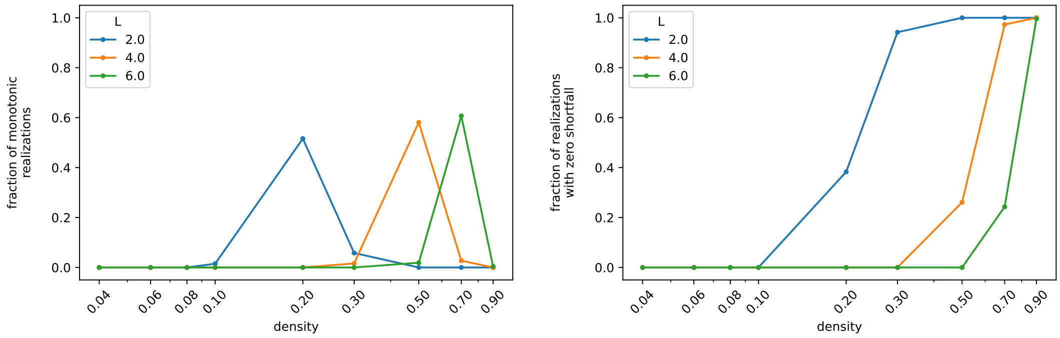

Figure 9: Fraction of realisations in the which aggregate shortfall is monotonic with respect to α (left panel) or in which is equal to zero for all values of α (right panel). $n = 100$, 1 000 realisations, and $L_i^{tot} = L$, for all i.

VM obligations L. $\alpha_{\min}$ is broadly around 50% and, within error bands, largely does not depend either on c or on L. We stress, however, that the variability of $\alpha_{\min}$ is substantial (note that error bands span one standard deviation), pointing to a strong dependence of $\alpha_{\min}$ on the specific realisation of the network of VM obligations.

Next, we look at the normalised aggregate shortfall at $\alpha_{\min}$, i.e. $\sum_i s_i(\alpha_{\min})/nL$. From the right panel of Figure 10 we can see that it is decreasing with the density c and increasing with total gross VM obligations L. Intuitively, as c increases the average number of counterparties increases, which leads to more opportunities for netting. Hence, we can expect shortfalls to be generally smaller for larger value of c. As the total gross VM obligation L increases, shortfalls become larger when the cash endowment is constant.

Finally, in order to gauge the economic significance of $\alpha_{\min}$, we look at the improvement of being at $\alpha_{\min}$ relative to the fully centrally cleared setting ($\alpha = 1$) or to the fully bilateral setting ($\alpha = 0$), i.e. $(\sum_i s_i(1) - \sum_i s_i(\alpha_{\min})) / \sum_i s_i(1)$ and $(\sum_i s_i(0) - \sum_i s_i(\alpha_{\min})) / \sum_i s_i(0)$. From Figure 11 we can see that the improvement relative to both $\alpha = 1$ and $\alpha = 0$ decreases with the total gross VM obligation L and increases with the density c. Mean improvements relative to $\alpha = 1$ are economically significant, ranging from around 20% to almost 100%. Similarly, for mean improvements

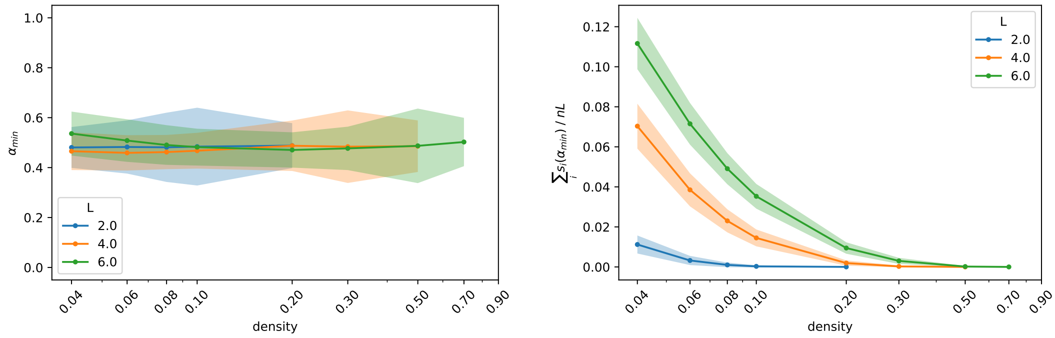

Figure 10: Mean value of α_{min}, the fraction of centrally cleared notional at which the normalised aggregate shortfall is minimal, (left panel) and mean normalised aggregate shortfall at α_{min} (right panel) for several values of c and L. Semi-transparent regions span one standard deviation. $n = 100$, 1 000 realisations, and $L_i^{tot} = L$, for all i. Only combinations of parameters for which the fraction of realisations in which the aggregate shortfall has a minimum is larger than 5% are shown.

relative to $\alpha = 0$, ranging from around 50% to almost 100%.

In Appendix B we broaden our analysis to the case in which gross total gross VM obligations L_i^{tot} are not equal for all banks. We draw L_i^{tot} from a Gaussian distribution with mean L and variance σ^2, examining the case of small ($\sigma = L/6$), medium ($\sigma = L/3$, and large ($\sigma = L$) levels of heterogeneity. Most of the findings illustrated for homogeneous VM obligations also hold for small and medium levels of heterogeneity. In particular, the aggregate shortfall (almost) always has a minimum when the density c is small (see Figure B.1 and the left panel of Figure B.3). When the density increases, fewer and fewer realisations display a minimum. As the level of heterogeneity increases, the region of the parameter space in which the fraction of realisations always has a minimum shrinks. When the heterogeneity is large, the number of realisations with a minimum decreases further (see the right panel of Figure B.3), and the fraction of realisations with a minimum is appreciably larger than zero only when the network of obligations is densely interconnected ($c \leq 0.1$).

Finally, we check whether the existence of a minimum in the aggregate shortfall can be attributed to the fact that payments are sequenced in our model (banks pay the CCP

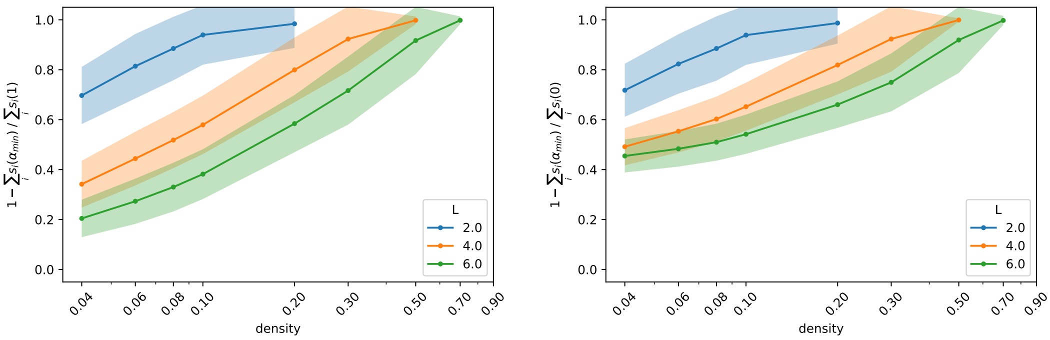

Figure 11: Improvements of being at $\alpha_{\min}$ relative to the fully centrally cleared setting (left panel) or to the fully bilateral setting (right panel). Semi-transparent regions span one standard deviation. $n = 100$, 1 000 realisations, and $L_i^{tot} = L$, for all i. Only combinations of parameters for which the fraction of realisations in which the aggregate shortfall has a minimum is larger than 5% are shown.

first, then the CCP pays banks, then banks settle bilateral obligations). In Appendix C we find that, broadly speaking, this is not the case. In fact, a substantial fraction of realisations in which the minimum in aggregate shortfall appears when introducing the sequencing is present only when total gross VM obligations and large ($L = 6$) and heterogeneous ($\sigma = L/3$). Even in those cases, aggregate shortfalls at $\alpha_{\min}$ with and without sequencing differ by relatively small amounts (for 95% of realisations normalised aggregate shortfalls at $\alpha_{\min}$ differ by less than 8 percentage points).

7 Conclusion

Central clearing is one of the pillars of the approach undertaken by regulators after the Global Financial Crisis to enhance financial stability. However, it is still very much debated whether introducing central clearing or extending its scope can have unintended consequences (Pirrong, 2011; Ghamami and Glasserman, 2017; Berndsen, 2020; Menkveld and Vuillemey, 2021; Bellia et al., 2024). While the default fund provides a mechanism to mutualise losses and insure clearing members against idiosyncratic failures, it can reduce incentives to select counterparties that are less risky (Biais et al., 2012, 2016) and in certain cases it might make it less desirable than bilateral clearing (Antinolfi et al., 2019). Similarly, while multilateral netting reduces the amount of collateral posted to the CCP within each asset class, it does not necessarily reduce the amount of collateral posted across all asset classes (Duffie and Zhu, 2011). We explicitly notice that central clearing also provides additional benefits. For example, the CCP can manage the default of clearing members in an orderly way by auctioning the positions of defaulted members to the surviving members, thereby mitigating the fire sales that would have ensued otherwise (Vuillemey, 2023).

Here we focus on liquidity shortfalls arising from the failure to meet variation margin calls. We simplify our model in order to remove the potential sources of inefficiency related to central clearing that have been previously studied in the literature. In fact, we consider only one asset class to avoid the tradeoff between multilateral netting within asset classes and bilateral netting across asset classes (Duffie and Zhu, 2011). We introduce only one CCP to avoid inefficiencies due to missed netting opportunities across different CCPs (Duffie et al., 2015; Veraart and Aldasoro, 2025). Finally, in our model the CCP always pays its obligations in full to avoid downstream consequences of its failure. Nevertheless, we find that increasing the fraction of centrally cleared notional does not necessarily lead to smaller liquidity shortfalls.

