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Abstract

This paper provides a method for comparing the performance of different models of bidding behaviour. It uses data on participants' bids but does not require data on their values. I find that a model of 'truthful bidding' – bidding one's true value for liquidity – outperforms a conventional model in which bidders shade their bids to maximise their expected surpluses, in the Bank of England's uniform-price divisible-good liquidity auctions. I provide two possible explanations for this result. First, when bidders are sufficiently risk averse, optimal strategies in the conventional model approximate truthful bidding. For the conventional model, I develop new identifying conditions which allow for risk aversion. I find that the degree of risk aversion required for truthful bidding to be approximately optimal is consistent with that found in studies that are the most similar to my setting. Second, the optimal strategy can be complicated. Truthful bidding is preferable, even for risk neutral bidders, if the cost of calculating what would otherwise be the optimal strategy exceeds around 5% of bidder surplus.

Key words: Auctions, bid shading, central bank liquidity provision, product mix auction.

JEL classification: D44, E58.

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1 Introduction

This paper provides a method to test the relative performance of alternative models of bidders' behaviour in auctions. It uses data on bids but does not require data on bidders' values. I analyse the Bank of England's Indexed Long-Term Repo (ILTR) auctions, in which the central bank provides liquidity insurance to financial institutions by lending a divisible quantity of funds against collateral. In these auctions, a bid expresses the price a bidder is willing to pay as a function of the quantity they demand. The payment rule is uniform-pricing. A bidder's bid for a marginal unit therefore may affect the price they pay for inframarginal units. This creates an incentive to bid differently from the bidder's true value for each unit of the good.

My main finding is that bidding behaviour is nevertheless better explained by a model of "truthful bidding" than by a conventional model in which bidders are both strategic and risk neutral. In a truthful bidding model, bidders submit bid functions corresponding to their true marginal valuation functions. Conversely, in a conventional strategic model, bidders best respond to their beliefs about the economic environment and other bidders' behaviour.

So the findings of this paper suggest that Milton Friedman (1960) may have been correct in advocating the uniform-price auction to issue U.S. Treasuries based on the view that bidders "need only know the maximum amount [they] are willing to pay for different quantities" (Friedman, 1991). He argued that bidders do not need to strategise so the uniform-price auction should reduce collusion and increase participation relative to the pay-as-bid auction.[1]

The difference between bidders' true values and their bids, i.e. their "bid shading", is difficult to measure because their values are rarely observed. Studies of divisible-good auctions therefore must assume a model of bidding when assessing the efficiency, revenue and other outcomes of alternative auction designs (see, e.g., surveys by Gentry et al., 2018; Hortaçsu and McAdams, 2018; Hortaçsu and Perrigne, 2021). The nature of bidding behaviour affects these market outcomes so the choice of model is critical to the analysis.

Truthful bidding can be justified as rational in two ways: risk aversion and costs of calculating the optimal strategy. First, when bidders are sufficiently risk averse, optimal strategies in the conventional model approximate truthful bidding, and I show that optimal strategies in the conventional model get closer to truthful bidding as risk aversion increases. (If bidding

[1] Uniform-price and pay-as-bid are the two payment rules most commonly used in practice. In pay-as-bid auctions, a bidder pays their bid for each unit that they win. Their bid therefore also affects both their probability of winning and the expected payment.

truthfully, bidders submit bids that correspond to their marginal values regardless of their degrees of risk aversion.) Second, truthful bidding is a best response when bidders face a cost of calculating what otherwise would be the optimal strategy and this cost outweighs the additional expected surplus generated by the strategy.

In this paper, I provide a test of competing models of bidding. It adapts Backus, Conlon and Sinkinson's (2021) test of firm conduct, which combines Rivers and Vuong's (2002) comparison of non-nested alternatives and Berry and Haile's (2014) exclusion restrictions. The procedure uses instrumental variables to evaluate the relative amount of bid shading that the different models predict. The instruments are variables that affect a bidder's bid shading but not their true value for liquidity. Under a correctly specified model, bidders' estimated values therefore will be uncorrelated with the amount of bid shading predicted by the instruments. Conversely, they will covary with the predicted bid shading under a misspecified model. Stronger covariance between the instruments and model-implied values is therefore evidence of worse model fit.

The key assumption for the test to be valid is that bidders have independent private values. Under this assumption, I use variables that measure the strength of competition, e.g. the number of other bidders in the auction, as instruments. One advantage of the approach is that, providing this assumption holds, the test remains valid even if both of the competing models are misspecified. This is because the procedure tests the models' *relative* performance. For example, if bidders are strategic but have different information to what is assumed in the conventional strategic model that I consider, the procedure will assess whether truthful bidding or the conventional model is a better approximation. While I focus on uniform-price divisible-good auctions, the method can be applied more broadly to auctions in which bidders' marginal values can be point identified and estimated under the candidate models of bidding behaviour, and valid instruments are available.

I compare truthful bidding to Kastl's (2011, 2012) conventional share auction model, which modifies Wilson's (1979) foundational model to reflect the fact that bid functions are step functions and which is now widely used in empirical studies (e.g. Cassola, Hortaçsu and Kastl, 2013; Hortaçsu, Kastl and Zhang, 2018; Allen, Kastl and Wittwer, 2022). This is the most appropriate benchmark to analyse behaviour in the Bank of England's liquidity auctions in which bidders submit a vector of ordered price-quantity pairs which constitute their stepped bid functions. In the auctions, bidders submit bid functions with very few steps, which is difficult to explain without Kastl's (2011) modification. Participants may optimally bid above or below their marginal values for liquidity in Kastl's (2011) model, so truthful bidding is an especially natural comparator.

I consider two explanations for why a model of truthful bidding outperforms this conventional model of behaviour, which assumes bidders are both risk neutral and strategic: risk aversion and costs of determining the optimal strategy.

Kastl's (2011) conventional model of bidding assumes that bidders are risk neutral. However, there are two reasons why risk aversion might be a more appropriate assumption in the Bank of England's liquidity auctions. First, in a principal agent framework, the manager tasked with bidding on behalf of the auction participant may have a concave utility function either because they are individually risk averse or because of the remuneration structure.[2] Second, the nature of the loans being allocated in the auctions suggests that financial institutions themselves might be risk averse—the auctions were introduced to respond to "the new demands for liquidity insurance that [the financial crisis] engendered" (Fisher, 2011a, p.15). Unsurprisingly, when bidders are risk averse, optimal strategies in the conventional model are closer to truthful bidding—if a bidder is sufficiently risk averse, they do not want to gamble on losing at prices at which they would prefer to win (or winning at prices at which they would prefer to lose) by submitting a bid that differs from their valuation. I build on Kastl (2011) by deriving novel identifying conditions, which allow for constant absolute risk aversion, in order to test the relative performance of the conventional model with varying degrees of risk aversion. I also show the best response gets closer to truthful bidding as risk aversion increases. The results show that truthful bidding can be rationalised within the conventional model by a degree of risk aversion which is consistent with that found in studies that are the most similar to my setting (Armantier and Sbaï, 2006; Boyarchenko, Lucca and Veldkamp, 2021).

Second, truthful bidding may be explained by a cost of determining the optimal strategy, i.e. a cost of "sophistication" (Hortaçsu and Puller, 2008), providing the cost exceeds the expected gains from what otherwise would be the optimal strategy. This resembles the finding that truthful bidding is an ε-equilibrium in Swinkels (2001) and Chakraborty and Englebrecht-Wiggans (2006): the loss from bidding truthfully rather than optimally becomes arbitrarily small as the number of bidders increases in multi-unit uniform-price auctions. I estimate a lower bound on the size of these costs by the difference between the surplus from what would be an approximate best response and the surplus from truthful bidding, assuming observed bids correspond to bidders' values. I find that the average lower bound is

[2]For example, this may result if the remuneration structure rewards not just higher expected profit for the participant but also winning at any price that the participant is willing to pay.

around 5% of bidder surplus, implying that, if bidders' actual strategies are to bid truthfully, they obtain up to 95% of the surplus that would have been generated by an approximate best response. In comparison, in Hortaçsu and Puller (2008), bidders' actual strategies only generate between 0% and 80% (excluding loss-making bidders) of the surplus generated by the optimal response.

This paper primarily contributes to the literature on testing equilibrium behaviour in divisible-good auctions. Wolak (2007) and McAdams (2008) use violations of over-identifying moment conditions, which rule out profitable unilateral deviations in bidders' strategies.[3] By contrast, I exploit exclusion restrictions to compare the goodness-of-fit of alternative models of behaviour. I therefore do not require over-identifying moments, but do require valid instruments. In this respect, the methods are complementary, allowing greater flexibility to the data available. By assessing relative rather than absolute model fit, my approach is also robust to model misspecification. It has been used to test firm conduct in product markets (Backus, Conlon and Sinkinson, 2021; Duarte, Magnolfi and Roncoroni, 2021; Starc and Wollmann, 2022) and labour markets (Roussille and Scuderi, 2022). To my knowledge, it has not yet been adapted to an auction setting.[4]

Other studies estimate marginal values using data outside the model and compare these directly to the observed bids. For example, Wolfram (1999) and Borenstein, Bushnell and Wolak (2002) cannot reject strategic bidding in British and Californian electricity markets, whereas Hortaçsu and Puller (2008) and Hortaçsu et al. (2019) find strategic heterogeneity in Texas. In these later studies, larger bidders submit bids close to those predicted by profit maximisation, yet smaller bidders persistently deviate from equilibrium bidding. Similar to my paper, Hortaçsu and Puller (2008) show that this can be rationalised by a cost of sophistication that only larger bidders are willing to incur.[5] In contrast to these studies, my testing procedure does not require data on marginal values, which are often unavailable.

[3] Wolak (2007) fails to reject profit maximisation by participants of Australian electricity markets, while Chapman, McAdams and Paarsch (2006) apply McAdams' (2008) method to the Bank of Canada's auctions of cash reserves and find that bids were "close to" best response (i.e. an ε-equilibrium).

[4] Roussille and Scuderi (2022) model their labour market setting as an auction, in which firms submit wage offers, or "bids", for job candidates on an online job board. The structure of their data and their choice of instruments are similar to mine: individual firms' wage offers (i.e. bids) are observed and the number of candidates relative to the number of competitors (i.e. auction supply relative to demand), which measures firms' expectations about competing bids, is used as the instrument.

[5] Hortaçsu et al. (2019) demonstrate that a cognitive hierarchy model provides a good fit in the Texan electricity market, with strategic sophistication increasing in firm size. The parameters of the cognitive hierarchy model are unidentified given only bidding data, so I cannot assess it with my approach. Truthful bidding is of course a special case in which Level-0 behaviour is defined by truthful bidding and adopted by all bidders.

My paper also contributes to the literature on risk aversion in financial markets. Armantier and Sbaï (2006) estimate the degree of risk aversion of bidders in French Treasury auctions and Boyarchenko, Lucca and Veldkamp (2021) calibrate the risk aversion parameter for bidders in US Treasury auctions. Applying the findings of these studies to the Bank of England's liquidity auctions would suggest that truthful bidding is approximately optimal in our context. In contrast, Allen and Wittwer (2023) estimate a much smaller degree of risk aversion when focusing exclusively on primary dealers in Canadian Treasury auctions.[6] Using data from Indian Treasury bill auctions, Gupta and Lamba (2023) use risk aversion to explain the combination of increased participation and higher uncertainty during the 2013 "taper tantrum". They build on Kastl's (2011) model by assuming bidders have constant relative risk aversion (CRRA), and identify values by assuming both that bidders' marginal valuation functions are step functions and that at most one bidder submits a bid equal to each possible auction price (i.e. bidders do not tie on the margin). I instead permit bidders to have constant absolute risk aversion (CARA) and allow for ties on the margin, which are highly probable in my context and in others in which the set of possible bid prices is discrete.[7]

The paper proceeds as follows. Section 2 provides background for the Bank of England's liquidity auctions and introduces the data. In Section 3, I describe the alternative models of behaviour and their implications for the amount that bids differ from bidders' values. The identification of bidders' values, the estimation method and the testing procedure are described in Section 4. Section 5 gives the results of the pairwise comparisons of models and shows how truthful bidding can be explained. Section 6 concludes.

[6] Outside of financial markets, Häfner's (2023) analysis of meat quota auctions is the only study, to my knowledge, of risk aversion in multi-unit auctions.

[7] CARA implies that a bidder's risk aversion does not vary with their net surplus from the auction (defined by the area between their marginal valuation function and the auction price, up to the quantity they are allocated), whereas Gupta and Lamba's (2023) CRRA specification implies that the bidder's risk aversion is decreasing in their net surplus from the auction. CARA seems more appropriate in my context for at least two reasons. First, the financial institution's net surplus from the auction is trivial relative to its total assets so CARA seems to be a more suitable approximation if the financial institution itself is risk averse. Second, if the manager tasked with bidding on behalf of the participant is individually risk averse, it is not clear why their risk aversion would be lower when the stakes were higher. Moreover, if the manager has a concave utility function because of the remuneration structure, CRRA would imply that the manager is rewarded more for winning in the auction when the participant's net surplus from the auction is smaller. It is unclear whether this would be an appropriate assumption.

2 Institutional setting and data

2.1 Indexed Long-Term Repo auction

My empirical setting is the Bank of England's (BoE) Indexed Long-Term Repo (ILTR) auction. This was introduced in 2010 to efficiently respond to "the new demands for liquidity insurance that [the financial crisis] engendered" (Fisher, 2011a, p.15) by lending funds to financial institutions against multiple types of collateral. It allocates the largest amount of funds among the range of BoE facilities that provide liquidity insurance. I study the period June 2010 – January 2014, in which the operations were held monthly.[8]

Loans of central bank reserves are issued at a spread over the BoE base rate (Bank Rate) for a 3- or 6-month term, and separate operations are run for different terms. The lending is collaterised. Within each operation, bidders may borrow the reserves against one of two types of collateral: "Level A", including highly liquid gilts, sterling Treasury bills and certain sovereign and central bank debt, and "Level B", including high quality, but less liquid, sovereign debt and certain asset-backed securities.[9] The ILTR operations therefore provide a "liquidity upgrade" to participants, allowing them to swap collateral for more liquid reserves. I refer to funds borrowed against each type of collateral by goods A and B respectively, and the spreads over Bank Rate at which the loans are allocated by the prices of goods A and B.

Each operation is a Product-Mix Auction (PMA), originally developed in Klemperer (2008) and further described in Klemperer (2010, 2018). It is a sealed-bid uniform-price auction in which goods are jointly allocated, as explained below. For each good, all winning bidders pay the auction price for that good.

Bidders Participants of the BoE's Sterling Monetary Framework (SMF) with access to the BoE's Open Market Operations (OMO) were eligible to register to participate in the ILTR auctions, including banks and building societies.[10] Nixon (2014) shows that the number of

[8] The auction design, supply curves representing the BoE's preferences between collateral sets, and eligibility for entry have evolved over time. These auctions are currently held weekly, with rules that have been modified since the sample period. Further details can be found at `https://www.bankofengland.co.uk/markets/bank-of-england-market-operations-guide/our-tools`.

[9] The classification of collateral sets reflected the relative liquidity of the assets (Fisher, 2011b), and the BoE imposed haircuts on the assets with the aim that differences between the collateral sets reflect only these liquidity premia, not credit risk premia. The haircuts applied to assets varied depending on, e.g. credit rating, interest rate, and maturity (Bank of England, 2010a). The classification was adjusted over the period (see Bank of England, 2011). The current classification can be found at `https://www.bankofengland.co.uk/-/media/boe/files/markets/eligible-collateral/summary-table-of-collateral.pdf`.

[10] Since 2014, after the sample period, broker-dealers and central counterparties are eligible to register for

institutions with OMO access was fairly stable from October 2009, before the ILTRs were introduced, to January 2014, ranging from 48 to 52. There was no formal obligation for registered bidders to participate in the auctions.

Bids Bidders can submit any number of sealed bids. A bid specifies the good (i.e. the type of collateral used), the quantity demanded, and the price (i.e. spread) the bidder is willing to pay. For example, a bid may specify demand of £50 million at a spread of 2 basis points (bps) for good A.[11] The minimum bid price is 0bps, with increments of 1bp. The minimum bid size is £5 million, with increments of £1 million, and the minimum unit of allocation is £100,000. A bidder may bid for a maximum of £1.5 billion and £0.75 billion of loans for 3-month and 6-month terms, respectively (Bank of England, 2010a).

Supply Prior to the auction, the BoE commits to the following supply preferences:

1. The maximum supply made available across goods A and B is publicly announced prior to the auction.

2. The BoE commits to a privately known "relative supply" curve. It is measured by the difference between the prices of goods B and A and is an increasing function of the quantity allocated of good B. This is "pinned down by [the BoE's] preferences" to provide liquidity insurance at prices which incentivise prudent liquidity management (Fisher, 2011a, p.12).

Market clearing The PMA uses the information from the submitted bids and auctioneer's supply curves to find the competitive equilibrium, assuming bids correspond to bidders' marginal values and the supply curves represent the prices the BoE is willing to accept. The quantity allocated of each good therefore depends on the submitted bids and BoE's supply preferences, and so is uncertain from the perspective of bidders (see Appendix A for an illustration).

For each good, the auction price is the maximum of the highest losing bid for that good and the minimum price the BoE is willing to accept, as expressed by the supply curves.[12]

the SMF (de Roure and McLaren, 2021).

[11]In the sample period, bidders were also permitted to submit "paired bids", which specify the quantity demanded and a price for each of goods A and B. For each bid, the bidder may be allocated the loan against either good A or good B. They are allocated the good which maximises their surplus, given the clearing prices and assuming bids reflect bidders' true preferences. For example, a paired bid may specify demand of £50 million at a spread of 2bps for good A or 25bps for good B. If the clearing prices are 1bps and 22bps respectively, the bidder is allocated £50 million of good B. In practice, bidders rarely made use of paired bids so I drop them from the sample.

[12]For good B, the minimum price the BoE is willing to accept is the sum of the auction price for good A

Bids strictly above the auction price are fully allocated and bids on the margin are rationed pro-rata for each good. Winning bidders pay the auction prices.

Information Certain information about the auction results is made public: the total quantity of bids for each good, total amounts allocated, clearing spreads and rationing coefficients. None of the number of bidders, the number of individual bids, and the range of bid prices are revealed.

2.2 Data

I use a unique proprietary dataset that consists of all bids submitted in the BoE's ILTR auctions, held monthly in June 2010 – January 2014, and the relative supply curves used by the BoE to determine the allocation across goods. To my knowledge, this data has only once been analysed.[13]

Supply The maximum supply, publicly announced prior to the auction, was £5 billion for the 3-month term auction and £2.5 billion for the 6-month term auctions. The minimum auction prices of goods A and B were 0bps and 5bps, except for the prices in the 6-month auctions from May 2011, in which the minimum for good B was 15bps. The relative supply curves were flat at the minimum spreads up to some fixed quantities of good B and beyond this were increasing; I cannot disclose their precise functional forms.

Bids The bidding data is de-identified. Across auctions, unique participants are indexed from 1 to P. For each participant, I observe the full set of bids that the individual submits on goods A and B, if any, in each of the 44 auctions. Each bid consists of the type of collateral the bidder will provide (the good), the spread in basis points over Bank Rate they are willing to pay (the bid price) and the quantity of liquidity they demand in £million. For each good, a bidder's bids imply a (price, quantity)-schedule, i.e. an individual bid function, which is a step function.

Aggregate demand The total quantity of funds demanded in the auctions decreased substantially in the second half of the sample period (see Figure 1) so I limit the analysis to the first 24 months of auctions (from June 2010 to May 2012).[14] Aggregate demand

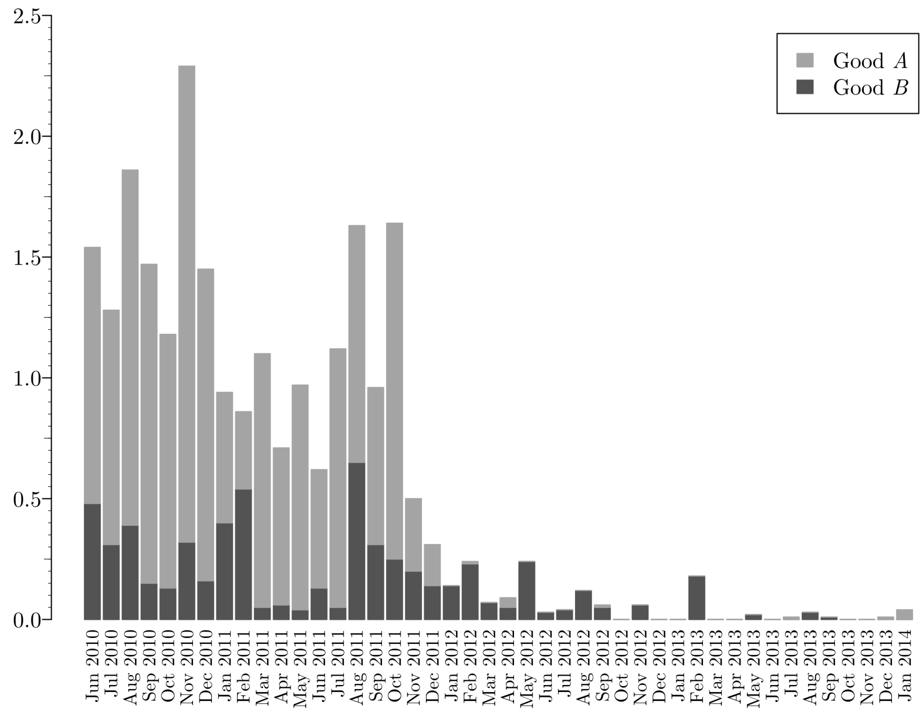
and the relative supply curve evaluated at the quantity allocated of good B. For good A, it is the difference between the auction price for good B and the relative supply curve evaluated at the quantity allocated of good B.

In practice, the precise price determination rules change slightly over the period for the rare cases where competitive equilibrium does not determine the prices uniquely (see Giese and Grace (2023) for details).

[13] de Roure and McLaren (2021) study the risk profile of participants in the ILTR auctions.

[14] Winters (2012) provides three explanations for this. First, an increase in the availability of funding from

Figure 1: Aggregate quantity demanded as a share of the maximum supply, June 2010–January 2014.



also varies over the period, which the estimation procedure described in Section 4.2 aims to account for.

Summary statistics Table 1 provides summary statistics of the data. The auction price of good A is typically close to its reserve of 0bps (the weighted average price paid is 1.14bps in the 3-month term auctions and 0.59bps in the 6-month term auctions), reflecting the high liquidity of assets in the Level A collateral set. In contrast, the assets in the Level B set are more heterogeneous and typically have limited secondary markets. This is reflected in much higher weighted average prices paid for good B of 21.67bps and 43.43bps in the 3-month and 6-month term auctions, respectively, and much greater variance in the auction prices than good A.

the BoE's quantitative easing programme and cheaper funding in the market than from the BoE reduced demand for good A. Second, a reduction in participants' holdings of assets in the Level B collateral set reduced demand for good B. And finally, funding allocated in the ILTR auctions was of relatively short maturity, so was not well-suited to meet participants' regulatory liquidity requirements. Increased availability of funds from the euro area and from alternative facilities in the BoE also likely played a role. For example, the Funding for Lending Scheme was introduced by the BoE in July 2012, allowing borrowing of up to four years. Another source is the BoE's Extended Collateral Term Repo (ECTR) operation, consisting of monthly auctions for 6-month term liquidity, although this was only activated once in June 2012.

Table 1: Summary statistics for ILTR auctions in June 2010 – May 2012

	3-month term		6-month term	
Number of auctions	16		8	
Both goods				
	Mean	Std. Dev.	Mean	Std. Dev.
Maximum supply (£billion)	5	0	2.5	0
Total quantity of funds demanded (£billion)	4.57	2.67	2.69	1.93
Number of bidders	8.13	3.07	7.63	3.85
Total quantity of funds allocated (£billion)	3.55	1.86	1.65	0.99
Good *A*				
	Mean	Std. Dev.	Mean	Std. Dev.
Quantity of funds demanded (% of allocation of good A)	125.72	22.68	171.98	66.62
Bid price (weighted, basis points)	2.51	2.85	3.06	4.31
Number of bids submitted per bidder (unweighted)	1.52	0.84	1.49	0.86
Number of bids submitted per bidder (weighted)	1.79	0.94	1.86	1.04
Quantity demanded by bidder (% of maximum supply)	13.65	11.87	13.93	12.30
Quantity of funds allocated (% of total allocated)	83.72	12.00	70.65	20.19
Price paid (weighted, basis points)	1.14	1.71	0.59	0.79
Good *B*				
	Mean	Std. Dev.	Mean	Std. Dev.
Quantity of funds demanded (% of allocation of good B)	151.26	58.15	202.74	91.49
Bid price (weighted, basis points)	23.61	9.96	38.96	16.15
Number of bids submitted per bidder (unweighted)	1.64	1.31	2.30	1.71
Number of bids submitted per bidder (weighted)	2.90	2.11	2.95	2.28
Quantity demanded by bidder (% of maximum supply)	4.30	6.41	9.69	9.87
Quantity of funds allocated (% of total allocated)	16.28	12.00	29.35	20.19
Price paid (weighted, basis points)	21.67	6.52	43.43	1.37

Weightings: Quantities of funds demanded and quantities of funds allocated per good are weighted by the aggregate quantities demanded in auctions; prices paid for each good are weighted by the aggregate quantities allocated of the good in auctions; bid prices are weighted by the quantities demanded at those prices by bidders; and the number of bids submitted per bidder is weighted by the total quantities demanded by bidders.

Table 2: Summary statistics by bidder size in the ILTR auctions in June 2010 – May 2012

	Good ***A***							
	3-month term				6-month term			
	Small bidders		Large bidders		Small bidders		Large bidders	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
Bid price (basis points)	3.22	2.13	2.51	2.85	5.33	5.20	3.03	4.29
Number of bids submitted per bidder	1.09	0.29	1.67	0.91	1.25	0.45	1.58	0.96

	Good ***B***							
	3-month term				6-month term			
	Small bidders		Large bidders		Small bidders		Large bidders	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
Bid price (basis points)	24.42	13.93	23.47	9.10	41.36	23.49	38.88	15.85
Number of bids submitted per bidder	1.28	0.72	2.14	1.74	1.89	1.17	2.50	1.92

Definitions: Small bidders are defined as those who demand less than 2.5% of the maximum supply in all auctions in the sample period; Large bidders are defined as those in the remaining set; bid prices are weighted by the quantities demanded at those prices by bidders, and numbers of submitted bids are unweighted.

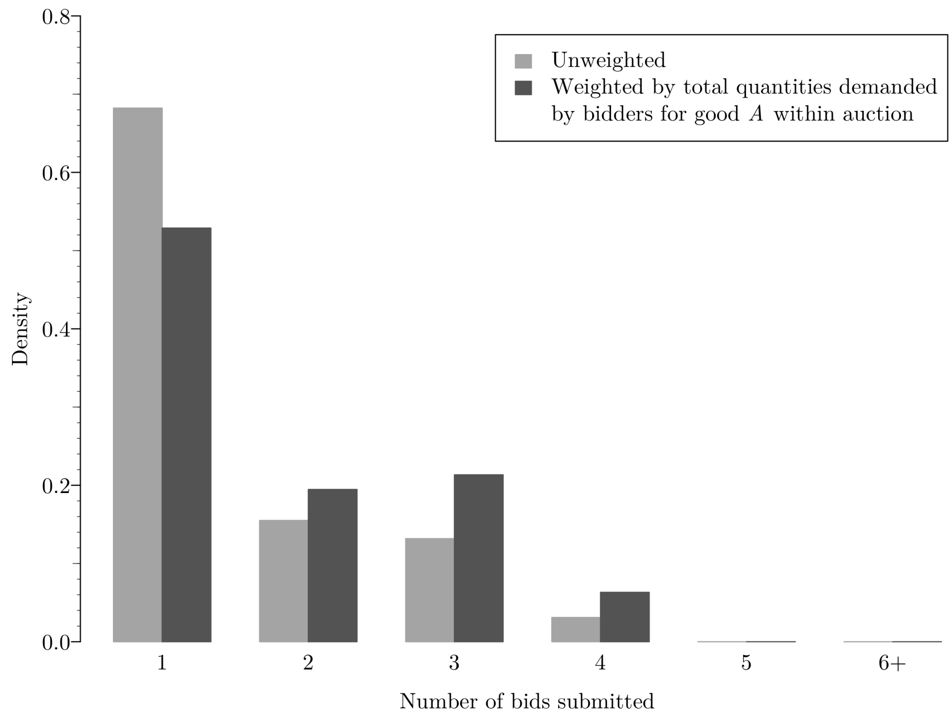
2.3 Bidding behaviour by bidder size

Bidders differ in the amount that they demand, and therefore in their ability to affect the auction price. In conventional models of bidding in uniform-price auctions, larger bidders have incentives which correspond to the quantity-shading incentives of oligopolists facing uncertain demand in Klemperer and Meyer (1989). A bidder can reduce the expected auction price paid for all units that they win by reducing the quantity that they demand at each price; the strength of this incentive is increasing in the quantity that the bidder demands. If bidders were acting strategically, we would therefore expect to observe larger bidders to *ceteris paribus* submit lower bid prices relative to their values.

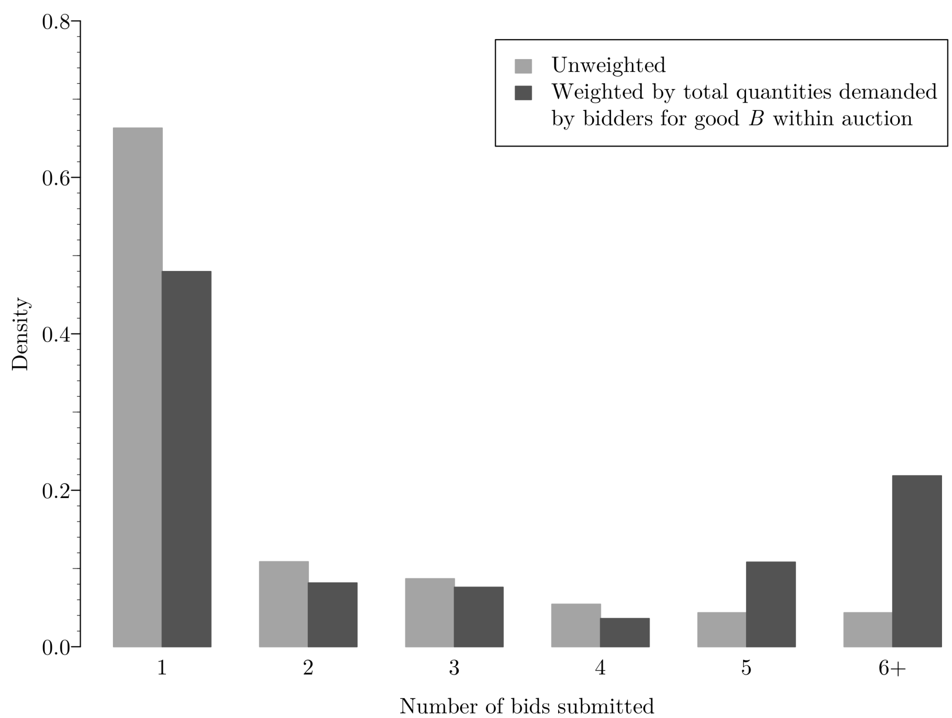
Table 2 shows the average bid prices of bidders who demand less than 2.5% of the maximum supply in all auctions, who are labelled "Small", and bidders in the remaining set, labelled "Large".[15] Larger bidders do submit lower bid prices in the ILTR auctions, but this does not isolate differences in strategic incentives from unobserved variation in bidders' values. It is possible, for example, that larger bidders in the ILTR auctions have greater access to alternative funding sources which affect their valuations. The main contribution of this paper is to employ a testing procedure, described in Section 4.3, which isolates the variation in bidders' strategic incentives from variation in their values in order to evaluate alternative models of behaviour.