We find that, at least when the underlying counterparties on centrally cleared and bilateral contracts are not exactly the same (more precisely when centrally cleared and bilateral exposures are independent), the aggregate shortfall is not minimal when all contracts are centrally cleared (or when all contracts are bilateral). Indeed, unless the network of counterparties is too interconnected or unless the gross VM obligations are too small, there exists a non-trivial optimal fraction of centrally cleared notional for which the aggregate liquidity shortfall is minimal. This suggests that increasing the scope for central clearing might not necessarily reduce the aggregate demand for collateral. In fact, it is true that increasing the fraction of centrally cleared notional leads to a reduction in net VM payment obligations due to the multilateral netting performed by the CCP. However, it also leads to the reduction in realised (bilateral) payments. When the fraction of centrally cleared notional is small the first effect prevails, but when it is large the second effect becomes dominant. Furthermore, the reduction in realised (bilateral) payments does not appear to be driven by the temporal constraints due to the sequencing of centrally cleared and bilateral payments.

In the cases considered here, the mean optimal fraction of centrally cleared notional ranges between 50% and 80%. Independently of the private incentives that individual institutions might face, it is unlikely that they would collectively reach optimality. The reason is that, in order to compute the optimal value of centrally cleared notional, one needs to simulate payments made by all institutions, which requires knowledge of all VM payments obligations, while typically each market participant only has knowledge about its outgoing and incoming VM payment obligations. In addition, the optimal value of centrally cleared notional depends strongly on the individual realisation of the network of payment obligations, i.e. on how the initial shock is distributed across market participants. Therefore, designing derivatives markets around a single optimal value of the fraction of notional to be centrally cleared would be difficult even for a regulator with access to full information.

When centrally cleared and bilateral exposures are perfectly correlated, and therefore the underlying counterparties on centrally cleared and bilateral contracts are exactly the same, increasing the fraction of centrally cleared notional is always weakly beneficial, in the sense that the aggregate liquidity shortfall weakly decreases with the fraction of centrally cleared notional. However, the aggregate shortfall starts decreasing only if a sufficiently large fraction of notional is centrally cleared. As a consequence, in this case, introducing central clearing might not reduce the demand for collateral, unless its scope is sufficiently large.

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A Proofs and lemmas

Theorem 1. *Let $\mathcal{S}(\mathbf{L}, \mathbf{e}^{(1)})$ be a family of clearing systems with perfectly correlated exposures. For all banks, the shortfall in the fully bilateral setting is larger than or equal to the shortfall in the fully centrally cleared setting:*

$$s_i(0) \geq s_i(1) \quad \forall i\,. \tag{A.1}$$

Proof. Let us divide banks into two groups, banks i with net obligations smaller than or equal to zero ($\sum_j L_{ij} - L_{ji} \leq 0$) and banks i with net obligations larger than zero ($\sum_j L_{ij} - L_{ji} > 0$). This property clearly does not depend on α. For banks such that $\sum_j L_{ij} - L_{ji} \leq 0$, from (4a) and (7) we have that: $s_i(1) = [(\sum_j L_{ij} - L_{ji})^+ - e_i^{(1)}]^+ = [-e_i^{(1)}]^+ = 0 \leq s_i(0)$. So, for all banks i that have net obligations smaller than or equal to zero the shortfall for $\alpha = 0$ is larger than or equal to the shortfall for $\alpha = 1$. For banks i with net obligations larger than zero we have: $s_i(1) = [(\sum_j L_{ij} - L_{ji})^+ - e_i^{(1)}]^+ = [(\sum_j L_{ij} - L_{ji}) - e_i^{(1)}]^+$. For $\alpha = 0$ the shortfall is only on bilateral contracts. Moreover, as no payment has been made in the first two rounds, $e_i^{(3)} = e_i^{(1)}$. If $s_i(0) > 0$, i.e. if $\bar{p}_i > p_i$, we have that $p_i = e_i^{(1)} + \sum_j \Pi_{ji} p_j$. If $s_i(0) \leq 0$, i.e. if $p_i = \bar{p}_i$, we have that

$\bar{p}_i \leq e_i^{(1)} + \sum_j \Pi_{ji} p_j$. Therefore, we can write:

$$
\begin{aligned}
s_i(0) = \bar{p}_i - p_i &= \left[\bar{p}_i - e_i^{(1)} - \sum_j \Pi_{ji} p_j\right]^+ \\
&\geq \left[\sum_j (L_{ij} - L_{ji})^+ - e_i^{(1)} - \sum_j \Pi_{ji} \bar{p}_j\right]^+ \\
&= \left[\sum_j (L_{ij} - L_{ji})^+ - e_i^{(1)} - \sum_j (L_{ji} - L_{ij})^+\right]^+ \\
&= \left[\sum_j (L_{ij} - L_{ji})^+ - e_i^{(1)} - \sum_j (L_{ij} - L_{ji})^-\right]^+ \\
&= \left[\sum_j (L_{ij} - L_{ji}) - e_i^{(1)}\right]^+ \\
&= s_i(1)\,.
\end{aligned}
\tag{A.2}
$$

Therefore, also for all banks i that have net obligations larger than zero the shortfall for $\alpha = 0$ is larger than or equal to the shortfall for $\alpha = 1$. □

Lemma 1 (Splitting of central clearing in two sub-stages)**.** *Let $\bar{p}_{i\to CCP}$ and $\bar{p}_{CCP\to i}$, for all i, be two vectors of net obligations to and from the CCP, defined as in (1), $\mathbf{e} \geq 0$ a vector of cash endowments, and $\beta \in [0,1]$. Let us introduce quantities in the first sub-stage:*

$$\bar{p}^{(1)}_{i\to CCP} = \beta \bar{p}_{i\to CCP} \tag{A.3a}$$

$$\bar{p}^{(1)}_{CCP\to i} = \beta \bar{p}_{CCP\to i} \tag{A.3b}$$

$$e_i^{(1)} = e_i\,, \tag{A.3c}$$

with payments $p^{(1)}_{i\to CCP}$ and shortfalls $s_i^{(1)}$ defined as in (6) and (7), for all i. Let us *introduce quantities in the second sub-stage:*

$$\bar{p}^{(2)}_{i\to CCP} = (1-\beta)\bar{p}_{i\to CCP} \tag{A.4a}$$

$$\bar{p}^{(2)}_{CCP\to i} = (1-\beta)\bar{p}_{CCP\to i} \tag{A.4b}$$

$$e^{(2)}_i = e^{(1)}_i - p^{(1)}_{i\to CCP} + \bar{p}^{(1)}_{CCP\to i}\,, \tag{A.4c}$$

with payments $p^{(2)}_{i\to CCP}$ *and shortfalls* $s^{(2)}_i$ *defined as in* (6) *and* (7)*, for all* i*. Finally, let us introduce cash endowment at the end of the second sub-stage:*

$$e^{(3)}_i = e^{(2)}_i - p^{(2)}_{i\to CCP} + \bar{p}^{(2)}_{CCP\to i}\,, \tag{A.5}$$

for all i*. Then we have that:*

$$p_{i\to CCP} = p^{(1)}_{i\to CCP} + p^{(2)}_{i\to CCP} \tag{A.6}$$

$$s^c_i = s^{c\,(1)}_i + s^{c\,(2)}_i \tag{A.7}$$

$$e^{(3)}_i = e_i - p_{i\to CCP} + \bar{p}_{CCP\to i}\,, \tag{A.8}$$

for all i*.*

Proof. Let us focus on bank i, as central clearing proceeds independently for each bank. First, we consider the case $\sum_j L_{ij} - L_{ji} > 0$, which implies that $\bar{p}_{i\to CCP} > 0$, $\bar{p}^{(1)}_{i\to CCP} > 0$, $\bar{p}^{(2)}_{i\to CCP} > 0$, and $\bar{p}_{CCP\to i} = \bar{p}^{(1)}_{CCP\to i} = \bar{p}^{(2)}_{CCP\to i} = 0$. Therefore, from by adding $p^{(1)}_{i\to CCP}$

on both sides of (6) for $p^{(2)}_{i\to CCP}$ we have:

$$\begin{aligned} p^{(1)}_{i\to CCP} + p^{(2)}_{i\to CCP} &= \min\left[e^{(2)}_i + p^{(1)}_{i\to CCP}, \bar{p}^{(2)}_{i\to CCP} + p^{(1)}_{i\to CCP}\right] \\ &= \min\left[e^{(1)}_i - p^{(1)}_{i\to CCP} + \bar{p}^{(1)}_{CCP\to i} + p^{(1)}_{i\to CCP}, \bar{p}^{(2)}_{i\to CCP} + p^{(1)}_{i\to CCP}\right] \\ &= \min\left[e^{(1)}_i, \bar{p}^{(2)}_{i\to CCP} + p^{(1)}_{i\to CCP}\right] . \end{aligned} \tag{A.9}$$

In the sub-case in which $e^{(1)}_i \geq \bar{p}^{(1)}_{i\to CCP}$ we have that $p^{(1)}_{i\to CCP} = \bar{p}^{(1)}_{i\to CCP}$ and (A.9) reads:

$$p^{(1)}_{i\to CCP} + p^{(2)}_{i\to CCP} = \min\left[e^{(1)}_i, \bar{p}^{(1)}_{i\to CCP} + \bar{p}^{(2)}_{i\to CCP}\right] . \tag{A.10}$$

In the sub-case in which $e^{(1)}_i < \bar{p}^{(1)}_{i\to CCP}$, we have that $p^{(1)}_{i\to CCP} = e^{(1)}_i$ and (A.9) reads:

$$\begin{aligned} p^{(1)}_{i\to CCP} + p^{(2)}_{i\to CCP} &= \min\left[e^{(1)}_i, e^{(1)}_i + \bar{p}^{(2)}_{i\to CCP}\right] \\ &= \min\left[e^{(1)}_i, \bar{p}^{(1)}_{i\to CCP} + \bar{p}^{(2)}_{i\to CCP}\right] . \end{aligned} \tag{A.11}$$

Second, we consider the case $\sum_j L_{ij} - L_{ji} \leq 0$, which implies that $\bar{p}_{i\to CCP} = \bar{p}^{(1)}_{i\to CCP} = \bar{p}^{(2)}_{i\to CCP} = 0$. Since there are no obligations to the CCP, also all payments to the CCP are equal to zero: $p_{i\to CCP} = p^{(1)}_{i\to CCP} = p^{(2)}_{i\to CCP} = 0$. This concludes the proof that $p_{i\to CCP} = p^{(1)}_{i\to CCP} + p^{(2)}_{i\to CCP}$ in all cases.