Figure 2: Number of bids submitted per bidder within auction on each good in June 2010 – May 2012.

(a) Good A



(b) Good B



2.4 Bids submitted per bidder

A striking feature of the data is that bidders submit few bids per auction (Figure 2). This characteristic of bidding is also seen in many other multi-unit auction settings.[16] It motivates the model of bidding behaviour, developed by Kastl (2011, 2012), which is now widely used in empirical studies (e.g. Cassola, Hortaçsu and Kastl, 2013; Hortaçsu, Kastl and Zhang, 2018; Allen, Kastl and Wittwer, 2022).

In the model, bidders face costs of submitting additional bids, which I refer to as "bidding costs". Given these costs, the number of bids to submit is also a strategic choice: a bidder must trade off the marginal bidding cost with the marginal benefit of fine-tuning their demand by submitting an additional bid. If the marginal benefits of fine-tuning are small, bidders optimally submit few bids.

Larger bidders submit more bids per good (see Table 2), but the number of bids they submit is still small. Without bidding costs or other frictions, this is difficult to explain within a strategic framework. Even if a bidder had a constant marginal value for the good, we would expect them to submit a large number of bids, as their incentive to strategically understate their value is increasing in the quantity that they demand (see Section 2.3).

It is therefore natural to use Kastl's (2011) model with bidding costs as a benchmark to compare alternative models of behaviour.

[15] Natural alternative definitions of bidder size give similar results.

[16] In Czech, Canadian and US Treasury auctions, bidders on average submit 2.3 bids (Kastl, 2011), 2.9 bids (Hortaçsu and Kastl, 2012) and between 3 and 5 bids (Hortaçsu, Kastl and Zhang, 2018), respectively. In wholesale electricity auctions, Hortaçsu and Puller (2008, p.106) document that bidders in Texas "do not make full use of the strategy space available to them, but rather use coarse-grained bidding strategies", and bidders only submit 4.4 bids on average in Spain (Reguant, 2014). This small number of bids is also observed in central bank operations: Cassola, Hortaçsu and Kastl (2013) find that participants submit 1.7 bids on average in the European Central Bank's auctions for short-term repurchase agreements, and 82% of participants in the Bank of Canada's cash management auctions submit only one bid (Chapman, McAdams and Paarsch, 2006).

Moreover, institutional constraints on the number of bids that a bidder may submit are almost never binding. For example, both Kastl (2011) and Hortaçsu and Puller (2008) document that not one bidder submits the maximum number of permitted bids (10 and 40, respectively). Even when this bound is relatively tight, as is the case in the cash-management auctions studied by Chapman et al.'s (2006) in which only 4 bids are permitted, only 2.5% of bidders submit the maximum number.

3 Candidate models of behaviour

This paper analyses competing models of behaviour of participants in the BoE's liquidity auctions using a testing procedure based on instrumental variables, described in Section 4.3. In this section, I first lay out the assumptions on the economic environment that are required for the test to be valid; the main assumption is that bidders have independent private values.

I then describe the competing models. The first is a "truthful bidding" model in which all bidders submit bids according to their true marginal values.

The second is a conventional strategic model of bidding (Kastl, 2011)—widely used in empirical studies of multi-unit auctions—which I generalise to allow for risk aversion. I focus on Kastl (2011) because it captures the key features of the BoE's liquidity auctions, namely that bidders make few bids per auction. One limitation is that the model does not permit bidders to be risk averse. Therefore I relax Kastl's (2011) assumption of risk neutrality. This generalisation shows that optimal strategies in the strategic model tend towards truthful bidding as the degree of risk aversion becomes large.

The correct model of course may differ from both of these two models; the procedure tests which of the two models is closer to the correct model.

The test is based on the amount of "bid shading"—the difference between bidders' marginal values and their bids—implied by the different models. Equipped with the setup of the two models, I show how bidders' marginal values, and therefore their bid shading, are identified in each model. In the truthful bidding model, bid shading is of course always zero, whereas in the conventional model it varies across bidders and across the quantities they demand. This variation in the amount of bid shading is key to the test's identification. The following description refers to one auction and I suppress the time subscript.

3.1 Assumptions on the economic environment

Two goods, A and B, are perfectly divisible, measured in the same units, and allocated by a Product-Mix Auction as described in Klemperer (2008) and Section 2.1.

There are N potential bidders. Prior to the auction, each bidder $i \in \mathcal{N} = \{1, ..., N\}$, receives a privately known, multi-dimensional signal, Θ_i. Bidder i's signal is distributed such that their value is above the reserve price for at most one of the goods. Each bidder will therefore only bid on at most one good. This greatly improves the tractability of the model and is appropriate for the majority of bidders (85%) in the data who only bid for one good. For the purpose of the analysis, it is also appropriate for an additional 4% of bidders per auction who bid on both goods, but for at least one of the goods, they submit only one bid and that bid is at the reserve price, so that it is irrelevant for the implications of the model.[17,18] Let $\boldsymbol{\Theta_{-i}} = \{\Theta_j\}_{\{j \neq i\}}$ denote the signals received by all bidders except i.

Motivated by the evidence of heterogeneity described in Section 2.3, bidders for each good are split into two groups, so the set of potential bidders for good $g \in \{A, B\}$ with bidder size $h \in \{1, 2\}$ is $\mathcal{N}^{g,h}$. This set has dimension $N^{g,h}$, where $\sum_{g=\{A,B\}} \sum_{h=\{1,2\}} N^{g,h} = N$. Bidders are assumed to be symmetric within group, that is, signals are identically distributed and marginal valuation functions are symmetric.[19]

The main assumption required for the testing procedure to be valid is that bidders have independent private values. Let $y_i \in \mathbb{R}_+$ be the quantity allocated to bidder i expressed as a share of the maximum supply.

The IPV Assumption *The marginal valuation of bidder $i \in \mathcal{N}^{g,h}$, $g \in \{A, B\}$, $h \in \{1, 2\}$, denoted $v^{g,h}(y_i, \Theta_i)$, is only a function of y_i and the bidder's signal Θ_i.*

Three reasons suggest that an independent private values framework is a reasonable approximation.

[17] The analysis uses local deviations in the quantity that a bidder demands to identify their values. If a bidder bids on both goods, but their bid is at the reserve for good A, then the bidder is always indifferent between winning and losing good A, so a deviation in the quantity they demand of good B does not affect their surplus from good A (and vice versa). We can therefore treat a bidder for both goods as if they are two bidders, one for each good.

[18] Moreover, for those bidders who do submit bids for both goods within an auction, additive separability of their utility functions in the quantities allocated of the two goods seems a reasonable assumption. This is because bidders have the option to submit "paired bids", which allow bidders to express rich one-for-one substitutes preferences between goods A and B (for details, see Klemperer, 2018), but they very rarely choose to submit such bids. The absence of these bids reveals that bidders do not view goods A and B as simple one-for-one substitutes. It is also reasonable to rule out complement and more complicated substitute preferences. This is because goods A and B represent liquidity borrowed against two different collateral sets. These can be interpreted as two goods measured in the same units that face different opportunity costs (the return on the next best use of the assets put up as collateral) but are used for broadly the same purpose of meeting liquidity needs. In this context, one-for-one or non-substitutability appear most plausible. Indeed, the BoE designed the ILTR auction anticipating bidders to express one-for-one substitutability between goods A and B, if at all.

[19] Bidders are financial institutions with diverse liquidity needs and motivations for participation, which suggests that bidders are drawing from different distributions of signals or have different marginal valuation functions. Section 2.3 shows one example of this heterogeneity: bidders systematically differ by the quantities that they demand. In line with the literature which assumes either one or few groups (see, e.g. Hortaçsu, 2002; Kastl, 2011; Hortaçsu, Kastl and Zhang 2018; Boneva, Kastl and Zikes, 2020), I group bidders by size of demand and assume i.i.d. signals within group to permit a sufficient sample size for estimation (see Section 4.2).

First, while the ILTR auctions were designed to enhance financial stability by meeting demands for liquidity insurance, the sample period of 2010 – 2014 was relatively stable. This suggests that the possible interdependence, which would arise from another bidder's private signal revealing information of perceived stress in the market, was less relevant in our sample.

Second, it is plausible that participation over the period studied was driven largely by idiosyncratic liquidity needs, rather than by speculation in the interbank market. The fairly short maturity of loans secured in the ILTR auctions limited incentives for speculation,[20] and this was especially the case for good B, because collateral assets in the Level B set were less liquid by definition. Moreover, the general collateral repo rate was a good proxy for the market value of good A, i.e. loans secured by highly liquid collateral. Because this rate was readily observable, the uncertainty about the future market value of good A was plausibly symmetric, which is consistent with an independent private values framework.[21]

Finally, neither Bindseil, Nyborg and Strebulaev (2009) nor Hortaçsu and Kastl (2012) find evidence for interdependent values in ECB repo auctions and Canadian Treasury auctions, respectively. It is not possible to perform these tests in our context, but their findings are informative given the similarity of our context to theirs.[22]

Bidders are also assumed to have marginal valuation functions that are non-increasing and continuous in the quantities they are allocated and bounded and strictly increasing in their own signals.[23]

[20] Hortaçsu and McAdams (2010) and Haile (2001) provide evidence, in the Turkish Treasury security market and timber market respectively, of limited scope for resale (i.e. speculation on the future price) at short time horizons.

[21] Tighter liquidity regulation, introduced by the UK Financial Services Authority in 2010, might explain participation in the ILTR auctions for 6-month loans. The regulation increased demand for high-quality liquid assets with more than 3-month maturity, and required banks to regularly participate in a number of different funding markets (Banerjee and Mio, 2018). Anecdotal evidence suggests that this was one motivation for participation, but unlikely to be the key driver given the relatively short maturity of the loans (Winters, 2012).

[22] Tests for interdependent values exist for single-unit auctions, exploiting random variation in the number of bidders (Athey and Haile, 2002; Haile, Hong and Shum, 2003), but are more difficult to construct for multi-unit auctions. Bindseil, Nyborg and Strebulaev's (2009) main test exploits variation in an interbank rate which can be seen as a benchmark for the auctions. We cannot construct this in our setting given the variation in assets used as collateral, especially in the Level B set. Similarly, Hortaçsu and Kastl's (2012) test, which relies on variation specific to their institutional setting, is not applicable to this context. In their setting, some bidders are dealers, observing a subset of the other bids prior to the auction, and variation in the behaviour of these bidders can be exploited to detect interdependent values. It is not possible to observe analogous variation in my data.

[23] The funds that a given bidder is allocated in the auction is one component of their larger stock of cash reserves. Collateral that can be used to back ILTR loans is typically deposited at the BoE prior to the auction, although it is possible to submit collateral for settlement following the operation (Alphandary, 2014). The setup costs are therefore mostly sunk, and there are no other obvious fixed costs associated with

Turning to the utility function, bidders are assumed to have utility functions which are quasi-linear in assets outside the auction.[24] I allow for bidders to be risk averse, with constant absolute risk aversion utility described by parameter $\rho^g \geq 0$, which is common to all bidders for good $g \in \{A, B\}$.

Let P^g be the auction price of good $g \in \{A, B\}$. The expected utility of type $\theta_i \in \Theta_i$ of bidder $i \in \mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$ given strategy profile $\boldsymbol{\sigma}(\boldsymbol{\Theta}) = \{\sigma_j(\Theta_j)\}_{\{j \in \mathcal{N}\}}$ in Model m is therefore

$$V_i(\theta_i) = \begin{cases} \mathbb{E}_{\boldsymbol{\Theta_{-i}}}\Big[U_i\Big(y_i(\boldsymbol{\sigma}(\boldsymbol{\Theta})), P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta}))|\theta_i\Big)\Big] & \text{if } \rho^g = 0 \\ \mathbb{E}_{\boldsymbol{\Theta_{-i}}}\Big[\frac{1}{\rho^g}\big(1 - e^{-\rho^g U_i(y_i(\boldsymbol{\sigma}(\boldsymbol{\Theta})), P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta}))|\theta_i)}\big)\Big] & \text{if } \rho^g > 0 \end{cases}$$

where

$$U_i\big(y_i(\boldsymbol{\sigma}(\boldsymbol{\Theta})), P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta}))|\theta_i\big) = \int_0^{y_i(\boldsymbol{\sigma}(\boldsymbol{\Theta})|\theta_i)} v^{g,h}(u, \theta_i)du - P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})|\theta_i)y_i(\boldsymbol{\sigma}(\boldsymbol{\Theta})|\theta_i).$$

In each auction, each bidder can submit multiple bids. These bids are summed horizontally in (price, quantity)-space to form an individual bid function, which is a step function. Hence, bidder i chooses the number of bids to submit, K_i, and, for each bid $k \in \{1, ..., K_i\}$, the bid price $b_{i,k}$, and cumulative quantity demanded, $q_{i,k}$, where the bids are ordered in increasing quantity. (The marginal quantity demanded at bid price $b_{i,k}$ is therefore $(q_{i,k} - q_{i,k-1})$.)

Corresponding to the ILTR auction rules, bids must be integers (i.e. bids must be whole basis points). To simplify the analysis, I approximate the grid of quantities permitted in the ILTR by assuming that quantities demanded are continuous in the unit interval.[25] Let $\boldsymbol{b_i} \in \mathbb{R}_+^{K_i}$ and $\boldsymbol{q_i} \in \mathbb{R}_+^{K_i}$ respectively denote the vector of bid prices and cumulative quantities demanded by bidder i. I refer to $(b_{i,k}, q_{i,k})$ as the bid price and quantity demanded at step k in bidder i's bid function.

accessing and using the liquidity. This suggests limited scope for increasing returns. If any allocated funds are put to its highest value use, a non-increasing marginal valuation function is a reasonable assumption. Chapman, McAdams and Paarsch (2006) corroborates this in a similar context of the Bank of Canada's liquidity auctions.

It is less clear whether the assumption of continuity is reasonable. It is supported by the fact that both the collateral used to back ILTR loans, and the funds obtained, constitute a small fraction of participants' total assets. This would be useful to explore in future work.

[24] Quasi-linearity is motivated by the fact that bidders use a negligible fraction of total assets as collateral for liquidity obtained in the ILTR auctions (de Roure and McLaren, 2021).

[25] In the ILTR auctions, the minimum permitted cumulative quantities demanded are 0.1% and 0.2% of the maximum supply in the 3-month and 6-month term auctions respectively, with increments of 0.02% and 0.04%, respectively.

3.2 The truthful bidding model

The first of the two competing models is a "truthful bidding" model, in which all bidders optimally submit bids which approximate their true demand for liquidity. Specifically, I assume that each bidder submits a bid function equal to the greatest integer function of their true marginal valuation function.[26]

Definition 1 (Model *T*) *In Model T, the IPV Assumption holds and each bidder submits a bid function that corresponds to the greatest integer function of their marginal valuation function (i.e. for all quantities, the largest integer that is weakly less than their marginal value).*

3.3 The conventional strategic model

The second model that I consider closely follows Kastl's (2011) modification of Wilson's (1979) share auction model, which is commonly used in the literature (e.g. Cassola, Hortaçsu and Kastl, 2013; Hortaçsu, Kastl and Zhang, 2018; Allen, Kastl and Wittwer, 2022). The main exception is that I relax Kastl's (2011) assumption of risk neutrality.

In this model, bidders choose their bids to maximise their expected surpluses, conditional on the information available to them at the time of bidding. The assumptions on bidders' information are as follows.

Bids in the ILTR auctions are submitted privately through an online platform, and so the number of other bidders is unknown, as are their bids. In the model, bidders are assumed to have common knowledge of the number of potential bidders, the joint distribution of signals, and the risk aversion parameters.[27]

In the ILTR setting, as in other studies in which a model of behaviour is assumed in order to recover bidders' values, the suitability of this common prior is unclear. Some information about the number of potential bidders, equal to the number of financial institutions with access to the BoE's OMOs, was publicly available during the period. For example, Winters

[26]This could be motivated in many ways. For example, each bidder faces an arbitrarily large cost of determining and submitting what would otherwise be the optimal bid function. Truthful bidding is an especially natural heuristic in the conventional strategic model that I consider because the optimal bid may be above or below a bidder's true value.

[27]Common knowledge of the number of potential bidders can be relaxed by instead assuming that the number of potential bidders is stochastic, but its distribution is common knowledge. Hortaçsu (2002) finds that this modification does not have a significant impact on his findings.

(2012) stated that there were 50 bidders signed up for OMOs at the time. Bidders also may have learned about the distribution of other bidders' values in auctions held by the BoE prior to our sample period, which had similar purpose to the ILTR auctions but differed in design.[28]

In addition, for simplicity, I assume that the BoE's maximum supply and relative supply curve are commonly known.[29]

Different assumptions on bidders' information would imply different optimal strategies. One of the benefits of the testing procedure described in Section 4.3 is that it tests the *relative* performance of the two models and so is robust to their misspecification. For example, if bidders are actually strategic but have different information from what is assumed here, the procedure tests which of the two models are a better approximation of this 'correct' model.

In this model, bidders face costs of submitting additional bids, which are increasing in the number of bids that they submit. Kastl (2011) introduces these "bidding costs", denoted $c_i(K_i)$, to rationalise the fact that bidders submit few bids per auction (see Section 2.4). There are various ways to interpret the bidding costs, including the physical and time costs of submitting bids or of fine-tuning them (see Appendix F for discussion).

The model is defined as follows.

Definition 2 (Model $\boldsymbol{S(\rho)}$) *In Model $S(\rho)$, the IPV Assumption holds; the number of potential bidders in each group, $N^{g,h}, g = \{A, B\}, h = \{1, 2\}$, the joint distribution of signals, the risk aversion parameters, the maximum supply and the relative supply curve are commonly known; $c_i(K_i)$ is the cost of submitting bids, where $c_i(K_i + 1) \geq c_i(K_i) \geq 0 \; \forall K_i \in [1, \bar{K} - 1]$; and the risk aversion parameters are equal to $\rho = (\rho^A, \rho^B)$.*

The solution concept is a group-symmetric Bayesian Nash Equilibrium, in which bidders within the same group who receive the same signal adopt the same strategy. An equilibrium is a strategy profile, $\boldsymbol{\sigma}(\boldsymbol{\Theta})$, such that for every bidder $i \in \mathcal{N}$ and almost every type θ_i, $\sigma_i(\theta_i)$ solves $\sigma_i(\theta_i) \in \arg\max_{\sigma_i(\theta_i) \in \mathcal{F}_i} (V_i(\theta_i) - c_i(K_i))$

[28] The ILTR auctions replaced the BoE's Long-Term Repo operations, which were pay-as-bid auctions in which a reserve price of 50bps was imposed for bids on good B (see Fisher (2011a) for further details).

[29] We could relax this assumption by assuming that the distribution of supply of each good is commonly known or we could assume that the BoE is a strategic actor with privately known preferences. Hortaçsu (2002) models the total supply in Turkish Treasury auctions as an AR(1) process, because bidders face some uncertainty in its precise quantity, but he finds this makes very little difference to his results. Our assumption may be more reasonable in later auctions in the sample, as some information about the relative supply function is revealed in the publicly released data of auction outcomes.

3.4 Bid shading in equilibrium

The testing procedure described in Section 4.3 exploits differences in the amount of "bid shading"—the difference between bidders' marginal valuations and their bids—implied by the alternative models of behaviour in order to compare their model fit.

Definition 3 (Bid shading) *The bid shading of bidder* $i \in \mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$ *at the quantity they demand at step* $k \in \{1, ..., K_i\}$ *is* $\mu_{i,k} = v^{g,h}(q_{i,k}) - b_{i,k}$.

In Model T, an equilibrium strategy profile is one in which each bidder submits a set of bids which correspond to the greatest integer function of their marginal valuation function. Because the marginal valuation function is assumed continuous, this implies that a bidder's bid at step $k \in \{1, ..., K_i\}$ is equal to their marginal value so that $\mu_{i,k} = 0 \; \forall k \in \{1, ..., K_i\}, i \in \mathcal{N}$ in Model T. This is the case regardless of the bidder's degree of risk aversion.

Bid shading in Model $S(\rho)$ is more complicated. A necessary condition for a strategy profile to be a Bayesian Nash Equilibrium in this model is that bidder $i \in \mathcal{N}$ cannot increase their expected utility by unilaterally deviating from their prescribed strategy, σ_i, given their beliefs. Each bidder's strategy is multi-dimensional—for each auction, and for each type $\theta_i \in \Theta_i$, the bidder must choose the number of steps to submit, and for each step, a bid price and quantity to demand—so there are many possible ways in which to deviate.

The main identifying condition rules out profitable deviations in the quantity that a bidder demands at a particular step for type $\theta_i \in \Theta_i$ in Model $S(\rho)$, holding the rest of their strategy constant.[30] Proposition 1 (below) states this condition, generalising Kastl's (2011) corresponding condition to allow for risk aversion.

The benefit of this deviation to the bidder is the difference between two effects. The first effect is the increase in expected utility from winning the marginal unit at step k, holding the distribution of the auction price constant. This effect is captured by the left-hand side of Equation 1 below. Following the standard Kastl (2011) decomposition, this distinguishes between the cases in which the bidder wins the full marginal unit, and the bidder "ties" with other bidders at step k and step $k + 1$ and so is rationed.

The second effect is a market power effect (the right-hand side of Equation 1 below). It mea-

[30] Appendix F presents a set of additional necessary conditions for equilibrium in Model $S(0)$, which rule out profitable unilateral deviations in the bid prices submitted by a bidder, holding the rest of their strategy constant.

sures the reduction in expected utility due to the bidder having to pay more in expectation for the units that they win, if the marginal increase in their demand increases the auction price.

A necessary condition for equilibrium in Model $S(\rho)$, stated in the following proposition, is that the bidder does not benefit from deviating, i.e. the two effects must be equal. This condition enables us to determine the optimal amount of bid shading at the quantity the bidder demands at step k, i.e. $\mu_{i,k} = v^{g,h}(q_{i,k}) - b_{i,k}$, which is a key input to the test.

Proposition 1 (Necessary condition on quantity deviations in Model $\boldsymbol{S(\rho)}$) *In Model $S(\rho)$ with $\rho = (\rho^A, \rho^B) \geq 0$, in any type-symmetric Bayesian Nash Equilibrium, for almost every type θ_i, every step k in the K_i-step bid function in strategy σ_i of bidder $i \in \mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$, must satisfy*

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{-\rho^g U_i(q_{i,k}, P^g|\theta_i)} \left(v^{g,h}(q_{i,k}, \theta_i) - P^g\right) \Big| b_{i,k} > P^g > b_{i,k+1}\right] \mathbb{P}(b_{i,k} > P^g > b_{i,k+1}) \\
&\quad + \mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{-\rho^g U_i(y_i, b_{i,k}|\theta_i)} \left(v^{g,h}(y_i, \theta_i) - b_{i,k}\right) \frac{\partial y_i}{\partial q_{i,k}} \Big| P^g = b_{i,k} \wedge Tie^g\right] \mathbb{P}(P^g = b_{i,k} \wedge Tie^g) \\
&\quad + \mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{-\rho^g U_i(y_i, b_{i,k+1}|\theta_i)} \left(v^{g,h}(y_i, \theta_i) - b_{i,k+1}\right) \frac{\partial y_i}{\partial q_{i,k}} \Big| P^g = b_{i,k+1} \wedge Tie^g\right] \mathbb{P}(P^g = b_{i,k+1} \wedge Tie^g) \\
&\quad = \begin{cases} q_{i,k} \frac{\partial}{\partial \epsilon} \left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}} \left[\tilde{P}^g(\epsilon) \mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right) \Big|_{\epsilon=0} & \textit{if } \rho^g = 0 \\ \frac{1}{\rho^g} \frac{\partial}{\partial \epsilon} \left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}} \left[e^{-\rho^g U_i(q_{i,k}, \tilde{P}^g(\epsilon)|\theta_i)} \mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right) \Big|_{\epsilon=0} & \textit{if } \rho^g > 0 \end{cases}
\end{aligned} \tag{1}
$$

where $P^g = P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})|\theta_i)$ and $y_i = y_i(\boldsymbol{\sigma}(\boldsymbol{\Theta})|\theta_i)$; $(P^g = p \wedge Tie^g)$ denotes the event that the auction price for good g is p and at least one other bidder submits a bid for good g with a bid price equal to p; $b_{i,K_i+1} = 0$; $\mathbb{I}(.)$ is the indicator function; and $\tilde{P}^g(\epsilon)$ is the auction price of good g if bidder i unilaterally deviates to a strategy in which type θ_i of bidder i demands a quantity $(q_{i,k} - \epsilon)$ for good g at step k and the rest of the strategy profile, including the rest of their strategy, is unchanged.

Proof. See Kastl (2011) for $\rho^g = 0$ and Appendix B for $\rho^g > 0$. ■

To understand this condition, it is helpful to first consider the simplest case in which a bidder takes the distribution of the auction price as given and is risk neutral. In this case (ignoring ties for convenience), Equation 1 simplifies to $v^{g,h}(q_{i,k}) = \mathbb{E}\big[P^g \big| b_{i,k} > P^g > b_{i,k+1}\big]$; the bidder demands a quantity at step k which equates their marginal value for the good with the expected auction price, conditional on winning. Because the auction price is uncertain, the bidder's demand for the marginal unit is equivalent to accepting a lottery with an expected payoff of zero.

If the price-taking bidder is instead risk averse (and may tie), winning the marginal unit at a loss (i.e. at a price above their marginal value) has a larger, negative, impact on the bidder's utility than winning the unit at a gain. For the same reason that risk aversion makes bidders more aggressive in single-unit first-price auctions (Krishna, 2009), risk aversion reduces the difference between the bid price that a bidder submits and their marginal value. With sufficiently high risk aversion, any lottery with the possibility of a negative payoff is undesirable, so a price-taking bidder who faces ties on the margin will bid truthfully.[31]

More generally a bidder has market power—the distribution of the auction price depends on the bids that they submit. Demanding a larger quantity weakly increases the price, which is paid for all units that they win.

If the bidder is risk neutral, market power creates an incentive for bid shading that corresponds to the quantity-shading incentive of an oligopolist facing uncertain demand in Klemperer and Meyer (1989). In my setting, the bidder has an incentive to bid for a quantity below the true amount that they demand at each price, in order to lower the auction price and reduce the total amount they must pay. The incentive to do so is increasing in the quantity that they demand.

At moderate levels, risk aversion's effect on the incentive to exert market power depends on the distribution of the auction price. Risk aversion weakens the incentive if the bidder's market power is weaker at higher prices. This is because exerting less market power would reduce the dispersion of the auction price, and hence of the bidder's utility. Because a risk averse bidder prefers a less dispersed distribution of the auction price, risk aversion therefore *ceteris paribus* reduces the amount of bid shading of a bidder with market power in this case. The converse holds if the bidder's market power is weaker at higher prices.

Nonetheless, as risk aversion increases, the difference between the bidder's bid and marginal value decreases for the same reasons as for the price-taking bidder. At a sufficiently high degree of risk aversion, a bidder bids truthfully, regardless of their market power.

Appendix B provides a more detailed explanation for Proposition 1.

[31]This result depends on the assumption of independent private values. In their study of Indian Treasury bill auctions, Gupta and Lamba (2023) assume that bidders' values have a common component. In their model, risk averse bidders increase their bid shading as uncertainty in the common value of the good rises to protect against the risk of a low ex post common value.

4 Econometric method

This section explains how the models are compared. I first describe the identification (Section 4.1) and estimation (Section 4.2) of bidders' marginal values, and therefore their bid shading, in each model. Section 4.3 then describes the testing procedure which is based on the relative amount of bid shading implied by the models.

4.1 Identification

In Model T, the marginal value of bidder $i \in \mathcal{N}$ at the quantity that they demand at step $k \in \{1, ..., K_i\}$ of their bid function is point identified by their bid at step k, $b_{i,k}$.

In Model $S(\rho)$, bidders' marginal values must satisfy Equation 1 in any type-symmetric Bayesian Nash Equilibrium. This condition can be used to recover bidders' unobserved marginal values at the quantities they demand at the steps of their bid functions from the observed bids, and hence to point identify their bid shading at these quantities.[32]

If bidder i does not expect to submit a bid at the same price as another bidder and therefore "tie" if their bid is marginal, their marginal value at the quantity they demand at step k can be readily estimated (see Equation 3 in Appendix D).

However, ties are important in our setting because bidders frequently submit bids at the same prices as other bidders in the ILTR auctions. This is especially the case for good A, for which the majority of bids (85.1% in June 2010 – May 2012) are submitted between 0 and 5bps. For goods A and B, respectively, 69% and 20% of bids in June 2010 – May 2012 are submitted at prices at which at least one other bidder bids in the same auction.

Allowing for ties makes identification more complex because the bidder is rationed pro rata if the auction price equals their bid. To evaluate the benefit of the quantity deviation, we therefore need to know the quantity the bidder will win if they do tie, which depends on the rationing coefficient, and their marginal value function between the quantity they win and the quantity they demand.

I therefore make two assumptions in order to identify bidders' marginal values in Model $S(\rho)$. The first assumption is that bidder i's marginal valuation function is flat around the quantity that they demand, so that their marginal value at the quantity that they win

[32] Equation 1 does not identify bidders' marginal values at quantities between the quantities demanded at the steps. In Appendix F, I provide additional necessary conditions for equilibrium in Model $S(0)$ which can be used to set-identify bidders' full marginal valuation functions.

when rationed is the same as their marginal value at the quantity they demand. For bidders who submit one bid (67% of bidders per auction in our sample), this corresponds to the assumption made by Kastl (2011) for identification when allowing for ties in Model $S(0)$.[33] Some natural alternative assumptions would achieve the same result, e.g. that the marginal value function is piecewise linear.