For shortfalls we have that:

$$\begin{aligned} s^c_i &= \bar{p}_{i\to CCP} - p_{i\to CCP} \\ &= \bar{p}^{(1)}_{i\to CCP} + \bar{p}^{(2)}_{i\to CCP} - p^{(1)}_{i\to CCP} - p^{(1)}_{i\to CCP} \\ &= s^{c\,(1)}_i + s^{c\,(2)}_i . \end{aligned} \tag{A.12}$$

Finally, we have:

$$
\begin{aligned}
e_i^{(3)} &= e_i^{(2)} - p_{i\to CCP}^{(2)} + \bar{p}_{CCP\to i}^{(2)} \\
&= e_i^{(1)} - p_{i\to CCP}^{(1)} + \bar{p}_{CCP\to i}^{(1)} - p_{i\to CCP}^{(2)} + \bar{p}_{CCP\to i}^{(2)} \\
&= e_i - p_{i\to CCP} + \bar{p}_{CCP\to i}\,.
\end{aligned}
\tag{A.13}
$$

□

Theorem 2. *Let $\mathcal{S}(\mathbf{L}, \mathbf{e}^{(1)})$ be a family of clearing systems with perfectly correlated exposures and let $\alpha \in [0, 1)$. For all banks, the shortfall at $\alpha < 1$ is larger than or equal to the shortfall in the fully centrally cleared setting:*

$$s_i(\alpha) \geq s_i(1) \quad \forall i\,. \tag{A.14}$$

Proof. At $\alpha = 1$ shortfalls from bilateral obligations are equal to zero, hence: $s_i(1) = s_i^c(1)$. At $\alpha = 1$ we clear centrally cleared obligations in two sub-stages. In the first sub-stage we clear centrally cleared obligations corresponding to a fraction α of notional, while in the second sub-stage we clear centrally cleared obligations corresponding to a fraction $1 - \alpha$ of notional (see Lemma 1). We denote the shortfalls that bank i records in those two sub-stages with $s_i^c(1|\alpha)$ and $s_i^c(1|1-\alpha)$. Therefore, by using Lemma 1 we have:

$$s_i(\alpha) = s_i^c(\alpha) + s_i^b(1-\alpha) \tag{A.15a}$$

$$s_i(1) = s_i^c(1|\alpha) + s_i^c(1|1-\alpha)\,. \tag{A.15b}$$

The first observation is that $s_i^c(\alpha) = s_i^c(1|\alpha)$, for all i. This descends from (7) because the cash endowment is equal to $e_i^{(1)}$ in both cases and net VM obligations are equal to $\bar{p}_{i\to CCP} = \alpha(\sum L_{ij} - L_{ji})^+$ in both cases (as we centrally clear the same fraction of notional α). From (10) it descends that also the cash endowment *after this stage* is the same.

The second observation is that $s_i^b(1-\alpha) \geq s_i^c(1|1-\alpha)$, for all i. Cash endowments are the same in both cases because are the cash endowments at the end of the previous stage. The matrix of *gross* VM obligations is $(1-\alpha)\mathbf{L}$ in both cases, but at α those cleared fully bilaterally, while at $\alpha = 1$ those are fully centrally cleared. Therefore, by using Theorem 1 we have that $s_i^b(1-\alpha) \geq s_i^c(1|1-\alpha)$, which implies $s_i(\alpha) \geq s_i(1)$. □

Lemma 2 (Splitting of Eisenberg and Noe in two sub-stages)**.** *Let $\bar{p}_{ij}$, for all i and j be a matrix of obligations defined as in (2), $\mathbf{e} \geq 0$ a vector of cash endowments, and $\beta \in [0,1]$. Let us introduce quantities in the first sub-stage:*

$$\bar{p}_{ij}^{(1)} = \beta \bar{p}_{ij} \tag{A.16a}$$

$$e_i^{(1)} = e_i\,, \tag{A.16b}$$

with total obligations $\bar{p}_i^{(1)}$ defined as in (11), relative liability matrix $\Pi_{ij}^{(1)}$ defined as in (12), payments $p_i^{(1)}$ defined as in (14) and shortfalls $s_i^{b\,(1)}$ defined as in (15), for all i and j. Let us introduce quantities in the second sub-stage:

$$\bar{p}_i^{(2)} = \bar{p}_i - p_i^{(1)} \tag{A.17a}$$

$$\Pi_{ij}^{(2)} = \Pi_{ij}^{(1)} \tag{A.17b}$$

$$\bar{p}_{ij}^{(2)} = \Pi_{ij}^{(2)} \bar{p}_i^{(2)} \tag{A.17c}$$

$$e_i^{(2)} = e_i^{(1)} - p_i^{(1)} + \sum_j \Pi_{ji}^{(1)} p_j^{(1)}\,, \tag{A.17d}$$

with payments $p_i^{(2)}$ defined as in (14), and shortfalls $s_i^{b\,(2)}$ defined as in (15), for all i and j. Then we have that:

$$\bar{p}_{ij}^{(2)} \geq (1-\beta)\bar{p}_{ij} \tag{A.18}$$

$$p_i = p_i^{(1)} + p_i^{(2)} \tag{A.19}$$

$$s_i^b = s_i^{b\,(1)} + s_i^{b\,(2)}\,, \tag{A.20}$$

for all i and j.

Proof. In order to prove (A.18) we note that in the case $\beta = 0$, $\bar{p}_{ij} = \bar{p}_{ij}^{(2)}$, as the clearing reduces to the second sub-stage. In the case $\beta > 0$ we have:

$$\begin{aligned}
\bar{p}_{ij}^{(2)} &= \Pi_{ij}^{(2)} \bar{p}_i^{(2)} \\
&= \Pi_{ij}^{(2)} \left(\bar{p}_i - p_i^{(1)}\right) \\
&\geq \Pi_{ij}^{(2)} \left(\bar{p}_i - \bar{p}_i^{(1)}\right) \\
&= \Pi_{ij}^{(2)} \left(\bar{p}_i - \beta \bar{p}_i\right) \\
&= (1-\beta)\Pi_{ij}\bar{p}_i \\
&= (1-\beta)\bar{p}_{ij}\,,
\end{aligned} \tag{A.21}$$

for all i and j.

From (A.17) we note that $\bar{p}_{ij} = \bar{p}_{ij}^{(1)} + \bar{p}_{ij}^{(2)}$, for all i. Using (15), this means that it is sufficient to prove $p_i = p_i^{(1)} + p_i^{(2)}$, for all i. The case $\beta = 0$ is easy to prove, as the clearing reduces to the second sub-stage. Let us then focus on the case in which $\beta > 0$. We start by observing that $\bar{p}_i = 0 \Leftrightarrow \bar{p}_i^{(1)} = 0$, meaning that the zero entries of the matrix $\mathbf{\Pi}$ coincide with the zero entries of the matrix $\mathbf{\Pi}^{(1)}$. As regards non-zero entries, we have:

$$\Pi_{ij}^{(1)} = \frac{\bar{p}_{ij}^{(1)}}{\bar{p}_i^{(1)}} = \frac{\beta \bar{p}_{ij}}{\beta \bar{p}_i} = \frac{\bar{p}_{ij}}{\bar{p}_i} = \Pi_{ij}\,, \tag{A.22}$$

so that $\mathbf{\Pi} = \mathbf{\Pi}^{(1)}$. From the definition of $p_i^{(2)}$ we have:

$$\begin{aligned} p_i^{(2)} &= \min\left(\bar{p}_i^{(2)}, e_i^{(2)} + \sum_j \Pi_{ji}^{(2)} p_j^{(2)}\right) \\ &= \min\left(\bar{p}_i - p_i^{(1)}, e_i^{(1)} - p_i^{(1)} + \sum_j \Pi_{ji}^{(1)} p_j^{(1)} + \sum_j \Pi_{ji}^{(2)} p_j^{(2)}\right) \\ &= \min\left(\bar{p}_i - p_i^{(1)}, e_i^{(1)} - p_i^{(1)} + \sum_j \Pi_{ji} (p_j^{(1)} + p_j^{(2)})\right) , \end{aligned} \tag{A.23}$$

for all i, or:

$$p_i^{(1)} + p_i^{(2)} = \min\left(\bar{p}_i, e_i + \sum_j \Pi_{ji} (p_j^{(1)} + p_j^{(2)})\right) , \tag{A.24}$$

for all i. Since:

$$p_i = \min\left(\bar{p}_i, e_i + \sum_j \Pi_{ji} p_j\right) , \tag{A.25}$$

for all i, by taking the least solution of both the previous equations, we have that $p_i = p_i^{(1)} + p_i^{(2)}$, for all i. □