The second assumption is that the rationing coefficient that bidder i expects to face at price $b_{i,k}$, conditional on the strategy profile in Model $S(\rho)$, is deterministic. This approximates a situation in which the bidder does not take the variation in their net utility from the auction into account when evaluating the marginal benefit of winning an additional unit at a auction price equal to their bid.[34]

Under these assumptions, the necessary condition for bidders' strategies to be a Bayesian Nash Equilibrium in Model $S(\rho)$ simplifies to

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}}\left(v^{g,h}(q_{i,k},\theta_i)-P^g\right)\Big|b_{i,k}>P^g>b_{i,k+1}\right]\mathbb{P}(b_{i,k}>P^g>b_{i,k+1})\\
&+\Bigg(e^{\rho^g(1-r_{i,k})(q_{i,k}-q_{i,k-1})v^{g,h}(q_{i,k},\theta_i)}e^{\rho^g b_{i,k}\left(q_{i,k-1}+r_{i,k}\left(q_{i,k}-q_{i,k-1}\right)\right)}\\
&\qquad\qquad\left(v^{g,h}\left(q_{i,k},\theta_i\right)-b_{i,k}\right)r_{i,k}\Bigg)\mathbb{P}(P^g=b_{i,k}\wedge Tie^g)\\
&+\Bigg(e^{-\rho^g r_{i,k+1}(q_{i,k+1}-q_{i,k})v^{g,h}(q_{i,k},\theta_i)}e^{\rho^g b_{i,k+1}\left(q_{i,k}+r_{i,k+1}\left(q_{i,k+1}-q_{i,k}\right)\right)}\\
&\qquad\qquad\left(v^{g,h}\left(q_{i,k+1},\theta_i\right)-b_{i,k+1}\right)\left(1-r^g_{i,k+1}\right)\Bigg)\mathbb{P}(P^g=b_{i,k+1}\wedge Tie^g)\\
&=\begin{cases}
q_{i,k}\frac{\partial}{\partial\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\tilde{P}^g(\epsilon)\mathbb{I}\left(b_{i,k}\geq\tilde{P}^g(\epsilon)\geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g=0\\
\frac{1}{\rho^g}\frac{\partial}{\partial\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g\tilde{P}^g(\epsilon)q_{i,k}}e\mathbb{I}\left(b_{i,k}\geq\tilde{P}^g(\epsilon)\geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g>0
\end{cases}
\end{aligned}\tag{2}$$

Appendix C provides additional details on identification in Model $S(\rho)$.

Equation 2 can be solved numerically to recover bidders' marginal values, by calibrating

[33] Kastl (2011) assumes that the expected marginal valuation function in the range $y_i \in [\mathbb{E}[y_i|P^g=b_{i,k}\wedge Tie^g], q_{i,k}]$ is constant at $v^{g,h}(q_{i,k},\theta_i)$. For bidders who submit more than one bid per auction, allowing for risk aversion also requires the stronger assumption that the marginal valuation function is also flat between the quantity that the bidder demands at step k and the quantity they are allocated when rationed at step $k+1$.

[34] We could relax the assumption by estimating the bidder's net utility for each realisation of the rationing coefficient and weighting by the probability of its occurrence. This would be difficult to estimate precisely given the available data.

the risk aversion parameter and using the bidding data to estimate bidders' beliefs over the distribution of the auction price and the impact of their own bids on this distribution.

It is not possible to identify the degree of risk aversion from Equation 2, so I test the relative performance of models which differ in this parameter in Section 4.3. I also estimate bidders' marginal values assuming no ties (see Appendix D).

I turn next to the estimation method for the remaining components of Equation 2.

4.2 Estimation

I estimate each bidder's marginal values at the quantities they demand at the steps of their bid function separately for each auction in which they participate. For ease of notation, I suppress time subscripts, except for when describing the resampling method for estimating a bidder's beliefs and the sample of other bidders used.

For Model T, the marginal value of bidder $i \in \mathcal{N}$ at the quantity they demand at step k of their bid function is trivially estimated by their bid at step k, $b_{i,k}$.

For Model $S(\rho)$, we must estimate bidders' beliefs over the distribution of the auction price in order to estimate their values. Under the assumption that bidders are playing a Bayesian Nash Equilibrium, each bidder forms beliefs over the distribution of auction prices that are consistent with other players' strategies in equilibrium, given their prior. Guerre, Perrigne and Vuong (2000) provide a technique to recover bidders' unobserved beliefs from the observed distribution of bids, which exploits the fact that the set of observed bids are the bidders' strategy profile in a pure-strategy equilibrium for a given realisation of signals. This strategy profile is determined by bidders' common prior of the number of potential bidders and the signal distribution, and the mapping from signals to strategies. Guerre, Perrigne and Vuong (2000) show that, assuming equilibrium, the realised bid distribution can be used to estimate bidders' beliefs without specifying a functional form or making distributional assumptions.

Hortaçsu (2002) proposes a resampling procedure to implement this technique for multi-unit auctions, which consistently estimates each bidder's beliefs of the distribution of the auction prices, conditional on their strategy. Intuitively, each bidder is best responding to the strategies of other bidders, and each observed bid function corresponds to another bidder's optimal strategy for a particular signal realisation. And so, holding a bidder's strategy fixed, the observed bid distribution reflects the bidder's beliefs over the signal distribution. (In line with the model described in Section 3.1, a bidder who submits bids on both goods is treated as two separate bidders.)

The resampling procedure is as follows. Fixing bidder i in auction t, where i belongs to group $\mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$, I construct a stratified sample of other bidders, denoted $\mathcal{O}_{i,t} = \{\mathcal{O}_{i,t}^{A,1}, \mathcal{O}_{i,t}^{A,2}, \mathcal{O}_{i,t}^{B,1}, \mathcal{O}_{i,t}^{B,2}\}$, from the bidding data as described below. The observation $(i', t') \in \mathcal{O}_{i,t}^{g,h}$ corresponds to the set of bids (i.e. individual bid function) submitted by bidder i' in group (g, h) in auction t' for a particular realisation of their signal. I draw $N^{g,h} - 1$ observations from $\mathcal{O}_{i,t}^{g,h}$ and $N^{g',h'}$ observations from $\mathcal{O}_{i,t}^{g',h'}, (g', h') \neq (g, h)$, with replacement. I calculate the equilibrium given the bids submitted by both the drawn bidders and the fixed bidder, and repeat this a large number of times for the fixed bidder. This yields an empirical distribution of auction prices, conditional on the bidder's own strategy.[35]

The resulting empirical distribution of auction prices for good g can be used to consistently estimate the expectation terms in Equation 2. The derivative term is estimated using a numerical derivative, as described in Kastl (2011).[36] Given these estimates, the marginal values at the quantities demanded at the steps of bidder i's bid function can be recovered iteratively, starting with the value at step $k = K_i^g$ (for which $q_{i,k+1} = 0$). I solve for $v^{g,h}(q_{i,k}, \theta_i)$ numerically.

Sample of other bidders The sample, $\mathcal{O}_{i,t}$, is a subset of all the bid functions submitted by the N potential bidders across the auctions. For the resampling procedure to yield a consistent estimator of bidder i's beliefs, two conditions on $\mathcal{O}_{i,t}$ must hold. First, the distribution of signals implied by $\mathcal{O}_{i,t}$ must be a consistent estimator for the true distribution of signals that bidder i faces in auction t. Second, the other bidders in $\mathcal{O}_{i,t}$ must face the same commonly known economic environment as bidder i faces in auction t and best respond to it.

In the main analysis, I define $\mathcal{O}_{i,t} = \{(i', t') : (i', t') \neq (i, t), t' \in Y_t\}$, where Y_t is the set of auctions including auction t and its two nearest neighbours of the same term, and stratify by

[35] The sample of other bidders is finite, so only a finite number of signal realisations is ever (implicitly) observed and the empirical distribution of auction prices may not have full support. This creates issues for calculating the conditional expectation terms and the derivative of the auction price. The former may not be identified and the latter may be numerically unstable. To address this, I use kernel density estimation to smooth the auction price distribution and ensure full support, discretised at integer prices.

[36] The quantity demanded by bidder i at step k is perturbed by $\epsilon > 0$, where ϵ is small, holding the rest of their strategy fixed. The auction price distribution is then recalculated for the bidder. The derivative term is the difference in expectations under the perturbed and unperturbed distributions, scaled by the size of the perturbation.

the grouping defined in Section 2.3.[37,38] This aims to maintain consistency in the economic environment and therefore signal distribution, while increasing precision by pooling bids from neighbouring auctions.

4.3 Testing procedure

Models $S(\rho)$ and T have different implications for bidders' marginal values and therefore for their bid shading. The testing procedure exploits these differences to compare the fit of the alternative models of behaviour.

Ideally we would compare the bid shading implied by each candidate model to the actual amount of bid shading. Naturally, the candidate model with the smaller prediction error would be the better approximation.

Because the actual amount of bid shading is unobserved, I follow Backus, Conlon and Sinkinson's (2021) approach, which uses instrumental variables to mimic this comparison. Bidders' true marginal values for liquidity are assumed to be mean independent of the instruments. Under this assumption, bidders' values are uncorrelated with the projection of their bid shading on the instruments (i.e. the variation in the bid shading explained by the variation in the instruments). A particular candidate model is therefore a better approximation if it generates a smaller covariance between its estimated values and the projection of its estimated bid shading. Broadly speaking, the testing procedure is based on the relative size of these covariances.

The test compares the fit of two competing models of behaviour. One advantage of this approach is that it remains valid even if the two models are misspecified because it only assesses their relative performance.[39] This is particularly relevant in the empirical auction literature, in which a model is often required in order to recover unknown parameters for

[37]The term to maturity of liquidity in each monthly auction follows a sequence. Two 3-month term auctions are followed by a 6-month term one. Anecdotal evidence suggests that some bidders participate in the ILTR auctions at regular 3- and 6-month intervals, with the intention to roll over the ILTR loan with the same piece of collateral. To eliminate the impact of correlated signals of this kind, I choose the two nearest neighbours, rather than three or more.

[38]Under the assumption of i.i.d. signals within group, as the sample becomes large, there is no reason to exclude observation (i, t) from the sample of other bidders, $\mathcal{O}_{i,t}$. Observation (i, t) represents the best response of one possible signal realisation of another bidder. However, with a finite sample including observation (i, t) in $\mathcal{O}_{i,t}$ artificially increases the estimated probability that bidder i ties on the margin in auction t, biasing the estimator of the bidder's beliefs. To address this, $(i, t) \notin \mathcal{O}_{i,t}$.

[39]In Appendix F, I analyse an external measure of fit of Model $S(0)$, by estimating lower bounds on the bidding costs implied by the model.

further analysis, but is necessarily a simplification of reality.[40] The question we ask is, among a set of candidate models of behaviour, which is the best approximation in our setting.

For a pair of models $m = \{1, 2\}$ and a measure of the model fit of each, $\mathcal{Q}_m$, the null hypothesis is that the two measures are equal, whereas the two alternatives are that the fit of one model is better than the other:

$$H_0 : \mathcal{Q}_1 = \mathcal{Q}_2$$
$$H_1 : \mathcal{Q}_1 < \mathcal{Q}_2,\ H_2 : \mathcal{Q}_1 > \mathcal{Q}_2$$

In our test, a larger $\mathcal{Q}_m$ implies a worse model fit. So H_1 implies that Model 1 is a better fit than Model 2, and H_2 is analogous for Model 2.

The test is based on the relationship between the model-implied marginal values and a function of a set of instruments, denoted $\boldsymbol{z}$, which are correlated with bidders' bid shading but uncorrelated with their true marginal values. One example of an instrument is the number of other bidders: with independent private values, the number of other bidders in the auction affects a bidder's market power but not their value. I define the true marginal value of bidder i at step k of their bid function in auction t by $v_{i,k,t}$, a function of a set of exogenous characteristics, $\boldsymbol{x}$, and an additively separable residual value, $\omega_{i,k,t}$. The critical assumption is that conditional on $\boldsymbol{x}$, the true residual value is mean independent of the instruments, $\boldsymbol{z}$, i.e. $\mathbb{E}\left[\omega_{i,k,t}|\boldsymbol{x}, \boldsymbol{z}\right] = 0$.

In each model, the bidder's marginal value is the sum of their observed bid and model-implied bid shading. Let $v^m_{i,k,t} = b_{i,k,t} + \mu^m_{i,k,t}$ be the marginal value of bidder i at step k in auction t under the assumptions of Model m, where $b_{i,k,t}$ is the bid price and $\mu^m_{i,k,t}$ is the bid shading. (The corresponding residual values implied by Model m are denoted $\omega^m_{i,k,t}$.) In Model T, $\mu^T_{i,k,t} = 0$ so the bidder's value is simply their bid, $v^T_{i,k,t} = b_{i,k,t}$. By contrast, bid shading is not necessarily zero in Model $S(\rho)$, and the model-implied value is $v^{S(\rho)}_{i,k,t} = b_{i,k,t} + \mu^{S(\rho)}_{i,k,t}$.

To provide intuition for the test, I first show how an incorrect model is falsified in a simple example in which it is compared to the correct one. For simplicity let $\boldsymbol{x}$ be a constant so that we can ignore the distinction between $v_{i,k,t}$ and $\omega_{i,k,t}$ in this example. Suppose

[40] For example, in the risk-neutral case, I compare Model $S(0)$, which assumes bidders have correct beliefs about the distribution of other bidders' bids, to Model T, in which bidders bid truthfully regardless of their beliefs. If neither model is correct, and the correct model is a version of Model $S(0)$, in which bidders have incorrect beliefs (or do not know the distribution of bids and their priors are incorrect), then the procedure tests whether Model T or Model $S(\rho)$ is a better approximation. For example, if bidders believe they face much stronger competition than in reality, Model T might be the better approximation.

that Model T is correct and Model $S(0)$ is incorrect (but the reasoning holds for each pair of models). The bidder's true marginal value is therefore equal to the marginal value implied by Model T, $v_{i,k,t} = v^T_{i,k,t} = b_{i,k,t}$. Moreover, because their true value is equal to their bid, the marginal value implied by Model $S(0)$ is the sum of the true marginal value and the bid shading, i.e. $v^{S(0)}_{i,k,t} = b_{i,k,t} + \mu^{S(0)}_{i,k,t} = v_{i,k,t} + \mu^{S(0)}_{i,k,t}$. So the covariance between the model-implied marginal valuations and a valid instrument, e.g. $z_{i,k,t}$, will be zero for Model T and non-zero for Model $S(0)$: $\text{Cov}(v^T_{i,k,t}, z_{i,k,t}) = \text{Cov}(v_{i,k,t}, z_{i,k,t}) = 0$ and $\text{Cov}(v^{S(0)}_{i,k,t}, z_{i,k,t}) = \text{Cov}(b_{i,k,t}, z_{i,k,t}) + \text{Cov}(\mu^{S(0)}_{i,k,t}, z_{i,k,t}) = \text{Cov}(v_{i,k,t}, z_{i,k,t}) + \text{Cov}(\mu^{S(0)}_{i,k,t}, z_{i,k,t}) = \text{Cov}(\mu^{S(0)}_{i,k,t}, z_{i,k,t}) \neq 0$.[41] The crux of the test is that the correct model would have zero covariance and a larger covariance implies a worse model fit. In general, neither model will be correct and, broadly speaking, the test compares the relative magnitude of the covariances.

The steps of the test modified from Backus, Conlon and Sinkinson (2021) are as follows.

1. For the two models $m = \{1, 2\}$,
   (a) For each step k of bidder i's bid function in auction t, estimate the bid shading, $\mu^m_{i,k,t}$, and calculate the implied values, $v^m_{i,k,t} = b_{i,k,t} + \mu^m_{i,k,t}$.
   (b) Estimate $v^m_{i,k,t}$ as a function, $h(\boldsymbol{x})$, of the set of exogenous characteristics $\boldsymbol{x}$, which are specified below, and obtain the residuals, $\hat{\omega}^m_{i,k,t}$, where $v^m_{i,k,t} = h(\boldsymbol{x}) + \omega^m_{i,k,t}$.
2. Estimate the difference in bid shading between the models as a function of $\boldsymbol{x}$ and the instruments, $\boldsymbol{z}$, i.e. $\Delta\mu_{i,k,t} = \mu^1_{i,k,t} - \mu^2_{i,k,t} = g(\boldsymbol{x}, \boldsymbol{z}) + \zeta_{i,k,t}$, where the instruments, $\boldsymbol{z}$, and the function, $g(.)$, are specified below. Obtain the predictions, $\Delta\hat{\mu}_{i,k,t}$.[42]
3. For each model, compute the value of the moment, $\hat{Q}^m = \left(\frac{1}{n}\sum_{i,k,t} \hat{\omega}^m_{i,k,t} \Delta\hat{\mu}_{i,k,t}\right)^2$, where n is the number of observations.
4. Estimate the standard error, $\frac{\hat{\sigma}}{\sqrt{n}}$, of the difference between the moments, $(\hat{Q}^1 - \hat{Q}^2)$, by repeating Steps 1–3 on bootstrapped samples.[43]

[41] It is straightforward to see the converse. Suppose that Model $S(0)$ is correct and Model T is incorrect. The bidder's true marginal value is therefore equal to the marginal value implied by Model $S(0)$, $v_{i,k,t} = v^{S(0)}_{i,k,t} = b_{i,k,t} + \mu^{S(0)}_{i,k,t}$. So the marginal value implied by Model T is $v^T_{i,k,t} = b_{i,k,t} = v^{S(0)}_{i,k,t} - \mu^{S(0)}_{i,k,t} = v_{i,k,t} - \mu^{S(0)}_{i,k,t}$, and the covariance between the model-implied marginal values and a valid instrument, $z_{i,k,t}$, will be zero for Model $S(0)$ and non-zero for Model T: $\text{Cov}(v^{S(0)}_{i,k,t}, z_{i,k,t}) = \text{Cov}(v_{i,k,t}, z_{i,k,t}) = 0$ and $\text{Cov}(v^T_{i,k,t}, z_{i,k,t}) = -\text{Cov}(\mu^{S(0)}_{i,k,t}, z_{i,k,t}) \neq 0$.

[42] This specification is chosen to maximise the power of the test (see Backus, Conlon and Sinkinson (2021) for details).

[43] I follow Backus, Conlon and Sinkinson (2021) and Roussille and Scuderi (2022) and treat the bid shading estimates for each model as data.

5. Compute the test statistic, $T = \frac{\sqrt{n}(\hat{Q}^1 - \hat{Q}^2)}{\hat{\sigma}}$, which is distributed $\mathcal{N}(0,1)$.

The set of exogenous characteristics, $\boldsymbol{x}$, which affect both bidders' values and their bid shading, includes auction-specific dummy variables, which control for heterogeneity in bidders' values across time, and a dummy variable equal to one if the bidder is large (as defined in Section 2.3.[44] The set $\boldsymbol{x}$ also includes a dummy variable equal to one if the bid corresponds to the last step in a bidder's bid function; this is intended to control for variation in bidders' values at the bid-bidder-auction level (our unit of observation), conditional on the bidder's size and on the auction.[45,46]

For the set of instruments, $\boldsymbol{z}$, I use measures of the strength of competition that a bidder faces in the auction (listed below). The main assumption for these instruments to satisfy the exogeneity condition, i.e. $\mathbb{E}[\omega_{i,k,t}|\boldsymbol{x}, \boldsymbol{z}] = 0$, is that bidders have independent private values (the IPV Assumption). Section 3.1 gives three reasons for why this assumption seems reasonable in my setting. For the instruments to be relevant, they must be correlated with the difference in bid shading between the competing models. Because one of the models is the conventional strategic model in which bidders are assumed to have consistent beliefs about the distribution of other bidders' bids, measures of the strength of competition are also relevant instruments.

There are many potential ways to measure the strength of competition. For a given bidder, this includes the number of bidders, or small or large bidders, for each good in the auction excluding the bidder themself (where size is defined in Section 2.3); the probability distribution of the auction price for each good that the bidder expects would occur if they did not bid at all, and moments of this distribution; and moments of the distribution of bids excluding the bidder's own bids. The IPV Assumption ensures that these statistics, which depend on other bidders' bids and their entry decisions, are valid instruments because bidder i's own valuation is mean independent of them.

Specifically, for bidder i in auction t, the set of instruments, which satisfy the exogeneity condition under the IPV Assumption, includes:

[44] Good-specific dummies are included in $\boldsymbol{x}$ in the test which pools bids across goods.

[45] The quantities demanded at the bids are excluded from $\boldsymbol{x}$ because, if Model $S(\rho)$ is the correct model of behaviour, they are an equilibrium outcome and therefore endogenous to bidders' values. In principle, we could alternatively find additional instruments for the quantities demanded, or estimate bidders' marginal values at all quantities (not just the quantities demanded at the steps of the bidders' bid functions), which would require additional identifying conditions (see Appendix F).

[46] Appendix D repeats the analysis excluding the last-step dummy from $\boldsymbol{x}$.

- For goods $g = \{A, B\}$, number of bidders, number of small bidders, and number of large bidders, each excluding bidder i, in auction t[47]

- The average bid in auction t, excluding the bids of bidder i

- For $P^g_{i,t}$, defined by the clearing price of good g in auction t, which is calculated using the resampling procedure described in Section 4.2 but excluding bidder i's bids when calculating the equilibrium distribution of the auction price,
  - The conditional expected price for good A, $\mathbb{E}\left[P^A \leq X\right]$ for $X = [1, 5, 10]$
  - The conditional expected price for good B, $\mathbb{E}\left[P^B \leq X\right]$ for $X = [6, ..., 40]$
  - The conditional expected price differences for B: $(X - \mathbb{E}\left[X - 5 \leq P^B \leq X\right])$ for $X = [6, ..., 40]$ and $(X - \mathbb{E}\left[X - 10 \leq P^B \leq X\right])$ for $X = [6, ..., 40]$[48]

- All of the above set interacted with good-term-specific dummies

This implies a large number of potential instruments (equal to 460). Following Carrasco (2012), Conlon (2017) and Backus, Conlon and Sinkinson (2021), I therefore project this set of instruments onto their principal components and select the three leading principal components as the three instruments to be used in the estimation; these explain 80% of the variation in the original to include in $\boldsymbol{z}$. The relationship between the instruments—measures of competition—and bid shading, are non-linear (for Model $S(\rho)$, this is shown by Equation 1). To capture these non-linearities, the functions $h(.)$ and $g(.)$ are estimated using a random forest.

I restrict the set of observations to bids strictly above the reserve prices.[49] Observations are weighted by the quantities demanded to ensure that the results are not driven by the fact that small bidders might submit bids far below their marginal values in Model $S(\rho)$ because the model implies that they are constrained in the number of bids that they submit.

[47]The model is estimated using a random forest so multicollinearity is not an issue.

[48]The bounds of the conditional expectations represent sensible points on the distribution of bids (see Table 1). They capture the range of auction prices that are likely to occur.

[49]Bids at the reserve prices might correspond to bidders who would have optimally submitted bids below the reserve (corresponding to larger bid shading) in Model $S(\rho)$ if negative bids were allowed.

5 Results

5.1 Model fit

I run the test for pairs of models including Model T and Model $S(\rho)$ with different degrees of risk aversion. Results are reported in Table 3. The first set of results includes bidders for both goods. For these tests, Model $S(\rho)$ is estimated with a common rationing coefficient for all bidders (written as a single parameter, ρ, for shorthand). The second and third sets of results only include bidders for good A and bidders for good B, respectively. For these cases, the test results for one good do not depend on the risk aversion of bidders for the other good, so I only specify the relevant risk aversion parameter (also written as a single parameter, ρ, for shorthand).

Each entry in Table 3 shows the test statistic for the row being Model 1 and the column being Model 2, so that a negative entry indicates that Model 1 has better model fit than Model 2 (and a positive entry indicates the converse). The test statistic is distributed $\mathcal{N}(0, 1)$ so the standard critical values apply: -1.645 for a 5% confidence level; -1.960 for 2.5%; -2.326 for 1%.

The test statistic when comparing Model T (Model 1) to Model $S(0)$ (Model 2) when pooling the two goods is -1.982, so Model T outperforms at the 2.5% significance level.

Table 3 shows that this is driven by Model T providing significantly better fit than Model $S(0)$ for good B, but the models fitting equally well for good A. Moreover, Model T outperforms Model $S(\rho)$ for low levels of risk aversion and Model $S(\rho)$ fits the data better for higher values of ρ for good B, but the fit across models cannot be differentiated for good A. There are two possible explanations for these differences between goods.

First, the definition of the two goods suggests that bidders for good B could plausibly be more risk averse than bidders for good A. Good B corresponds to funds lent against less liquid collateral assets, which typically have fairly illiquid secondary markets. A bidder for good B, who does not win in the BoE's auctions may have limited opportunities to obtain the loan elsewhere, unlike bidders for good A, who typically face a liquid secondary market.

Second, we have less power to discriminate between the different models for good A. 85.1% of bids for good A are between 0 and 5 basis points (and have to be a whole basis point), and the auction price is rarely above 2 basis points. This means that there is less variation in the observed bids, and therefore estimated values. This lack of variation makes it difficult to discriminate between models.

Table 3: Testing results

Both goods							
Model 1	Model 2						
	$S(0)$	$S(0.1)$	$S(0.2)$	$S(0.3)$	$S(0.4)$	$S(0.5)$	$S(1)$
T	-1.982	-1.758	-1.662	-1.551	-1.487	-1.310	-1.085
$S(0)$		0.054	0.071	1.556	1.646	1.739	1.885
Good *A*							
Model 1	Model 2						
	$S(0)$	$S(0.1)$	$S(0.2)$	$S(0.3)$	$S(0.4)$	$S(0.5)$	$S(1)$
T	0.215	0.024	0.024	0.024	0.024	0.024	0.193
$S(0)$		0.014	0.011	0.009	0.006	-0.002	-0.040
Good *B*							
Model 1	Model 2						
	$S(0)$	$S(0.1)$	$S(0.2)$	$S(0.3)$	$S(0.4)$	$S(0.5)$	$S(1)$
T	-2.574	-2.318	-2.112	-2.101	-1.882	-1.770	-1.447
$S(0)$		0.279	1.359	1.790	2.011	2.394	2.574

Each entry shows the test statistic for the row being Model 1 and the column being Model 2, so that a negative entry indicates that the row model has better model fit than the column model (and a positive entry indicates the converse). The test statistic is distributed $\mathcal{N}(0, 1)$ so the standard critical values apply: -1.645 for a 5% confidence level; -1.960 for 2.5%; -2.326 for 1%.

The results for good B show that Model T outperforms Model $S(\rho)$ for low degrees of risk aversion, suggesting that truthful bidding is a better approximation than the conventional risk-neutral strategic model. Moreover, Model $S(\rho)$ with high degrees of risk aversion outperforms Model $S(0)$, so risk aversion improves the fit of the conventional model.

Appendix D shows that the test results are very similar when Model $S(\rho)$ is estimated under the assumption that bidders do not tie.

In Appendix F, I analyse an external measure of fit of Model $S(0)$. The measure is the size of the bidding costs required to explain the data under the assumptions of the model. Larger bidding costs suggest a worse model fit because the physical costs of submitting bids are trivial in the BoE's ILTR auctions.[50] I estimate a lower bound on the bidding costs by the marginal benefit of submitting an additional bid.[51] For good A, the estimated bidding costs

[50]Interpreting the costs as cognitive costs does not seem to improve the model fit (see Appendix F.1).

[51]For a bidder who submits K bids, the bidding cost of the $(K + 1)$th bid must be larger than the marginal benefit of "fine-tuning" their bid function, which is equal to the difference in their expected utility from submitting what would be the optimal $(K+1)$-step bid function and what would be the optimal K-step

are trivial, suggesting that the model characterises bidding behaviour well by this measure. This is not the case for good B, at least for the 67% of bidders for good B who submit only one bid. This complements the main analysis, which finds an improvement on Model $S(0)$.

5.2 Rationalising truthful bidding

The results of the pairwise tests show that if bidders are sufficiently risk averse, both Model T, i.e. truthful bidding, and Model $S(\rho)$, i.e. strategic bidding, fit the data equally well. This is because bidders' optimal strategies in both models are to bid approximately according to their true values if ρ is sufficiently high. Alternatively, if bidders have a low degree of risk aversion, Model T fits the data significantly better than Model $S(\rho)$. Moreover, truthful bidding is the most natural comparator to the conventional strategic model because bidders' optimal strategies in Model $S(\rho)$ may be to bid above or below their values. This suggests that a model in which bids correspond to bidders' true marginal values might be a reasonable approximation.

I consider two ways to rationalise this "truthful bidding". First, bidders are sufficiently risk averse that they bid truthfully even if they are best responding to correct beliefs about the economic environment and other participants' behaviour, i.e. the assumptions of Model $S(\rho)$ hold. Second, regardless of their degree of risk aversion, bidders face sufficiently high costs of calculating the optimal strategy that it is more profitable to avoid this cost by bidding in a different way—simply bidding according to one's value is a natural alternative.[52,53] I consider these two explanations in turn.

Risk aversion With a sufficiently high degree of risk aversion, Model $S(\rho)$ predicts that bidders will bid according to their true marginal values, i.e. Model $S(\rho)$ and Model T will predict the same behaviour. This explains the results of the pairwise test in Table 3. The difference in model fit between Model T and Model $S(\rho)$ is insignificant if $\rho \geq 1$ (i.e. $\log(\rho) \geq 0$) for good B, and for all ρ for good A, implying that the models explain the data

one. This marginal benefit of fine-tuning, which provides a lower bound on the bidding cost of the $(K+1)$th bid.

[52]In a more general model, a bidder might choose to avoid the cost of precisely calculating the optimal strategy by choosing an approximation to it. In Wilson's (1979) classical model, truthful bidding is optimal for small bidders anyway but larger bidders optimally submit a bid function below their marginal valuation function in order to exploit market power, in which case, a small amount of bid shading might be a better approximation than truthful bidding. This is not the case in Model $S(\rho)$, in which the optimal amount of bid shading can be positive or negative.

[53]The two explanations are not mutually exclusive. The cost of calculating the optimal strategy required to rationalise truthful bidding is decreasing in the degree of risk aversion.

equally well.

There are two potential reasons why risk aversion may be a suitable assumption in the BoE's liquidity auctions. First, the incentives of the manager tasked with bidding on behalf of the participant may lead to risk-averse behaviour within a principal-agent framework.[54] Second, the financial institutions themselves may be risk averse, given the auctions are intended to provide liquidity insurance, suggesting bidders may be willing to pay higher prices to reduce the uncertainty in their allocations. Because the sample period turned out to be relatively stable and participants' bids did not appear to reflect acute liquidity needs, the principal-agent interpretation seems more appropriate.

To examine the degree of risk aversion required for bidders to bid truthfully in Model $S(\rho)$, Figure 3 plots the absolute value of bid shading implied by Model $S(\rho)$ against the log of the risk aversion parameter, $\log(\rho)$.[55] It shows that almost all bidders bid approximately truthfully in Model $S(\rho)$ if $\log(\rho) \geq 4$. In this range, the mean and 90th percentile of absolute bid shading for good B (weighted by quantity demanded) are less than 0.43 and 1 basis point, respectively. The mean and 90th percentile for good A are both less than 0.15 basis points.

To put this in context, a bidder in the 3-month auctions with $\log(\rho) = 4$ facing a 50-50 bet to lose £100,000 or gain X would accept if $X >$ £100,109. (A bidder in the 6-month auctions would accept if $X >$ £100,209.)[56] This suggests that a fairly low degree of risk aversion is needed for truthful bidding to be optimal in the ILTR auctions.

The degree of risk aversion that is sufficient for truthful bidding to be approximately optimal in Model $S(\rho)$ is consistent with the degree of risk aversion found for bidders in Treasury auctions, which are the closest settings to my study in which risk aversion has been estimated. Armantier and Sbaï (2006) estimate CARA parameters for small and large bidders in French Treasury auctions, which are approximately equivalent to $\log(\rho) = 9.83$ and $\log(\rho) = 5.04$, respectively. In US Treasury auctions, Boyarchenko, Lucca and Veldkamp (2021) calibrate a parameter which is on average approximately equivalent to $\log(\rho) = 4.48$. Using any of these

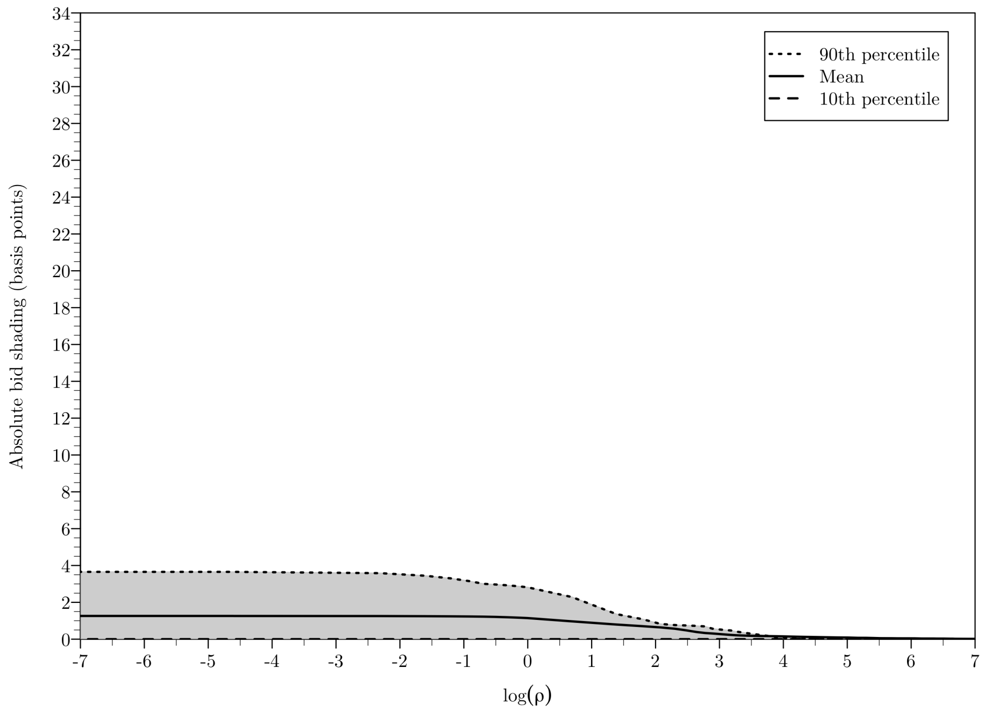
[54]For example, Gordy (1999) describes a manager tasked with bidding in the auction, who has a concave payoff function either because of their own risk preferences or because of the remuneration structure.

[55]The results show that a small amount of risk aversion has only a very small effect on optimal bid shading—bid shading is roughly constant as a function of ρ in the range $\log(\rho) < -2.5$ ($\rho < 0.082$). This is largely driven by the fact that the optimal amount of bid shading for all bidders depends on the value of a marginal unit (see Section B.1.1), and the rate at which this value declines as a function of ρ is increasing if ρ is small.

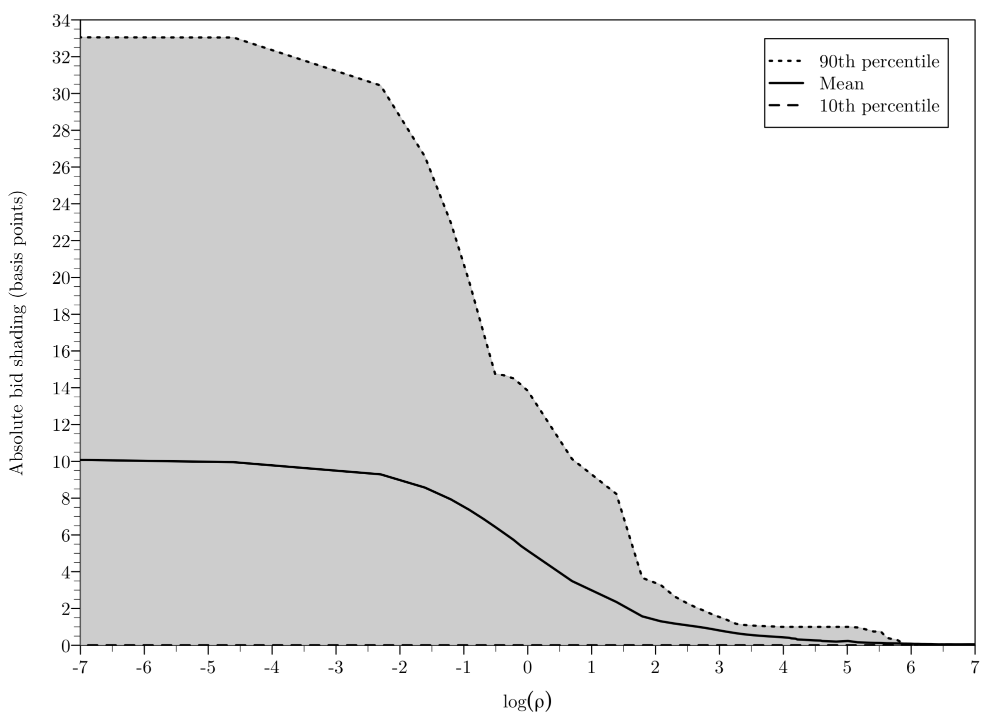
[56]Aggregate bidder surplus per auction, assuming bidders bid truthfully, is £220,461 in the 3-month auctions and £326,687 in the 6-month auctions on average (weighted by the quantities allocated per auction).

Figure 3: Absolute value of bid shading in Model $S(\rho)$ for $\log(\rho) \in [-10, 10]$, June 2010 – May 2012 (observations are weighted by the quantities demanded)

(a) Good A



(b) Good B



values in Model $S(\rho)$ in my setting would suggest that truthful bidding can be rationalised by the fact that bidders are risk averse.

However, the degree of risk aversion estimated in auctions varies across studies. When focusing exclusively on primary dealers in Canadian Treasury auctions, Allen and Wittwer (2023) estimate a much smaller degree of risk aversion, approximately equivalent to $\log(\rho) = -4.57$. Unlike in Treasury auctions, bidders are not designated a "primary dealer" status in the BoE's liquidity auctions, and I do not have the necessary data to condition the risk aversion parameter on other bidder characteristics. However, it is plausible that risk aversion varies across bidders in my context. While my results suggest that risk aversion can explain truthful bidding when setting a common parameter ρ for all bidders, it may not be a sufficient explanation for individual bidders.

Appendix E provides the calculations for the results in this section.

Cost of sophistication An alternative way to explain truthful bidding is that a more sophisticated strategy is complicated and calculating it requires too many resources to be worthwhile. If the cost of calculating what would otherwise be the optimal strategy, i.e. the "cost of sophistication", is larger than the difference in expected utility between this strategy and bidding according to the bidder's true value for liquidity, the bidder prefers to bid truthfully. Hortaçsu and Puller (2008) find that a cost of sophistication best explains the fact that smaller bidders in Texan electricity markets persistently deviate from profit-maximising behaviour.

This interpretation of truthful bidding corresponds to Swinkels (2001) and Chakraborty and Englebrecht-Wiggans (2006), which show that the loss from bidding truthfully rather than strategically becomes arbitrarily small as the number of bidders increases in discrete multi-unit auctions. In these studies, truthful bidding is an ε-equilibrium, in which the "ε" can be interpreted as the cost of determining what otherwise would be the optimal strategy. If the incremental profit from this strategy is small, only a small cost of calculating it is required for truthful bidding to be a best response.

In the ILTR auctions, I estimate the incremental profit from what otherwise would be the optimal strategy for each bidder in the absence of a cost of sophistication, assuming bidders bid truthfully. If a bidder chooses to bids truthfully, the cost must be larger than the incremental profit. The estimate therefore provides a lower bound on the cost of sophistication that can explain why bidders bid truthfully.

Specifically, I compare the expected surplus for each bidder from submitting the optimal bid function with two steps, to submitting a bid function which corresponds to their true marginal valuation function, under the assumption that bidders bid truthfully and are risk neutral.[57,58] The expected surplus from the optimal strategy would be larger if we estimated the optimal bid function with three or more steps rather than two (which would increase the lower bound), but it is likely that the difference would be small.[59]

I find that the average lower bound on the cost of sophistication is 4.28% of bidder surplus for bidders for good A and 5.11% for bidders for good B. This suggests that a relatively small cost of determining the optimal strategy can rationalise truthful bidding. Equivalently, if bidders' actual strategies are to bid truthfully, they obtain up to 95% of the surplus that would have been generated by the two-step optimal bid function. This suggests that a relatively small amount of surplus is 'left on the table' by bidding truthfully. By comparison, Hortaçsu and Puller (2008) find that bidders' actual strategies generate between 0% and 80% (excluding loss-making bidders) of that generated by the optimal response (based on the information available to the bidders at the time of bidding).

6 Conclusion

The BoE introduced its liquidity auctions both to efficiently provide liquidity insurance to financial institutions and to provide better information to the BoE about market conditions. The design—a Product-Mix Auction—is efficient if bids correspond to bidders' true marginal values for loans. Moreover, Paul Fisher (then Executive Director at the BoE) noted at the time that the pattern of bids gives a signal of market stress "because bids in the auctions should provide accurate information on individual banks' demand for liquidity and the prices they are willing to pay for it" (Fisher, 2011a pp.11). That is, bids were understood as

[57] I estimate the surpluses and optimal two-step bid function by estimating the distribution of other bidders' bids and implementing a grid search across all bid prices and quantity increments of 0.25% of the maximum supply. For bidders whose bid functions do not lie on this grid, I take the lower envelope of their bid functions as their "truthful" bid function (for bidders who submit more than two bids, I take the lower envelope of the highest two bids they submit). This is analogous to the approach described in Appendix F.

[58] Allowing for risk aversion would reduce the incremental profit from what otherwise would be the optimal strategy. This would reduce the estimated lower bound on the cost of sophistication that explains why bidders bid truthfully

[59] One reason is that Appendix F estimates the surpluses of optimal bid functions with different numbers of steps and finds that the benefit of a second step can be large whereas the benefit of a third step is small. While these benefits are estimated under slightly different assumptions, namely that the bidder is bidding optimally rather than truthfully, this suggests that the results would not change much if we estimated a three-step bid function. Moreover, Kastl (2012) shows that the loss from using a step function rather than a continuous bid function decreases at a quadratic rate, suggesting that the additional surplus from additional steps would also be small. Estimating the optimal bid function becomes very computationally intensive with many steps, so I only estimate the two-step optimal bid function.

expressing bidders' true valuations for loans.

If the bids were instead viewed through the model of behaviour which is more conventional in the literature—bidders choose to submit bids which may differ from their true values in order to maximise their own expected surplus—the measured efficiency of, and information gleaned from, the auctions would change.

I therefore compare the relative performance of these alternative models of behaviour. I find that bidding behaviour is better explained by a model of "truthful bidding" than by a conventional model in which bidders are both strategic and risk neutral, using a testing procedure based on model fit developed by Backus, Conlon and Sinkinson (2021). Moreover, I build on Kastl's (2011) framework to allow for risk aversion, and I find that the degree of risk aversion required for truthful bidding to be approximately optimal within the conventional model is consistent with that found in studies of risk aversion that are the most similar to my setting. Alternatively, truthful bidding can be rationalised by a cost of determining what otherwise would be the optimal strategy, and so can be interpreted as an ε-equilibrium as in Swinkels (2001) and Chakraborty and Englebrecht-Wiggans (2006).

Importantly, I have not confirmed that bidders do indeed bid truthfully in the BoE's liquidity auctions but only that this simple model of behaviour—truthful bidding—is an improvement on the conventional model. There may be alternative models that I have not considered that fit the data even better.

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Appendices

A The PMA design in the Bank of England's liquidity auctions

Figure 4: Equilibrium prices, (p_A, p_B), and allocations, (q_A, q_B), in the PMA.

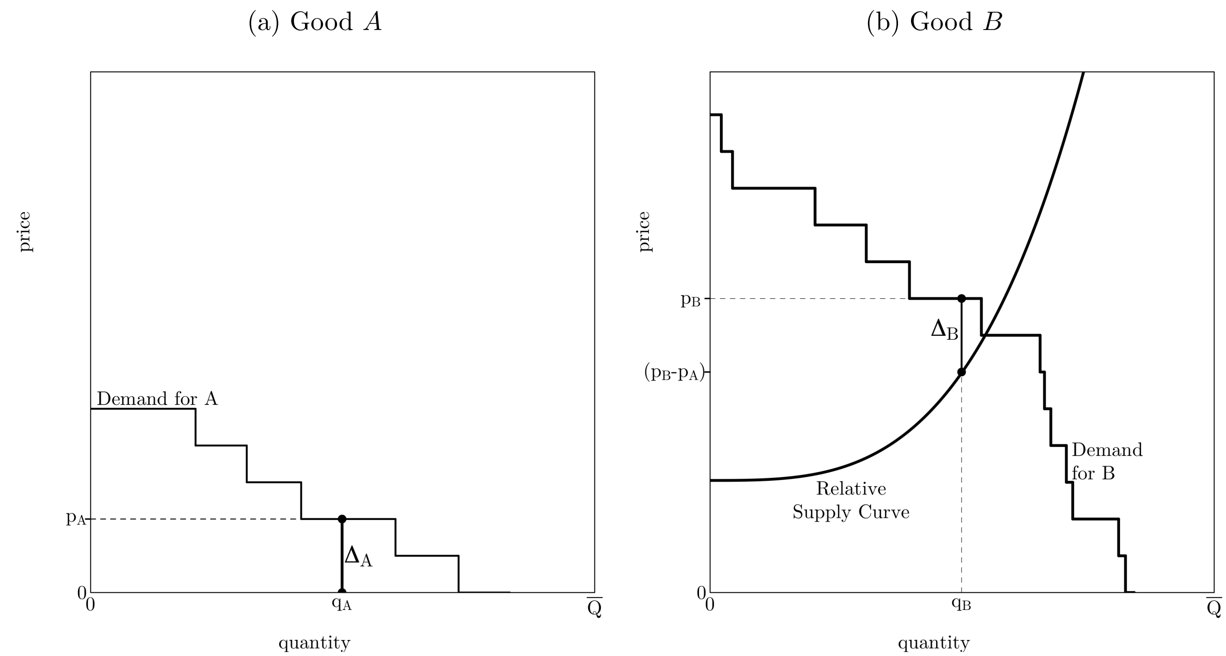


Figure 4 shows the equilibrium prices and allocations in the PMA for a pair of illustrative demand curves, where each curve is the sum of submitted bids for that good. The PMA finds the competitive equilibrium, assuming bids correspond to bidders' marginal values and the supply curves represent the prices the BoE is willing to accept. Given the demand curves ("Demand for A" and "Demand for B"),the BoE's maximum supply ($\bar{Q}$), and the BoE's "Relative Supply Curve", the PMA therefore finds the quantities, (q_A, q_B), such that the benefit of allocating a marginal unit of good A is equal to the benefit of allocating a marginal unit of good B, i.e. $\Delta_A = \Delta_B$. At this allocation, the auction prices are the highest losing bids for the two goods, i.e. (p_A, p_B).

Equivalently, the bidders' marginal willingness to pay for good B relative to good A (i.e. the difference in the auction prices, $(p_B - p_A)$) is equal to the price the BoE is willing to accept for good B relative to good A (i.e. the height of the relative supply curve at the equilibrium allocation).

Figure 5: New equilibrium prices, (p'_A, p'_B), and allocations, (q'_A, q'_B), in the PMA.

(a) Good A (b) Good B

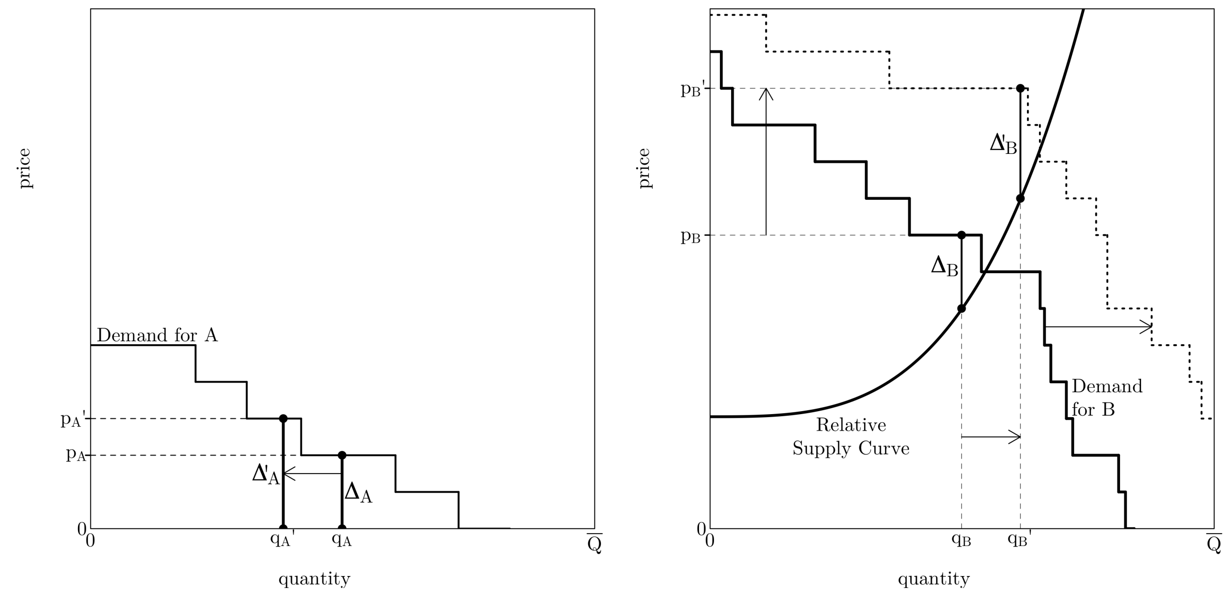


Figure 5 shows how the PMA automatically adjusts the allocations to find the competitive equilibrium when demand (or supply) changes. Suppose the demand for good B shifts out from "Demand for B" to the dotted line. The equilibrium allocation of good B increases from q_B to q'_B, and the allocation of good A correspondingly decreases from q_A to q'_A, so that the benefit of allocating a marginal unit to each good remains equal, i.e. $\Delta'_A = \Delta'_B$. The equilibrium prices also adjust to p'_B and p'_A to implement this allocation. The relative price of good B has increased, from $(p_B - p_A)$ to $(p'_B - p'_A)$, because the relative demand for good B has increased.

B Further details for Proposition 1

B.1 A more detailed explanation of Proposition 1

To understand Proposition 1 and the implications of the degree of risk aversion for bid shading in equilibrium of Model $S(\rho)$, it is helpful to consider the special case in which there are no ties on the margin for bidder i. In this case, Equation 1 implies bid shading of

$$\mu_{i,k} = \frac{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}} P^g \middle| b_{i,k} > P^g > b_{i,k+1}\right]}{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}} \middle| b_{i,k} > P^g > b_{i,k+1}\right]} - b_{i,k}$$
$$+ \begin{cases} \Omega q_{i,k} \frac{\partial}{\partial \epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\tilde{P}^g(\epsilon)\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g = 0 \\ \Omega \frac{\frac{\partial}{\partial \epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g \tilde{P}^g(\epsilon) q_{i,k}}\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0}}{\rho^g \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}} \middle| b_{i,k} > P^g > b_{i,k+1}\right]} & \text{if } \rho^g > 0 \end{cases} \tag{3}$$

where $\Omega = \frac{1}{\mathbb{P}(b_{i,k} > P^g > b_{i,k+1})}$ (see Appendix B.2.1).

To interpret this bid shading, I consider bidders who take the distribution of the auction price as given and those with market power in turn.

B.1.1 Price-taking bidders

In Model $S(\rho)$, if a bidder is sufficiently small relative to the market, marginally increasing the quantity that they demand at step k has no impact on the auction price (the last term of the right-hand side of Equation 3 is zero).

In the case of no ties on the margin, the deviation only changes their expected utility if the market clears at a price strictly below their bid price at step k, $b_{i,k}$, but strictly above their bid price at step $k+1$, $b_{i,k+1}$. In this range of auction prices, the bidder wins precisely the cumulative quantity that they demand at step k. The change in their expected utility from deviating is the expected utility of winning the marginal unit, conditional on the auction price being in this range.[60]

If the bidder is risk neutral, they demand a quantity at step k which equates their marginal value for the good with the conditional expected auction price, i.e. $v^{g,h}(q_{i,k}) = \mathbb{E}\left[P^g \middle| b_{i,k} > P^g > b_{i,k+1}\right]$. This trades off the possibility of winning the unit at a price at which the bidder

[60]The discussion is more complex with ties (see Kastl, 2011). If the price is strictly between $b_{i,k+1}$ and $b_{i,k}$, the logic of the no ties case is identical. If, instead, the price equals the bidder's bid at step k, they are rationed, and their deviation only changes their allocation if another bidder is *also* rationed at step k, i.e. there is a "tie". (If the bidder's bid is the only one at price $b_{i,k}$, they are allocated the entire quantity supplied on the margin so their allocation is independent of the quantity that they demand.) Similarly, if the price equals their bid at step $k+1$, they are rationed and their deviation increases the quantity they are allocated only in the event of a tie, because the deviation increases the quantity they demand strictly above the auction price and correspondingly reduces the quantity they demand on the margin.

would like to win, $P^g \leq v^{g,h}(q_{i,k})$, with winning at a price at which they would prefer to lose, $v^{g,h}(q_{i,k}) < P^g$. In equilibrium, their demand for the marginal unit is equivalent to accepting a lottery with an expected payoff of zero. This implies that they bid above their marginal value at the quantity they demand at step k so their bid shading is negative, $\mu_{i,k} < 0$.[61]

If the bidder is instead risk averse, they dislike the possibility of a negative payoff and are only willing to accept the lottery if its expected payoff is sufficiently high. Risk aversion leads the bidder to reduce the difference between the bid price they submit and their marginal value.

With sufficiently high risk aversion, any lottery with the possibility of a negative payoff is undesirable. The bidder therefore demands the maximum quantity at bid price $b_{i,k}$ for which the payoff from winning the marginal unit is non-negative. If the bidder may tie on the margin (so that $\mathbb{P}(P^g = b_{i,k} \wedge Tie^g) > 0$), this implies that the bidder will bid truthfully, because the bidder's payoff from winning the marginal unit is only non-negative if their bid is weakly below their marginal value at the quantity they demand. So, in the case of ties on the margin, the bidder will not shade their bid at all.

The case of no ties on the margin is more complex. When their bid is marginal, the quantity that the bidder demands does not determine their allocation so that a marginal increase in their demand has no impact on their utility. This means that the bidder's payoff from winning the marginal unit is non-negative when their bid is marginal, as well as when their bid is weakly below their marginal value. When the bidder is sufficiently risk averse, they demand the maximum quantity at bid price $b_{i,k}$ for which the payoff from winning the marginal unit is non-negative. With a continuous bid function, this implies that they will bid one basis point above their value (assuming the auction price distribution has full support). That is, the bidder's bid shading is -1.

B.1.2 Bidders with market power

More generally, a bidder has market power—the distribution of the auction price depends on the bids that they submit. Demanding a larger quantity weakly increases the price, which is paid for all units that they win.

[61] Kastl (2011) discusses this somewhat surprising implication, which arises from the bidders' restricted strategy sets in Model $S(0)$. In contrast, a price-taking bidder would bid according to their marginal valuation function in Wilson's (1979) classical model in which bidders submit continuous bid functions.

The last term of the right-hand side of Equation 3 captures the effect of this market power on the profitability of a marginal increase in the quantity demanded by bidder i at step k in Model $S(\rho)$. Holding the rest of their strategy constant, the deviation increases aggregate demand at prices between $b_{i,k}$ and $b_{i,k+1}$, which weakly increases auction prices in this range.

To isolate the effects of risk aversion, it is helpful to approximate the impact of the deviation by a locally differentiable function.[62] Doing so, we can rewrite the last term of Equation 3 for $\rho^g \geq 0$ as:[63]

$$q_{i,k}\Omega\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{\partial P^g}{\partial q_{i,k}}\mathbb{I}\left(b_{i,k} > P^g \geq b_{i,k+1}\right)\right] + C + D \tag{4}$$

where

$$\begin{aligned} C &= q_{i,k}\Omega\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{\partial P^g}{\partial q_{i,k}}\mathbb{I}\left(b_{i,k} > P^g \geq b_{i,k+1}\right)\right]\left(\frac{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}}\middle|b_{i,k} > P^g \geq b_{i,k+1}\right]}{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}}\middle|b_{i,k} > P^g > b_{i,k+1}\right]} - 1\right) \\ D &= \frac{q_{i,k}\Omega Cov\left(e^{\rho^g P^g q_{i,k}}, \frac{\partial P^g}{\partial q_{i,k}}\middle|b_{i,k} > P^g \geq b_{i,k+1}\right)\mathbb{P}\left(b_{i,k} > P^g \geq b_{i,k+1}\right)}{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}}\middle|b_{i,k} > P^g > b_{i,k+1}\right]} \end{aligned} \tag{5}$$

See Appendix B.2.2.

If the bidder is risk neutral, $C = D = 0$. In this case, market power creates an incentive for bid shading that corresponds to the quantity-shading incentive of an oligopolist facing uncertain demand in Klemperer and Meyer (1989). In my setting, the bidder has an incentive to bid for a quantity below the true amount that they demand at each price, in order to lower the auction price and reduce the total amount they must pay. The incentive to do so is increasing in the quantity that they demand.

If the bidder is risk averse, $\rho^g > 0$ so $C \leq 0$ and the sign of D depends on their beliefs over the distribution of the auction price.

The term C accounts for the fact that the marginal utility of a risk averse bidder is decreasing; they place greater weight on low-utility outcomes than they do on high-utility outcomes. In particular, they place disproportionate weight on the outcome where the original auction

[62] This is only an approximation because the bid functions of other bidders are also step functions.

[63] This uses the facts that $\mathbb{I}(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}) = \mathbb{I}\left(b_{i,k} \geq P^g \geq b_{i,k+1}\right)$ and that $\frac{\partial P^g}{\partial q_{i,k}} = 0$ for $P^g = b_{i,k}$ (because the bidder's deviation does not change the probability of $P^g \in [b_{i,k+1}, b_{i,k}]$ and the auction price is an increasing function of $q_{i,k}$) so the upper bound in the indicator function can be written strictly.

price equals their bid at step $k+1$ because their deviation would unambiguously reduce their utility: it would not change their allocation (assuming no ties) but might increase the auction price.[64] As risk aversion increases, $C \to -q_{i,k}\Omega\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{\partial P^g}{\partial q_{i,k}}\mathbb{I}\left(b_{i,k} > P^g \geq b_{i,k+1}\right)\right]$, cancelling out the incentive of the risk neutral bidder to exploit market power (the first term of Equation 4), so that, if $D = 0$, the bidder does not shade their bid in order to exploit their market power.

The term D is explained by the fact that a risk averse bidder prefers a less dispersed distribution of utility. If their market power is weaker at higher prices (so that $D < 0$), a reduction in the quantity that they demand reduces higher prices by less than it reduces lower prices. This deviation therefore increases the dispersion of the auction price distribution, and consequently increases the dispersion of the distribution of their utility. This weakens the incentive of the bidder to exert market power and reduces their bid shading at step k, relative to a comparable risk neutral bidder. Conversely, if their market power is stronger at higher prices, reducing the quantity they demand reduces the dispersion of the distribution of their utility, increasing their bid shading *ceteris paribus*.

The relationship between the auction price, P^g, and its derivative, $\frac{\partial P^g}{\partial q_{i,k}}$, depends on the bidder's beliefs over the distribution of other bidders' signals, as well as on the bid price that they submit.[65] And so, at moderate levels of risk aversion, D may be positive or negative and the relationship between bid shading and risk aversion is an empirical matter. Nonetheless, as risk aversion increases, $D \to 0$ for the same reasons why $C \to -q_{i,k}\Omega\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{\partial P^g}{\partial q_{i,k}}\mathbb{I}\left(b_{i,k} > P^g \geq b_{i,k+1}\right)\right]$. Truthful bidding is therefore approximately optimal if the bidder is sufficiently risk averse.

B.2 Proof of Proposition 1

Kastl (2011) proves Proposition 1 for $\rho^g = 0$. I follow his approach for the case of $\rho^g > 0$.

With a slight abuse of notation, it is helpful to define $P^g(0)$ and $y_i(0)$ by the equilibrium auction price of good g and bidder i's equilibrium allocation, respectively, under the original strategy profile, $\boldsymbol{\sigma}(\boldsymbol{\Theta})$; and to define $P^g(\epsilon)$ and $y_i(\epsilon)$ by the equilibrium auction price of good g and bidder i's equilibrium allocation, respectively, if bidder i unilaterally deviates to

[64] In the knife-edge case that the bidder does not expect their bid at step $k+1$ to ever set the auction price, i.e. $\mathbb{P}(P^g = b_{i,k+1}) = 0$, then $C = 0$.

[65] The distribution of other bidders' signals implies a distribution of residual supply functions that the bidder faces. If the residual supply functions were vertical translations of one another and continuous, the relationship would be positive if the possible residual supply curves were convex (so $D > 0$), and negative if they were concave ($D < 0$).

a strategy in which their type θ_i demands a quantity $(q_{i,k} - \epsilon), \epsilon > 0$, at step k and the rest of the strategy profile, including the rest of their strategy, is unchanged.

For $\rho^g > 0$, we aim to evaluate the limit:

$$\lim_{\epsilon \to 0} \frac{\mathbb{E}_{\boldsymbol{\Theta_{-i}}} \left[\frac{1}{\rho^g} \left(1 - e^{-\rho^g U_i(y_i(\epsilon), P^g(\epsilon))} \right) \right] - \mathbb{E}_{\boldsymbol{\Theta_{-i}}} \left[\frac{1}{\rho^g} \left(1 - e^{-\rho^g U_i(y_i(0), P^g(0))} \right) \right]}{\epsilon} \tag{6}$$

where I write $U_i(.|\theta_i)$ as $U_i(.)$ for convenience.

Equation 1 divides the impact of bidder i's deviation into the impact on the quantity allocated to bidder i and the impact on the expected auction price conditional on winning. To do this, I follow Kastl's (2011) approach, which partitions the state space as follows

$$\begin{aligned}
\theta_{1k}(x) &= \{\boldsymbol{\Theta_{-i}} : b_{i,k+1} < P^g(x) \leq b_{i,k},\ y_i(x) = q_{i,k} - x\} \\
\theta_{2k}(x) &= \left\{\boldsymbol{\Theta_{-i}} : P^g(x) = b_{i,k},\ y_i(x) = y_i^{RAT}(q_{i,k} - x - q_{i,k-1}) : q_{i,k-1} < y_i^{RAT} < q_{i,k} - x\right\} \\
\theta_{3k}(x) &= \left\{\boldsymbol{\Theta_{-i}} : P^g(x) = b_{i,k+1},\ y_i(x) = y_i^{RAT}(q_{i,k+1} - q_{i,k} + x) : q_{i,k} - x < y_i^{RAT} < q_{i,k+1}\right\} \\
\theta_{4k}(x) &= \{\boldsymbol{\Theta_{-i}} : b_{i,k} < P^g(x),\ y_i(x) \leq q_{i,k-1}\} \\
\theta_{5k}(x) &= \{\boldsymbol{\Theta_{-i}} : P^g(x) < b_{i,k+1},\ q_{i,k+1} \leq y_i(x)\}
\end{aligned}$$

for $x \in \{0, \epsilon\}$, where $\epsilon > 0$ and $y_i^{RAT}(q)$ is the quantity that bidder i is allocated when rationed, given the quantity that i demands at the margin is q.