Theorem 3. *Let $\mathcal{S}(\mathbf{L}, \mathbf{e}^{(1)})$ be a family of clearing systems with perfectly correlated exposures and let $\alpha_1, \alpha_2 \in [0,1]$, with $\alpha_1 \leq \alpha_2$. The aggregate shortfall is a decreasing function of α:*

$$\sum_i s_i(\alpha_1) \geq \sum_i s_i(\alpha_2) . \tag{A.26}$$

Proof. Let $\alpha_1 \leq \alpha_2$. The strategy we follow here is based on a specific decomposition of shortfalls at α_1 and α_2. Both at α_1 and α_2 we have

$$s_i(\alpha_1) = s_i^c(\alpha_1) + s_i^b(\alpha_1) \tag{A.27a}$$

$$s_i(\alpha_2) = s_i^c(\alpha_2) + s_i^b(\alpha_2) . \tag{A.27b}$$

At α_1, a fraction $1-\alpha_1$ of notional is in bilateral obligations, which yield the shortfall $s_i^b(\alpha_1)$. We further split the stage in which bilateral obligations are cleared in two sub-stages. In the first sub-stage we clear bilateral obligations corresponding to a fraction $\alpha_2-\alpha_1$ of notional, while in the second sub-stage we clear the residual bilateral obligations (see Lemma 2). In the second sub-stage, obligations will be *larger than or equal to* obligations corresponding to a fraction $1-\alpha_2$ of notional, which are the bilateral obligations cleared at α_2. We denote the shortfalls that bank i records in those two sub-stages with $s_i^b(\alpha_1|\alpha_2-\alpha_1)$ and $s_i^b(\alpha_1|\geq 1-\alpha_2)$. Similarly, at α_2 a fraction α_2 of notional is in centrally cleared obligations, which yields the shortfall $s_i^c(\alpha_2)$. Here we clear centrally cleared obligations in two sub-stages. In the first sub-stage we clear centrally cleared obligations corresponding to a fraction α_1 of notional, while in the second sub-stage we clear centrally cleared obligations corresponding to a fraction $\alpha_2-\alpha_1$ of notional (see Lemma 1). We denote the shortfalls that bank i records in those two sub-stages with $s_i^c(\alpha_2|\alpha_1)$ and $s_i^c(\alpha_2|\alpha_2-\alpha_1)$. By using Lemmas 1 and 2 we can rewrite (A.27) as:

$$s_i(\alpha_1) = s_i^c(\alpha_1) + s_i^b(\alpha_1|\alpha_2-\alpha_1) + s_i^b(\alpha_1|\geq 1-\alpha_2) \tag{A.28a}$$

$$s_i(\alpha_2) = s_i^c(\alpha_2|\alpha_1) + s_i^c(\alpha_2|\alpha_2-\alpha_1) + s_i^b(\alpha_2)\,. \tag{A.28b}$$

The first observation is that $s_i^c(\alpha_1) = s_i^c(\alpha_2|\alpha_1)$, for all i. This descends from (7) because the cash endowment is equal to $e_i^{(1)}$ in both cases and net VM obligations are equal to $\bar{p}_{i\to CCP} = \alpha_1(\sum L_{ij} - L_{ji})^+$ in both cases (as we centrally clear the same fraction of notional α_1). From (10) it descends that also the cash endowment *after this stage* is the same, let us denote it with $\tilde{\mathbf{e}}$.

The second observation is that $s_i^b(\alpha_1|\alpha_2-\alpha_1) \geq s_i^c(\alpha_2|\alpha_2-\alpha_1)$, for all i. Cash endowments are the same in both cases because are the cash endowments at the end of the previous stage. The matrix of *gross* VM obligations is $(\alpha_2-\alpha_1)\mathbf{L}$ in both cases, but at α_1 those cleared fully bilaterally, while at α_2 those are fully centrally cleared.

Therefore, by using Theorem 1 we have that $s_i^b(\alpha_1|\alpha_2-\alpha_1) \geq s_i^c(\alpha_2|\alpha_2-\alpha_1)$.

The final step is to compare $\sum_i s_i^b(\alpha_1| \geq 1-\alpha_2)$ with $\sum_i s_i^b(\alpha_2)$. Both at α_1 and α_2 obligations are cleared bilaterally. At α_2 we clear bilaterally the matrix of *gross* VM obligations $\mathbf{L}'' = (1-\alpha_2)\mathbf{L}$, corresponding to *net* VM obligations $\bar{p}''_{ij}(1-\alpha_2)\bar{p}_{ij}$, for all i and j. At α_2 the cash available is the cash after the second sub-stage of central clearing (see Lemma 1), which we denote with $\mathbf{e}''$. At α_1 we clear bilaterally *net* VM obligations $\bar{p}'_{ij} \geq (1-\alpha_2)\bar{p}_{ij}$. To see this, it is sufficient to use (A.18) in Lemma 2 and noting that at α_1 we clear bilaterally a fraction of notional $1-\alpha_1$, further split in two sub-stages with $\beta = (\alpha_2-\alpha_1)/(1-\alpha_1)$ (and therefore $1-\beta = (1-\alpha_2)/(1-\alpha_1)$). At α_1 the cash available is the cash after the first sub-stage of bilateral clearing (see Lemma 2), which we denote with $\mathbf{e}'$. We now show that $e''_i \geq e'_i$, for all i. In order to see this, let us remind that:

$$e''_i = \left[\tilde{e}_i - (\alpha_2-\alpha_1)\Big(\sum_j L_{ij} - L_{ji}\Big)^+\right]^+ + (\alpha_2-\alpha_1)\Big(\sum_j L_{ij} - L_{ji}\Big)^- . \tag{A.29}$$

If $\sum_j L_{ij} - L_{ji} \geq 0$, then $\Big(\sum_j L_{ij} - L_{ji}\Big)^+ = \sum_j L_{ij} - L_{ji}$ and $\Big(\sum_j L_{ij} - L_{ji}\Big)^- = 0$, so that:

$$e''_i = \left[\tilde{e}_i - (\alpha_2-\alpha_1)\Big(\sum_j L_{ij} - L_{ji}\Big)\right]^+ . \tag{A.30}$$

If $\sum_j L_{ij} - L_{ji} < 0$, then $\Big(\sum_j L_{ij} - L_{ji}\Big)^+ = 0$ and $\Big(\sum_j L_{ij} - L_{ji}\Big)^- = -\Big(\sum_j L_{ij} - L_{ji}\Big) > 0$, meaning that

$$\begin{aligned} e''_i &= \tilde{e}_i - (\alpha_2-\alpha_1)\Big(\sum_j L_{ij} - L_{ji}\Big) \\ &= \left[\tilde{e}_i - (\alpha_2-\alpha_1)\sum_j (L_{ij} - L_{ji})\right]^+ , \end{aligned} \tag{A.31}$$

which then holds in all cases. Instead:

$$e'_i = \tilde{e}_i - p_i(\alpha_1|\alpha_2 - \alpha_1) + \sum_j \Pi_{ji} p_j(\alpha_1|\alpha_2 - \alpha_1)\,, \tag{A.32}$$

where with $p_i(\alpha_1|\alpha_2 - \alpha_1)$ we denote the payment made by bank i in the first sub-stage of bilateral clearing at α_1. Now, payments in the second sub-stage at α_1 are:

$$p_i(\alpha_1|\alpha_2 - \alpha_1) = \min\left[\bar{p}_i(\alpha_1|\alpha_2 - \alpha_1), \tilde{e}_i + \sum_j \Pi_{ji} p_i(\alpha_1|\alpha_2 - \alpha_1)\right]\,, \tag{A.33}$$

where with $\bar{p}_i(\alpha_1|\alpha_2 - \alpha_1)$ we denote the net VM obligation of bank i in the first sub-stage of bilateral clearing at α_1. This leaves us with two cases. Either: $p_i(\alpha_1|\alpha_2 - \alpha_1) = \tilde{e}_i + \sum_j \Pi_{ji} p_j(\alpha_1|\alpha_2 - \alpha_1)$, and therefore $e'_i = 0$, which immediately implies $e'_i \leq e''_i$. Or: $p_i(\alpha_1|\alpha_2 - \alpha_1) = \bar{p}_i(\alpha_1|\alpha_2 - \alpha_1)$, and therefore:

$$\begin{aligned}
e'_i &= \tilde{e}_i - \bar{p}_i(\alpha_1|\alpha_2 - \alpha_1) + \sum_j \Pi_{ji} p_j(\alpha_1|\alpha_2 - \alpha_1) \\
&\leq \tilde{e}_i - \bar{p}_i(\alpha_1|\alpha_2 - \alpha_1) + \sum_j \Pi_{ji} \bar{p}_j(\alpha_1|\alpha_2 - \alpha_1) \\
&\leq \tilde{e}_i - (\alpha_2 - \alpha_1) \sum_j (L_{ij} - L_{ji})^+ + (\alpha_2 - \alpha_1) \sum_j (L_{ji} - L_{ij})^+ \\
&= \tilde{e}_i - (\alpha_2 - \alpha_1) \sum_j (L_{ij} - L_{ji})^+ + (\alpha_2 - \alpha_1) \sum_j (L_{ij} - L_{ji})^- \\
&= \tilde{e}_i - (\alpha_2 - \alpha_1) \sum_j (L_{ij} - L_{ji}) \\
&\leq e''_i\,.
\end{aligned} \tag{A.34}$$

To summarise, $\sum_i s_i^b(\alpha_1| \geq 1 - \alpha_2)$ is the aggregate shortfall resulting from the least solution of the Eisenberg and Noe algorithm with obligations $\bar{\mathbf{p}}'$ and cash $\mathbf{e}'$, while $\sum_i s_i^b(\alpha_2)$ is the aggregate shortfall resulting from the least solution of the Eisenberg and Noe algorithm with obligations $\bar{\mathbf{p}}''$ and cash $\mathbf{e}''$. In both case the matrix of relative liabilities is Π. Moreover, $\bar{\mathbf{p}}' \geq \bar{\mathbf{p}}''$ and $\mathbf{e}' \leq \mathbf{e}''$. Since the least solution of the Eisenberg and Noe algorithm (and payments therefore shortfalls) depend only on obligations and cash (as the matrix of relative liabilities is Π in all cases), for the remainder of the proof we will denote with $p_i(\mathbf{e}, \bar{\mathbf{p}}$ the payment of bank i when cash is $\mathbf{e}$ and obligations are $\bar{\mathbf{p}}$, and analogously for shortfalls. We have: $s_i^b(\alpha_1| \geq 1 - \alpha_2) = s_i^b(\mathbf{e}', \bar{\mathbf{p}}')$, and $s_i^b(\alpha_2) = s_i^b(\mathbf{e}'', \bar{\mathbf{p}}'')$, for all i. By using Lemma 5 in Eisenberg and Noe (2001) we have that:

$$\sum_i \left|\bar{p}_i' - \bar{p}_i''\right| \geq \sum_i \left|p_i(\mathbf{e}', \bar{\mathbf{p}}') - p_i(\mathbf{e}', \bar{\mathbf{p}}'')\right| \tag{A.35}$$

but from Lemma 5 in Eisenberg and Noe (2001) we also have that $p_i(\mathbf{e}', \bar{\mathbf{p}}') \geq p_i(\mathbf{e}', \bar{\mathbf{p}}'')$, for all i. Since $\bar{\mathbf{p}}' \geq \bar{\mathbf{p}}''$, we have:

$$\sum_i \bar{p}_i' - \bar{p}_i'' \geq \sum_i p_i(\mathbf{e}', \bar{\mathbf{p}}') - p_i(\mathbf{e}', \bar{\mathbf{p}}'') \tag{A.36}$$

or, by re-arranging terms:

$$\begin{aligned} \sum_i \bar{p}_i' - p_i(\mathbf{e}', \bar{\mathbf{p}}') &\geq \sum_i \bar{p}_i'' - p_i(\mathbf{e}', \bar{\mathbf{p}}'') \\ \sum_i s_i^b(\mathbf{e}', \bar{\mathbf{p}}') &\geq \sum_i s_i^b(\mathbf{e}', \bar{\mathbf{p}}'') \,. \end{aligned} \tag{A.37}$$