The set $\theta_{1k}(x)$ corresponds to the outcomes in which bidder i is allocated precisely the quantity they demand at step k; $\theta_{2k}(x)$ and $\theta_{3k}(x)$ correspond to the outcomes in which bidder i is rationed at step k and step $k+1$, respectively (the distinction between tying with another bidder or not when rationed is made later in the proof); $\theta_{4k}(x)$ and $\theta_{5k}(x)$ correspond to the outcomes in which bidder i is allocated strictly less and strictly more, respectively, than the quantity they demand at step k. We then define the following subsets of θ_{2k} and θ_{3k} as

$$\begin{aligned}
\omega_{1k}(\epsilon) &= \{\boldsymbol{\Theta_{-i}} : \boldsymbol{\Theta_{-i}} \in \theta_{2k}(0) \cap \theta_{1k}(\epsilon)\} \\
\omega_{2k}(\epsilon) &= \{\boldsymbol{\Theta_{-i}} : \boldsymbol{\Theta_{-i}} \in \theta_{2k}(0) \cap \theta_{3k}(\epsilon)\} \\
\omega_{3k}(\epsilon) &= \{\boldsymbol{\Theta_{-i}} : \boldsymbol{\Theta_{-i}} \in \theta_{1k}(0) \cap \theta_{3k}(\epsilon)\}
\end{aligned}$$

We have that $\sum_{j=1}^{3} \mathbb{P}\left(\theta_{jk}(0)\right) = \sum_{j=1}^{3} \mathbb{P}\left(\theta_{jk}(\epsilon)\right)$, because the deviation only changes bidder i's outcome in cases in which the auction price is weakly between their bid prices at steps k

and $k+1$. By the Law of Total Probability, we can evaluate the utility function piecewise using the definition of the partition:

$$\begin{aligned}
&\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i(\epsilon),P^g(\epsilon))}\right)\right]-\mathbb{E}_{\mathbf{\Theta}_{-i}}\left(\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i(0),P^g(0))}\right)\right]\\
=&\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k}-\epsilon,P^g(\epsilon))}\right);\theta_{1k}(\epsilon)\right]-\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(0))}\right);\theta_{1k}(0)\right]\\
&+\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k}-\epsilon-q_{i,k-1}),b_{i,k})}\right);\theta_{2k}(\epsilon)\right]\\
&-\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k}-q_{i,k-1}),b_{i,k})}\right);\theta_{2k}(0)\right]\\
&+\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k+1}-q_{i,k}+\epsilon),b_{i,k+1})}\right);\theta_{3k}(\epsilon)\right]\\
&-\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k+1}-q_{i,k}),b_{i,k+1})}\right);\theta_{3k}(0)\right]\\
=&\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k}-\epsilon,P^g(\epsilon))}\right);\theta_{1k}(0)\right]\\
&+\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k}-\epsilon-q_{i,k-1}),b_{i,k})}\right);\theta_{2k}(0)\right]\\
&-\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k}-q_{i,k-1}),b_{i,k})}\right);\theta_{2k}(0)\right]\\
&+\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k+1}-q_{i,k}+\epsilon),b_{i,k+1})}\right);\theta_{3k}(0)\right]\\
&-\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k+1}-q_{i,k}),b_{i,k+1})}\right);\theta_{3k}(0)\right]\\
&-\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(0))}\right);\theta_{1k}(0)\right]\\
&-\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k}-\epsilon-q_{i,k-1}),b_{i,k})}\right);\omega_{1k}(\epsilon)\right]\\
&-\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k}-\epsilon-q_{i,k-1}),b_{i,k})}\right);\omega_{2k}(\epsilon)\right]\\
&+\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k+1}-q_{i,k}+\epsilon),b_{i,k+1})}\right);\omega_{3k}(\epsilon)\right]\\
&+\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k+1}-q_{i,k}+\epsilon),b_{i,k+1})}\right);\omega_{2k}(\epsilon)\right]\\
&+\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k}-\epsilon,P^g(\epsilon))}\right);\omega_{1k}(\epsilon)\right]-\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k}-\epsilon,P^g(\epsilon))}\right);\omega_{3k}(\epsilon)\right]
\end{aligned} \tag{7}$$

where the last line follows from the facts that

$$\begin{aligned}
\mathbb{P}\left(\theta_{1k}(\epsilon)\right) &= \mathbb{P}\left(\theta_{1k}(0)\right) + \mathbb{P}\left(\omega_{1k}(\epsilon)\right) - \mathbb{P}\left(\omega_{3k}(\epsilon)\right) \\
\mathbb{P}\left(\theta_{2k}(\epsilon)\right) &= \mathbb{P}\left(\theta_{2k}(0)\right) - \mathbb{P}\left(\omega_{1k}(\epsilon)\right) - \mathbb{P}\left(\omega_{2k}(\epsilon)\right) \\
\mathbb{P}\left(\theta_{3k}(\epsilon)\right) &= \mathbb{P}\left(\theta_{3k}(0)\right) + \mathbb{P}\left(\omega_{3k}(\epsilon)\right) + \mathbb{P}\left(\omega_{2k}(\epsilon)\right)
\end{aligned}$$

Note that

$$\begin{aligned}
&\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\theta_{2k}(0)\right] - \mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\theta_{2k}(\epsilon)\right] \\
&\quad - \mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\omega_{1k}(\epsilon)\right] - \mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\omega_{2k}(\epsilon)\right] = 0
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\theta_{3k}(0)\right] - \mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\theta_{3k}(\epsilon)\right] \\
&\quad + \mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\omega_{3k}(\epsilon)\right] + \mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\omega_{2k}(\epsilon)\right] = 0
\end{aligned} \tag{9}$$

Summing Equations 7-9, adding and subtracting

$$\begin{aligned}
&\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\theta_{1k}(0)\right], \mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(0))}\right);\theta_{2k}(0)\right], \\
&\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(0))}\right);\theta_{3k}(0)\right], \mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},b_{i,k})}\right);\omega_{1k}(\epsilon)\right], \\
&\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},b_{i,k})}\right);\omega_{2k}(\epsilon)\right], \mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},b_{i,k+1})}\right);\omega_{3k}(\epsilon)\right], \\
&\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},b_{i,k+1})}\right);\omega_{2k}(\epsilon)\right],
\end{aligned}$$

and collecting terms, we therefore aim to evaluate

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i(\epsilon),P^g(\epsilon))}\right)\right]-\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left(\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i(0),P^g(0))}\right)\right]\right)\right)\\
=&\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k}-\epsilon,P^g(\epsilon))}\right);\theta_{1k}(0)\right]-\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\theta_{1k}(0)\right]\right)\right)\\
&+\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k}-\epsilon-q_{i,k-1}),b_{i,k})}\right);\theta_{2k}(0)\right]\right.\right.\\
&\left.\left.-\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k}-q_{i,k-1}),b_{i,k})}\right);\theta_{2k}(0)\right]\right)\right)\\
&+\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k+1}-q_{i,k}+\epsilon),b_{i,k+1})}\right);\theta_{3k}(0)\right]\right.\right.\\
&\left.\left.-\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k+1}-q_{i,k}),b_{i,k+1})}\right);\theta_{3k}(0)\right]\right)\right)\\
&+\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right)-\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(0))}\right);\bigcup_{j=1}^{3}\theta_{jk}(0)\right]\right)\right)\\
&+\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},b_{i,k})}\right);\omega_{1k}(\epsilon)\right]-\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k}-\epsilon-q_{i,k-1}),b_{i,k})}\right);\omega_{1k}(\epsilon)\right]\right.\right.\\
&+\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},b_{i,k})}\right);\omega_{2k}(\epsilon)\right]-\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k}-\epsilon-q_{i,k-1}),b_{i,k})}\right);\omega_{2k}(\epsilon)\right]\\
&+\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k+1}-q_{i,k}+\epsilon),b_{i,k+1})}\right);\omega_{3k}(\epsilon)\right]-\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},b_{i,k+1})}\right);\omega_{3k}(\epsilon)\right]\\
&+\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k+1}-q_{i,k}+\epsilon),b_{i,k+1})}\right);\omega_{2k}(\epsilon)\right]-\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},b_{i,k+1})}\right);\omega_{2k}(\epsilon)\right]\\
&+\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k}-\epsilon,P^g(\epsilon))}\right);\omega_{1k}(\epsilon)\right]-\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\omega_{1k}(\epsilon)\right]\\
&\left.\left.-\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k}-\epsilon,P^g(\epsilon))}\right);\omega_{3k}(\epsilon)\right]+\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\omega_{3k}(\epsilon)\right]\right)\right)
\end{aligned}
\tag{10}
$$

Recalling the definition of $U_i(y_i(x), P^g(x))$, we have that

$$
\frac{\partial U_i(y_i(x),P^g(x))}{y_i(x)} = v^{g,h}(y_i(x),\theta_i) - P^g(x)
$$

Applying L'Hôpital's rule, the first three limits are

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k}-\epsilon,P^g(\epsilon))}\right);\theta_{1k}(0)\right]-\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\theta_{1k}(0)\right]\right)$$

$$=\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[e^{-\rho^g U_i(q_{i,k},P^g(0))}\left(v^{g,h}(q_{i,k},\theta_i)-P^g(0)\right);\theta_{1k}(0)\right]$$

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k}-\epsilon-q_{i,k-1}),b_{i,k})}\right);\theta_{2k}(0)\right]\right.$$

$$\left.-\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k}-q_{i,k-1}),b_{i,k})}\right);\theta_{2k}(0)\right]\right)$$

$$=\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[e^{-\rho^g U_i(y_i^{RAT}(q_{i,k}-q_{i,k-1}),b_{i,k})}\left(v^{g,h}(y_i^{RAT}(q_{i,k}-q_{i,k-1}),\theta_i)-b_{i,k}\right);\theta_{2k}(0)\right]$$

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k+1}-q_{i,k}+\epsilon),b_{i,k+1})}\right);\theta_{3k}(0)\right]\right.$$

$$\left.-\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(y_i^{RAT}(q_{i,k+1}-q_{i,k}),b_{i,k+1})}\right);\theta_{3k}(0)\right]\right)$$

$$=\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[e^{-\rho^g U_i(y_i^{RAT}(q_{i,k+1}-q_{i,k}),b_{i,k+1})}\left(v^{g,h}(y_i^{RAT}(q_{i,k+1}-q_{i,k}),\theta_i)-b_{i,k+1}\right);\theta_{3k}(0)\right]$$

Now consider the fourth limit. We can show that the expectation term is continuous in ϵ by first partitioning $\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right);\bigcup_{j=1}^{3}\theta_{jk}(0)\right]$, into

$$\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right)\middle|\theta_{1k}(0)\right]\mathbb{P}\left(\theta_{1k}(0)\right)$$

$$+\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},b_{i,k})}\right)\middle|\theta_{2k}(0)\right]\mathbb{P}\left(\theta_{2k}(0)\right)+\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},b_{i,k+1}}\right)\middle|\theta_{3k}(0)\right]\mathbb{P}\left(\theta_{3k}(0)\right).$$

Because both $\left[P^g(\epsilon)\middle|\theta_{1k}(0)\right]\in[b_{i,k+1},b_{i,k}]$ (so that $\left[U_i(q_{i,k},P^g(\epsilon)))\middle|\theta_{1k}(0)\right]\geq 0$) and $\rho^g>0$, it follows that the exponential is bounded, i.e. $\left[e^{\rho^g U_i(q_{i,k},P^g(\epsilon))}\middle|\theta_{1k}(0)\right]\in[0,1]$. Therefore by the same reasoning as Kastl's (2011) Lemmas A3 and A4, $\mathbb{E}_{\mathbf{\Theta}_{-i}}\left[\frac{1}{\rho^g}\left(1-e^{-\rho^g U_i(q_{i,k},P^g(\epsilon))}\right)\middle|\theta_{1k}(0)\right]$ is continuous in ϵ at $\epsilon=0$ for a.e. $\theta_i\in\Theta_i$, is of bounded variation and satisifes the Luzin N property and so is locally differentiable with respect to ϵ at $\epsilon=0$ for a.e. $\theta_i\in\Theta_i$.

And so, applying L'Hôpital's rule to the fourth limit, we have

$$\lim_{\epsilon \to 0} \left(\frac{1}{\epsilon} \left(\mathbb{E}_{\mathbf{\Theta}_{-i}} \left[\frac{1}{\rho^g} \left(1 - e^{-\rho^g U_i(q_{i,k}, P^g(\epsilon))}\right) - \frac{1}{\rho^g} \left(1 - e^{-\rho^g U_i(q_{i,k}, P^g(0))}\right); \bigcup_{j=1}^{3} \theta_{jk}(0) \right] \right) \right)$$
$$= \frac{\partial}{\partial \epsilon} \left(\mathbb{E}_{\mathbf{\Theta}_{-i}} \left[\frac{1}{\rho^g} \left(1 - e^{-\rho^g U_i(q_{i,k}, P^g(\epsilon))}\right); \bigcup_{j=1}^{3} \theta_{jk}(0) \right] \right) \Bigg|_{\epsilon=0}$$

And finally, we can express the expectations in the last limit in terms of their conditional expectations, i.e.

$$\mathbb{E}_{\mathbf{\Theta}_{-i}} \left[\frac{1}{\rho^g} \left(1 - e^{-\rho^g U_i(y^g_{i,k}(x), P^g(x))}\right); \omega_{jk}(\epsilon) \right] = \mathbb{E}_{\mathbf{\Theta}_{-i}} \left[\frac{1}{\rho^g} \left(1 - e^{-\rho^g U_i(y^g_{i,k}(x), P^g(x))}\right) \Big| \omega_{jk}(\epsilon) \right] \mathbb{P}\left(\omega_{jk}(\epsilon)\right)$$

for $j \in \{1,2,3\}$ and note that $\lim_{\epsilon \to 0} \left(y_i^{RAT}(q_{i,k} - \epsilon - q_{i,k-1}) \Big| \omega_{jk}(\epsilon) \right) = q_{i,k}$; $\lim_{\epsilon \to 0} \left(y_i^{RAT}(q_{i,k+1} - q_{i,k-1} + \epsilon) \Big| \omega_{jk}(\epsilon) \right) = q_{i,k}$; $\lim_{\epsilon \to 0} \left(q_{i,k-1} - \epsilon \Big| \omega_{jk}(\epsilon) \right) = q_{i,k}$; and, because Kastl's (2011) Lemma 1 holds, $\lim_{\epsilon \to 0} \left(\mathbb{P}\left(\omega_{jk}(\epsilon)\right)\right) = 0$. It follows from L'Hôpital's rule that the last limit is zero (see Kastl (2011) for details).

Combining these results, recalling the definitions of the θ_{jk}s, observing both that $\mathbb{I}(b_{i,k} \geq P^g(\epsilon) \geq b_{i,k+1}) = \mathbb{I}\left(b_{i,k} \geq P^g(0) \geq b_{i,k+1}\right)$ and that the deviation only affects the quantity allocated to bidder i in the event of being rationed (i.e. in the sets θ_{2k} and θ_{3k}) if another bidder is also rationed (i.e. bidder i "ties" on the margin), the necessary condition for equilibrium, for $\rho^g > 0$, is the second case of Equation 1.

B.2.1 Proof for the case of no ties

In the case that there are no ties on the margin for bidder i, the condition which rules out profitable local deviations in the quantity that they demand is particularly simple. The bidder's gross utility, $\int_0^{q_{i,k}} v^{g,h}(u, \theta_i) du$, is deterministic, conditional on winning, as are their bidding costs, $c_i(K_i)$, so these elements do not enter into the bidder's tradeoff, even if they are risk averse. The only random component in their utility function is the price that they must pay for the units that they win. Simple manipulation as follows shows that Equation 1 simplifies to Equation 3 under these assumptions.

If there are no ties then $\mathbb{P}(P^g = b_{i,k+1} \wedge Tie^g) = \mathbb{P}(P^g = b_{i,k} \wedge Tie^g) = 0$, Equation 1 becomes

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{-\rho^g U_i(q_{i,k},P^g|\theta_i)}\left(v^{g,h}(q_{i,k},\theta_i) - P^g\right)\Big| b_{i,k} > P^g > b_{i,k+1}\right]\mathbb{P}(b_{i,k} > P^g > b_{i,k+1}) \\
&\quad = \begin{cases} q_{i,k}\frac{\partial}{\partial \epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[\tilde{P}^g(\epsilon)\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g = 0 \\ \frac{1}{\rho^g}\frac{\partial}{\partial \epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{-\rho^g U_i(q_{i,k},\tilde{P}^g(\epsilon)|\theta_i)}\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g > 0 \end{cases} \\
&\Longrightarrow e^{-\rho^g \int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du} e^{\rho^g c_i(K_i)}\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{\rho^g P^g q_{i,k}}\left(v^{g,h}(q_{i,k},\theta_i) - P^g\right)\Big| b_{i,k} > P^g > b_{i,k+1}\right]\mathbb{P}(b_{i,k} > P^g > b_{i,k+1}) \\
&\quad = \begin{cases} q_{i,k}\frac{\partial}{\partial \epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[\tilde{P}^g(\epsilon)\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g = 0 \\ e^{-\rho^g \int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du} e^{\rho^g c_i(K_i)}\frac{1}{\rho^g}\frac{\partial}{\partial \epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{\rho^g \tilde{P}^g(\epsilon) q_{i,k}}\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g > 0 \end{cases} \\
&\Longrightarrow v^{g,h}(q_{i,k},\theta_i)\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{\rho^g P^g q_{i,k}}\Big| b_{i,k} > P^g > b_{i,k+1}\right]\mathbb{P}(b_{i,k} > P^g > b_{i,k+1}) \\
&\qquad - \mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{\rho^g P^g q_{i,k}} P^g\Big| b_{i,k} > P^g > b_{i,k+1}\right]\mathbb{P}(b_{i,k} > P^g > b_{i,k+1}) \\
&\quad = \begin{cases} q_{i,k}\frac{\partial}{\partial \epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[\tilde{P}^g(\epsilon)\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g = 0 \\ \frac{1}{\rho^g}\frac{\partial}{\partial \epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{\rho^g \tilde{P}^g(\epsilon) q_{i,k}}\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g > 0 \end{cases} \\
&\Longrightarrow \mu_{i,k} = v^{g,h}(q_{i,k},\theta_i) - b_{i,k} \\
&\quad = \frac{\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{\rho^g P^g q_{i,k}} P^g\Big| b_{i,k} > P^g > b_{i,k+1}\right]}{\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{\rho^g P^g q_{i,k}}\Big| b_{i,k} > P^g > b_{i,k+1}\right]} - b_{i,k} \\
&\qquad + \begin{cases} \Omega q_{i,k}\frac{\partial}{\partial \epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[\tilde{P}^g(\epsilon)\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g = 0 \\ \Omega\frac{\frac{\partial}{\partial \epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{\rho^g \tilde{P}^g(\epsilon) q_{i,k}}\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0}}{\rho^g \mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{\rho^g P^g q_{i,k}}\Big| b_{i,k} > P^g > b_{i,k+1}\right]} & \text{if } \rho^g > 0 \end{cases}
\end{aligned}
$$

where $\Omega = \frac{1}{\mathbb{P}(b_{i,k} > P^g > b_{i,k+1})}$, which corresponds to Equation 3.

B.2.2 Proof for Equation 4, the market power component

The following shows that Equation 4 holds. The first line follows because $\frac{\partial P^g}{\partial q_{i,k}} = 0$ for $P^g = b_{i,k}$,

$$
\begin{aligned}
&q_{i,k}\Omega\frac{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}}\frac{\partial P^g}{\partial q_{i,k}}\mathbb{I}\left(b_{i,k} \geq P^g \geq b_{i,k+1}\right)\right]}{\mathbb{E}\left[e^{\rho^g P^g q_{i,k}}\middle|b_{i,k} > P^g > b_{i,k+1}\right]}\\
=&q_{i,k}\Omega\frac{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}}\frac{\partial P^g}{\partial q_{i,k}}\mathbb{I}\left(b_{i,k} > P^g \geq b_{i,k+1}\right)\right]}{\mathbb{E}\left[e^{\rho^g P^g q_{i,k}}\middle|b_{i,k} > P^g > b_{i,k+1}\right]}\\
=&q_{i,k}\Omega\frac{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}}\frac{\partial P^g}{\partial q_{i,k}}\middle|b_{i,k} > P^g \geq b_{i,k+1}\right]\mathbb{P}\left(b_{i,k} > P^g \geq b_{i,k+1}\right)}{\mathbb{E}\left[e^{\rho^g P^g q_{i,k}}\middle|b_{i,k} > P^g > b_{i,k+1}\right]}\\
=&q_{i,k}\Omega\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{\partial P^g}{\partial q_{i,k}}\middle|b_{i,k} > P^g \geq b_{i,k+1}\right]\mathbb{P}\left(b_{i,k} > P^g \geq b_{i,k+1}\right)\\
&+q_{i,k}\Omega\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{\partial P^g}{\partial q_{i,k}}\middle|b_{i,k} > P^g \geq b_{i,k+1}\right]\mathbb{P}\left(b_{i,k} > P^g \geq b_{i,k+1}\right)\left(\frac{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}}\middle|b_{i,k} > P^g \geq b_{i,k+1}\right]}{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}}\middle|b_{i,k} > P^g > b_{i,k+1}\right]}-1\right)\\
&+\frac{q_{i,k}\Omega Cov\left(e^{\rho^g P^g q_{i,k}},\frac{\partial P^g}{\partial q_{i,k}}\middle|b_{i,k} > P^g \geq b_{i,k+1}\right)\mathbb{P}\left(b_{i,k} > P^g \geq b_{i,k+1}\right)}{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}}\middle|b_{i,k} > P^g > b_{i,k+1}\right]}\\
=&q_{i,k}\Omega\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{\partial P^g}{\partial q_{i,k}}\mathbb{I}\left(b_{i,k} > P^g \geq b_{i,k+1}\right)\right]\\
&+q_{i,k}\Omega\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\frac{\partial P^g}{\partial q_{i,k}}\mathbb{I}\left(b_{i,k} > P^g \geq b_{i,k+1}\right)\right]\left(\frac{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}}\middle|b_{i,k} > P^g \geq b_{i,k+1}\right]}{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}}\middle|b_{i,k} > P^g > b_{i,k+1}\right]}-1\right)\\
&+\frac{q_{i,k}\Omega Cov\left(e^{\rho^g P^g q_{i,k}},\frac{\partial P^g}{\partial q_{i,k}}\middle|b_{i,k} > P^g \geq b_{i,k+1}\right)\mathbb{P}\left(b_{i,k} > P^g \geq b_{i,k+1}\right)}{\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[e^{\rho^g P^g q_{i,k}}\middle|b_{i,k} > P^g > b_{i,k+1}\right]}
\end{aligned}
$$

which corresponds to Equation 4.

C Further details on identification in Model $S(\rho)$

This section shows how Equation 2, which identifies bidders' marginal values, is derived from Equation 1, which is the necessary condition for equilibrium in Model $S(\rho)$.

The main complexity comes from allowing for ties. If both a bidder's bid equals the auction price and another bidder submits a bid at that same price, a marginal increase in the quantity the bidder demands at step k increases the quantity they are allocated by the product of the marginal quantity they demand at step k and the rationing coefficient.[66] The bidder's deviation in the quantity they demand therefore affects their utility differently in the case that their bid is marginal relative to the case that their bid is strictly above the auction price.

To estimate a bidder's marginal value at step k using Equation 1, we therefore require three further pieces of information beyond what is required in the case of no ties:

1. The rationing coefficient at step k, which determines the quantity the bidder wins when rationed at step k.
2. The bidder's marginal value at the quantity they win when rationed at step k, which is the marginal benefit of winning a additional unit at step k when the bidder is risk neutral, *ceteris paribus*.
3. The bidder's marginal value function between the quantity they win when rationed at step k and the quantity they demand at step k. When the bidder is risk averse, this impacts the bidder's overall net utility from the auction and therefore the marginal benefit of winning an additional unit, *ceteris paribus*.

Bidders beliefs about the size of the rationing coefficient can be estimated from the data (see Section 4.2). As described in Section 4.1, I make two additional assumptions to recover the two remaining pieces of information. Alternative natural assumptions would achieve the same result.

Assumption 1 *For bidder $i \in \mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$,*

$$v^{g,h}(y_i, \theta_i) = v^{g,h}(q_{i,k}, \theta_i) \;\; \forall y_i \in \Big[q_{i,k-1} + r_{i,k}(q_{i,k} - q_{i,k-1}), q_{i,k} + r_{i,k+1}(q_{i,k+1} - q_{i,k})\Big).$$

Assumption 2 *The rationing coefficient for type $\theta_i \in \Theta_i$ of bidder $i \in \mathcal{N}$ at step $k \in \{1, ..., K_i\}$, conditional on the strategy profile $\boldsymbol{\sigma}(\boldsymbol{\Theta})$, is deterministic and denoted $r_{i,k}$.*

[66] The bidder is allocated the entire quantity they demand at step $k-1$ as well as an amount which is proportional to the marginal quantity they demand at step k. If the bidder ties, the deviation increases their allocation by the product of the marginal quantity they demand at step k and the rationing coefficient because bids are rationed pro-rata. If the bidder does not tie, they are allocated the entire quantity supplied on the margin, so a deviation in the quantity they demand has no impact on their allocation.

Under Assumptions 1 and 2, Equation 1 simplifies as follows. First note that Assumption 2 implies that $\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[\frac{\partial y_i}{\partial q_{i,k}}\middle| P^g = b_{i,k+1} \wedge Tie^g\right] = (1 - r_{i,k})$. Splitting up $U_i(q_{i,k}, P^g|\theta_i)$ and $U_i(q_{i,k}, \tilde{P}^g(\epsilon)|\theta_i)$ into their constituent parts, we have

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{-\rho^g \int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du} e^{\rho^g P^g q_{i,k}} e^{-\rho^g c_i(K_i)} \left(v^{g,h}(q_{i,k},\theta_i) - P^g\right) \middle| b_{i,k} > P^g > b_{i,k+1}\right] \mathbb{P}(b_{i,k} > P^g > b_{i,k+1}) \\
&+ \mathbb{E}_{\boldsymbol{\Theta_{-i}}}\Bigg[e^{-\rho^g \int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du} e^{\rho^g \int_{(q_{i,k-1}+r_{i,k}(q_{i,k}-q_{i,k-1}))}^{q_{i,k}} v^{g,h}(u,\theta_i)du} e^{\rho^g b_{i,k}\left(q_{i,k-1}+r_{i,k}\left(q_{i,k}-q_{i,k-1}\right)\right)} e^{-\rho^g c_i(K_i)} \\
&\qquad\qquad \left(v^{g,h}\left(q_{i,k-1} + r_{i,k}\left(q_{i,k} - q_{i,k-1}\right), \theta_i\right) - b_{i,k}\right) r_{i,k}\Bigg] \mathbb{P}(P^g = b_{i,k} \wedge Tie^g) \\
&+ \mathbb{E}_{\boldsymbol{\Theta_{-i}}}\Bigg[e^{-\rho^g \int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du} e^{-\rho^g \int_{q_{i,k}}^{(q_{i,k}+r_{i,k+1}(q_{i,k+1}-q_{i,k}))} v^{g,h}(u,\theta_i)du} e^{\rho^g b_{i,k+1}\left(q_{i,k}+r_{i,k+1}\left(q_{i,k+1}-q_{i,k}\right)\right)} e^{-\rho^g c_i(K_i)} \\
&\qquad\qquad \left(v^{g,h}\left(q_{i,k} + r_{i,k+1}\left(q_{i,k+1} - q_{i,k}\right), \theta_i\right) - b_{i,k+1}\right)\left(1 - r^g_{i,k+1}\right)\Bigg] \mathbb{P}(P^g = b_{i,k+1} \wedge Tie^g) \\
&= \begin{cases} q_{i,k}\frac{\partial}{\partial\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[\tilde{P}^g(\epsilon)\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g = 0 \\ \frac{1}{\rho^g}\frac{\partial}{\partial\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{-\rho^g \int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du} e^{\rho^g \tilde{P}^g(\epsilon) q_{i,k}} e^{-\rho^g c_i(K_i)}\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g > 0 \end{cases}
\end{aligned}
$$

Then, cancelling the deterministic components of the bidder's expected utility,

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{\rho^g P^g q_{i,k}} \left(v^{g,h}(q_{i,k},\theta_i) - P^g\right) \middle| b_{i,k} > P^g > b_{i,k+1}\right] \mathbb{P}(b_{i,k} > P^g > b_{i,k+1}) \\
&+ \Bigg(e^{\rho^g \int_{(q_{i,k-1}+r_{i,k}(q_{i,k}-q_{i,k-1}))}^{q_{i,k}} v^{g,h}(u,\theta_i)du} e^{\rho^g b_{i,k}\left(q_{i,k-1}+r_{i,k}\left(q_{i,k}-q_{i,k-1}\right)\right)} \\
&\qquad\qquad \left(v^{g,h}\left(q_{i,k-1} + r_{i,k}\left(q_{i,k} - q_{i,k-1}\right), \theta_i\right) - b_{i,k}\right) r_{i,k}\Bigg) \mathbb{P}(P^g = b_{i,k} \wedge Tie^g) \\
&+ \Bigg(e^{-\rho^g \int_{q_{i,k}}^{(q_{i,k}+r_{i,k+1}(q_{i,k+1}-q_{i,k}))} v^{g,h}(u,\theta_i)du} e^{\rho^g b_{i,k+1}\left(q_{i,k}+r_{i,k+1}\left(q_{i,k+1}-q_{i,k}\right)\right)} \\
&\qquad\qquad \left(v^{g,h}\left(q_{i,k} + r_{i,k+1}\left(q_{i,k+1} - q_{i,k}\right), \theta_i\right) - b_{i,k+1}\right)\left(1 - r^g_{i,k+1}\right)\Bigg) \mathbb{P}(P^g = b_{i,k+1} \wedge Tie^g) \\
&= \begin{cases} q_{i,k}\frac{\partial}{\partial\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[\tilde{P}^g(\epsilon)\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g = 0 \\ \frac{1}{\rho^g}\frac{\partial}{\partial\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{\rho^g \tilde{P}^g(\epsilon) q_{i,k}} e\mathbb{I}\left(b_{i,k} \geq \tilde{P}^g(\epsilon) \geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g > 0 \end{cases}
\end{aligned}
$$

Finally, Assumption 1 on the shape of the marginal valuation function implies

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\Big[e^{\rho^g P^g q_{i,k}}\left(v^{g,h}(q_{i,k},\theta_i)-P^g\right)\Big|b_{i,k}>P^g>b_{i,k+1}\Big]\mathbb{P}(b_{i,k}>P^g>b_{i,k+1})\\
&+\Bigg(e^{\rho^g(1-r_{i,k})(q_{i,k}-q_{i,k-1})v^{g,h}(q_{i,k},\theta_i)}e^{\rho^g b_{i,k}\left(q_{i,k-1}+r_{i,k}\left(q_{i,k}-q_{i,k-1}\right)\right)}\\
&\qquad\qquad\left(v^{g,h}\left(q_{i,k},\theta_i\right)-b_{i,k}\right)r_{i,k}\Bigg)\mathbb{P}(P^g=b_{i,k}\wedge Tie^g)\\
&+\Bigg(e^{-\rho^g r_{i,k+1}(q_{i,k+1}-q_{i,k})v^{g,h}(q_{i,k},\theta_i)}e^{\rho^g b_{i,k+1}\left(q_{i,k}+r_{i,k+1}\left(q_{i,k+1}-q_{i,k}\right)\right)}\\
&\qquad\qquad\left(v^{g,h}\left(q_{i,k+1},\theta_i\right)-b_{i,k+1}\right)\left(1-r^g_{i,k+1}\right)\Bigg)\mathbb{P}(P^g=b_{i,k+1}\wedge Tie^g)\\
=&\begin{cases}q_{i,k}\frac{\partial}{\partial\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[\tilde{P}^g(\epsilon)\mathbb{I}\left(b_{i,k}\geq\tilde{P}^g(\epsilon)\geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g=0\\ \frac{1}{\rho^g}\frac{\partial}{\partial\epsilon}\left(\mathbb{E}_{\boldsymbol{\Theta_{-i}}}\left[e^{\rho^g\tilde{P}^g(\epsilon)q_{i,k}}e\mathbb{I}\left(b_{i,k}\geq\tilde{P}^g(\epsilon)\geq b_{i,k+1}\right)\right]\right)\Big|_{\epsilon=0} & \text{if } \rho^g>0\end{cases}
\end{aligned}$$

which corresponds to Equation 2.