Moreover, using again Lemma 5 in Eisenberg and Noe (2001), since $\mathbf{e}'' \geq \mathbf{e}'$, we have that $p_i(\mathbf{e}'', \bar{\mathbf{p}}'') \geq p_i(\mathbf{e}', \bar{\mathbf{p}}'')$, for all i, or:

$$s_i^b(\mathbf{e}'', \bar{\mathbf{p}}'') = \bar{p}_i'' - p_i(\mathbf{e}'', \bar{\mathbf{p}}'') \leq \bar{p}_i'' - p_i(\mathbf{e}', \bar{\mathbf{p}}'') = s_i^b(\mathbf{e}', \bar{\mathbf{p}}'') \,, \tag{A.38}$$

for all i. Therefore, from (A.37) we have:

$$\sum_i s_i^b(\mathbf{e}', \bar{\mathbf{p}}') \geq \sum_i s_i^b(\mathbf{e}', \bar{\mathbf{p}}'') \geq \sum_i s_i^b(\mathbf{e}'', \bar{\mathbf{p}}'') \tag{A.39}$$

or, by remembering the definitions:

$$\sum_i s_i^b(\alpha_1| \geq 1-\alpha_2) \geq \sum_i s_i^b(\alpha_2)\,, \tag{A.40}$$

which concludes the proof. □

Theorem 4. *Let $\mathcal{S}(\mathbf{L}, \mathbf{e}^{(1)})$ be a family of clearing systems with perfectly correlated exposures and let:*

$$\alpha^* = \min_i \frac{e_i^{(1)}}{\left(\sum_j L_{ij} - L_{ji}\right)^+}\,. \tag{A.41}$$

Then, for all $\alpha < \alpha^*$, $s_i(\alpha)$ is independent of α, i.e. $s_i(0) = s_i(\alpha)$, for all i.

Proof. Let us start by observing that:

$$e_i^{(3)} = \left[e_i^{(1)} - \alpha\Big(\sum_j L_{ij} - L_{ji}\Big)^+\right]^+ + \alpha\Big(\sum_j L_{ij} - L_{ji}\Big)^-\,. \tag{A.42}$$

If $\sum_j L_{ij} - L_{ji} \geq 0$, then $\left(\sum_j L_{ij} - L_{ji}\right)^+ = \sum_j L_{ij} - L_{ji}$ and $\left(\sum_j L_{ij} - L_{ji}\right)^- = 0$. Therefore, for $\alpha \leq \alpha^*$:

$$e_i^{(3)} = e_i^{(1)} - \alpha \sum_j \Big(L_{ij} - L_{ji}\Big)\,. \tag{A.43}$$

If $\sum_j L_{ij} - L_{ji} < 0$, then $\left(\sum_j L_{ij} - L_{ji}\right)^+ = 0$ and $\left(\sum_j L_{ij} - L_{ji}\right)^- = -\Big(\sum_j L_{ij} - L_{ji}\Big) > 0$, meaning that:

$$e_i^{(3)} = e_i^{(1)} - \alpha \sum_j \Big(L_{ij} - L_{ji}\Big)\,. \tag{A.44}$$

which then holds for all i and for $\alpha \leq \alpha^*$.

For the remainder of the proof, in order to make our notation more compact we introduce:

$$b_{ij} = L_{ij} - L_{ji} \tag{A.45}$$

and we briefly note that $b_{ij} = b^+_{ij} - b^-_{ij}$ and that $b^+_{ij} = b^-_{ji}$. In order to prove that shortfalls do not depend on α we will check that all the terms that multiply α (which we refer to as the α terms) are equal to zero. We use the symbol $\overset{\alpha}{\simeq}$ to indicate that we are keeping only the α terms or the terms that may contain α.

Let us denote with $\mathcal{S}$ the set of banks that do not default in the bilateral round and with $\mathcal{D}$ the set of banks that default in the bilateral round. All banks in $\mathcal{S}$ pay in full and have zero shortfall. The realized payments of banks in $\mathcal{D}$ are:

$$
\begin{aligned}
p^*_i &= e^{(3)}_i + \sum_j \Pi_{ji} p^*_j \\
&= e^{(1)}_i - \alpha \sum_j b_{ij} + \sum_{j\in\mathcal{S}} \Pi_{ji}\bar{p}_j + \sum_{j\in\mathcal{D}} \Pi_{ji} p^*_j \\
&= e^{(1)}_i - \alpha \sum_{j\in\mathcal{S}} b_{ij} - \alpha \sum_{j\in\mathcal{D}} b_{ij} + (1-\alpha)\sum_{j\in\mathcal{S}} b^+_{ji} + \sum_{j\in\mathcal{D}} \Pi_{ji} p^*_j \,,
\end{aligned}
\tag{A.46}
$$

while their shortfall is:

$$
\begin{aligned}
s_i &= (1-\alpha)\sum_j b^+_{ij} - p^*_i \\
&= (1-\alpha)\sum_{j\in\mathcal{S}} b^+_{ij} + (1-\alpha)\sum_{j\in\mathcal{D}} b^+_{ij} - p^*_i \\
&\overset{\alpha}{\simeq} -\alpha \sum_{j\in\mathcal{S}} b^+_{ij} - \alpha \sum_{j\in\mathcal{D}} b^+_{ij} + \alpha \sum_{j\in\mathcal{S}} b_{ij} + \alpha \sum_{j\in\mathcal{D}} b_{ij} + \alpha \sum_{j\in\mathcal{S}} b^+_{ji} - \sum_{j\in\mathcal{D}} \Pi_{ji} p^*_j \\
&= \alpha \sum_{j\in\mathcal{S}} \left(-b^+_{ij} + b_{ij} + b^+_{ji}\right) + \alpha \sum_{j\in\mathcal{D}} \left(b_{ij} - b^+_{ij}\right) - \sum_{j\in\mathcal{D}} \Pi_{ji} p^*_j \\
&= \alpha \sum_{j\in\mathcal{S}} \left(b_{ij} - (b^+_{ij} - b^-_{ij})\right) - \alpha \sum_{j\in\mathcal{D}} b^-_{ij} - \sum_{j\in\mathcal{D}} \Pi_{ji} p^*_j \\
&= -\alpha \sum_{j\in\mathcal{D}} b^-_{ij} - \sum_{j\in\mathcal{D}} \Pi_{ji} p^*_j \,.
\end{aligned}
\tag{A.47}
$$

As a consequence, we are left to prove that all the α terms in:

$$-s_i \overset{\alpha}{\simeq} \alpha \sum_{j\in\mathcal{D}} b^-_{ij} + \sum_{j\in\mathcal{D}} \Pi_{ji} p^*_j \tag{A.48}$$

sum to zero. To this effect let us re-write (A.48) as:

$$\begin{aligned} -s_i &\overset{\alpha}{\simeq} \alpha \sum_{j\in\mathcal{D}} b^-_{ij} + \sum_{j\in\mathcal{D}} \Pi_{ji} p^*_j \\ &= \alpha \sum_{j\in\mathcal{D}} b^-_{ij} + \sum_{j\in\mathcal{D}} \frac{b^+_{ji}}{\sum_k b^+_{jk}} p^*_j \\ &= \alpha \sum_{j\in\mathcal{D}} b^-_{ij} + \sum_{j\in\mathcal{D}} \frac{b^-_{ij}}{\sum_k b^+_{jk}} p^*_j \\ &= \sum_{j\in\mathcal{D}} b^-_{ij} \left(\alpha + \frac{p^*_j}{\sum_k b^+_{jk}} \right) \end{aligned} \tag{A.49}$$

Since all js in the summation above are in $\mathcal{D}$, we can use (A.46) and keep only the α terms:

$$\begin{aligned} p^*_j &\overset{\alpha}{\simeq} -\alpha \sum_{k\in\mathcal{S}} b_{jk} - \alpha \sum_{k\in\mathcal{D}} b_{jk} - \alpha \sum_{k\in\mathcal{S}} b^+_{kj} + \sum_{k\in\mathcal{D}} \Pi_{kj} p^*_k \\ &= -\alpha \sum_{k\in\mathcal{S}} \left(b_{jk} + b^-_{jk} \right) - \alpha \sum_{k\in\mathcal{D}} b_{jk} + \sum_{k\in\mathcal{D}} \Pi_{kj} p^*_k \\ &= -\alpha \sum_{k\in\mathcal{S}} b^+_{jk} - \alpha \sum_{k\in\mathcal{D}} b_{jk} + \sum_{k\in\mathcal{D}} \Pi_{kj} p^*_k \\ &= -\alpha \sum_{k\in\mathcal{S}} b^+_{jk} - \alpha \sum_{k\in\mathcal{D}} b^+_{jk} + \alpha \sum_{k\in\mathcal{D}} b^-_{jk} + \sum_{k\in\mathcal{D}} \Pi_{kj} p^*_k \\ &= -\alpha \sum_{k} b^+_{jk} + \alpha \sum_{k\in\mathcal{D}} b^-_{jk} + \sum_{k\in\mathcal{D}} \Pi_{kj} p^*_k \,, \end{aligned} \tag{A.50}$$

which we can now plug into (A.49):

$$
\begin{aligned}
-s_i &\overset{\alpha}{\simeq} \sum_{j\in\mathcal{D}} b^-_{ij}\left[\alpha + \frac{1}{\sum_k b^+_{jk}}\left(-\alpha\sum_k b^+_{jk} + \alpha\sum_{k\in\mathcal{D}} b^-_{jk} + \sum_{k\in\mathcal{D}}\Pi_{kj}p^*_k\right)\right] \\
&= \sum_{j\in\mathcal{D}} \frac{b^-_{ij}}{\sum_k b^+_{jk}}\left(\alpha\sum_{k\in\mathcal{D}} b^-_{jk} + \sum_{k\in\mathcal{D}}\Pi_{kj}p^*_k\right) \\
&= \sum_{j\in\mathcal{D}} \frac{b^+_{ji}}{\sum_k b^+_{jk}}\left(\alpha\sum_{k\in\mathcal{D}} b^-_{jk} + \sum_{k\in\mathcal{D}}\Pi_{kj}p^*_k\right) \\
&= \sum_{j\in\mathcal{D}} \Pi_{ji}\left(\alpha\sum_{k\in\mathcal{D}} b^-_{jk} + \sum_{k\in\mathcal{D}}\Pi_{kj}p^*_k\right),
\end{aligned}
\tag{A.51}
$$

where the coefficients $\frac{b^-_{ij}}{\sum_k b^+_{jk}}$ do not depend on α, while the terms in parentheses have the same form of (A.48). The important observation here is that, when computing the α terms for the shortfall of banks in $\mathcal{D}$, we are left with only with summations over banks in $\mathcal{D}$. If we proceed iteratively by plugging in the analogous of (A.50) for p^*_k we arrive at an analogous expression where $\sum_{j\in\mathcal{D}}\Pi_{ji}$ is replaced by $\sum_{j,k\in\mathcal{D}}\Pi_{kj}\Pi_{ji}$ and the terms in parentheses correspond to the neighbors of k that are in $\mathcal{D}$. Eventually, if we keep iterating this procedure, the only terms left correspond to cycles of banks in $\mathcal{D}$. (This immediately implies that, if we only have two defaulted banks, shortfalls do not depend on α, as bilateral netting means that we cannot have 2-cycles.) Therefore, if we denote with $\mathcal{C}_i$ the set of cycles of i and with ℓ_c the length of the cycle c, we have:

$$
-s_i \overset{\alpha}{\simeq} \sum_{c\in\mathcal{C}_i} \Pi_{ij_{\ell_c}}\dots\Pi_{j_2j_1}\Pi_{j_1i}\left(\alpha\sum_{j_1\in\mathcal{D}} b^-_{ij_1} + \sum_{j_1\in\mathcal{D}}\Pi_{j_1i}p^*_{j_1}\right). \tag{A.52}
$$

We can now go through each cycle an arbitrary number of times, say n_c for cycle c:

$$
-s_i \overset{\alpha}{\simeq} \sum_{c\in\mathcal{C}_i} \left(\Pi_{ij_{\ell_c}}\dots\Pi_{j_2j_1}\Pi_{j_1i}\right)^{n_c}\left(\alpha\sum_{j_1\in\mathcal{D}} b^-_{ij_1} + \sum_{j_1\in\mathcal{D}}\Pi_{j_1i}p^*_{j_1}\right). \tag{A.53}
$$

The term in parentheses does not depend on n_c and is finite (realized payments) cannot exceed the payment obligations. On the other hand, as long as one of the $\Pi_{j_m,j_{m+1}}$ is strictly smaller than one, in the limit $n_c \to \infty$ we are left with no α terms. If all $\Pi_{j_m,j_{m+1}}$ are equal to zero, it means that i is part of an isolated cycle (i.e. a closed chain in which all banks do not have obligations to any other bank) in which all banks are in $\mathcal{D}$, which is the only case that is left to prove. However, in the Eisenberg and Noe algorithm, as long as $e_i^{(3)}$ is strictly larger than zero for all i, there cannot be a closed cycle of banks in $\mathcal{D}$. In fact, banks will make partial payments and at least the link with the smallest payment obligation will disappear. This means that we are only requiring that $e_i^{(3)} > 0$, for all i, which is true as long as $\alpha < \alpha^*$. □

B Heterogeneous VM obligations

Here we analyse the case in which total gross VM obligations L_i^{tot} are not equal for all banks. We draw L_i^{tot} from a Gaussian distribution with mean L and variance σ^2:

$$L_i^{tot} \sim \mathcal{N}(L, \sigma^2)\,. \tag{B.1}$$

In principle, (B.1) can yield negative values of L_i^{tot}. Since total gross VM obligations of bank i cannot be negative, we set L_i^{tot} to zero in those cases. We start with $\sigma = L/6$. As for the case of homogeneous VM obligations, the aggregate shortfall has always a minimum when the density c is sufficiently small and, by increasing the the density, the fraction with a minimum decreases (see Figure B.1). However, now the fraction of realisations with a minimum initially increases with the mean total gross VM obligations (e.g. when L goes from 2 to 4) and then decreases (e.g. when L goes from 4 to 6). In this case realisations in which the aggregate shortfall is zero for all values of α occur only for smaller values of the mean total gross VM obligations (see the right panel of Figure B.2). The remaining realisations, i.e. those that do not have one minimum and in which the aggregate shortfall is not zero for all values of α are not necessarily monotonic in

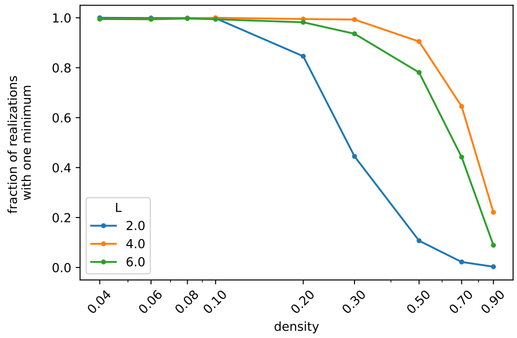

Figure B.1: Fraction of realisations in which the aggregate shortfall has one minimum with respect to α. $n = 100$, and 1 000 realisations, and L_i^{tot} drawn from a Gaussian distribution with mean L and variance $(L/6)^2$.

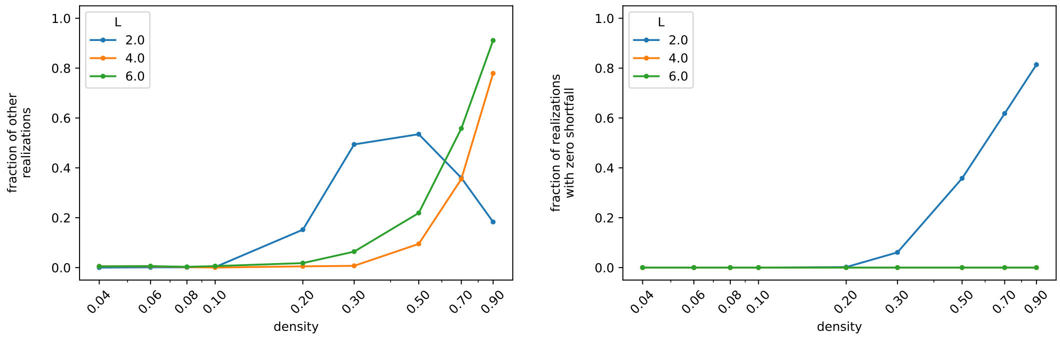

Figure B.2: Fraction of realisations in the which the aggregate shortfall is equal to zero for all values of α (right panel) or in which it neither has one minimum with respect to α nor is always equal to zero (left panel). $n = 100$, 1 000 realisations, and L_i^{tot} drawn from a Gaussian distribution with mean L and variance $(L/6)^2$.

this case and can exhibit multiple stationary points (see the left panel of Figure B.2).

What happens when we increase σ further? Here we look at the cases $\sigma = L/3$ and $\sigma = L$. From Figure B.3 we can see that the fraction of realisations in which the aggregate shortfall has one minimum with respect to α is decreasing with the density c and with the mean total gross VM obligations L. By comparing Figures B.1 and B.3 it appears that the region of the parameter space in which the fraction of realisations with a minimum is equal to one shrinks as the heterogeneity in total gross VM obligations increases. For $\sigma = L$ for no combination of parameters the fraction of realisations with

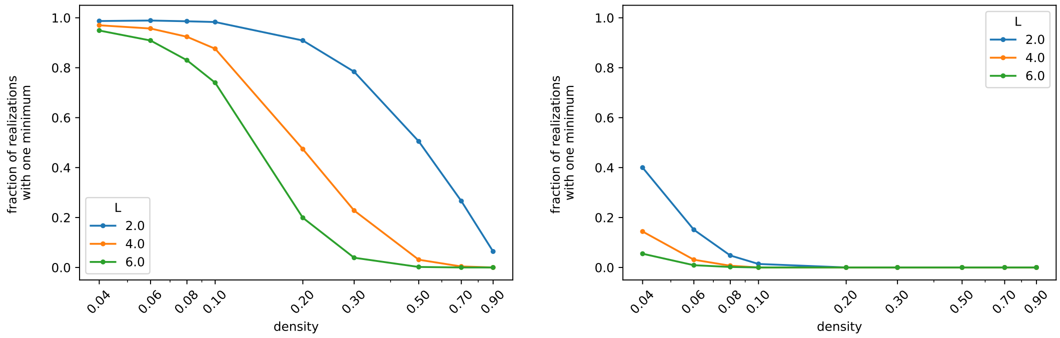

Figure B.3: Fraction of realisations in which the aggregate shortfall has one minimum with respect to α. $n = 100$, and 1 000 realisations, and L_i^{tot} drawn from a Gaussian distribution with mean L and variance $(L/3)^2$ (left panel) and L^2 (right panel).

a minimum is equal to one. For $\sigma = L/3$ and $\sigma = L$ there are no realisations for which the aggregate shortfall is equal to zero for all values of α. Similarly to the case $\sigma = L/6$, realisations that do not have one minimum are not necessarily monotonic.