D Robustness of the test results

Table 4 shows the test results for each pair of models, with the bid shading in Model $S(\rho)$ estimated under the assumption that bidders do not tie, i.e. estimated from Equation 3, and the testing specification identical to the main analysis. Figure 6 plots the absolute value of bid shading implied by Model $S(\rho)$ against the log of the risk aversion parameter, $\log(\rho)$, assuming bidders do not tie.

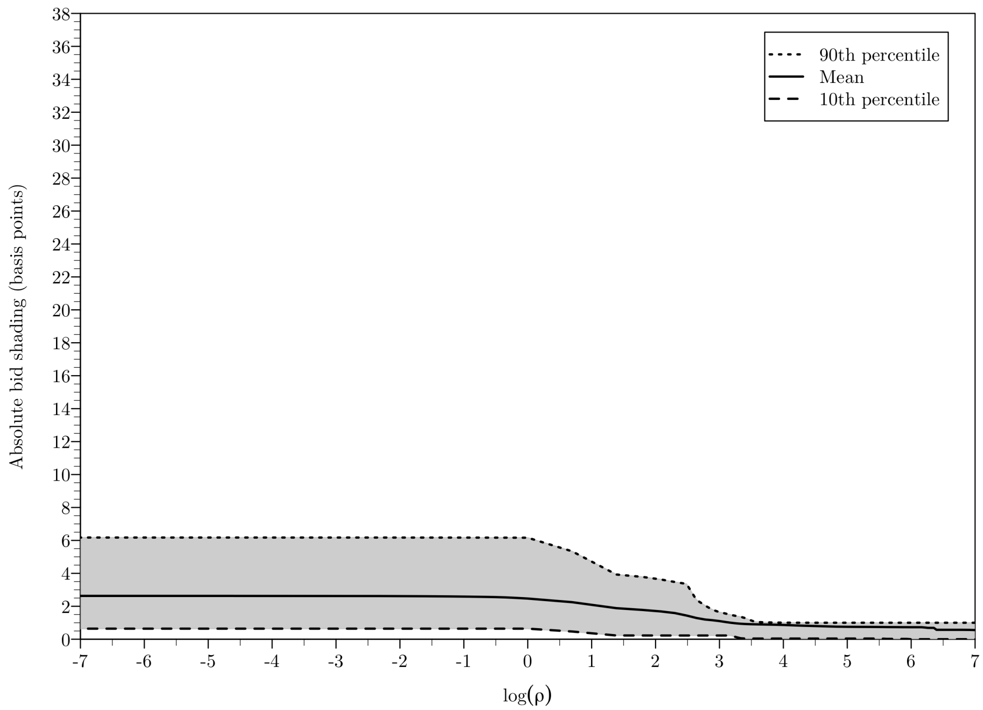
Table 4: Testing results assuming no ties

			Both goods				
Model 1	Model 2						
	$S(0)$	$S(0.1)$	$S(0.2)$	$S(0.3)$	$S(0.4)$	$S(0.5)$	$S(1)$
T	-1.764	-1.701	-1.655	-1.456	-1.408	-1.375	-1.243
$S(0)$		0.064	0.879	1.145	1.281	1.428	1.670
			Good *A*				
Model 1	Model 2						
	$S(0)$	$S(0.1)$	$S(0.2)$	$S(0.3)$	$S(0.4)$	$S(0.5)$	$S(1)$
T	0.019	0.018	0.018	0.017	0.017	0.017	0.014
$S(0)$		0.004	0.004	0.004	0.004	0.004	-0.046
			Good *B*				
Model 1	Model 2						
	$S(0)$	$S(0.1)$	$S(0.2)$	$S(0.3)$	$S(0.4)$	$S(0.5)$	$S(1)$
T	-2.322	-2.223	-2.174	-2.094	-2.014	-1.941	-1.770
$S(0)$		0.053	0.088	0.120	0.149	1.673	2.125

Each entry shows the test statistic for the row being Model 1 and the column being Model 2, so that a negative entry indicates that the row model has better model fit than the column model (and a positive entry indicates the converse). The test statistic is distributed $\mathcal{N}(0,1)$ so the standard critical values apply: -1.645 for a 5% confidence level; -1.960 for 2.5%; -2.326 for 1%.

Figure 6: Absolute value of bid shading in Model $S(\rho)$ for $\log(\rho) \in [-10, 10]$, June 2010 – May 2012, assuming no ties (observations are weighted by the quantities demanded)

(a) Good A



(b) Good B

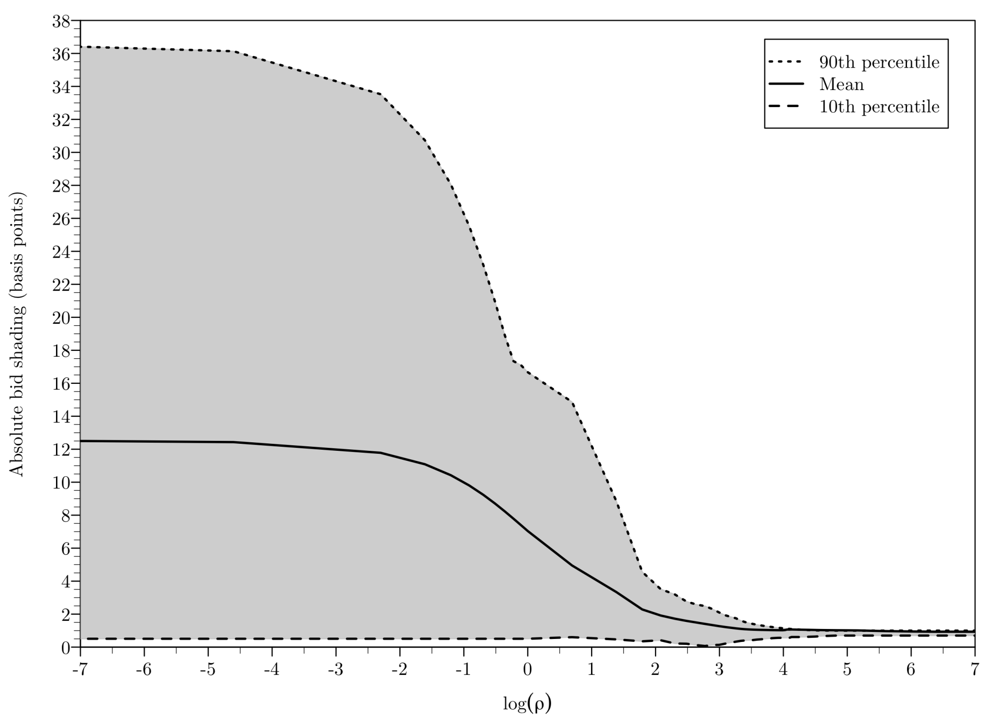


Table 5 shows the test results for each pair of models, with the bid shading in Model $S(\rho)$ estimated under the assumption of the main model and the testing specification identical to the main analysis except that the dummy variable for a last step in a bidder's bid function is excluded from the set of characteristics, $\boldsymbol{x}$.

Table 5: Testing results excluding the dummy for last step from $\boldsymbol{x}$

Both goods							
Model 1	Model 2						
	$S(0)$	$S(0.1)$	$S(0.2)$	$S(0.3)$	$S(0.4)$	$S(0.5)$	$S(1)$
T	-1.609	-1.511	-1.464	-1.307	1.295	-1.101	-1.000
$S(0)$		0.060	0.085	0.094	0.099	1.342	1.724
Good *A*							
Model 1	Model 2						
	$S(0)$	$S(0.1)$	$S(0.2)$	$S(0.3)$	$S(0.4)$	$S(0.5)$	$S(1)$
T	0.024	0.024	0.025	0.025	0.024	0.023	0.023
$S(0)$		0.005	0.006	0.006	0.005	0.004	-0.024
Good *B*							
Model 1	Model 2						
	$S(0)$	$S(0.1)$	$S(0.2)$	$S(0.3)$	$S(0.4)$	$S(0.5)$	$S(1)$
T	-2.144	-2.032	-1.952	-1.875	-1.798	-1.739	-1.494
$S(0)$		0.137	0.152	0.175	1.770	1.875	2.103

Each entry shows the test statistic for the row being Model 1 and the column being Model 2, so that a negative entry indicates that the row model has better model fit than the column model (and a positive entry indicates the converse). The test statistic is distributed $\mathcal{N}(0,1)$ so the standard critical values apply: -1.645 for a 5% confidence level; -1.960 for 2.5%; -2.326 for 1%.

E Calculations for Section 5.2

The 50-50 bet

In the model, the risk aversion parameter, ρ, measures the rate at which a bidder's marginal utility decreases when their net utility from the auction (or their "wealth") increases by one unit of the maximum supply. Maximum supply is £5 billion in the 3-month term auctions and £2.5 billion in the 6-month term auctions so the risk aversion parameters measured in £ would be $\rho^M = (0.2 \times 10^{-9} \times \rho)$ and $\rho^M = (0.4 \times 10^{-9} \times \rho)$, respectively.

Let W be the bidder's initial wealth. A bidder will be indifferent to the 50-50 bet to lose Y and win X if $\frac{1}{2}\frac{1}{\rho^M}\left(1 - \exp^{-\rho^M(W+X)}\right) + \frac{1}{2}\frac{1}{\rho^M}\left(1 - \exp^{-\rho^M(W-Y)}\right) = \frac{1}{\rho^M}\left(1 - \exp^{-\rho^M(W)}\right)$, then $X = -\frac{\log(2-\exp^{\rho^M Y})}{\rho^M}$. And so, if $Y = 100,000$ and $\log(\rho) = 4$ for a bidder in the 3-month auctions, so that $\rho^M = 1.091963 \times 10^{-8}$, then $X =$ £100,109. For a bidder in the 6-month auctions, i.e. $\rho^M = 2.18393 \times 10^{-8}$, then $X =$ £100,209.

Estimates of CARA parameters

Armantier and Sbaï (2006) estimate CARA parameters of 6.907×10^{-6} and 5.732×10^{-8}, respectively, in euros in auctions held in May 1998 – December 2000. I convert these at the average EUR/GBP exchange rate in May 1998 – December 2000 of 1.55 (where the EUR/GBP exchange rate in 1998 is calculated by (1/6.55957)FRF/GBP). The average maximum supply in my study is £4.17 billion, so Armantier and Sbaï's (2006) estimates imply $\rho = \frac{(6.907\times10^{-6})}{1.55} \times 4.17 \times 10^9$ and $\rho = \frac{(5.732\times10^{-8})}{1.55} \times 4.17 \times 10^9$, respectively.

Boyarchenko, Lucca and Veldkamp (2021) calibrate a CARA parameter of 366.61 given a total auction supply normalised to one, which implies $\rho = 366.61\hat{Q}$, where $\hat{Q}$ is the ratio of the average maximum supply in my model to the average issuance in their study. The average maximum supply in my study is £4.17 billion; their sample includes US Treasury auctions of 2-, 3-, 5-, 7-, and 10-year notes in September 2004 – June 2014, of which the average issuance was £17.3 billion (converted from US dollars at the exchange rate on issuance date), which implies $\hat{Q} = 0.24$.

Allen and Wittwer (2023) provide a median estimate for the risk aversion parameter in their setting equal to 0.006, for a total auction supply normalised to one. The auction supply is on average Can\$4.12 billion, which is approximately £2.43 billion (converted from Canadian dollars at an exchange rate of 0.59). Following the same approach as above, this implies $\hat{Q} = \frac{4.17}{2.43} = 1.72$, and therefore $\rho \approx 0.006 * 1.72 = 0.01032$.

F Bidding costs

In the main analysis, I evaluate the relative performance of alternative models of behaviour, including Model $S(0)$, a conventional model in which bidders are both strategic and risk neutral. However, the approach gives no insight to absolute model fit so this appendix analyses an external measure of fit of Model $S(0)$. The model assumes that bidders face "bidding costs", unobserved to the researcher, which deter bidders from making a large number of bids and which explain the few number of bids submitted in practice. The measure of fit that I focus on is the size of these bidding costs required to explain the data under the assumptions of the model. In the BoE's ILTR auctions, the physical costs of submitting bids are trivial.[67] Under this interpretation, I find that the estimated bidding costs for bidders for good B are relatively large, which suggests evidence against this model.

To estimate the bidding costs, I make use of the fact that the number of bids submitted is a strategic choice for the bidder. If they submit K bids, the bidding cost of the $(K+1)$th bid must be larger than the marginal benefit of "fine-tuning" their bid function, which is equal to the difference in their expected utility from submitting what would be the optimal $(K+1)$-step bid function and what would be the optimal K-step one. Having estimated a bidder's full marginal valuation function, we can estimate this marginal benefit of fine-tuning, which provides a lower bound on the bidding cost of the $(K+1)$th bid.

This exercise requires estimates of bidders' full marginal valuation functions, but the condition which rules out profitable deviations in the quantities that bidders demand (Equation 2 in the main text) only identifies bidders' marginal values at the quantities demanded at the steps of their bid functions. I therefore provide a set of novel additional necessary conditions which rule out profitable unilateral deviations in the bid prices submitted by a bidder, holding the rest of their strategy constant. Kastl (2011) derives similar conditions for a deviation in bid price, under the conditions that the support of the auction price is continuous and bidders do not tie on the margin. I instead derive conditions for the case in which the set of possible auction prices is discrete and bidders may ties on the margin, both of which characterise my context. These additional conditions set identify the full marginal valuation function. (It is not possible to point identify the full marginal valuation function without further assumptions.) I combine these with the necessary conditions which point identify the marginal valuation function at the quantities demanded at the steps of the bid function (Equation 2 with $\rho = 0$), and use a statistical interpolant to point estimate bidders' full marginal valuation functions in a way which is consistent with these conditions.

[67] The following section discusses alternative interpretations in terms of cognitive costs.

I then estimate the expected marginal benefit of submitting an additional bid by the difference in a bidder's expected utility from submitting the optimal $(K+1)$-step bid function and from submitting the optimal K-step bid function. This provides a lower bound on the bidding costs implied by the model.[68] Because the estimation procedure is computationally intensive, I obtain results for bidders who submit one or two bids per auction, which covers 81% of observations.

For a range of marginal valuation functions consistent with the necessary conditions for equilibrium, I find that the average lower bound on the bidding costs for bidders who submit one bid per auction for funds against Level B collateral (67% of observations for B), ranges from £1,800 to £12,000, equivalent to around 3% of average bidder surplus, and is significantly different from zero. I also find that the estimated lower bounds are of similar magnitude when using the same estimation method for the marginal valuation functions as Kastl (2011). These bidding costs seem large when interpreted as the physical costs of submitting bids. These seemingly large costs are explained by the fact that bidders make very few bids in the BoE's liquidity auctions but could increase their surplus by submitting additional bids. The results suggest that the conventional strategic model does not characterise this aspect of bidding behaviour well, at least for the 67% of bidders for good B who submit only one bid. This complements the findings that the simpler model of truthful bidding is a better approximation.

In contrast, the results show that the average lower bound on bidding costs for bidders who submit one bid per auction for funds against Level A collateral (68% of observations for A), the estimated lower bounds on bidding costs are very small, equal to approximately £100, equivalent to around 0.2% of average bidder surplus. This is unsurprising given the auction price for these funds is stable across auctions so there are limited gains from further fine-tuning one's bid function; it is also consistent with the findings in the main text that we cannot discriminate between alternative models of behaviour for bidders for A. Similarly, the average lower bound on bidding costs for bidders who submit two bids per auction for funds is very small for both goods. This is consistent with Kastl's (2012) finding that even a small cost of submitting additional bids can make it optimal to submit few bids.

This appendix proceeds as follows. Section F.1 first discusses possible interpretations of the bidding costs. Section F.2 provides necessary conditions for equilibrium that, together with Equation 2 in the main text, set identify bidders' marginal valuation functions. Estimating

[68] An analogous method (which is not relevant to the external measure of fit) provides an upper bound on the bidding costs, equal to the difference in a bidder's expected utility from submitting the optimal K-step bid function and from submitting the optimal $(K-1)$-step bid function.

the marginal valuation functions and bidding costs is described in Sections F.3 and F.4. Section F.5 shows the results.

F.1 Interpretation

Evaluating Model $S(0)$ based on the size of the bidding costs requires an interpretation of these costs.

The most natural interpretation is that they represent the costs of physically submitting the bids. In the ILTR auctions, as in many other settings, bids are submitted through an online platform, called BTender, which is used by bidders to participate in many other BoE operations. So the financial and time costs of submitting bids are trivial.[69]

Alternatively, the bidding costs could be interpreted cognitively. For example, if we assume bidders have infinite experience in participating in the auctions, then they know the average expected utility from submitting a given number of steps based on prior experience, and decide how many steps to submit given the bidding cost of each. Given their choice of number of steps, K_i, a bidder then determines their optimal bids. Under this interpretation, the bidding cost may be interpreted as the cost of calculating the optimal K_i-step bid function. However, this requires a long history of a stable environment, which is not directly applicable to the ILTR setting.

An alternative cognitive interpretation would be that bidders have unlimited cognitive ability, calculate the maximum expected surplus that can be obtained for every possible number of submitted bids, and then choose the optimal number to submit. This would imply that only negligible bidding costs could rationalise the few bids per bidder that are seen in the data. Both cognitive interpretations seem extreme if taken literally, and it is unclear which is preferable.[70]

[69]There is a fixed cost of registering to participate in the ILTR auctions. The bidder must register to become a participant of the BoE's Sterling Monetary Framework and Open Market Operations and typically pledge collateral prior to the auction. These costs are sunk at the time of bidding.

[70]Because bidders can submit any number of bids in the ILTR auctions, we cannot interpret these bidding costs as a constraint imposed by the auction rules. (In this interpretation, the cost of submitting bids of number greater than $\bar{K}$ would be infinite: $c_i(K_i) \to \infty \, \forall K_i > \bar{K}$.) Upper bounds on the number of bids are observed in other settings, but they are never binding as far as I know. For example, in Kastl's (2011) Czech Treasury auctions, the upper bound is 10 and the maximum number of bids by an individual bidder is 9. Similarly, in Hortaçsu and Puller's (2008) study of Texan electricity auctions, "only one firm ever used the maximum number of steps, and that only occurred once for that firm" (McAdams, 2008, citing private communication with Steven Puller).

F.2 Set identification of the marginal valuation function

In equilibrium of Model $S(0)$, the observed number of bids submitted by a bidder is optimal, chosen to trade off the marginal cost of an additional bid with the marginal benefit of fine-tuning the bidder's bid function, i.e. the additional expected utility net of bidding costs from submitting an additional bid.

For type θ_i of bidder i, the marginal benefit is equal to the difference in expected utility (net of bidding costs) between submitting the $(K_i + 1)$-step bid function which maximises expected utility conditional on submitting (K_i+1) bids, and submitting the observed K_i-step bid function, which is optimal conditional on submitting K_i bids. This difference therefore provides an estimate of the lower bound of bidder i's bidding cost of an additional bid, $c_i(K_i + 1) - c_i(K_i)$.

To estimate this difference in expected utility, we must first estimate the bidder's unobserved marginal valuation function.

While point identification in a single-unit auction is feasible under certain conditions, identification in multi-unit auctions is more challenging (McAdams, 2008). Equation 2 point identifies a bidder's marginal values at the quantities they demand at the steps of their bid function by ruling out profitable unilateral deviations in the quantity they demand at each step, holding the rest of their strategy constant.

I derive a set of novel necessary conditions that rule out profitable local deviations in the *bid prices* that a bidder submits, in order to set identify the marginal valuation function between the quantities a bidder demands at the steps of their bid function. I also bound the bidder's marginal value for the first unit (i.e. the intercept of the marginal valuation function) from below.

Equilibrium in Model $S(0)$ requires that a bidder cannot profit from a unilateral deviation in the bid price that they submit at a particular step, holding the rest of their strategy fixed. I consider single basis point deviations, above and below the original bid price, as these are the smallest permitted deviations in price that a bidder may make. Equilibrium requires that the bidder's expected utility given each of these unilateral local deviations is weakly less than their expected utility given their original strategy. The change in expected utility depends on whether the bidder has market power, and the amount of rationing the bidder expects to face if the auction price is set equal to their bid.

To build intuition, I first consider two examples in which a bidder submits a single bid. In both examples, the bidder is a price taker, so their deviation does not affect the distribution of the auction price. In the first example, there is no rationing, so all bids weakly above the auction price are fully allocated, whereas the second example allows for rationing.

In Proposition 2, I then state the pair of conditions that are necessary for equilibrium in Model $S(0)$, allowing for multiple bids and market power. This pair of conditions provides set identification of each bidder's marginal valuation function at all quantities (not just at the quantities they demand at the steps of their bid function). To identify the marginal valuation functions in Model $S(0)$, I assume that the rationing coefficient that bidder i expects to face at a particular price, conditional on the strategy profile, is deterministic. To my knowledge, the constraints imposed by Proposition 2 have not yet been applied in the literature.

By ruling out a profitable unilateral downward deviation in bid price by 1 basis point, Corollary 1 provides set identification of each bidder's marginal value for the first unit (i.e. intercept of their marginal valuation function). This also makes the identifying assumption of a deterministic rationing coefficient, conditional on the strategy profile. I am unaware of any existing studies that have recognised this bound.

F.2.1 Example: price-taking bidder, no rationing

I first consider a bidder who submits a single bid on good $g \in \{A, B\}$ at bid price b for a quantity q. In this example, the bidder is a price taker (so they take the distribution of the auction price as fixed) and there is no rationing (so all bids weakly above the auction price are fully allocated).

I consider two unilateral deviations in bid price to $b - 1$ and $b + 1$, holding the quantity demanded by the bidder (and the number of bids, equal to one) constant.

Equilibrium requires that the bidder cannot profit from either of these unilateral deviations. That is, their expected utility if they follow their original strategy must weakly exceed their expected utility if they instead deviate.

Since they are a price taker and there is no rationing, the downward deviation only changes their allocation in states in which the auction price is b: they win the entire quantity that they demand under their original strategy, but win nothing if they deviate. Equilibrium requires that they weakly prefer to win rather than lose in this case, and so requires that their average marginal value for those units is greater than the price paid, b. The quantity that they win at all other possible auction prices is unchanged.

Analogously, equilibrium requires that the bidder would prefer to lose at price $b+1$ rather than win the quantity q that they originally demand at price b. They would win at this price given the upward deviation but lose under their original strategy. So equilibrium requires that their average marginal value for those units is less than the price that would be paid, $b+1$.

Combining these two conditions, the bidder's average marginal value for the units up to the quantity that they demand is bounded by their bid price and the integer above their bid price:

$$b \leq \underbrace{\frac{1}{q}\int_0^q v^{g,h}(u,\theta_i)du}_{\text{Average marginal value for quantity demanded}} \leq b+1 \tag{11}$$

This constraint is illustrated in Figure 7 in an example in which the bidder's marginal valuation function is linear. Equilibrium requires that Area G $\geq$ Area H in panel (a) and Area I $\leq$ Area J in panel (b).

F.2.2 Example: price-taking, allowing for rationing

For the same price-taking bidder, I now allow for the possibility that the bidder is rationed if the auction price is set equal to their bid. Suppose that the rationing coefficient is deterministic at each auction price, conditional on the strategy profile: $r^g_{i,b}$ is the rationing coefficient at auction price $P^g = b$, conditional on the original strategy profile.[71]

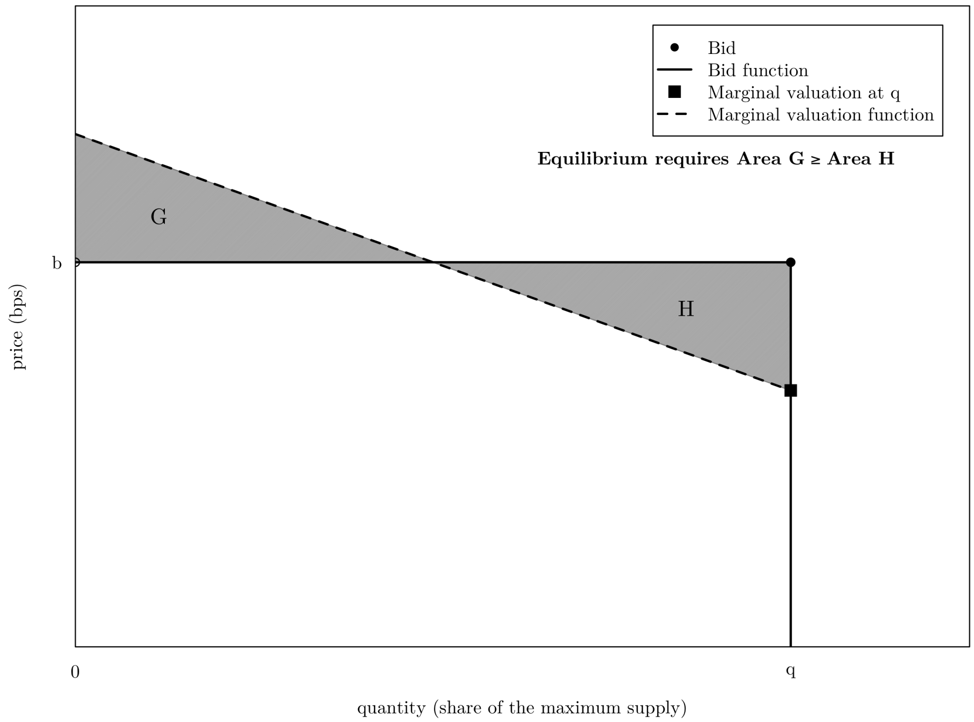
For this bidder, I consider the same two unilateral deviations in bid price to $b-1$ and $b+1$. I denote the respective rationing coefficients at prices $b-1$ and $b+1$ by $r^g_{i,b-1}$ and $r^g_{i,b+1}$.

First consider the downward deviation to $b-1$. The bidder is a price taker, so the deviation only changes their allocation in states in which the auction price is either $b-1$ or b. In the case that $P^g = b-1$, they are fully allocated the quantity that they demand under their original strategy, but rationed if they deviate. In the case that $P^g = b$, they are rationed under their original strategy, and strictly loses if they deviate. So their deviation weakly *reduces* the quantity they are allocated at each of the auction prices, $b-1$ and b.

[71] One interpretation is that there is very little variation in the rationing coefficient at each auction price, so the bidder believes it to be approximately constant.