Also in the case of heterogeneous VM obligations, both total net VM obligations and total payments decrease strictly with α.[12] Using Proposition 2, this implies that the existence of the minimum in aggregate shortfall is due to the fact that, when the fraction of centrally cleared notional is sufficiently large, gains from multilateral netting are not sufficient to offset the losses due to reduced delivered payments. In almost all (99.51% for $\sigma = L/6$ and 99.98% for $\sigma = L/3$) realisations with a minimum, payments are delivered more efficiently in a fully bilateral market than in a fully centrally cleared market ($\epsilon^b(0) > \epsilon^c(1)$).

Similarly to the case of homogeneous total gross VM obligations, for $\sigma = L/6$ the mean value of $\alpha_{\min}$ does not depend on the density c, it depends only weakly on the mean total gross VM obligations L, and, within the error bands, it is not significantly different from 50% (see the left panel of Figure B.4). Instead, for $\sigma = L/3$ the mean value of $\alpha_{\min}$ increases with the mean total gross VM obligations L and, for $L = 4$ and 6,

[12]This happens for all realisations with the partial exception of the case $\sigma = L$ in which total bilateral payments are not decreasing in very few (less than 1%) of the realisations that display the minimum in aggregate shortfall.

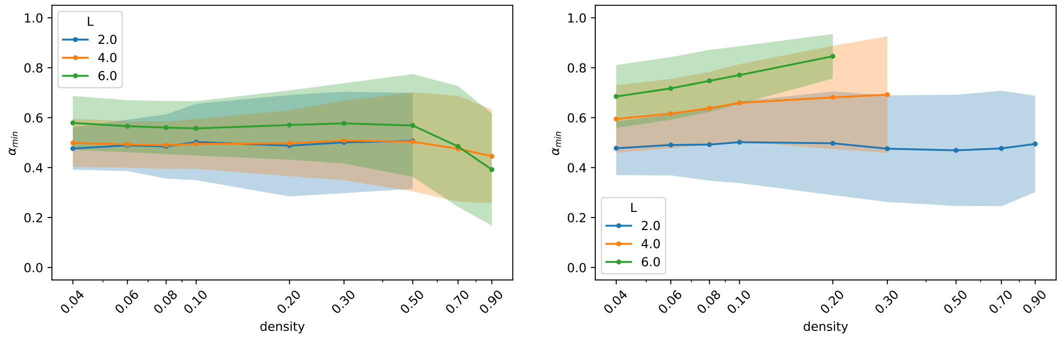

Figure B.4: Mean value of $\alpha_{\min}$, the fraction of centrally cleared notional at which the normalised aggregate shortfall is minimal for several values of c and L. Semi-transparent regions span one standard deviation. $n = 100$, 1 000 realisations, and L_i^{tot} drawn from a Gaussian distribution with mean L and variance $(L/6)^2$ (left panel) and $(L/3)^2$ (right panel). Only combinations of parameters for which the fraction of realisations in which the aggregate shortfall has a minimum is larger than 5% are shown.

also with the density c (see the right panel of Figure B.4). In all cases $\alpha_{\min}$ is above 50%, getting as large as 80%. Here $\alpha_{\min}$ has a larger variability when compared to the case of homogeneous VM obligations, suggesting again a strong dependence on the individual realisation of the network of VM obligations.

From Figure B.5 we can see that, as in the case in which VM obligations are homogeneous, normalised aggregate shortfall at $\alpha_{\min}$ is decreasing with the density c and increasing with the mean total gross VM obligations L.

In Figure B.6 we show the improvement of being at $\alpha_{\min}$ relative to the fully centrally cleared setting ($\alpha = 1$) or to the fully bilateral setting ($\alpha = 0$). In all cases improvements decrease with the mean total gross VM obligations L. For $\sigma = L/6$, improvements increase with the density c for $L = 2$. For larger values of L, improvements initially increase with the density, but they eventually decrease. For $\sigma = L/3$, improvements decrease with the density. As for the case of homogeneous VM obligations, mean improvements both with respect to $\alpha = 1$ and $\alpha = 0$ are economically significant, reaching up to around 80% for $\sigma = L/6$ and up to around 40% for $\sigma = L/3$. However, differently from the case of homogeneous VM obligations, mean improvements become small when

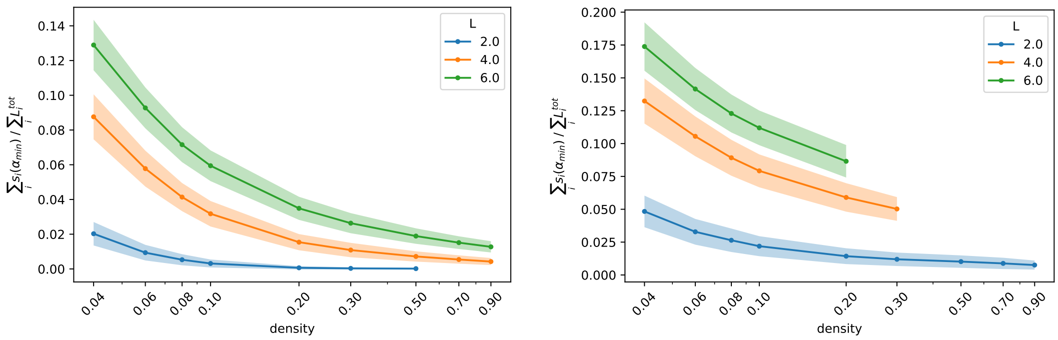

Figure B.5: Mean normalised aggregate shortfall at $\alpha_{\min}$ for several values of c and L. Semi-transparent regions span one standard deviation. $n = 100$, 1 000 realisations, and L_i^{tot} drawn from a Gaussian distribution with mean L and variance $(L/6)^2$ (left panel) and $(L/3)^2$ (right panel). Only combinations of parameters for which the fraction of realisations in which the aggregate shortfall has a minimum is larger than 5% are shown.

the network of obligations is very densely interconnected, except for $L = 2$ and $\sigma = L/6$.

C The role of payment sequencing

In this section we investigate whether the sequencing of payments — banks pay the CCP first, then the CCP pays banks, and only afterwards banks settle bilateral obligations among themselves — is a possible reason for the existence of an optimal fraction of centrally cleared notional. The argument would be that the sequencing payments introduces temporal constraints that might lead to inefficiencies. For example, let us imagine that one bank is a net payer to the CCP and a net receiver from bilateral counterparties. If payments were *not* sequenced, that bank could redirect the payments received from bilateral counterparties to the CCP. Instead, if payments are sequenced and its cash buffer is not sufficient to cover the payment obligation due to the CCP, that bank has to source the gap in order to be able to pay the CCP. We have already discussed how increasing the fraction of notional that is centrally cleared generates two competing forces — VM payment obligations decrease, but (bilateral) payments also decrease — and how the minimum in aggregate shortfall results from the relationship between those. Increasing

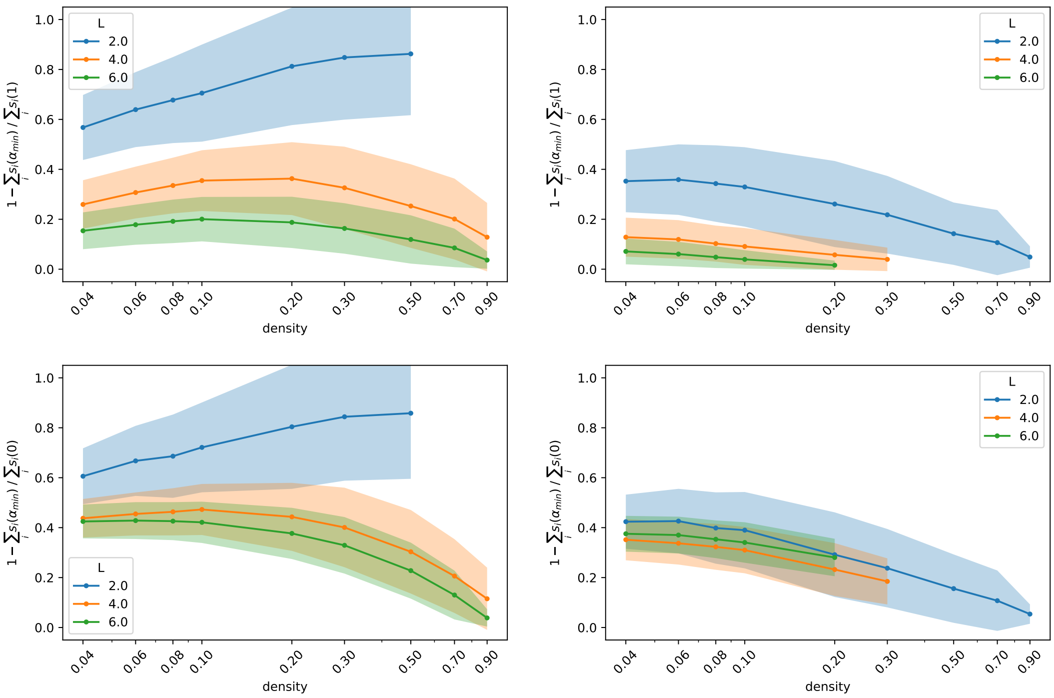

Figure B.6: Improvements of being at α_{min} relative to the fully centrally cleared setting (top panels) or to the fully bilateral setting (bottom panels). Semi-transparent regions span one standard deviation. $n = 100$, 1 000 realisations, and L_i^{tot} drawn from a Gaussian distribution with mean L and variance $(L/6)^2$ (left panels) and $(L/3)^2$ (right panels). Only combinations of parameters for which the fraction of realisations in which the aggregate shortfall has a minimum is larger than 5% are shown.

the fraction of notional that is centrally cleared would also increase the payments subject to the temporal constraints, and that would be why bilateral payments would decrease.