Figure 7: Constraints on the marginal valuation function for a single-step price-taking bidder, without rationing

(a) Downward Deviation



(b) Upward Deviation

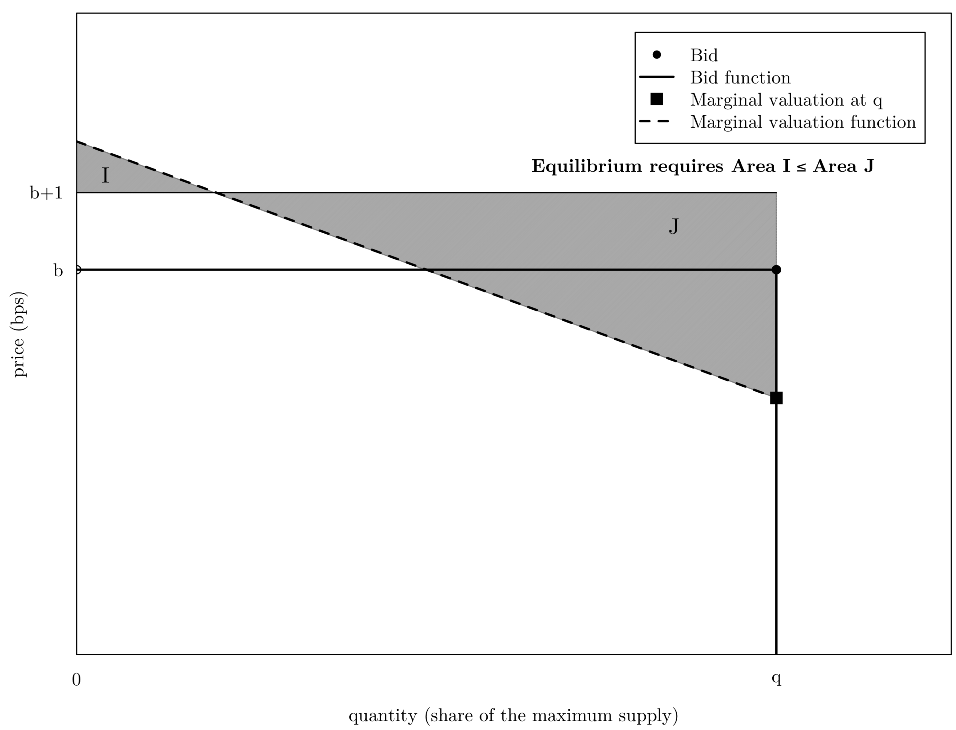
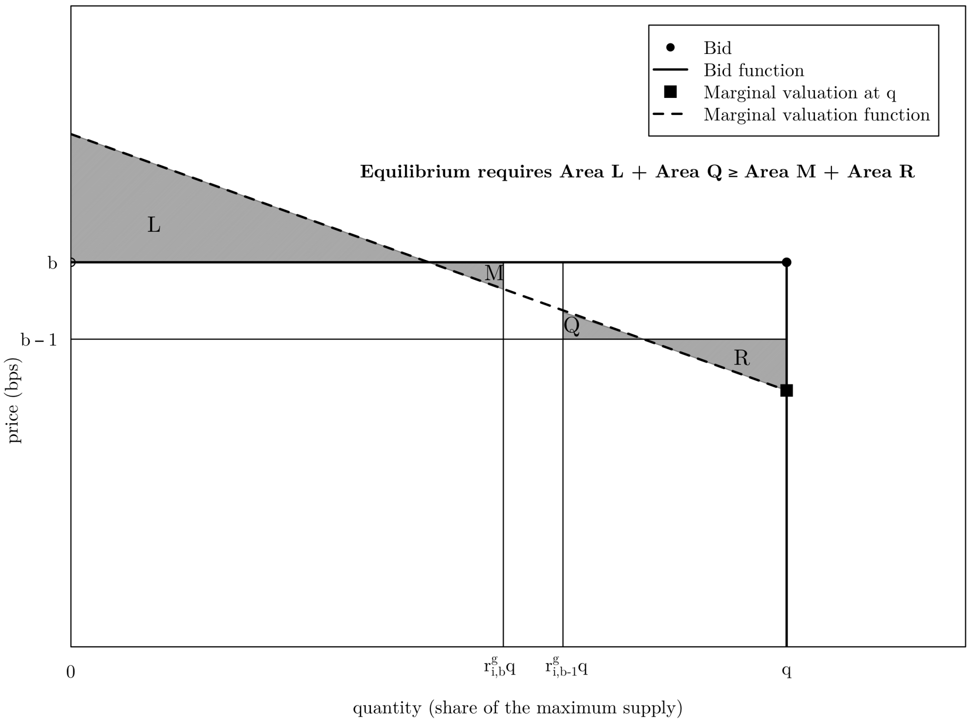
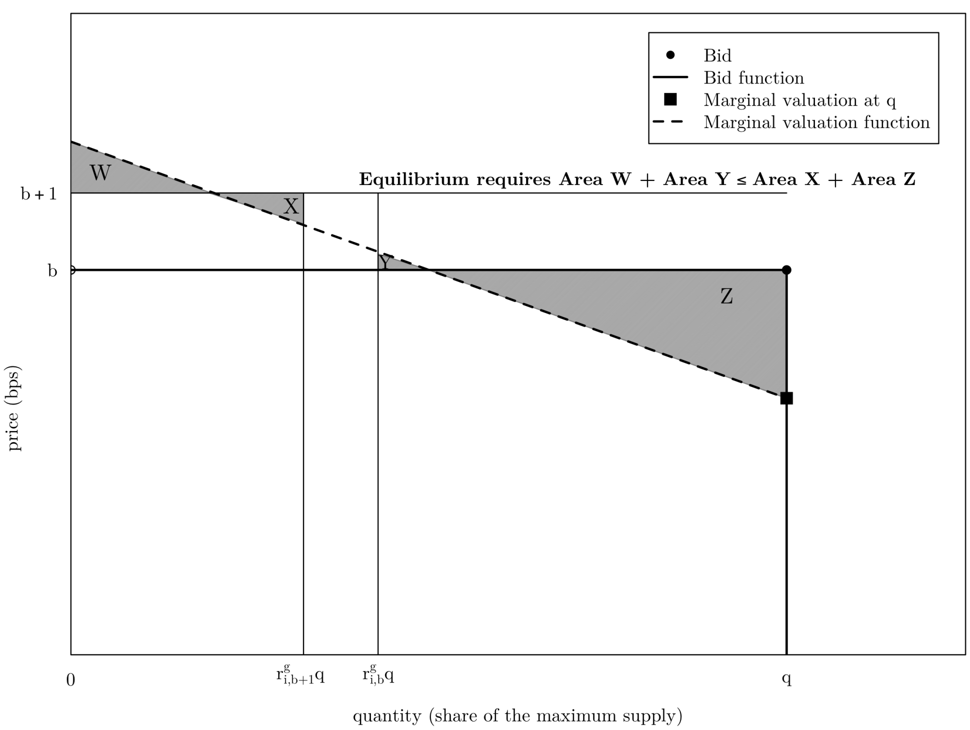


Figure 8: Constraints on the marginal valuation function for a single-step price-taking bidder, with rationing

(a) Downward Deviation



(b) Upward Deviation



Equilibrium requires that deviating from the bidder's original strategy reduces their expected utility. So their value for the additional units that they are allocated under their original strategy, weighted by the probability of the auction prices occurring, must be weakly greater than the increase in total expected payment that they would make, that is,

$$\underbrace{\mathbb{P}(P^g = b)\int_0^{qr^g_{i,b}} v^{g,h}(u,\theta_i)du + \mathbb{P}(P^g = b-1)\int_{qr^g_{i,b-1}}^{q} v^{g,h}(u,\theta_i)du}_{\text{Expected increase in valuation given demand } q\text{, and bid price } b\text{, relative to } b-1}$$
$$\geq \quad \underbrace{\mathbb{P}(P^g = b)bqr^g_{i,b} + \mathbb{P}(P^g = b-1)(b-1)q\left(1 - r^g_{i,b-1}\right)}_{\text{Increase in expected payment given demand } q\text{, and bid price } b\text{, relative to } b-1} \tag{12}$$

Analogously, equilibrium requires that the bidder cannot profit from a unilateral upward deviation in bid price by one unit to $b+1$. The deviation only changes their allocation in states in which the auction price is either b or $b+1$, weakly increasing it in both cases. Given $P^g = b$, they are rationed under their original strategy but fully allocated if they deviate. Given $P^g = b+1$, they strictly lose under their original strategy but are rationed if they deviate.

Equilibrium requires that the bidder's value for the additional units that they are allocated if they deviate, weighted by the probability of the auction prices occurring, is weakly *lower* than the expected increase in total payment that they would make, that is,

$$\underbrace{\mathbb{P}(P^g = b+1)(b+1)qr^g_{i,b+1} + \mathbb{P}(P^g = b)bq\left(1 - r^g_{i,b}\right)}_{\text{Increase in expected payment given demand } q\text{, and bid price } b+1\text{, relative to } b}$$
$$\geq \underbrace{\mathbb{P}(P^g = b+1)\int_0^{qr^g_{i,b+1}} v^{g,h}(u,\theta_i)du + \mathbb{P}(P^g = b)\int_{qr^g_{i,b}}^{q} v^{g,h}(u,\theta_i)du}_{\text{Expected increase in valuation given demand } q\text{, and bid price } b+1\text{, relative to } b} \tag{13}$$

These two conditions are illustrated in Figure 8, in an example in which the bidder's marginal valuation function is linear and the distribution of the auction price is uniform. Equilibrium requires that Area L + Area Q $\geq$ Area M + Area R in panel (a) and that Area W + Area Y $\leq$ Area X + Area Z in panel (b).

F.2.3 Equilibrium conditions ruling out profitable deviations in bid price

For the price-taking bidder considered above, the incentive to deviate depends only on the effect on the quantity that they are allocated at each auction price. Equilibrium requires that the expected change in total payment from a unilateral local deviation in bid price exceeds the expected change in valuation, which deters the bidder from deviating.

Proposition 2 generalises this condition to also account for the fact that each bidder may submit multiple bids, and may have market power so that their deviation affects the distribution of the auction price. I consider two, single basis point deviations: one below the original bid price, and one above. Equations 14 and 15 specify necessary conditions for equilibrium, which rule out that these unilateral deviations are profitable.

In Model $S(0)$, the equilibrium strategy profile is $\boldsymbol{\sigma}(\boldsymbol{\Theta})$. Let $\boldsymbol{\sigma}'(\boldsymbol{\Theta})$ and $\boldsymbol{\sigma}''(\boldsymbol{\Theta})$ denote the strategy profiles which are identical to $\boldsymbol{\sigma}(\boldsymbol{\Theta})$ except that the bid prices submitted by type θ_i of bidder $i \in \mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$ at step k, in their K_i-step bid function, are equal to $b_{i,k} - 1$ and $b_{i,k} + 1$, respectively. For ease of notation, I write $\boldsymbol{\sigma}(\boldsymbol{\Theta}|\theta_i) = \boldsymbol{\sigma}(\boldsymbol{\Theta})$, and similarly for $\boldsymbol{\sigma}'(\boldsymbol{\Theta}|\theta_i)$ and $\boldsymbol{\sigma}''(\boldsymbol{\Theta}|\theta_i)$.

To permit identification, I extend Assumption 2 to the strategy profiles, $\boldsymbol{\sigma}'(\boldsymbol{\Theta}|\theta_i)$ and $\boldsymbol{\sigma}''(\boldsymbol{\Theta}|\theta_i)$, i.e. the rationing coefficient is deterministic at each auction price, conditional on each strategy profile.[72]

Assumption 3 *The rationing coefficients for type $\theta_i \in \Theta_i$ of bidder $i \in \mathcal{N}$ at step $k \in \{1, ..., K_i\}$, conditional on the strategy profiles $\boldsymbol{\sigma}(\boldsymbol{\Theta}|\theta_i), \boldsymbol{\sigma}'(\boldsymbol{\Theta}|\theta_i)$ and $\boldsymbol{\sigma}''(\boldsymbol{\Theta}|\theta_i)$ are deterministic and are denoted $r_{i,k}, r'_{i,k}$, and $r''_{i,k}$, respectively.*

Proposition 2 follows from this assumption and the assumptions of Model $S(0)$.

Proposition 2 (Necessary condition on bid price deviations in Model $\boldsymbol{S(0)}$) *In Model $S(0)$ under Assumption 3, in any type-symmetric Bayesian Nash Equilibrium, for type $\theta_i \in \Theta_i$ of bidder $i \in \mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$, every step $k \in \{1, ..., K_i\}$ in the K_i-*

[72] The necessary conditions for equilibrium without the identifying assumptions are Equations 22 and 25 in Appendix G.

step bid function in $\sigma_i(\theta_i)$ *where* $b_{i,k}-1>b_{i,k+1}$ *and* $b_{i,k}+1<b^g_{i,k-1}$*, must satisfy*[73]

$$\begin{aligned}
&\left(\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r_{i,k}} v^{g,h}(u,\theta_i)du - b_{i,k}\left(q_{i,k-1}+(q_{i,k}-q_{i,k-1})r_{i,k}\right)\right)\mathbb{P}\left(P^g=b_{i,k}\middle|\boldsymbol{\sigma}(\boldsymbol{\Theta})\right)\\
&\quad+\left(\int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du-(b_{i,k}-1)q_{i,k}\right)\mathbb{P}\left(P^g=b_{i,k}-1\middle|\boldsymbol{\sigma}(\boldsymbol{\Theta})\right)\\
&\geq\left(\int_0^{q_{i,k-1}} v^{g,h}(u,\theta_i)du-b_{i,k}q_{i,k-1}\right)\mathbb{P}\left(P^g=b_{i,k}\middle|\boldsymbol{\sigma}'(\boldsymbol{\Theta})\right)\\
&\quad+\left(\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r'_{i,k}} v^{g,h}(u,\theta_i)du\right.\\
&\qquad\left.-(b_{i,k}-1)\left(q_{i,k-1}+(q_{i,k}-q_{i,k-1})r'_{i,k}\right)\right)\mathbb{P}\left(P^g=b_{i,k}-1\middle|\boldsymbol{\sigma}'(\boldsymbol{\Theta})\right) \qquad (14)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_0^{q_{i,k-1}} v^{g,h}(u,\theta_i)du-(b_{i,k}+1)q_{i,k-1}\right)\mathbb{P}\left(P^g=b_{i,k}+1\middle|\boldsymbol{\sigma}(\boldsymbol{\Theta})\right)\\
&\quad+\left(\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r_{i,k}} v^{g,h}(u,\theta_i)du\right.\\
&\qquad\left.-b_{i,k}\left(q_{i,k-1}+(q_{i,k}-q_{i,k-1})r_{i,k}\right)\right)\mathbb{P}\left(P^g=b_{i,k}\middle|\boldsymbol{\sigma}(\boldsymbol{\Theta})\right)\\
&\geq\left(\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r''_{i,k}} v^{g,h}(u,\theta_i)du\right.\\
&\qquad\left.-(b_{i,k}+1)\left(q_{i,k-1}+(q_{i,k}-q_{i,k-1})r''_{i,k}\right)\right)\mathbb{P}\left(P^g=b_{i,k}+1\middle|\boldsymbol{\sigma}''(\boldsymbol{\Theta})\right)\\
&\quad+\left(\int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du-b_{i,k}q_{i,k}\right)\mathbb{P}\left(P^g=b_{i,k}\middle|\boldsymbol{\sigma}''(\boldsymbol{\Theta})\right) \qquad (15)
\end{aligned}$$

Proof. See Appendix G. ■

As discussed in Section 4.2, each of the probabilities and rationing coefficients in Equations 14 and 15 can be estimated and $b_{i,k}$, $q_{i,k}$ and $q_{i,k-1}$ are observed. Approximating the integrals with summations over an arbitrarily fine quantity grid, these two equations therefore impose linear constraints on the marginal valuation functions of bidders that rationalise observed behaviour in Model $S(0)$.

[73] The conditions for bidders who submit adjacent bid prices are analogous, and shown in Appendix G.3.

By ruling out profitable unilateral deviations in bid price, it is also possible to bound each bidder's marginal value for the first unit (i.e. intercept of their marginal valuation function) from below.

Corollary 1 (Necessary condition on marginal valuation function in Model $\boldsymbol{S(0)}$) *In Model $S(0)$ under Assumption 3, in any type-symmetric Bayesian Nash Equilibrium, for type $\theta_i \in \Theta_i$ of bidder $i \in \mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$, step $k = 1$ in the K_i-step bid function in $\sigma_i(\theta_i)$ must satisfy*

$$v^{g,h}(0, \theta_i) \geq b^g_{i,1} - 1 \tag{16}$$

under both the assumptions that $\mathbb{P}\left(P^g = b^g_{i,1} \middle| \boldsymbol{\sigma}(\boldsymbol{\Theta})\right) > 0$ or $\mathbb{P}\left(P^g = b^g_{i,1} - 1 \middle| \boldsymbol{\sigma}'(\boldsymbol{\Theta})\right) > 0$ and that $r_{i,1} \neq 0$.

Proof. See Appendix G.4. ■

Equation 16 places a further restriction on the set of marginal valuation functions that rationalise observed behaviour. This constraint is slack for a bidder who submits bids at prices below their marginal value at the quantity they demand at the first step of their bid function because the marginal valuation function is downward sloping (see Section 3.1), i.e. for whom $\mu^g_{i,1} \geq 0$ (which can be determined by Equation 2). However, for bidders who submit bids at prices above their marginal value at the quantity they demand at the first step of their bid functions, i.e. $\mu^g_{i,1} < 0$, Corollary 1 restricts the set of marginal valuation functions consistent with equilibrium in Model $S(0)$.

F.3 Estimating marginal valuation functions

Equations 2 and 14-16 describe necessary relationships between a bidder's observed bids and their unobserved marginal valuation function in equilibrium of Model $S(0)$, for a particular realisation of their signal. As described in Section 4.2, I use the first equation to obtain point estimates of each bidder's marginal values at the quantities they demand at the steps of their bid function. The points of a bidder's marginal valuation function between the quantities they demand at the steps cannot be point identified, but are set identified. I therefore interpolate the remaining parts of the bidder's marginal valuation function in a way which is consistent with Proposition 2 and Corollary 1.

There is a range of possible marginal valuation functions that satisfy Equations 14-16: the intercept for bidder i could be any value greater than $(b^g_{i,1} - 1)$ and, for each of these intercepts, there is potentially a large range of marginal valuation functions consistent with equilibrium in Model $S(0)$.

If the point estimate of the bidder's marginal value at the quantity they demand at the first step of their bid function is greater than their bid, i.e. $\mu^g_{i,1} \geq 0$, I set the intercept equal to this point estimate. For all other observations, the lower bound specified in Equation 16 is binding so I produce three estimates corresponding to three different intercepts: the price of the first step of the bidder's bid function plus a "wedge", $w \in \{1, 5, 10\}$.[74]

$$\hat{v}_i(0) = \max\{b_{i,1} + w, \hat{v}_i(q_{i,1})\} \text{ for } \ w = \{1, 5, 10\} \tag{17}$$

where $\hat{v}_i(q)$ is the estimated marginal value of bidder i at quantity q.

For each of these intercepts, I estimate a marginal valuation function that passes through the point estimates at the quantities demanded at the steps of the bid function, using Gaussian Process Regression (GPR). This is a Bayesian non-parametric statistical interpolant that assumes a Gaussian prior, updated using the point estimates of the marginal values and the specified intercept (Rasmussen and Williams, 2006).[75] It is more suitable than other methods such as linear regression as there is no prima facie measurement error in the point estimates. It also provides more flexibility as there is no strong reason to believe that the marginal valuation function is linear and the prior is defined over the entire function, rather than the parameters. Intuitively, it chooses the most likely function that passes through the specified points.[76]

I then use minimum-distance estimation to find the marginal valuation function nearest to the GPR estimate that is consistent with the monotonicity assumption (see Section 3.1) and Proposition 2; I denote the estimator by MD-GPR.[77]

[74]The intercept of the marginal valuation function is not point identified, and there is no appropriate loss function to guide the choice of intercept within the set specified in Corollary 1. I therefore use three specifications, with the values of w motivated by the standard deviation of bids submitted in the ILTR auctions.

[75]For an arbitrarily fine grid of quantities given by $\boldsymbol{q} = (q_1, q_2, \ldots, q_N)$, where $q_1 < q_2 < ... < q_N, q_1 = 0, q_N \geq q_{i,K_i}$, and $q_{i,k} \in \{q_1, ..., q_N\}$ $\forall k \in \{1, ..., K_i\}$, the GPR prior probability density of the vector of function values $\boldsymbol{v_i}(\boldsymbol{q}) = (v_i(q_1), v_i(q_2), ..., v_i(q_N))$ is jointly Gaussian, with a covariance matrix defined by a kernel function. The smaller is the distance $|q - q_{i,k}|$, the greater is the covariance between the predicted marginal value at quantity q and the marginal value point estimate, $\hat{v}_i(q_{i,k}), k = \{1, \ldots, K_i\}$.

[76]Alternative GPR specifications, including different kernel functions for the covariance matrix that determine the smoothness of the function, make little difference to the results.

[77]This finds the estimate, nearest to the GPR estimate, that is consistent with the inequality constraints specified in Equations 14 and 15, approximating the integrals sums over quantity units of 0.1% of the maximum supply.

I assume that the bidder's marginal value for quantities larger than the total quantity that they demand equals zero, that is, $\hat{v}_i(q, \theta_i) = 0 \, \forall \, q > q_{i,K_i}$. This reduces the expected utility from submitting an additional bid and so means the results are conservative estimates of the lower bounds of the bidding costs implied by Model $S(0)$.

To compare my results to Kastl (2011), I also estimate the marginal valuation function by a step function which is an upper envelope of the point estimates of the bidder's marginal values at the quantities demanded at the steps of the bid function.[78] In this specification, the marginal values for all quantities across the first step of the bidder's bid function are set equal to the point estimate of the marginal value at the quantity demanded at the first step. And so, for bidders who bid above their marginal value (by more than 1 basis point) at the first step of their bid function, this estimate violates the conditions specified in Proposition 2 and Corollary 1.[79] Under the identifying assumption that I make of a deterministic rationing coefficient, Kastl's (2011) method underestimates the average steepness of the marginal valuation function across the first step of the bid function.

An example of the four estimated marginal valuation functions—three estimated by MD-GPR and the fourth estimated by the upper envelope which is stepped—is shown for a bidder who submits a bid function with two steps for good B in Figure 9. The first three estimates all satisfy the necessary conditions for equilibrium in Model $S(0)$ described in Propositions 1 and 2.

F.4 Estimating bidding costs

In Model $S(0)$, a bidder who submits K_i bids has chosen to do so to maximise their expected utility: it must be that the marginal cost of submitting an additional bid, equal to $c_i(K_i + 1) - c_i(K_i)$, is greater than the marginal benefit of fine-tuning, equal to the gain in expected utility (net of bidding costs) from submitting the $(K_i + 1)$-step bid function which maximises expected utility conditional on submitting $(K_i + 1)$ bids, rather than the K_i-step bid function that they use. This condition allows us to estimate a lower bound on the size of the bidding costs implied by Model $S(0)$.

For a bidder who submits K_i bids, the K_i-step bid function which maximises expected utility conditional on submitting K_i bids is their observed bid function under the assumptions of

[78] Specifically, this is a step function with the quantities at each step coinciding with those of the bid function. The marginal values at the quantities demanded at the steps of the bid function are point identified by Equation 2 (with $\rho = 0$) and the marginal values for quantities greater than the total quantity demanded by the bidder are equal to zero.

[79] Both Kastl (2011) and I find this to be economically relevant.

Model $S(0)$. I estimate the bidder's expected utility given their strategy by estimating the distribution of the auction price using the resampling technique described in Section 4.2. I use this distribution to estimate the bidder's expected utility (net of bidding costs) of using a K_i-step bid function for each of the estimates of the marginal valuation functions.

The $(K_i + 1)$-step bid function which maximises their expected utility conditional on submitting $(K_i + 1)$ bids is unobserved. I implement a grid search across all bid prices and quantity increments of 0.25% to estimate this optimal $(K_i + 1)$-step bid function. For each of the (K_i+1)-step bid functions on this grid, and for each estimate of the bidder's marginal valuation function, I estimate the bidder's beliefs over the distribution of the auction price conditional on the bid function, and then use this distribution to estimate the bidder's expected utility (net of bidding costs). The estimate of the optimal $(K_i + 1)$-step bid function is the one which maximises the bidder's expected utility.

The difference in the estimated expected utilities from the (K_i+1)- and K_i-step optimal bid functions provides an estimate of the lower bound on bidding costs for each estimate of the bidder's marginal valuation function.[80]

To estimate the standard errors, I follow Backus, Conlon and Sinkinson (2021) and Roussille and Scuderi (2022) by treating the optimal bid function for each number of steps as known.

F.5 Results

Table 6 shows the estimated lower bounds on the bidding costs of submitting additional bids for goods A and B.[81] These lower bounds are estimated for the three MD-GPR estimates of the marginal valuation function: $\max\left\{b^g_{i,1} + w, \hat{v}^g_i(q^g_{i,1})\right\}$ for $w = \{1, 5, 10\}$, and for the step function which is the upper envelope of the point estimates of the bidder's marginal values at the quantities they demand at the steps of their bid function.

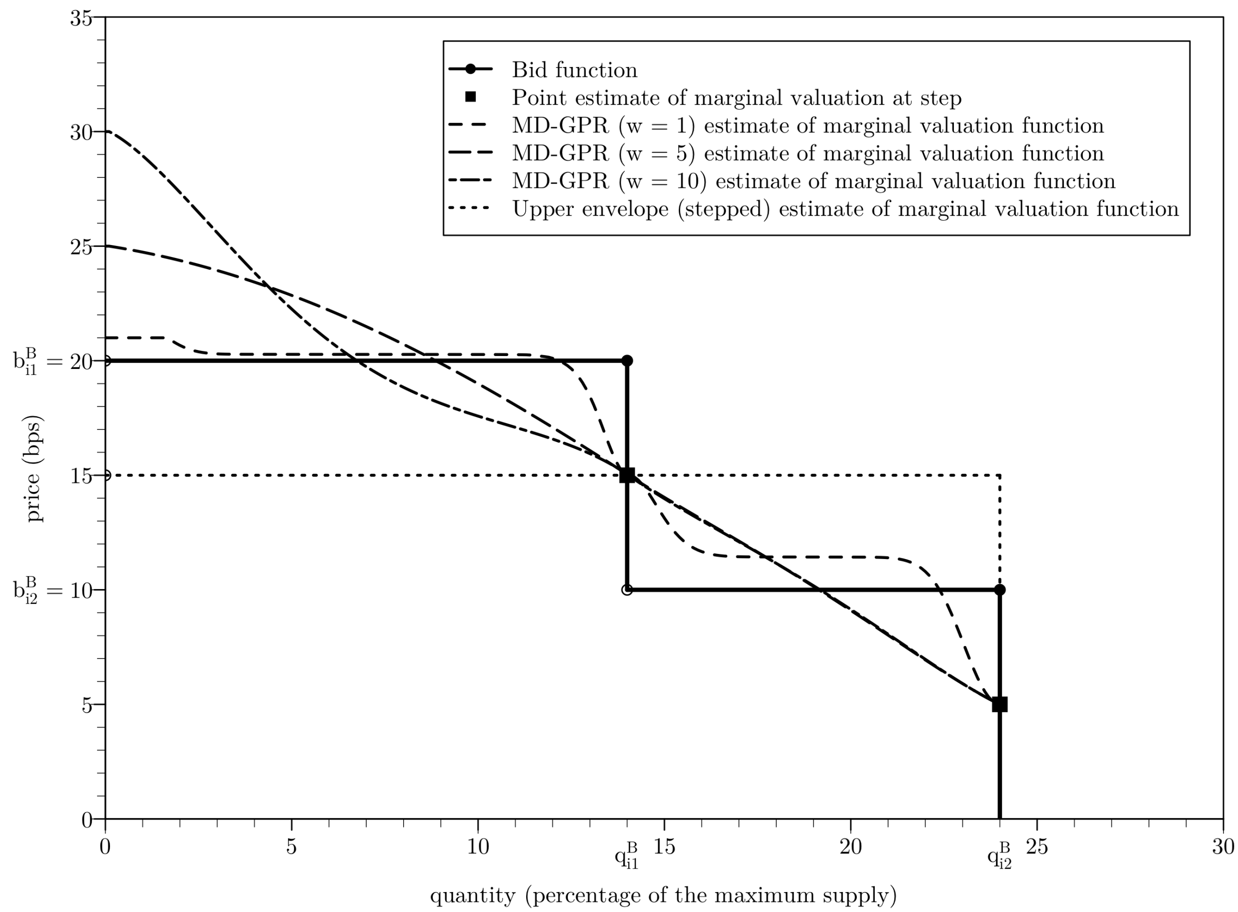
The estimated bidding costs vary across goods, numbers of steps, and intercepts, and within these subgroups (shown by the large standard deviations). This variation can be interpreted in terms of the variation in the expected benefit of fine-tuning one's bid function from submitting additional bids.[82]

[80] We can also estimate an upper bound on the bidding cost of the K_ith bid, by comparing the estimated expected utilities (net of bidding costs) of submitting the optimal K_i- and $(K_i + 1)$-step bid functions.

[81] The lower bounds for the second step are estimated for every bidder, for every auction in which they participate and submit a bid function with one step, in June 2010 – May 2012. The lower bounds for the third step are estimated, analogously, for every bidder, for every auction in which they participate and submit a bid function with two steps.

[82] The heterogeneity itself if difficult to explain if the costs are interpreted as the physical costs of sub-

Figure 9: Example of four estimated marginal valuation functions of a bidder, with using MD-GPR and an upper envelope.



The results show that the average lower bound on the bidding cost of submitting a second step for good B is large: bidders who submit one bid for good B are giving up between £1,800 and £12,707 on average by only submitting one bid. If the costs are interpreted as physical costs, which are trivial in practice, this suggests that Model $S(0)$, which assumes both that bidders are strategic and that they are risk neutral, does not provide a good characterisation of the behaviour of these bidders. This is consistent with the results in the main analysis: for bidders for good B, a "truthful bidding" model in which bidders' bids correspond to their true marginal values for liquidity fit the data better than Model $S(0)$.

In contrast, the average lower bound on bidding costs for bidders for good A are very small, even for bidders who submit one bid. This can be explained by the fact that there are limited benefits for these bidders from submitting additional bids, because the range of possible clearing prices is so narrow. The average clearing price for A in June 2010 – May 2012 is 1.38bps and its standard deviation is 1.82bps in the 3-month term auctions (and these statistics are respectively 0.38bps and 0.74bps in the 6-month term auctions), so bidders gain little from fine-tuning.

Finally, the bidding cost of submitting a third step is also small. This suggests that bidders may only forego a small amount of surplus by submitting a two-step bid function as an approximation to what would be the optimal bid function in the absence of bidding costs. This follows similar intuition to Wilson's (1993) finding that a monopolist can approximate an optimal non-linear multi-part tariff with four or five two-part tariffs.

The findings for good B, including the results for the marginal valuation functions estimated as step functions, differ from those found by Kastl (2011), in a study of Czech Treasury auctions. In that context, he estimates the lower bound on the bidding costs of submitting a second step to be between \$2 and \$150. There are some clear institutional differences which might explain why Model $S(0)$ seems to better characterise bidding behaviour in the Czech Treasury auctions. For example, bidders appear to face less uncertainty in Kastl's (2011) setting, in which only one good is auctioned and supply uncertainty (generated by non-competitive bids) is limited. However, there are some comparable features, including the average number of bids submitted per bidder.[83]

mitting bids, which are constant across bidders and bids in the ILTR auctions.

[83]In Kastl's (2011) setting, the average is 1.41 for "Small" bidders (defined as demanding less than 5% of the auction) and is 2.95 for "Large" bidders (the remaining set).

Table 6: Estimated lower bound on bidding costs implied by Model $S(0)$, £, June 2010 – May 2012

Lower bound of cost of submitting 2nd step (£)				
	Good A		Good B	
Estimation type	Mean	Std. Dev.	Mean	Std. Dev.
MD-GPR ($w = 1$)	103.7	206.0	9557.6	41734.4
	(24.0)	(39.8)	(4776.7)	(13107.6)
MD-GPR ($w = 5$)	183.6	365.5	1976.7	5884.7
	(41.9)	(71.2)	(700.6)	(1423.8)
MD-GPR ($w = 10$)	129.1	260.3	1839.1	5222.0
	(28.4)	(44.3)	(640.4)	(1387.9)
Upper envelope (stepped)	61.9	138.8	12707.5	41325.1
	(19.6)	(34.2)	(6680.2)	(13797.1)
Lower bound of cost of submitting 3rd step (£)				
	Good A		Good B	
Estimation type	Mean	Std. Dev.	Mean	Std. Dev.
MD-GPR ($w = 1$)	0.9	2.5	42.3	72.1
	(0.6)	(0.9)	(34.6)	(33.8)
MD-GPR ($w = 5$)	0.5	2.1	106.8	168.7
	(0.5)	(1.3)	(81.5)	(75.6)
MD-GPR ($w = 10$)	0.3	1.4	110.3	134.9
	(0.3)	(0.9)	(64.3)	(47.6)
Upper envelope (stepped)	0.0	0.0	0.0	0.0
	(0.0)	(0.0)	(0.0)	(0.0)

Notes: For each "wedge", $w \in \{1, 5, 10\}$, the intercept of bidder i's marginal valuation function for good $g \in \{A, B\}$ is equal to $\hat{v}_i^g(0) = \max\left\{b_{i,1}^g + w, \hat{v}_i^g(q_{i,1}^g)\right\}$. Standard errors in parentheses.

Table 7: Estimated lower bound on bidding costs implied by Model $S(0)$, percentage of average bidding surplus, June 2010 – May 2012

Lower bound of cost of submitting 2nd step (% of average bidder surplus)				
	Good A		Good B	
Estimation type	Mean	Std. Dev.	Mean	Std. Dev.
MD-GPR ($w = 1$)	0.379	0.547	3.262	6.728
	(0.071)	(0.098)	(1.062)	(1.902)
MD-GPR ($w = 5$)	0.316	0.466	2.336	6.278
	(0.059)	(0.071)	(0.998)	(3.299)
MD-GPR ($w = 10$)	0.106	0.166	2.232	4.816
	(0.021)	(0.029)	(0.776)	(2.342)
Upper envelope (stepped)	0.144	0.326	3.377	8.344
	(0.047)	(0.079)	(1.380)	(3.405)
Lower bound of cost of submitting 3rd step (% of average bidder surplus)				
	Good A		Good B	
Estimation type	Mean	Std. Dev.	Mean	Std. Dev.
MD-GPR ($w = 1$)	0.003	0.009	0.096	0.151
	(0.002)	(0.004)	(0.073)	(0.066)
MD-GPR ($w = 5$)	0.001	0.004	0.237	0.330
	(0.001)	(0.003)	(0.160)	(0.134)
MD-GPR ($w = 10$)	0.002	0.007	0.171	0.234
	(0.002)	(0.004)	(0.113)	(0.097)
Upper envelope (stepped)	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)

Notes: For each "wedge", $w \in \{1, 5, 10\}$, the intercept of bidder i's marginal valuation function for good $g \in \{A, B\}$ is equal to $\hat{v}_i^g(0) = \max\left\{b_{i,1}^g + w, \hat{v}_i^g(q_{i,1}^g)\right\}$. Standard errors in parentheses.

G Proof of Proposition 2

In Model $S(0)$, the equilibrium strategy profile is $\boldsymbol{\sigma}(\boldsymbol{\Theta})$. Let $\boldsymbol{\sigma}'(\boldsymbol{\Theta})$ and $\boldsymbol{\sigma}''(\boldsymbol{\Theta})$ denote the strategy profiles which are identical to $\boldsymbol{\sigma}(\boldsymbol{\Theta})$ except that the bid prices submitted by type θ_i of bidder $i \in \mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$ at step k, in their K_i-step bid function for good g, are equal to $b_{i,k} - 1$ and $b_{i,k} + 1$, respectively. For ease of notation, I write $\boldsymbol{\sigma}(\boldsymbol{\Theta}|\theta_i) = \boldsymbol{\sigma}(\boldsymbol{\Theta})$, and similarly for $\boldsymbol{\sigma}'(\boldsymbol{\Theta}|\theta_i)$ and $\boldsymbol{\sigma}''(\boldsymbol{\Theta}|\theta_i)$.

I first derive Equation 14, ruling out a profitable unilateral *downward* deviation in bid price of one unit. I then derive Equation 15, ruling out a profitable analogous *upward* deviation.

G.1 Downward deviation

Equilibrium of Model $S(0)$ requires that the deviation of bidder $i \in \mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$ at step k to a bid price of $b_{i,k} - 1$ rather than $b_{i,k}$ weakly reduces their expected utility, and so requires

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[U_i(\boldsymbol{\sigma}(\boldsymbol{\Theta})|\theta_i)\right] \geq \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[U_i(\boldsymbol{\sigma}'(\boldsymbol{\Theta})|\theta_i)\right] \\
\implies\ &\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\int_0^{y_i^g(\boldsymbol{\sigma}(\boldsymbol{\Theta}))} v^{g,h}(u,\theta_i)du - P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta}))y_i^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) - c_i(K_i)\right] \\
&\geq \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\int_0^{y_i^g(\boldsymbol{\sigma}'(\boldsymbol{\Theta}))} v^{g,h}(u,\theta_i)du - P^g(\boldsymbol{\sigma}'(\boldsymbol{\Theta}))y_i^g(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) - c_i(K_i)\right] \qquad (18)
\end{aligned}$$

By the Law of Iterated Expectations (LIE),

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[U_i(\boldsymbol{\sigma}(\boldsymbol{\Theta})|\theta_i)\right] \\
&= \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[U_i\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\middle|P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) \notin \{b_{i,k}-1, b_{i,k}\}\right)\right]\mathbb{P}\left(P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) \notin \{b_{i,k}-1, b_{i,k}\}\right) \\
&\quad + \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[U_i\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\middle|P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) = b_{i,k}\right)\right]\mathbb{P}\left(P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) = b_{i,k}\right) \\
&\quad + \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[U_i\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\middle|P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) = b_{i,k}-1\right)\right]\mathbb{P}\left(P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) = b_{i,k}-1\right) \qquad (19)
\end{aligned}$$

and $\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[U_i(\boldsymbol{\sigma}'(\boldsymbol{\Theta})|\theta_i)\right]$ can be decomposed analogously.

First suppose that the bid prices at steps k and $k-1$ are non-adjacent so that $b_{i,k}-1 > b_{i,k+1}$. I consider the case in which $b_{i,k} - 1 = b_{i,k+1}$ in Appendix G.3. Using the LIE, the bidder's expected utilities given their strategy and deviation, respectively, can be written as

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[U_i(\boldsymbol{\sigma}(\boldsymbol{\Theta})|\theta_i)\right] \\
&= \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\Bigg[\int_0^{y_i(\boldsymbol{\sigma}(\boldsymbol{\Theta}))} v^{g,h}(u,\theta_i)du \\
&\qquad - P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta}))y_i(\boldsymbol{\sigma}(\boldsymbol{\Theta}))\Bigg| P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) \notin \{b_{i,k}-1, b_{i,k}\}\Bigg]\mathbb{P}\left(P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) \notin \{b_{i,k}-1, b_{i,k}\}\right) \\
&+ \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\Bigg[\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r^g(\boldsymbol{\sigma}(\boldsymbol{\Theta}))} v^{g,h}(u,\theta_i)du \\
&\qquad - b_{i,k}\left(q_{i,k-1}+(q_{i,k}-q_{i,k-1})r^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right)\right)\Bigg| P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) = b_{i,k}\Bigg]\mathbb{P}\left(P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) = b_{i,k}\right) \\
&+ \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du - (b_{i,k}-1)q_{i,k}\middle| P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) = b_{i,k}-1\right]\mathbb{P}\left(P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) = b_{i,k}-1\right) \\
&- c_i(K_i)
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[U_i(\boldsymbol{\sigma}'(\boldsymbol{\Theta})|\theta_i)\right] \\
&= \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\Bigg[\int_0^{y_i(\boldsymbol{\sigma}'(\boldsymbol{\Theta}))} v^{g,h}(u,\theta_i)du \\
&\qquad - P^g(\boldsymbol{\sigma}'(\boldsymbol{\Theta}))y_i(\boldsymbol{\sigma}'(\boldsymbol{\Theta}))\Bigg| P^g(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) \notin \{b_{i,k}-1, b_{i,k}\}\Bigg]\mathbb{P}\left(P^g(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) \notin \{b_{i,k}-1, b_{i,k}\}\right) \\
&+ \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\int_0^{q_{i,k-1}} v^{g,h}(u,\theta_i)du - b_{i,k}q_{i,k-1}\middle| P^g(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) = b_{i,k}\right]\mathbb{P}\left(P^g(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) = b_{i,k}\right) \\
&+ \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\Bigg[\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r^g(\boldsymbol{\sigma}'(\boldsymbol{\Theta}))} v^{g,h}(u,\theta_i)du - (b_{i,k}-1)\Big(q_{i,k-1}+ \\
&\qquad (q_{i,k}-q_{i,k-1})r^g\left(\boldsymbol{\sigma}'(\boldsymbol{\Theta})\right)\Big)\Bigg| P^g(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) = b_{i,k}-1\Bigg]\mathbb{P}\left(P^g(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) = b_{i,k}-1\right) \\
&- c_i(K_i)
\end{aligned}
\tag{21}
$$

where $r^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right)$ and $r^g\left(\boldsymbol{\sigma}'(\boldsymbol{\Theta})\right)$ are the rationing coefficients for good g given strategy profiles $\boldsymbol{\sigma}(\boldsymbol{\Theta})$ and $\boldsymbol{\sigma}'(\boldsymbol{\Theta})$ (and $\Theta_i = \theta_i$); $q_0 = 0$.

First note that the probability of the event $P^g \in \{b_{i,k}-1, b_{i,k}\}$, and the allocation given each auction price $P^g \notin \{b_{i,k} - 1, b_{i,k}\}$ are identical under $\boldsymbol{\sigma}(\boldsymbol{\Theta})$ and $\boldsymbol{\sigma}'(\boldsymbol{\Theta})$.

To show this, I split the distribution of the auction price of good g into three collectively exhaustive sets, $(P^g < b_{i,k}-1)$, $(b_{i,k} < P^g)$ and $P^g \in \{b_{i,k}-1, b_{i,k}\}$, and consider each in turn. The Product-Mix Auction uses the information from the submitted bids and the BoE's supply curves to find the competitive equilibrium, assuming bids correspond to bidders' marginal values and the supply curves represent the BoE's marginal costs of supply (see Section 2.1). The auction price of good g is the maximum of the highest losing bid on good g and the marginal cost of supply of good g at the quantity allocated (equal to 0 for good A and equal to the sum of P^A and the relative supply of B, evaluated at the quantity allocated of B).

For all signal realisations such that $P^g < b_{i,k} - 1$ under $\boldsymbol{\sigma}(\boldsymbol{\Theta})$, bidder i's deviation to a bid price of $b_{i,k} - 1$ at step k does not affect the competitive equilibrium, so the outcome is identical under the two strategy profiles, $\boldsymbol{\sigma}(\boldsymbol{\Theta})$ and $\boldsymbol{\sigma}'(\boldsymbol{\Theta})$.

Similarly, for all signal realisations such that $P^g > b_{i,k}$ under $\boldsymbol{\sigma}(\boldsymbol{\Theta})$, bidder i's deviation to a bid price of $b_{i,k} - 1$ at step k also does not affect the competitive equilibrium, so the outcome is identical under the two strategy profiles, $\boldsymbol{\sigma}(\boldsymbol{\Theta})$ and $\boldsymbol{\sigma}'(\boldsymbol{\Theta})$. Now consider signal realisations such that $P^g = b_{i,k}$ under $\boldsymbol{\sigma}(\boldsymbol{\Theta})$.

First consider $g = A$. The auction price of A is the maximum of the marginal losing bid on A and 0. If $P^A(\boldsymbol{\sigma}(\boldsymbol{\Theta})) = b_{i,k} > 0$, then bidder i's bid at step k is marginal under $\boldsymbol{\sigma}(\boldsymbol{\Theta})$. It follows that, under $\boldsymbol{\sigma}'(\boldsymbol{\Theta})$, his bid at step k is either marginal (in which case $P^A(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) = b_{i,k} - 1$) or his bid at step k is strictly unallocated (in which case $P^A(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) = b_{i,k}$). So if $P^A(\boldsymbol{\sigma}(\boldsymbol{\Theta})) = b_{i,k}$ then $P^A(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) \in \{b_{i,k} - 1, b_{i,k}\}$.

Now consider $g = B$. There are three subcases:

(i) If bidder i's deviation has no effect on the auction price of good A, i.e. $P^A(\boldsymbol{\sigma}(\boldsymbol{\Theta})) = P^A(\boldsymbol{\sigma}'(\boldsymbol{\Theta}))$, the result immediately follows: B's auction price is determined by the intersection of the aggregate bid function for good B and the difference between the relative supply of B and P^A. Bidder i's deviation has no effect on this intersection (if the quantity he demands at step k is less than the amount of unallocated demand at P^B under $\boldsymbol{\sigma}(\boldsymbol{\Theta})$) or his deviation reduces the intersection to $P^g = b_{i,k} - 1$.

(ii) If i's deviation reduces the auction price of good A, i.e. $P^A(\boldsymbol{\sigma}(\boldsymbol{\Theta})) > P^A(\boldsymbol{\sigma}'(\boldsymbol{\Theta}))$, then the allocation of good A is strictly larger under $\boldsymbol{\sigma}'(\boldsymbol{\Theta})$ than under $\boldsymbol{\sigma}(\boldsymbol{\Theta})$. (The reason is that P^A is the maximum of the marginal losing bid on A and 0. So a bid on A must have been unallocated under $\boldsymbol{\sigma}(\boldsymbol{\Theta})$ but allocated under $\boldsymbol{\sigma}'(\boldsymbol{\Theta})$.) If $P^B(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) < b_{i,k}-1$, then the quantity allocated of good B under $\boldsymbol{\sigma}'(\boldsymbol{\Theta})$ is also weakly larger than the quantity allocated of good B under $\boldsymbol{\sigma}(\boldsymbol{\Theta})$, so the total allocation across both goods strictly increases. But $P^A(\boldsymbol{\sigma}(\boldsymbol{\Theta})) > P^A(\boldsymbol{\sigma}'(\boldsymbol{\Theta}))$ implies $P^A(\boldsymbol{\sigma}(\boldsymbol{\Theta})) > 0$, which can only occur if the entire maximum supply is allocated under $\boldsymbol{\sigma}(\boldsymbol{\Theta})$. So we have a contradiction: if $P^A(\boldsymbol{\sigma}(\boldsymbol{\Theta})) > P^A(\boldsymbol{\sigma}'(\boldsymbol{\Theta}))$, it is not possible that $P^B(\boldsymbol{\sigma}'(\boldsymbol{\Theta}))) < b_{i,k} - 1$. Moreover, bidder i's deviation weakly reduces demand at each pair of auction prices (P^A, P^B) so cannot increase the price of good B. Overall, if $P^A(\boldsymbol{\sigma}(\boldsymbol{\Theta})) > P^A(\boldsymbol{\sigma}'(\boldsymbol{\Theta}))$, then $P^B(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) \in \{b_{i,k} - 1, b_{i,k}\}$.

(iii) Bidder i's deviation weakly reduces demand at each pair of auction prices (P^A, P^B), so cannot increase the auction price of good A.

It follows from these three subcases that if $P^B(\boldsymbol{\sigma}(\boldsymbol{\Theta})) = b_{i,k}$ then $P^B(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) \in \{b_{i,k}-1, b_{i,k}\}$.

For analogous reasoning, if $P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) = b_{i,k} - 1$ then $P^g(\boldsymbol{\sigma}'(\boldsymbol{\Theta})) = b_{i,k} - 1$ for $g \in \{A, B\}$. Taken together, the probability of the event $P^g \notin \{b_{i,k} - 1, b_{i,k}\}$, and the allocation of good g given each auction price $P^g \notin \{b_{i,k} - 1, b_{i,k}\}$ are identical under $\boldsymbol{\sigma}(\boldsymbol{\Theta})$ and $\boldsymbol{\sigma}'(\boldsymbol{\Theta})$.

Equilibrium therefore requires

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\Bigg[\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r^g(\boldsymbol{\sigma}(\boldsymbol{\Theta}))} v^{g,h}(u,\theta_i)du \\
&\qquad - b_{i,k}\left(q_{i,k-1}+(q_{i,k}-q_{i,k-1})r^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right)\right)\Bigg| P^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right) = b_{i,k}\Bigg]\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right) = b_{i,k}\right) \\
&+\left(\int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du - (b_{i,k}-1)q_{i,k}\right)\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right) = b_{i,k}-1\right) \\
&\geq \left(\int_0^{q_{i,k-1}} v^{g,h}(u,\theta_i)du - b_{i,k}q_{i,k-1}\right)\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}'(\boldsymbol{\Theta})\right) = b_{i,k}\right) \\
&+\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\Bigg[\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r^g(\boldsymbol{\sigma}'(\boldsymbol{\Theta}))} v^{g,h}(u,\theta_i)du \\
&\qquad - (b_{i,k}-1)\left(q_{i,k-1}+(q_{i,k}-q_{i,k-1})r^g\left(\boldsymbol{\sigma}'(\boldsymbol{\Theta})\right)\right)\Bigg| P^g\left(\boldsymbol{\sigma}'(\boldsymbol{\Theta})\right) = b_{i,k}-1\Bigg]\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}'(\boldsymbol{\Theta})\right) = b_{i,k}-1\right)
\end{aligned} \tag{22}$$

Equation 22 cannot be readily estimated as it requires evaluation of an integral with a random interval. I therefore make Assumption 3 that the rationing coefficient is deterministic at each auction price, conditional on the strategy profile. Under this assumption, the changes in the bidder's gross utility, at each auction price, conditional on their strategy, are deterministic, and can be estimated.

Under this assumption, equilibrium requires

$$
\begin{aligned}
&\left(\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r_{i,k}} v^{g,h}(u,\theta_i)du \right.\\
&\qquad \left. - b_{i,k}\left(q_{i,k-1}+(q_{i,k}-q_{i,k-1})r_{i,k}\right)\right)\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right)=b_{i,k}\right)\\
&+\left(\int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du-(b_{i,k}-1)q_{i,k}\right)\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right)=b_{i,k}-1\right)\\
&\geq\left(\int_0^{q_{i,k-1}} v^{g,h}(u,\theta_i)du-b_{i,k}q_{i,k-1}\right)\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}'(\boldsymbol{\Theta})\right)=b_{i,k}\right)\\
&+\left(\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r'_{i,k}} v^{g,h}(u,\theta_i)du\right.\\
&\qquad \left. -(b_{i,k}-1)\left(q_{i,k-1}+(q_{i,k}-q_{i,k-1})r'_{i,k}\right)\right)\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}'(\boldsymbol{\Theta})\right)=b_{i,k}-1\right) \qquad (23)
\end{aligned}
$$

G.2 Upward deviation

Equilibrium of Model $S(0)$ also requires that the deviation of bidder $i \in \mathcal{N}^{g,h}$, $g \in \{A,B\}$, $h \in \{1,2\}$ at step k to a bid price of $b_{i,k}+1$ rather than $b_{i,k}$ weakly reduces their expected utility, and so requires

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[U_i(\boldsymbol{\sigma}(\boldsymbol{\Theta})|\theta_i)\right] \geq \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[U_i(\boldsymbol{\sigma}''(\boldsymbol{\Theta})|\theta_i)\right]\\
&\implies \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\int_0^{y_i^g(\boldsymbol{\sigma}(\boldsymbol{\Theta}))} v^{g,h}(u,\theta_i)du - P^g(\boldsymbol{\sigma}(\boldsymbol{\Theta}))y_i^g(\boldsymbol{\sigma}(\boldsymbol{\Theta})) - c_i(K_i)\right]\\
&\qquad \geq \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\int_0^{y_i^g(\boldsymbol{\sigma}''(\boldsymbol{\Theta}))} v^{g,h}(u,\theta_i)du - P^g(\boldsymbol{\sigma}''(\boldsymbol{\Theta}))y_i^g(\boldsymbol{\sigma}''(\boldsymbol{\Theta})) - c_i(K_i)\right] \qquad (24)
\end{aligned}
$$

By the Law of Iterated Expectations, $\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[U_i(\boldsymbol{\sigma}''(\boldsymbol{\Theta})|\theta_i)\right]$ can be decomposed analogously to $\mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[U_i(\boldsymbol{\sigma}(\boldsymbol{\Theta})|\theta_i)\right]$ in Equation 19.

First suppose that the bid prices are non-adjacent so that $b_{i,k}+1 < b^g_{i,k-1}$. I consider the case in which $b_{i,k}+1 = b^g_{i,k-1}$ in Appendix G.3.

By analogous reasoning to the case of the deviation to $(b_{i,k}-1)$, it follows that the probability of the event $P^g \in \{b_{i,k}, b_{i,k}+1\}$, and the allocation given each auction price $P^g \notin \{b_{i,k}, b_{i,k}+1\}$ are identical under $\boldsymbol{\sigma}(\boldsymbol{\Theta})$ and $\boldsymbol{\sigma}''(\boldsymbol{\Theta})$. Equilibrium therefore requires

$$\begin{aligned}
&\left(\int_0^{q_{i,k-1}} v^{g,h}(u,\theta_i)du - (b_{i,k}+1)q_{i,k-1}\right)\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right) = b_{i,k}+1\right) \\
&+ \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r^g(\boldsymbol{\sigma}(\boldsymbol{\Theta}))} v^{g,h}(u,\theta_i)du \right.\\
&\qquad \left. - b_{i,k}\left(q_{i,k-1} + (q_{i,k}-q_{i,k-1})r^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right)\right) \middle| P^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right) = b_{i,k}\right]\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right) = b_{i,k}\right) \\
&\geq \mathbb{E}_{\boldsymbol{\Theta}_{-i}}\left[\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r^g(\boldsymbol{\sigma}''(\boldsymbol{\Theta}))} v^{g,h}(u,\theta_i)du \right.\\
&\qquad \left. - (b_{i,k}+1)\left(q_{i,k-1} + (q_{i,k}-q_{i,k-1})r^g\left(\boldsymbol{\sigma}''(\boldsymbol{\Theta})\right)\right) \middle| P^g\left(\boldsymbol{\sigma}''(\boldsymbol{\Theta})\right) = b_{i,k}+1\right]\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}''(\boldsymbol{\Theta})\right) = b_{i,k}+1\right) \\
&+ \left(\int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du - b_{i,k}q_{i,k}\right)\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}''(\boldsymbol{\Theta})\right) = b_{i,k}\right) \qquad (25)
\end{aligned}$$

Under Assumption 3 that the rationing coefficient is deterministic at each auction price, conditional on the strategy profile, equilibrium therefore requires

$$\begin{aligned}
&\left(\int_0^{q_{i,k-1}} v^{g,h}(u,\theta_i)du - (b_{i,k}+1)q_{i,k-1}\right)\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right) = b_{i,k}+1\right) \\
&+ \left(\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r_{i,k}} v^{g,h}(u,\theta_i)du \right.\\
&\qquad \left. - b_{i,k}\left(q_{i,k-1} + (q_{i,k}-q_{i,k-1})r_{i,k}\right)\right)\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}(\boldsymbol{\Theta})\right) = b_{i,k}\right) \\
&\geq \left(\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r''_{i,k}} v^{g,h}(u,\theta_i)du \right.\\
&\qquad \left. - (b_{i,k}+1)\left(q_{i,k-1} + (q_{i,k}-q_{i,k-1})r''_{i,k}\right)\right)\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}''(\boldsymbol{\Theta})\right) = b_{i,k}+1\right) \\
&+ \left(\int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du - b_{i,k}q_{i,k}\right)\mathbb{P}\left(P^g\left(\boldsymbol{\sigma}''(\boldsymbol{\Theta})\right) = b_{i,k}\right) \qquad (26)
\end{aligned}$$

G.3 Proposition 2 for adjacent bid prices

The following two conditions adapt Proposition 2 to the case in which bidder $i \in \mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$ submits bids with adjacent bid prices on good $g \in \{A, B\}$.

As in Proposition 2, the equilibrium strategy profile is $\boldsymbol{\sigma}(\boldsymbol{\Theta})$, and $\boldsymbol{\sigma}'(\boldsymbol{\Theta})$ and $\boldsymbol{\sigma}''(\boldsymbol{\Theta})$ denote the strategy profiles which are identical to $\boldsymbol{\sigma}(\boldsymbol{\Theta})$ except that the bid prices submitted by type θ_i of bidder $i \in \mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$ at step k, in their K_i-step bid function, are equal to $b_{i,k} - 1$ and $b_{i,k} + 1$, respectively.

Adjacent bid prices at steps $k-1$ and k only. In Model $S(0)$, under Assumption 3, in any type-symmetric Bayesian Nash Equilibrium, for type $\theta_i \in \Theta_i$ of bidder $i \in \mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$, every step $k \in \{1, ..., K_i\}$ in the K_i-step bid function in $\sigma_i(\theta_i)$ where $b_{i,k} - 1 = b_{i,k+1}$ and $b_{i,k} + 1 < b^g_{i,k-1}$, must satisfy

$$\begin{aligned}
&\left(\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r_{i,k}} v^{g,h}(u,\theta_i)du - b_{i,k}\left(q_{i,k-1}+(q_{i,k}-q_{i,k-1})r_{i,k}\right)\right)\mathbb{P}\left(P^g = b_{i,k}\middle|\boldsymbol{\sigma}(\boldsymbol{\Theta})\right) \\
&+\Bigg(\int_0^{q_{i,k}+(q_{i,k+1}-q_{i,k})r_{i,k}} v^{g,h}(u,\theta_i)du \\
&\qquad - (b_{i,k}-1)\left(q_{i,k}+(q_{i,k+1}-q_{i,k})r_{i,k}\right)\Bigg)\mathbb{P}\left(P^g = b_{i,k}-1\middle|\boldsymbol{\sigma}(\boldsymbol{\Theta})\right) \\
&\geq \left(\int_0^{q_{i,k-1}} v^{g,h}(u,\theta_i)du - b_{i,k}q_{i,k-1}\right)\mathbb{P}\left(P^g = b_{i,k}\middle|\boldsymbol{\sigma}'(\boldsymbol{\Theta})\right) \\
&+\Bigg(\int_0^{q_{i,k-1}+(q_{i,k+1}-q_{i,k-1})r'_{i,k}} v^{g,h}(u,\theta_i)du \\
&\qquad - (b_{i,k}-1)\left(q_{i,k-1}+(q_{i,k+1}-q_{i,k-1})r'_{i,k}\right)\Bigg)\mathbb{P}\left(P^g = b_{i,k}-1\middle|\boldsymbol{\sigma}'(\boldsymbol{\Theta})\right) \qquad (27)
\end{aligned}$$

and Equation 15, where $q^g_{i,0} = 0$.

Adjacent bid prices at steps k and $k+1$ only. In Model $S(0)$, under Assumption 3, in any type-symmetric Bayesian Nash Equilibrium, for type $\theta_i \in \Theta_i$ of bidder $i \in \mathcal{N}^{g,h}, g \in \{A, B\}, h \in \{1, 2\}$, every step $k \in \{1, ..., K_i\}$ in the K_i-step bid function in $\sigma_i(\theta_i)$ where

$b_{i,k} - 1 > b_{i,k+1}$ and $b_{i,k} + 1 = b^g_{i,k-1}$, must satisfy

$$\begin{aligned}
&\Bigg(\int_0^{q^g_{i,k-2}+(q_{i,k-1}-q^g_{i,k-2})r_{i,k}} v^{g,h}(u,\theta_i)du \\
&\qquad\qquad - (b_{i,k}+1)\left(q^g_{i,k-2} + (q_{i,k-1} - q^g_{i,k-2})r_{i,k}\right) \Bigg) \mathbb{P}\left(P^g = b_{i,k}+1 \middle| \boldsymbol{\sigma}(\boldsymbol{\Theta})\right) \\
&\quad + \Bigg(\int_0^{q_{i,k-1}+(q_{i,k}-q_{i,k-1})r_{i,k}} v^{g,h}(u,\theta_i)du \\
&\qquad\qquad - b_{i,k}\left(q_{i,k-1} + (q_{i,k} - q_{i,k-1})r_{i,k}\right) \Bigg) \mathbb{P}\left(P^g = b_{i,k} \middle| \boldsymbol{\sigma}(\boldsymbol{\Theta})\right) \\
&\quad \geq \Bigg(\int_0^{q^g_{i,k-2}+(q_{i,k}-q^g_{i,k-2})r''_{i,k}} v^{g,h}(u,\theta_i)du \\
&\qquad\qquad - (b_{i,k}+1)\left(q^g_{i,k-2} + (q_{i,k} - q^g_{i,k-2})r''_{i,k}\right) \Bigg) \mathbb{P}\left(P^g = b_{i,k}+1 \middle| \boldsymbol{\sigma}''(\boldsymbol{\Theta})\right) \\
&\quad + \left(\int_0^{q_{i,k}} v^{g,h}(u,\theta_i)du - b_{i,k}q_{i,k} \right) \mathbb{P}\left(P^g = b_{i,k} \middle| \boldsymbol{\sigma}''(\boldsymbol{\Theta})\right)
\end{aligned} \tag{28}$$

and Equation 14, where $q^g_{i,0} = q^g_{i,-1} = 0$.

Adjacent bid prices at steps $k-1$, k and $k+1$. In Model $S(0)$, under Assumption 3, in any type-symmetric Bayesian Nash Equilibrium, for type $\theta_i \in \Theta_i$ of bidder $i \in \mathcal{N}^{g,h}, g \in \{A,B\}, h \in \{1,2\}$, every step $k \in \{1,...,K_i\}$ in the K_i-step bid function in $\sigma_i(\theta_i)$ where $b_{i,k} - 1 = b_{i,k+1}$ and $b_{i,k} + 1 = b^g_{i,k-1}$, must satisfy Equations 27 and 28.

G.4 Proof of Corollary 1

At step $k = 1$ of the K_i-step bid function in $\sigma_i(\theta_i)$ of type $\theta_i \in \Theta_i$ of bidder $i \in \mathcal{N}^{g,h}, g \in \{A,B\}, h \in \{1,2\}$, Equation 14 becomes

$$\begin{aligned}
&W\,\mathbb{P}\left(P^g = b^g_{i,1} \middle| \sigma_i(\theta_i)\right) + X\,\mathbb{P}\left(P^g = b^g_{i,1} - 1 \middle| \boldsymbol{\sigma}(\boldsymbol{\Theta})\right) \\
&\quad \geq Y\,\mathbb{P}\left(P^g = b^g_{i,1} \middle| \sigma'_i\right) + Z\,\mathbb{P}\left(P^g = b^g_{i,1} - 1 \middle| \boldsymbol{\sigma}'(\boldsymbol{\Theta})\right)
\end{aligned} \tag{29}$$

where

$$
\begin{aligned}
W &:= \left(\int_0^{q_{i,1}^g r_{i,1}} v^{g,h}(u,\theta_i)du - b_{i,1}^g q_{i,1}^g r_{i,1} \right) \\
X &:= \left(\int_0^{q_{i,1}^g} v^{g,h}(u,\theta_i)du - (b_{i,1}^g - 1) q_{i,1}^g \right) \\
Y &:= 0 \\
Z &:= \left(\int_0^{q_{i,1}^g r'_{i,1}} v^{g,h}(u,\theta_i)du - (b_{i,1}^g - 1) q_{i,1}^g r'_{i,1} \right)
\end{aligned}
$$

Suppose $v^{g,h}(0,\theta_i) < b_{i,1}^g - 1$. Because the bidder's marginal valuation function is monotonic (see Section 3.1), it follows that the bidder's marginal valuation function for good g is everywhere below $b_{i,1}^g - 1$. So, except in the knife-edge case in which $r_{i,1} = 0$, then $W < Y = 0$; and $X \leq Z \leq 0$.

As discussed in Appendix B.2,

$$
\mathbb{P}\left(P^g = b_{i,1}^g \middle| \boldsymbol{\sigma}(\boldsymbol{\Theta})\right) \geq \mathbb{P}\left(P^g = b_{i,1}^g \middle| \boldsymbol{\sigma}'(\boldsymbol{\Theta})\right) \geq 0 \tag{30}
$$

$$
0 \leq \mathbb{P}\left(P^g = b_{i,1}^g - 1 \middle| \boldsymbol{\sigma}(\boldsymbol{\Theta})\right) \leq \mathbb{P}\left(P^g = b_{i,1}^g - 1 \middle| \boldsymbol{\sigma}'(\boldsymbol{\Theta})\right) \tag{31}
$$

If $\mathbb{P}\left(P^g = b_{i,1}^g \middle| \boldsymbol{\sigma}(\boldsymbol{\Theta})\right) > 0$, then $W\,\mathbb{P}\left(P^g = b_{i,1}^g \middle| \boldsymbol{\sigma}(\boldsymbol{\Theta})\right) < Y\,\mathbb{P}\left(P^g = b_{i,1}^g \middle| \boldsymbol{\sigma}'(\boldsymbol{\Theta})\right)$.

If $\mathbb{P}\left(P^g = b_{i,1}^g - 1 \middle| \boldsymbol{\sigma}'(\boldsymbol{\Theta})\right) > 0$, then $X\,\mathbb{P}\left(P^g = b_{i,1}^g - 1 \middle| \boldsymbol{\sigma}(\boldsymbol{\Theta})\right) < Z\,\mathbb{P}\left(P^g = b_{i,1}^g - 1 \middle| \boldsymbol{\sigma}'(\boldsymbol{\Theta})\right)$.

So, providing $\mathbb{P}\left(P^g = b_{i,1}^g \middle| \boldsymbol{\sigma}(\boldsymbol{\Theta})\right) > 0$ or $\mathbb{P}\left(P^g = b_{i,1}^g - 1 \middle| \boldsymbol{\sigma}'(\boldsymbol{\Theta})\right) > 0$, and $r_{i,1} \neq 0$, these inequalities contradict Equation 29.