In this section we show that, in most cases, this is not the case and that in practice payment sequencing plays only a limited role in the existence of an optimal fraction of centrally cleared notional. To this end we compare the results of the simulations above, in which payments are sequenced, with analogous simulations in which payments are not sequenced and take place in a single round. This means that all payments occur in the third round of the payment algorithm described in Section 2.4, i.e. by using the Eisenberg and Noe model for all payment obligations. In the version of the model with payment sequencing the CCP is always able to pay its obligations to banks in full. Therefore, in order to keep the comparison fair, in the version of the model without payment sequencing we assign a very large cash buffer to the CCP, so that also in this case it is always able to pay its obligations in full.

When VM obligations are homogeneous, by comparing the results for the model with and without payment sequencing, we find that the realisations with a minimum are exactly the same. As a consequence, payment sequencing appears to be irrelevant for the *existence* of a minimum. But it could impact the value of the aggregate shortfall at the minimum. In Figure C.1 we show the cumulative distribution (across all realisations and values of parameters) of the difference between aggregate shortfalls at $\alpha_{\min}$ with and without payment sequencing. Positive (negative) values indicate that shortfalls are larger with payment sequencing. We find that in more than 97% of instances introducing the sequencing leads to larger aggregate shortfalls at $\alpha_{\min}$. However, the effect is small as the median of the difference between aggregate shortfalls at $\alpha_{\min}$ is around 0.16 percentage points, and the 95th percentile of the difference is equal to 1.09 percentage point.

For heterogeneous VM obligations we find that the fraction of realisations with a minimum is very similar in the cases with and without sequencing, see Figure C.2. Only for $L = 6$ and $\sigma = L/3$ the fraction of realisations with a minimum is materially larger

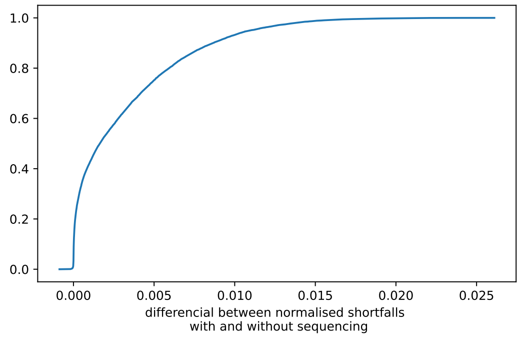

Figure C.1: Cumulative distribution of the difference between normalised shortfalls with and without payment sequencing. Positive (negative) values indicate that shortfalls are larger (smaller) with payment sequencing. $n = 100$, 1 000 realisations, and $L_i^{tot} = L$, for all i.

in the case with payment sequencing. Nevertheless, the realisations in which there is minimum might not be necessarily the same. Hence, we exclude all realisations in which there is a minimum *both* in the case with and without sequencing and we focus on the realisation in which there is a minimum *only* in the case with sequencing and *only* in the case without sequencing. For $L = 6$ there are considerably more realisations in which there is a minimum only in the case with sequencing, (see Figure C.3). Also for $\sigma = L/3$ and $L < 6$ there are sizeable differences between the case with and without sequencing, but those are smaller. The cumulative distributions (across all realisations and values of parameters) of the difference between aggregate shortfalls at α_{min} with and without payment sequencing are shown in Figure C.4. Introducing the sequencing leads to larger aggregate shortfalls at α_{min} in around 87% and 85% of instances, for $\sigma = L/6$ and $\sigma = L/3$ respectively. The medians of the difference between aggregate shortfalls at α_{min} are equal to about 0.07 and 0.11 percentage points, while the 95th percentiles of the difference are equal to 1.01 and 8.32 percentage points, for $\sigma = L/6$ and $\sigma = L/3$ respectively.

Overall, those results show that payment sequencing plays a minor role in the existence of an optimal value of centrally cleared notional, except when VM obligations

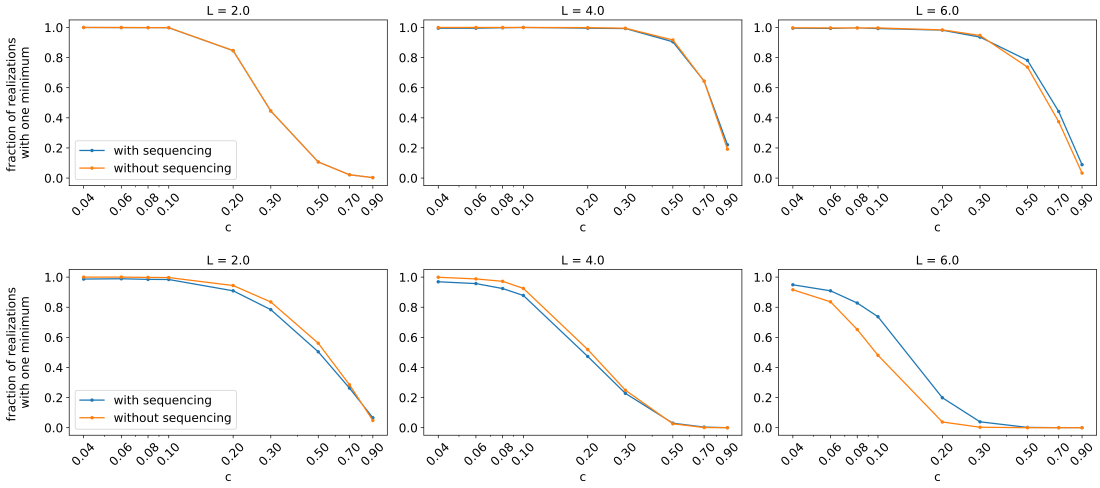

Figure C.2: Fraction of realisations in which the aggregate shortfall has one minimum with respect to α. $n = 100$, and 1 000 realisations, and L_i^{tot} drawn from a Gaussian distribution with mean L and variance $(L/6)^2$ (top panel) and $(L/3)^2$ (bottom panel).

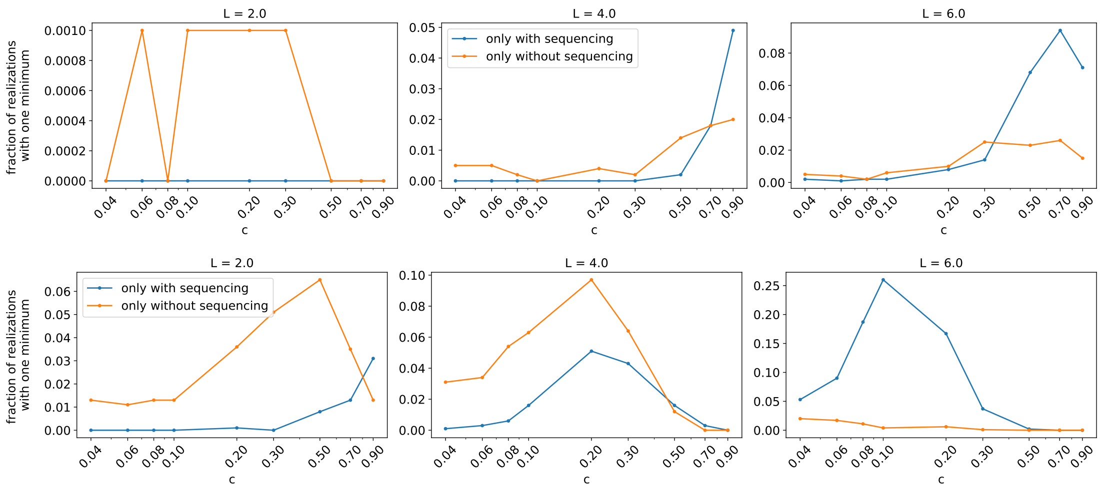

Figure C.3: Fraction of realisations in which the aggregate shortfall has one minimum only with or without sequencing with respect to α. $n = 100$, and 1 000 realisations, and L_i^{tot} drawn from a Gaussian distribution with mean L and variance $(L/6)^2$ (top panel) and $(L/3)^2$ (bottom panel).

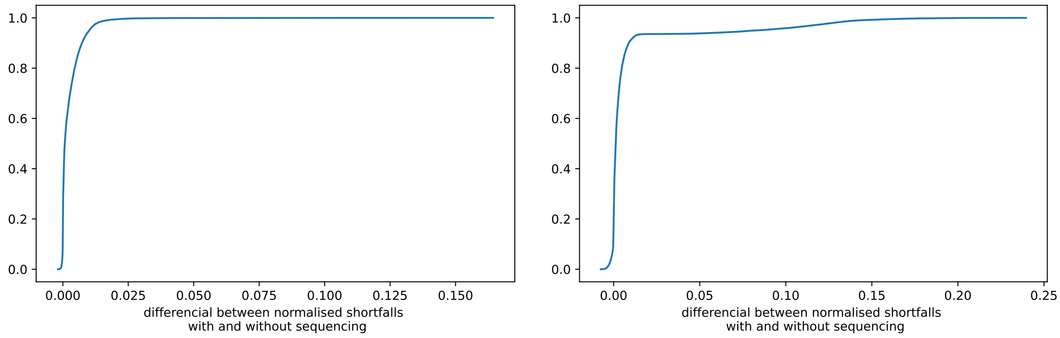

Figure C.4: Cumulative distribution of the difference between normalised shortfalls with and without payment sequencing. Positive (negative) values indicate that shortfalls are larger (smaller) with payment sequencing. $n = 100$, 1 000 realisations, and L_i^{tot} drawn from a Gaussian distribution with mean L and variance $(L/6)^2$ (left panel) and $(L/3)^2$ (right panel).

are large and heterogeneous. Even in those cases, the quantitative difference between aggregate shortfalls with and without sequencing when the value of centrally cleared notional is optimal is relatively small.